APPENDIX TO THE PAPER: PLANE CURVES WHICH ARE QUANTUM HOMOGENEOUS SPACES

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1. Introduction

This appendix contains detailed proofs of the cases (ii) $t = 2$ and (iii) $t = 3$ of Proposition 2.8 of [1]. For all notation and motivation, see in particular [1, §2.3]. All references to lemmas, etc are to [1].

2. The case $t = 2$.

This section contains the proof of [1, Proposition 2.8] when we assume hypothesis (ii), that $t = 2$. Since $t = 2$ we may assume that $n \geq 4$. The issue is to resolve the ambiguities

$$(\sigma_{j+2}, \sigma_j, a^2, a^j x^{n-j-2}, x^2),$$

in the three cases: (a) $j = 1$, (b) $j = n-3$, and (c) $1 < j < n-3$. We use the following identities throughout the proofs:

$$(1) \quad P(r,s) = P(r-1,s-1)xa + P(r-1,s-1)ax + P(r-2,s)a^2 + P(r,s-2)x^2,$$

$$(2) \quad P(r,s) = xaP(r-1,s-1) + axP(r-1,s-1) + a^2P(r-2,s) + x^2P(r,s-2),$$

$$(3) \quad P(r,s) = xP(r-1,s-1)a + aP(r-1,s-1)x + aP(r-2,s)a + xP(r,s-2)x.$$

2.1. **Case (a): $j = 1$.** For $j = 1$, the ambiguity arises from the two routes to resolve the word

$$a^2\omega_1 = a^2(ax^{n-1}) = (a^3x^{n-3})x^2 = \omega_3 x^2$$

in the free algebra $k\langle a, x\rangle$ using the relations σ_1 and σ_3. Considering first σ_1, use (1) to write it as

$$\omega_1 = ax^{n-1} \to -(\sum_{i=2}^{n-1} r_i(P(0,i-2)ax + P(0,i-2)xa) + \sum_{i=3}^{n-1} r_iP(1,i-3)x^2 +$$

$$Q(1,n-3)x^2 + P(0,n-2)ax + P(0,n-2)xa + r_1a) + r_1a^n$$

$$= -(\sum_{i=2}^{n} r_i(P(0,i-2)ax + P(0,i-2)xa) + \sum_{i=3}^{n-1} r_iP(1,i-3)x^2 + Q(1,n-3)x^2 + r_1a) + r_1a^n.$$

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Premultiply this by a^2, and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yielding
(α)

$$a^2\omega_1 \to -(\sum_{i=2}^{n} r_i(a^2P(0,i-2)ax+a^2P(0,i-2)xa)+\sum_{i=3}^{n-1} r_i a^2P(1,i-3)x^2+\underline{a^2Q(1,n-3)ax}+r_1\underline{a^3})+r_1\underline{a^{n+2}}.$$

The following words in (α) of length $n+2$ are reducible:

$$-a^2x^{n-2}ax \qquad \text{and} \qquad -a^2x^{n-2}xa.$$

To reduce $-a^2x^{n-2}ax$, first, using (2), we write σ_2 as

$$-a^2x^{n-2} \to \sum_{i=3}^{n-1} r_i(\underline{xaP(1,i-3)} + \underline{axP(1,i-3)}) + \sum_{i=4}^{n-1} r_i\underline{x^2P(2,i-4)} + \sum_{i=2}^{n-1} r_i\underline{a^2P(0,i-2)}$$

$$+\underline{xaP(1,n-3)} + \underline{axP(1,n-3)} + \underline{x^2P(2,n-4)} - r_2\underline{a^n}$$

$$= \sum_{i=3}^{n} r_i(\underline{xaP(1,i-3)} + \underline{axP(1,i-3)}) + \sum_{i=4}^{n} r_i\underline{x^2P(2,i-4)} + \sum_{i=2}^{n-1} r_i\underline{a^2P(0,i-2)} - r_2\underline{a^n}.$$

Post multiply this by ax and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words, noting also that a^n is central in $A_0(x,a,g)$. This yields

$$(\beta) \qquad -a^2x^{n-2}ax \to \sum_{i=3}^{n} r_i(\underline{xaP(1,i-3)ax} + \underline{axP(1,i-3)ax}) + \sum_{i=4}^{n} r_i\underline{x^2P(2,i-4)ax} +$$

$$\sum_{i=2}^{n-1} r_i\underline{a^2P(0,i-2)ax} - r_2\underline{axa^n}$$

Similarly, for $-a^2x^{n-2}xa$, post multiplying the relation for $-a^2x^{n-2}$ above with xa and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$(\tau) \qquad -a^2x^{n-2}xa \to \sum_{i=3}^{n} r_i(xaP(1,i-3)xa + \underline{axP(1,i-3)xa}) + \sum_{i=4}^{n} r_i\underline{x^2P(2,i-4)xa} +$$

$$\sum_{i=2}^{n-1} r_i a^2P(0,i-2)xa - r_2\underline{xa^{n+1}}.$$

To reduce the word $xa^2x^{n-2}a$ in (τ), note first that using (3), the left hand side of σ_2 reduces to

$$a^2x^{n-2} \to -(\sum_{i=2}^{n-1} r_i\underline{aP(0,i-2)a} + \sum_{i=3}^{n-1} r_i(\underline{xP(1,i-3)a} + \underline{aP(1,i-3)x}) + \sum_{i=4}^{n-1} r_i\underline{xP(2,i-4)x}$$

$$+\underline{xP(1,n-3)a} + \underline{aQ(1,n-3)x} + \underline{xP(2,n-4)x} + \underline{aP(0,n-2)a}) + r_2\underline{a^n}.$$

$$= -(\sum_{i=2}^{n} r_i\underline{aP(0,i-2)a} + \sum_{i=3}^{n-1} r_i(\underline{xP(1,i-3)a} + \underline{aP(1,i-3)x}) + \sum_{i=4}^{n} r_i\underline{xP(2,i-4)x}$$

$$+\underline{xP(1,n-3)a} + \underline{aQ(1,n-3)x}) + r_2\underline{a^n}.$$

Pre and post multiplying this by x and a respectively and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$(\gamma) \qquad xa^2x^{n-2}a \to -(\sum_{i=2}^{n} r_i\underline{xaP(0,i-2)a^2} + \sum_{i=3}^{n-1} r_i(\underline{x^2P(1,i-3)a^2} + \underline{xaP(1,i-3)xa})$$

$$+\sum_{i=4}^{n} r_i\underline{x^2P(2,i-4)xa} + \underline{x^2P(1,n-3)a^2} + \underline{xaQ(1,n-3)xa}) + r_2\underline{xa^{n+1}}.$$

Thus, the reduction process ends here. Substituting (γ) into (τ), and then with (β) into (α), and simplifying, yields
(χ)

$$a^2\omega_1 \to (\sum_{i=3}^{n} r_i(\underline{xaP(1,i-3)ax} + \underline{axP(1,i-3)ax} + \underline{axP(1,n-3)xa}) + \sum_{i=4}^{n} r_i\underline{x^2P(2,i-4)ax} + r_1\underline{a^{n+2}}) - (\sum_{i=2}^{n} r_i\underline{xaP(0,i-2)a^2} + \sum_{i=3}^{n-1} r_i a^2P(1,i-3)x^2 + \sum_{i=3}^{n} r_i\underline{x^2P(1,i-3)a^2} + \underline{a^2Q(1,n-3)x^2} + r_1\underline{a^3} + r_2\underline{axa^n}).$$

Turning now to $\omega_3 x^2$, use (2) to expand the right hand side of σ_3. This yields

$$\omega_3 \to -(\sum_{i=3}^{n-1} r_i\underline{a^2P(1,i-3)} + \underline{a^2Q(1,n-3)} + \sum_{i=4}^{n-1} r_i(\underline{xaP(2,i-4)} + \underline{axP(2,i-4)}) + \underline{xaP(2,n-4)} + \underline{axP(2,n-4)} + \sum_{i=5}^{n-1} r_i\underline{x^2P(3,i-5)} + \underline{x^2P(3,n-5)}) + r_3\underline{a^n}$$

$$= -(\sum_{i=3}^{n-1} r_i\underline{a^2P(1,i-3)} + \underline{a^2Q(1,n-3)} + \sum_{i=4}^{n} r_i(\underline{xaP(2,i-4)} + \underline{axP(2,i-4)}) + \sum_{i=5}^{n} r_i\underline{x^2P(3,i-5)}) + r_3\underline{a^n}.$$

When we post multiply this by x^2 and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words, we obtain

$$(\alpha') \quad \omega_3 x^2 \to -(\sum_{i=3}^{n-1} r_i a^2P(1,i-3)x^2 + \underline{a^2Q(1,n-3)x^2} + \sum_{i=4}^{n} r_i(xaP(2,i-4)x^2 + axP(2,i-4)x^2) + \sum_{i=5}^{n} r_i x^2P(3,i-5)x^2) + r_3\underline{x^2a^n}.$$

We find the following reducible words of length $n+2$ in (α'), from $xaP(2,n-4)x^2$, $axP(2,n-4)x^2$ and $x^2P(3,n-5)x^2$ respectively:

$$xa^3x^{n-2}, \quad axa^2x^{n-2}, \quad x^2a^3x^{n-3}$$

To deal with these, first use (1), to write σ_2 as

$$a^2x^{n-2} \to -(\sum_{i=2}^{n-1} r_i\underline{P(0,i-2)a^2} + \sum_{i=3}^{n-1} r_i(\underline{P(1,i-3)ax} + \underline{P(1,i-3)xa}) + \sum_{i=4}^{n-1} r_i\underline{P(2,i-4)x^2} + \underline{Q(2,n-4)x^2} + \underline{P(0,n-2)a^2} + \underline{P(1,n-3)ax} + \underline{P(1,n-3)xa}) + r_2\underline{a^n}$$

$$= -(\sum_{i=2}^{n} r_i\underline{P(0,i-2)a^2} + \sum_{i=3}^{n} r_i(\underline{P(1,i-3)ax} + \underline{P(1,i-3)xa}) + \sum_{i=4}^{n-1} r_i\underline{P(2,i-4)x^2} + \underline{Q(2,n-4)x^2}) + r_2\underline{a^n}.$$

Premultiplying this by xa and ax and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields respectively

$$(\beta') \qquad -xaa^2x^{n-2} \to (\sum_{i=2}^{n} r_i\underline{xaP(0,i-2)a^2} + \sum_{i=3}^{n} r_i(\underline{xaP(1,i-3)ax} + xaP(1,i-3)xa) +$$

$$\sum_{i=4}^{n-1} r_i xaP(2,i-4)x^2 + \underline{xaQ(2,n-4)x^2}) - r_2\underline{xa^{n+1}}$$

and

$$(\gamma') \qquad -axa^2x^{n-2} \to (\sum_{i=2}^{n} r_i axP(0,i-2)a^2 + \sum_{i=3}^{n} r_i(\underline{axP(1,i-3)ax} + \underline{axP(1,i-3)xa})+$$

$$\sum_{i=4}^{n-1} r_i\underline{axP(2,i-4)x^2} + \underline{axQ(2,n-4)x^2}) - r_2\underline{axa^n}.$$

Recall from (γ) above that the reducible word $xa^2x^{n-2}a$ occurring in $(\beta')$is reduced :by

$$(\eta') \qquad xa^2x^{n-2}a \to -(\sum_{i=3}^{n-1} r_i(\underline{x^2P(1,i-3)a^2} + \underline{xaP(1,i-3)xa}) + \sum_{i=4}^{n} \underline{x^2P(2,i-4)xa}+$$

$$\sum_{i=2}^{n} \underline{xaP(0,i-2)a^2} + \underline{x^2P(1,n-3)a^2} + \underline{xaQ(1,n-3)xa}) + r_2\underline{xa^{n+1}}.$$

To deal with the reducible word $ax^{n-1}a^2$ in (γ'), express σ_1 using (1) as

$$ax^{n-1} \to -(\sum_{i=2}^{n-1} r_i(\underline{axP(0,i-2)} + \underline{xaP(0,i-2)}) + \underline{xaP(0,n-2)} + \sum_{i=3}^{n-1} r_i\underline{x^2P(1,i-3)}+$$

$$\underline{x^2P(1,n-3)} + r_1\underline{a}) + r_1\underline{a^n}.$$

Therefore

$$(\tau') \qquad ax^{n-1}a^2 \to -(\sum_{i=2}^{n-1} r_i(\underline{axP(0,i-2)a^2} + \underline{xaP(0,i-2)a^2}) + \underline{xaP(0,n-2)a^2}+$$

$$\sum_{i=3}^{n} r_i\underline{x^2P(1,i-3)a^2} + r_1\underline{a^3}) + r_1\underline{a^{n+2}}.$$

Using (1), σ_3 can be stated as

$$a^3x^{n-3} \to -(\sum_{i=4}^{n-1} r_i\underline{P(1,i-3)a^2} + \sum_{i=4}^{n-1} r_i(\underline{P(2,i-4)ax} + \underline{P(2,i-4)xa}) + \sum_{i=5}^{n-1} r_i\underline{P(3,i-5)x^2}$$

$$+\underline{Q(3,n-5)x^2} + \underline{P(1,n-3)a^2} + \underline{P(2,n-4)ax} + \underline{P(2,n-4)xa} + r_3\underline{a^3}) + r_3\underline{a^n}$$

$$= -(\sum_{i=4}^{n} r_i\underline{P(1,i-3)a^2} + \sum_{i=4}^{n} r_i(\underline{P(2,i-4)ax} + \underline{P(2,i-4)xa}) + \sum_{i=5}^{n-1} r_i\underline{P(3,i-5)x^2}$$

$$+\underline{Q(3,n-5)x^2} + r_3\underline{a^3}) + r_3\underline{a^n}.$$

Premultiplying this by x^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$(\chi') \qquad -x^2a^3x^{n-3} \to (\sum_{i=4}^{n} r_i\underline{x^2P(1,i-3)a^2} + \sum_{i=4}^{n} r_i(\underline{x^2P(2,i-4)ax} + \underline{x^2P(2,i-4)xa})+$$

$$\sum_{i=5}^{n-1} r_i\underline{x^2P(3,i-5)x^2} + \underline{x^2Q(3,n-5)x^2} + r_3\underline{x^2a^3}) + r_3\underline{x^2a^n}.$$

Thus the reduction process ends here. Substituting (β'), (γ'), (η'), (τ') and (χ') into (α') and simplifying yields
(ξ')

$$\omega_3 x^2 \to (\sum_{i=3}^{n} r_i(\underline{xaP(1,i-3)ax} + \underline{axP(1,i-3)ax} + \underline{axP(1,n-3)xa}) + \sum_{i=4}^{n} r_i \underline{x^2P(2,i-4)ax} +$$

$$r_1\underline{a^{n+2}}) - (\sum_{i=2}^{n} r_i \underline{xaP(0,i-2)a^2} + \sum_{i=3}^{n-1} r_i a^2 P(1,i-3)x^2 + \sum_{i=3}^{n} r_i \underline{x^2P(1,i-3)a^2} +$$

$$\underline{a^2Q(1,n-3)x^2} + r_1\underline{a^3} + r_2\underline{axa^n}).$$

Comparing (χ) and (ξ') we conclude that the overlap ambiguity $(\sigma_3, \sigma_1, a^2, ax^{n-3}, x^2)$ is resolvable.

2.2. **Case (b):** $j = n-3$**.** We consider here the overlap ambiguity $(\sigma_{n-3}, \sigma_{n-1}, a^2, a^{n-3}x, x^2)$. That is, we resolve the two ways to reduce the overlap ambiguity

$$a^2\omega_{n-3} = a^2(a^{n-3}x^3) = (a^{n-1}x)x^2 = \omega_{n-1}x^2.$$

We may assume without loss of generality that $n \geq 5$ since we have dealt with $j = 1 = 4-3$ in Proposition 2.8(i). Using (1), we expand the right hand side of σ_{n-3} as

$$a^{n-3}x^3 \to -(\sum_{i=n-3}^{n-1} r_i \underline{P(n-5, i-(n-3))a^2}$$

$$+ \sum_{i=n-2}^{n-1} r_i(\underline{P(n-4, i-(n-2))xa} + \underline{P(n-4, i-(n-2))ax})$$

$$+r_{n-1}\underline{P(n-3,0)x^2} + \underline{Q(n-3,1)x^2} + \underline{P(n-5,3)a^2} + \underline{P(n-4,2)xa} + \underline{P(n-4,2)ax}) + r_{n-3}\underline{a^n}$$

$$= -(\sum_{i=n-3}^{n} r_i \underline{P(n-5, i-(n-3))a^2} + \sum_{i=n-2}^{n} r_i(\underline{P(n-4, i-(n-2))xa} + \underline{P(n-4, i-(n-2))ax})$$

$$+r_{n-1}\underline{P(n-3,0)x^2} + \underline{Q(n-3,1)x^2}) + r_{n-3}\underline{a^n}.$$

Premultiplying this by a^2 and separating reducible and irreducible words using Lemmas 2.6 and 2.7, yields

(I) $$a^2\omega_{n-3} \to -(\sum_{i=n-3}^{n} r_i a^2 P(n-5, i-(n-3))a^2 + \sum_{i=n-2}^{n} r_i(a^2P(n-4, i-(n-2))xa +$$

$$a^2P(n-4, i-(n-2))ax) + r_{n-1}a^2P(n-3,0)x^2 + \underline{a^2Q(n-3,1)x^2}) + r_{n-3}\underline{a^{n+2}}.$$

The reducible words of length $n+2$ above are :

$$a^{n-3}x^3a^2, \qquad a^{n-2}x^2ax, \qquad a^{n-2}x^2xa,$$

from $a^2P(n-5,3)a^2$, $a^2P(n-4,2)ax$ and $a^2P(n-4,2)xa$ respectively.

To deal with $a^{n-3}x^3a^2$ use (2) to expand the right hand side of σ_{n-3} as

$$a^{n-3}x^3 \to -(\sum_{i=n-3}^{n-1} r_i \underline{a^2P(n-5, i-(n-3))} + \sum_{i=n-2}^{n-1} r_i(\underline{axP(n-4, i-(n-2))} + \underline{xaP(n-4, i-(n-2))})$$

$$+r_{n-1}\underline{x^2P(n-3,0)} + \underline{a^2Q(n-5,3)} + \underline{axP(n-4,2)} + \underline{xaP(n-4,2)} + \underline{x^2P(n-3,1)}) + r_{n-3}\underline{a^n}$$

$$= -(\sum_{i=n-3}^{n-1} r_i \underline{a^2P(n-5,i-(n-3))} + \sum_{i=n-2}^{n} r_i(\underline{axP(n-4,i-(n-2))} + \underline{xaP(n-4,i-(n-2))})$$

$$+\sum_{i=n-1}^{n} r_i \underline{x^2P(n-3,i-(n-1))} + \underline{a^2Q(n-5,3)}) + r_{n-3}\underline{a^n}.$$

Post multiplying this by a^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$\textbf{(II)} \quad -a^{n-3}x^3a^2 \to (\sum_{i=n-3}^{n-1} r_i \underline{a^2P(n-5,i-(n-3))a^2} + \sum_{i=n-2}^{n} r_i(\underline{axP(n-4,i-(n-2))a^2} +$$

$$\underline{xaP(n-4,i-(n-2))a^2}) + \sum_{i=n-1}^{n} r_i \underline{x^2P(n-3,i-(n-1))a^2} + \underline{a^2Q(n-5,3)a^2}) - r_{n-3}\underline{a^{n+2}}.$$

For $a^{n-2}x^2ax$ and $a^{n-2}x^2xa$, using (2), the right hand side of σ_{n-2} becomes

$$a^{n-2}x^2 \to -(\sum_{i=n-2}^{n-1} r_i \underline{a^2P(n-4,i-(n-2))} + r_{n-1}(\underline{xaP(n-3,i-(n-1))} + \underline{axP(n-3,i-(n-1))})$$

$$+\underline{xaP(n-3,1)} + \underline{axP(n-3,1)} + \underline{x^2P(n-2,0)} + \underline{a^2Q(n-4,2)}) + r_{n-2}\underline{a^n}.$$

Postmultiplying this by (respectively) ax and xa and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$\textbf{(III)} \quad -a^{n-2}x^2ax \to (\sum_{i=n-2}^{n-1} r_i a^2P(n-4,i-(n-2))ax + \sum_{i=n-1}^{n} r_i(xaP(n-3,i-(n-1))ax$$

$$+\underline{axP(n-3,i-(n-1))ax}) + x^2P(n-2,0)ax + \underline{a^2Q(n-4,2)ax}) - r_{n-2}\underline{axa^n},$$

and

$$\textbf{(IV)} \quad -a^{n-2}x^2xa \to (\sum_{i=n-2}^{n-1} r_i a^2P(n-4,i-(n-2))xa + \sum_{i=n-1}^{n} r_i(xaP(n-3,i-(n-1))xa$$

$$+\underline{axP(n-3,i-(n-1))xa}) + \underline{x^2P(n-2,0)xa} + \underline{a^2Q(n-4,2)xa}) - r_{n-2}\underline{xa^{n+1}}.$$

To handle the reducible word $x^2a^{n-1}x$ in (**III**), using (1), the right hand side of σ_{n-1} becomes

$$a^{n-1}x \to -\Big(\underline{P(n-2,0)xa} + \underline{P(n-3,1)a^2} + r_{n-1}\underline{a^{n-1}}\Big) + r_{n-1}\underline{a^n}.$$

Premultiplying this by x^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$\textbf{(V)} \qquad x^2a^{n-1}x \to -\Big(\underline{x^2P(n-2,0)xa} + \underline{x^2P(n-3,1)a^2} + r_{n-1}\underline{x^2a^{n-1}}\Big) + r_{n-1}\underline{x^2a^n}.$$

Returning to (**IV**), we reduce the word $xa^{n-2}x^2a \in xaP(n-3,1)xa$ as follows. Use (3) to write the right hand side of σ_{n-2} as

$$a^{n-2}x^2 \to -(\sum_{i=n-2}^{n-1} r_i \underline{aP(n-4,i-(n-2))a} + r_{n-1}(\underline{aP(n-3,0)x} + \underline{xP(n-3,0)a})$$

$$+\underline{xP(n-2,0)x} + \underline{aQ(n-3,1)x} + \underline{aP(n-4,2)a} + \underline{xP(n-3,1)a}) + r_{n-2}\underline{a^n}$$

$$= -(\sum_{i=n-2}^{n} r_i \underline{aP(n-4,i-(n-2))a} + r_{n-1}\underline{aP(n-3,0)x} + \sum_{i=n-1}^{n} r_i \underline{xP(n-3,i-(n-1))a}$$

$$+\underline{xP(n-2,0)x} + \underline{aQ(n-3,1)x}) + r_{n-2}\underline{a^n}.$$

Therefore, premultiplying by x and postmultiplying by a, while using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

(VI) $$xa^{n-2}x^2a \to -(\sum_{i=n-2}^{n} r_i\underline{xaP(n-4,i-(n-2))a^2} + r_{n-1}\underline{xaP(n-3,0)xa}+$$

$$\sum_{i=n-1}^{n} r_i\underline{x^2P(n-3,i-(n-1))a^2} + \underline{x^2P(n-2,0)xa} + \underline{xaQ(n-3,1)xa}) + r_{n-2}\underline{xa^{n+1}}.$$

The reduction process stops here and we get the following after assembling (**I**), (**II**), (**III**), (**IV**), (**V**), (**VI**) and simplifying:

(VII)

$$a^2\omega_{n-3} \to (\sum_{i=n-1}^{n} r_i(xaP(n-3,i-(n-1))ax + \underline{axP(n-3,i-(n-1))ax} + \underline{axP(n-3,i-(n-1))xa})$$

$$+\sum_{i=n-2}^{n} r_i\underline{axP(n-4,i-(n-2))a^2} + r_{n-1}\underline{x^2a^n}) - (r_{n-2}\underline{axa^n} + r_{n-1}a^2P(n-3,0)x^2+$$

$$\underline{a^2Q(n-3,1)x^2} + \underline{x^2P(n-2,0)xa} + \sum_{i=n-1}^{n} r_i\underline{x^2P(n-3,i-(n-1))a^2}).$$

Now, consider the alternate grouping of the overlap ambiguity, namely $\omega_{n-1}x^2 = (a^{n-1}x)x^2$. Using (2), the words on the right hand side of σ_{n-1} are written as

$$\omega_{n-1} \to -\left(\underline{a^2Q(n-3,1)} + \underline{axP(n-2,0)} + \underline{xaP(n-2,0)} + r_{n-1}\underline{a^{n-1}}\right) + r_{n-1}\underline{a^n}.$$

Postmultiplying this by x^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

(I')

$$\omega_{n-1}x^2 \to -\left(\underline{a^2Q(n-3,1)x^2} + axP(n-2,0)x^2 + xaP(n-2,0)x^2 + r_{n-1}a^{n-1}x^2\right) + r_{n-1}\underline{x^2a^n}.$$

The reducible words of length $n+2$ above are

$$axa^{n-2}x^2 \qquad \text{and} \qquad xaa^{n-2}x^2.$$

To handle these, use (1) to expand the right side of σ_{n-2} as

$$a^{n-2}x^2 \to -(r_{n-2}a^{n-2} + r_{n-1}P(n-4,1)a^2 + r_{n-1}a^{n-2}x$$

$$+r_{n-1}a^{n-3}xa + P(n-4,2)a^2 + P(n-3,1)ax + P(n-3,1)xa) + r_{n-2}a^n.$$

Premultiplying this by ax and xa gives, respectively,

(II') $$-axa^{n-2}x^2 \to r_{n-2}\underline{axa^{n-2}} + r_{n-1}\underline{axP(n-4,1)a^2} + r_{n-1}\underline{axa^{n-2}x}$$

$$+r_{n-1}\underline{axa^{n-3}xa} + \underline{axP(n-4,2)a^2} + \underline{axP(n-3,1)ax} + \underline{axP(n-3,1)xa} - r_{n-2}\underline{axa^n},$$

and

(III') $$-xaa^{n-2}x^2 \to r_{n-2}\underline{xaa^{n-2}} + r_{n-1}\underline{xaP(n-4,1)a^2} + r_{n-1}xaa^{n-2}x$$

$$+r_{n-1}\underline{xaa^{n-3}xa} + \underline{xaP(n-4,2)a^2} + \underline{xaP(n-3,1)ax} + xaP(n-3,1)xa - r_{n-2}\underline{xaa^n}.$$

In (**III'**), the reducible word $xa^{n-2}x^2a \in xaP(n-3,1)xa$ can be reduced using (**VI**). Thus, the reduction process ends here and assembling (**I'**), (**II'**), (**III'**) and (**VI**), we obtain
(**IV'**)

$$\omega_{n-1}x^2 \to \Big(\sum_{i=n-1}^{n} r_i(xaP(n-3,i-(n-1))ax+\underline{axP(n-3,i-(n-1))ax}+\underline{axP(n-3,i-(n-1))xa})$$

$$+\sum_{i=n-2}^{n} r_i\underline{axP(n-4,i-(n-2))a^2}+r_{n-1}\underline{x^2a^n}\Big)-\big(r_{n-2}\underline{axa^n}+r_{n-1}a^2P(n-3,0)x^2+$$

$$\underline{a^2Q(n-3,1)x^2}+\underline{x^2P(n-2,0)xa}+\sum_{i=n-1}^{n} r_i\underline{x^2P(n-3,i-(n-1))a^2}\big).$$

Therefore, comparing (**VII**) and (**IV'**), we conclude that the overlap ambiguity $(\sigma_{n-3},\sigma_{n-1},a^2,a^{n-3}x,x^2)$ is resolvable.

2.3. **Case (c):** $1<j<n-3$. In this case, $n\geq 6$. Consider the overlap ambiguity $(\sigma_j,\sigma_{j+2},a^2,a^jx^{n-(j+2)},x^2)$. We use (1) to write the right hand side of σ_j as

$$\begin{aligned}\omega_j \quad \to \quad & -\Big(\sum_{i=j}^{n-1} r_i\underline{P(j-2,i-j)a^2}+\sum_{i=j+1}^{n-1} r_i\big(\underline{P(j-1,i-j-1)ax}\\
&+\underline{P(j-1,i-j-1)xa}\big)+\sum_{i=j+2}^{n-1} r_i\underline{P(j,i-j-2)x^2}+\underline{Q(j,n-j-2)x^2}\\
&+\underline{P(j-2,n-j)a^2}+\underline{P(j-1,n-j-1)ax}+\underline{P(j-1,n-j-1)xa}\Big)+r_j\underline{a^n}\end{aligned}$$

$$=-\Big(\sum_{i=j}^{n} r_i\underline{P(j-2,i-j)a^2}+\sum_{i=j+1}^{n} r_i\Big(\underline{P(j-1,i-j-1)ax}+\underline{P(j-1,i-j-1)xa}\Big)+$$

$$\sum_{i=j+2}^{n-1} r_i\underline{P(j,i-j-2)x^2}+\underline{Q(j,n-j-2)x^2}\Big)+r_j\underline{a^n}$$

Premultiplying this by a^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields
(**A**)

$$a^2\omega_j \to -\Big(\sum_{i=j}^{n} r_ia^2P(j-2,i-j)a^2+\sum_{i=j+1}^{n} r_i\left(a^2P(j-1,i-j-1)ax+a^2P(j-1,i-j-1)xa\right)$$

$$+\sum_{i=j+2}^{n-1} r_ia^2P(j,i-j-2)x^2+\underline{a^2Q(j,n-j-2)x^2}\Big)+r_j\underline{a^{n+2}}$$

This produces the following reducible words of length $n+2$ from (**A**):

(**B**) $$a^{j+1}x^{n-j-1}ax,\ a^{j+1}x^{n-j-1}xa,\ a^jx^{n-j}a^2,$$

respectively from $a^2P(j-1,n-j-1)ax$, $a^2P(j-1,n-j-1)xa$ and $a^2P(j-2,n-j)a^2$. The second of these words, $a^{j+1}x^{n-j-1}xa$, can be bracketed as $(a^{j+1}x^{n-j-1})xa$ or as $a(a^jx^{n-j})a$. So it involves an overlap ambiguity, but for $t=1$ and this overlap ambiguity has been resolved in [1, Proposition 2.8(i)]. Thus resolving via either relation will lead to the same result.

We treat the reducible words in (**B**) in turn, starting with $a^{j+1}x^{n-j-1}ax$ and $a^{j+1}x^{n-j-1}xa$. Expand σ_{j+1} using (2) as

$$a^{j+1}x^{n-j-1} \to -\Big(\sum_{i=j+1}^{n-1} r_i \underline{a^2P(j-1,i-j-1)} + \underline{a^2Q(j-1,n-j-1)}$$

$$+\sum_{i=j+2}^{n-1} r_i \left(\underline{axP(j,i-j-2)} + \underline{xaP(j,i-j-2)}\right) + \underline{axP(j,n-j-2)} + \underline{xaP(j,n-j-2)}$$

$$+\sum_{i=j+3}^{n-1} r_i \underline{x^2P(j+1,i-j-3)} + \underline{x^2P(j+1,n-j-3)}\Big) + r_{j+1}\underline{a^n}$$

$$= -\Big(\sum_{i=j+1}^{n-1} r_i \underline{a^2P(j-1,i-j-1)} + \underline{a^2Q(j-1,n-j-1)}+$$

$$\sum_{i=j+2}^{n} r_i \left(\underline{axP(j,i-j-2)} + \underline{xaP(j,i-j-2)}\right) + \sum_{i=j+3}^{n} r_i \underline{x^2P(j+1,i-j-3)}\Big) + r_{j+1}\underline{a^n}.$$

Postmultiplying this by ax and xa and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields the following two reductions,(**C**) and (**D**):

(**C**) $$-a^{j+1}x^{n-j-1}ax \to \Big(\sum_{i=j+1}^{n-1} r_i \underline{a^2P(j-1,i-j-1)ax} + \underline{a^2Q(j-1,n-j-1)ax}$$

$$+\sum_{i=j+2}^{n} r_i \left(\underline{axP(j,i-j-2)ax} + \underline{xaP(j,i-j-2)ax}\right) + \sum_{i=j+3}^{n} r_i \underline{x^2P(j+1,i-j-3)ax}\Big) - r_{j+1}\underline{axa^n};$$

and

(**D**) $$-a^{j+1}x^{n-j-1}xa \to \Big(\sum_{i=j+1}^{n-1} r_i \underline{a^2P(j-1,i-j-1)xa} + \underline{a^2Q(j-1,n-j-1)xa}+$$

$$\sum_{i=j+2}^{n} r_i \left(\underline{axP(j,i-j-2)xa} + xaP(j,i-j-2)xa\right) + \sum_{i=j+3}^{n} r_i \underline{x^2P(j+1,i-j-3)xa}\Big) - r_{j+1}\underline{xa^{n+1}}.$$

To deal with the reduction of $xa^{j+1}x^{n-j-1}a$ in (**D**), use (3) to expand σ_{j+1} as

$$a^{j+1}x^{n-j-1} \to -\Big(\sum_{i=j+1}^{n-1} r_i \underline{aP(j-1,i-j-1)a} + \underline{aP(j-1,n-j-1)a} + \sum_{i=j+2}^{n-1} r_i \underline{aP(j,i-j-2)x}+$$

$$\underline{aQ(j,n-j-2)x} + \sum_{i=j+2}^{n-1} r_i \underline{xP(j,i-j-2)a} + \underline{xP(j,n-j-2)a} + \sum_{i=j+3}^{n} r_i \underline{xP(j+1,i-j-3)x}$$

$$+\underline{xP(j+1,n-j-3)x}\Big) + r_{j+1}\underline{a^n}$$

$$= -\Big(\sum_{i=j+1}^{n} r_i \underline{aP(j-1,i-j-1)a} + \sum_{i=j+2}^{n-1} r_i \underline{aP(j,i-j-2)x} + \underline{aQ(j,n-j-2)x}$$

$$+\sum_{i=j+2}^{n} r_i \underline{xP(j,i-j-2)a} + \sum_{i=j+3}^{n} r_i \underline{xP(j+1,i-j-3)x}\Big) + r_{j+1}\underline{a^n}.$$

Premultiplying the above by x and postmultiplying it by a, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, produces

$$(\mathbf{E}) \qquad xa^{j+1}x^{n-j-1}a \to -\Big(\sum_{i=j+1}^{n} r_i\underline{xaP(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n-1} r_i\underline{xaP(j,i-j-2)xa}$$

$$+\underline{xaQ(j,n-j-2)xa} + \sum_{i=j+2}^{n} r_i\underline{x^2P(j,i-j-2)a^2} + \sum_{i=j+3}^{n} r_i\underline{x^2P(j+1,i-j-3)xa}\Big) + r_{j+1}\underline{xa^{n+1}}.$$

To deal with $a^jx^{n-j}a^2$, the third reducible word listed in $(\mathbf{B})$, use (2) to expand σ_j as

$$a^jx^{n-j} \to -\Big(\sum_{i=j}^{n-1} r_i\underline{a^2P(j-2,i-j)} + \underline{a^2Q(j-2,n-j)} + \sum_{i=j+1}^{n-1} r_i\big(\underline{axP(j-1,i-j-1)}$$

$$+\underline{xaP(j-1,i-j-1)}\big) + \underline{axP(j-1,n-j-1)} + \underline{xaP(j-1,n-j-1)} + \sum_{i=j+2}^{n-1} r_i\underline{x^2P(j,i-j-2)}$$

$$+\underline{x^2P(j,n-j-2)}\Big) + r_j\underline{a^n}$$

$$= -\Big(\sum_{i=j}^{n-1} r_i\underline{a^2P(j-2,i-j)} + \underline{a^2Q(j-2,n-j)} + \sum_{i=j+1}^{n} r_i\big(\underline{axP(j-1,i-j-1)}$$

$$+\underline{xaP(j-1,i-j-1)}\big) + \sum_{i=j+2}^{n} r_i\underline{x^2P(j,i-j-2)}\Big) + r_j\underline{a^n}.$$

Postmultiplying this by a^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields

$(\mathbf{F})$

$$-a^jx^{n-j}a^2 \to \Big(\sum_{i=j}^{n-1} r_i\underline{a^2P(j-2,i-j)a^2} + \underline{a^2Q(j-2,n-j)a^2} + \sum_{i=j+1}^{n} r_i\big(\underline{axP(j-1,i-j-1)a^2}$$

$$+\underline{xaP(j-1,i-j-1)a^2}\big) + \sum_{i=j+2}^{n} r_i\underline{x^2P(j,i-j-2)a^2}\Big) - r_j\underline{a^{n+2}}.$$

Hence the reduction process ends here and assembling $(\mathbf{A})$, $(\mathbf{B})$, $(\mathbf{C})$, $(\mathbf{D})$, $(\mathbf{E})$ and $(\mathbf{F})$ we obtain:

$(\mathbf{G})$

$$a^2\omega_j \to \Big(\sum_{i=j+1}^{n} r_i\underline{axP(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i\big(\underline{xaP(j,i-j-2)ax} + \underline{axP(j,i-j-2)ax}$$

$$+\underline{axP(j,i-j-2)xa}\big) + \sum_{i=j+3}^{n} r_i\underline{x^2P(j+1,i-j-3)ax}\Big) - \Big(\sum_{i=j+2}^{n-1} r_ia^2P(j,i-j-2)x^2 +$$

$$\underline{a^2Q(j,n-j-2)x^2} + r_{j+1}\underline{axa^n}\Big).$$

Consider now the second component of the overlap ambiguity. Use (2) to expand σ_{j+2} as

$$a^{j+2}x^{n-(j+2)} \to -\Big(\sum_{i=j+2}^{n-1} r_i\underline{a^2P(j,i-j-2)} + \underline{a^2Q(j,n-j-2)} + \sum_{i=j+3}^{n-1} r_i\big(\underline{axP(j+1,i-j-3)} +$$

$$\underline{xaP(j+1,i-j-3)}\big) + \underline{axP(j+1,n-j-3)} + \underline{xaP(j+1,n-j-3)} + \sum_{i=j+4}^{n-1} r_i\underline{x^2P(j+2,i-j-4)}$$

$$+\underline{x^2P(j+2,n-j-4)}\big) + \underline{r_{j+2}a^n}$$

$$= -\Big(\sum_{i=j+2}^{n-1} r_i \underline{a^2P(j,i-j-2)} + \underline{a^2Q(j,n-j-2)} + \sum_{i=j+3}^{n} r_i\big(\underline{axP(j+1,i-j-3)}$$

$$+\underline{xaP(j+1,i-j-3)}\big) + \sum_{i=j+4}^{n} r_i\underline{x^2P(j+2,i-j-4)}\big) + r_{j+2}\underline{a^n}.$$

Thus, post multiplying this by x^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields

$$\textbf{(H)}\ \ \omega_{j+2}x^2 \to -\Big(\sum_{i=j+2}^{n-1} r_i a^2P(j,i-j-2)x^2 + \underline{a^2Q(j,n-j-2)x^2} + \sum_{i=j+3}^{n} r_i\big(axP(j+1,i-j-3)x^2$$

$$+xaP(j+1,i-j-3)x^2\big) + \sum_{i=j+4}^{n} r_i x^2P(j+2,i-j-4)x^2\big) + r_{j+2}\underline{a^nx^2}.$$

The reducible words in (**H**) of length $n+2$ are

$$\textbf{(J)}\qquad xaa^{j+1}x^{n-j-1},\ axa^{j+1}x^{n-j-1},\ x^2a^{j+2}x^{n-j-2}$$

which respectively belong to $xaP(j+1,n-j-3)x^2$, $axP(j+1,n-j-3)x^2$ and $x^2P(j+2,n-j-4)x^2$. The word $xaa^{j+1}x^{n-j-1}$ can be grouped as $xa(a^{j+1}x^{n-j-1})$ or as $x(a^{j+2}x^{n-j-2})x$. Thus it involves an overlap ambiguity, but for $t=1$. So this overlap ambiguity has been resolved in [1, Proposition 2.8(i)].

To deal with the first two reducible words listed in (**J**), first use (1) to expand σ_{j+1} as

$$a^{j+1}x^{n-j-1} \to -\Big(\sum_{i=j+1}^{n-1} r_i\underline{P(j-1,i-j-1)a^2} + \underline{P(j-1,n-j-1)a^2} + \sum_{i=j+2}^{n-1} r_i\big(\underline{P(j,i-j-2)ax} +$$

$$\underline{P(j,i-j-2)xa}\big) + \underline{P(j,n-j-2)ax} + \underline{P(j,n-j-2)xa} + \sum_{i=j+3}^{n-1} \underline{P(j+1,i-j-3)x^2}$$

$$\underline{Q(j+1,n-j-3)x^2}\big) + r_{j+1}\underline{a^n}$$

$$= -\Big(\sum_{i=j+1}^{n} r_i\underline{P(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i\left(\underline{P(j,i-j-2)ax} + \underline{P(j,i-j-2)xa}\right) +$$

$$\sum_{i=j+3}^{n-1} \underline{P(j+1,i-j-3)x^2} + \underline{Q(j+1,n-j-3)x^2}\big) + r_{j+1}\underline{a^n}.$$

Now, premultiplying this by xa and ax and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yield

(**K**)

$$-xaa^{j+1}x^{n-j-1} \to \Big(\sum_{i=j+1}^{n} r_i\underline{xaP(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i\left(\underline{xaP(j,i-j-2)ax} + xaP(j,i-j-2)xa\right)$$

$$+\sum_{i=j+3}^{n-1} xaP(j+1,i-j-3)x^2 + \underline{xaQ(j+1,n-j-3)x^2}\big) - r_{j+1}\underline{xa^{n+1}};$$

and
(**L**)

$$-axa^{j+1}x^{n-j-1} \to \Big(\sum_{i=j+1}^{n} r_i \underline{axP(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i \left(\underline{axP(j,i-j-2)ax} + \underline{axP(j,i-j-2)xa} \right)$$

$$+ \sum_{i=j+3}^{n-1} \underline{axP(j+1,i-j-3)x^2} + \underline{axQ(j+1,n-j-3)x^2} \Big) - r_{j+1}\underline{axa^n}$$

respectively. Observe that $xa^{j+1}x^{n-j-1}a \in xaP(j,n-j-2)xa$ appears in (**K**); its reduction to a linear combination of irreducible words is given in (**E**).

To reduce $x^2a^{j+2}x^{n-j-2}$, the third word listed in (**J**), first use (1) to expand σ_{j+2} as

$$a^{j+2}x^{n-j-2} \to -\Big(\sum_{i=j+2}^{n-1} r_i \underline{P(j,i-j-2)a^2} + \underline{P(j,n-j-2)a^2} + \sum_{i=j+3}^{n} r_i \big(\underline{P(j+1,i-j-3)ax} +$$

$$\underline{P(j+1,i-j-3)xa} \big) + \underline{P(j+1,n-j-3)ax} + \underline{P(j+1,n-j-3)xa} + \sum_{i=j+4}^{n-1} r_i \underline{P(j+2,i-j-4)x^2}$$

$$+ \underline{Q(j+2,n-j-4)x^2} \Big) + \underline{r_{j+2}a^n}$$

$$= -\Big(\sum_{i=j+2}^{n} r_i \underline{P(j,i-j-2)a^2} + \sum_{i=j+3}^{n} r_i \left(\underline{P(j+1,i-j-3)ax} + \underline{P(j+1,i-j-3)xa} \right) +$$

$$\sum_{i=j+4}^{n-1} r_i \underline{P(j+2,i-j-4)x^2} + \underline{Q(j+2,n-j-4)x^2} \Big) + r_{j+2}\underline{a^n}.$$

Premultiplying this by x^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields

(**M**) $$-x^2a^{j+2}x^{n-j-2} \to \Big(\sum_{i=j+2}^{n} r_i \underline{x^2P(j,i-j-2)a^2} + \sum_{i=j+3}^{n} r_i \big(\underline{x^2P(j+1,i-j-3)ax} +$$

$$\underline{x^2P(j+1,i-j-3)xa} \big) + \sum_{i=j+4}^{n-1} r_i \underline{x^2P(j+2,i-j-4)x^2} + \underline{x^2Q(j+2,n-j-4)x^2} \Big) + r_{j+2}\underline{x^2a^n}.$$

Thus, the reduction process ends here and when we assemble (**H**), (**K**), (**L**), (**E**) and (**M**) we obtain
(**N**)

$$\omega_{j+2}x^2 \to \Big(\sum_{i=j+1}^{n} r_i \underline{axP(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i \big(\underline{xaP(j,i-j-2)ax} + \underline{axP(j,i-j-2)ax} +$$

$$\underline{axP(j,i-j-2)xa} \big) + \sum_{i=j+3}^{n} r_i \underline{x^2P(j+1,i-j-3)ax} \big) - \Big(\sum_{i=j+2}^{n-1} r_i a^2P(j,i-j-2)x^2 +$$

$$\underline{a^2Q(j,n-j-2)x^2} + r_{j+1}\underline{axa^n} \big).$$

Comparing (**G**) and (**N**) we conclude that the overlap ambiguity $(\sigma_j, \sigma_{j+2}, a^2, a^jx^{n-(j+2)}, x^2)$ is resolvable for all j with $1 < j < n-3$, thus completing the proof of Case (c), and with it the proof of Proposition 2.8(ii).

3. THE CASE $t = 3$.

This is hypothesis (iii) in [1, Proposition 2.8]. Here we let $t = 3$ and so $n \geq 5$. We resolve the ambiguities

$$(\sigma_{j+3}, \sigma_j, a^3, a^j x^{n-j-3}, x^3),$$

That is, an overlap ambiguity occurs through the two routes to resolve the word

$$a^3\omega_j = a^3(a^j x^{n-j}) = (a^{j+3}x^{n-j-3})x^3 = \omega_{j+3}x^3$$

in the free algebra $k\langle a, x\rangle$ using the relations σ_j and σ_{j+3} for $1 \leq j \leq n-4$. The argument splits into four sub-cases, namely (a) $j = 1$, (b) $j = 2$, (c) $3 \leq j < n-4$ and (d) $j = n-4$. Throughout the proofs, we use the following identities:

(4)
$$P(r,s) = P(r-3,s)a^3 + P(r-2,s-1)(a^2x+axa+xa^2) + P(r-1,s-2)(ax^2+xax+x^2a) + P(r,s-3)x^3$$

(5)
$$P(r,s) = a^3P(r-3,s) + (a^2x+axa+xa^2)P(r-2,s-1) + (ax^2+xax+x^2a)P(r-1,s-2) + x^3P(r,s-3)$$

(6)
$$P(r,s) = \big(x^2P(r-1,s-2) + axP(r-2,s-1) + xaP(r-2,s-1) + a^2P(r-3,s)\big)\, a + \big(x^2P(r,s-3) + axP(r-1,s-2) + xaP(r-1,s-2) + a^2P(r-2,s-1)\big)\, x$$

(7)
$$P(r,s) = a\left(P(r-1,s-2)x^2 + P(r-2,s-1)ax + P(r-2,s-1)xa + P(r-3,s)a^2\right) + x\left(P(r,s-3)x^2 + P(r-1,s-2)ax + P(r-1,s-2)xa + P(r-2,s-1)a^2\right).$$

3.1. **Case (a):** $j = 1$: We consider in this subsection the overlap ambiguity $(\sigma_4, \sigma_1, a^3, ax^{n-4}, x^3)$. Use (4) to expand the right side of σ_1 as

$$ax^{n-1} \to -(\sum_{i=3}^{n-1} r_i\underline{P(0,i-3)(x^2a+xax+ax^2)} + \underline{P(0,n-3)(x^2a+xax+ax^2)}$$
$$+\underline{Q(1,n-4)x^3} + \sum_{i=4}^{n-1} r_i\underline{P(1,i-4)x^3} + r_2\underline{P(1,1)} + r_1\underline{a}) + r_1\underline{a^n}$$
$$= -\big(\sum_{i=3}^{n} r_i\underline{P(0,i-3)(x^2a+xax+ax^2)} + \sum_{i=4}^{n-1} r_i\underline{P(1,i-4)x^3} + \underline{Q(1,n-4)x^3} + r_2\underline{P(1,1)} + r_1\underline{a}\big) + r_1\underline{a^n}.$$

Thus, premultiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

(8)
$$a^3\omega_1 \to -(\sum_{i=3}^{n} r_i a^3P(0,i-3)(x^2a+xax+ax^2) + \sum_{i=4}^{n-1} r_i\underline{a^3P(1,i-4)x^3} +$$
$$\underline{a^3Q(1,n-4)x^3} + r_2\underline{a^3P(1,1)} + r_1\underline{a^4}) + r_1\underline{a^{n+3}}.$$

The following words in (8) of length $n+3$ are reducible:

$$a^3x^{n-1}a,\ a^3x^{n-3}ax^2,\ a^3x^{n-2}ax.$$

We first reduce $a^3x^{n-1}a$ as follows. Use (5) to expand σ_3 as

$$a^3x^{n-3} \to -(\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)} + \sum_{i=4}^{n-1} r_i\underline{(xa^2+axa+a^2x)P(1,i-4)} +$$

$$\underline{(xa^2+axa+a^2x)P(1,n-4)}+\sum_{i=5}^{n-1} r_i\underline{(x^2a+xax+ax^2)P(2,i-5)}+$$

$$\underline{(x^2a+xax+ax^2)P(2,n-5)}+\sum_{i=6}^{n-1} r_i\underline{x^3P(3,i-6)})+\underline{x^3P(3,n-6)})+r_3\underline{a^n}$$

$$=-(\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)}+\sum_{i=4}^{n} r_i\underline{(xa^2+axa+a^2x)P(1,i-4)}+$$

$$\sum_{i=5}^{n} r_i\underline{(x^2a+xax+ax^2)P(2,i-5)}+\sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)})+r_3\underline{a^n}$$

Postmultiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields respectively

$$-a^3x^{n-1}a \to (\sum_{i=3}^{n-1} r_ia^3P(0,i-3)x^2a+\sum_{i=4}^{n} r_i(xa^2+axa+a^2x)P(1,i-4)x^2a+ \tag{9}$$

$$\sum_{i=5}^{n} r_i\underline{(x^2a+xax+ax^2)P(2,i-5)x^2a}+\sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)x^2a})-r_3\underline{x^2a^{n+1}},$$

$$-a^3x^{n-2}ax \to (\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)xax}+\sum_{i=4}^{n} r_i(xa^2+axa+a^2x)P(1,i-4)xax+ \tag{10}$$

$$\sum_{i=5}^{n} r_i\underline{(x^2a+xax+ax^2)P(2,i-5)xax}+\sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)xax})-r_3\underline{xaxa^n}$$

and

$$-a^3x^{n-3}ax^2 \to (\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)ax^2}+\sum_{i=4}^{n} r_i\underline{(xa^2+axa+a^2x)P(1,i-4)ax^2}+ \tag{11}$$

$$\sum_{i=5}^{n} r_i\underline{(x^2a+xax+ax^2)P(2,i-5)ax^2}+\sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)ax^2})-r_3\underline{ax^2a^n}$$

respectively. The reducible words in (9)-(11) of length $n+3$ are as follows:

- **(I)** $axa^2x^{n-2}a \in axaP(1,n-4)x^2a$,
- **(II)** $x^2a^3x^{n-3}a \in x^2aP(2,n-5)x^2a$,
- **(III)** $xa^3x^{n-3}xa \in xa^2P(1,n-4)x^2a$,
- **(IV)** $xa^3x^{n-3}ax \in xa^2P(1,n-4)xax$.

considering first the reduction of **(I)** $axa^2x^{n-2}a$, use (7) to expand σ_2 as

$$a^2x^{n-2} \to -(\sum_{i=3}^{n-1} r_i(\underline{aP(0,i-3)ax}+\underline{aP(0,i-3)xa}+\underline{xP(0,i-3)a^2})+\underline{aP(0,n-3)ax}+$$

$$\underline{aP(0,n-3)xa}+\underline{xP(0,n-3)a^2}+\sum_{i=4}^{n-1} r_i\underline{aP(1,i-4)x^2}+\underline{aQ(1,n-4)x^2}+$$

$$\sum_{i=4}^{n-1} r_i(\underline{xP(1,i-4)ax}+\underline{xP(1,i-4)xa})+\underline{xP(1,n-4)ax}+\underline{xP(1,n-4)xa}+$$

$$\sum_{i=5}^{n-1} r_i \underline{xP(2,i-5)x^2} + \underline{xP(2,n-5)x^2} + r_2\underline{a^2}) + r_2\underline{a^n}$$

$$= -(\sum_{i=3}^{n} r_i(\underline{aP(0,i-3)ax}+\underline{aP(0,i-3)xa}+\underline{xP(0,i-3)a^2})+\sum_{i=4}^{n-1} r_i\underline{aP(1,i-4)x^2}+\underline{aQ(1,n-4)x^2}$$

$$+\sum_{i=4}^{n} r_i(\underline{xP(1,i-4)ax}+\underline{xP(1,i-4)xa})+\sum_{i=5}^{n} r_i\underline{xP(2,i-5)x^2}+r_2\underline{a^2})+r_2\underline{a^n}.$$

Premultiply and postmultiply this by ax and a respectively and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words to get

(12) $$axa^2x^{n-2}a \to -(\sum_{i=3}^{n} r_i(\underline{axaP(0,i-3)axa}+\underline{axaP(0,i-3)xa^2}+ax^2P(0,i-3)a^3)+$$

$$\sum_{i=4}^{n-1} r_i\underline{axaP(1,i-4)x^2a}+\underline{axaQ(1,n-4)x^2a}+\sum_{i=4}^{n} r_i(\underline{ax^2P(1,i-4)axa}+$$

$$\underline{ax^2P(1,i-4)xa^2})+\sum_{i=5}^{n} r_i\underline{ax^2P(2,i-5)x^2a}+r_2\underline{axa^3})+r_2\underline{axa^{n+1}}.$$

The only reducible word above is $ax^{n-1}a^3 \in ax^2P(0,n-3)a^3$. To reduce $ax^{n-1}a^3$, use (5) to expand σ_1 as

$$ax^{n-1} \to -(\sum_{i=4}^{n-1} r_i\underline{x^3P(1,i-4)}+\underline{x^3P(1,n-4)}+\sum_{i=3}^{n-1} r_i\underline{(x^2a+xax+ax^2)P(0,i-3)}$$

$$+r_2\underline{P(1,1)}+r_1\underline{a})+r_1\underline{a^n}$$

$$= -(\sum_{i=4}^{n} r_i\underline{x^3P(1,i-4)}+\sum_{i=3}^{n-1} r_i\underline{(x^2a+xax+ax^2)P(0,i-3)}+r_2\underline{P(1,1)}+r_1\underline{a})+r_1\underline{a^n}.$$

Post multiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

(13)

$$-ax^{n-1}a^3 \to (\sum_{i=4}^{n} r_i\underline{x^3P(1,i-4)a^3}+\sum_{i=3}^{n-1} r_i\underline{(x^2a+xax+ax^2)P(0,i-3)a^3}+r_2\underline{P(1,1)a^3}+r_1\underline{a^4})$$

$$-r_1\underline{a^{n+3}}.$$

Turning now to the reduction of (**IV**) $xa^3x^{n-3}ax$ and (**III**) $xa^3x^{n-3}xa$, use (6) to expand σ_3 as

$$a^3x^{n-3} \to -(\sum_{i=3}^{n-1} r_i\underline{a^2P(0,i-3)a}+\underline{a^2P(0,n-3)a}+\sum_{i=4}^{n-1} r_i\underline{a^2P(1,i-4)x}+\underline{a^2Q(1,n-4)x}$$

$$+\sum_{i=4}^{n-1} r_i(\underline{axP(1,i-4)a}+\underline{xaP(1,i-4)a})+\underline{axP(1,n-4)a}+\underline{xaP(1,n-4)a}+$$

$$\sum_{i=5}^{n-1} r_i(\underline{x^2P(2,i-5)a}+\underline{xaP(2,i-5)x}+\underline{axP(2,i-5)x})+\underline{x^2P(2,n-5)a}+$$

$$\underline{xaP(2,n-5)x}+\underline{axP(2,n-5)x}+\sum_{i=6}^{n-1} r_i\underline{x^2P(3,i-6)x}+\underline{x^2P(3,n-6)x})+r_3\underline{a^n}$$

$$= -(\sum_{i=3}^{n} r_i \underline{a^2P(0,i-3)a} + \sum_{i=4}^{n-1} r_i \underline{a^2P(1,i-4)x} + \underline{a^2Q(1,n-4)x} + \sum_{i=4}^{n} r_i(\underline{axP(1,i-4)a}$$

$$+\underline{xaP(1,i-4)a}) + \sum_{i=5}^{n} r_i(\underline{x^2P(2,i-5)a} + \underline{xaP(2,i-5)x} + \underline{axP(2,i-5)x}) +$$

$$\sum_{i=6}^{n} r_i \underline{x^2P(3,i-6)x}) + r_3\underline{a^n}.$$

Premultiplying this by x and post multiplying by ax and xa, using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields respectively

$$(14)\ \ xa^3x^{n-3}ax \to -(\sum_{i=3}^{n} r_i \underline{xa^2P(0,i-3)a^2x} + \sum_{i=4}^{n-1} r_i \underline{xa^2P(1,i-4)xax} + \underline{xa^2Q(1,n-4)xax} +$$

$$\sum_{i=4}^{n} r_i(\underline{xaxP(1,i-4)a^2x} + \underline{x^2aP(1,i-4)a^2x}) + \sum_{i=5}^{n} r_i(\underline{x^3P(2,i-5)a^2x} + \underline{x^2aP(2,i-5)xax}$$

$$+\underline{xaxP(2,i-5)xax}) + \sum_{i=6}^{n} r_i \underline{x^3P(3,i-6)xax}) + r_3\underline{xaxa^n};$$

and

$$(15)\ \ xa^3x^{n-3}xa \to -(\sum_{i=3}^{n} r_i \underline{xa^2P(0,i-3)axa} + \sum_{i=4}^{n-1} r_i \underline{xa^2P(1,i-4)x^2a} + \underline{xa^2Q(1,n-4)x^2a} +$$

$$\sum_{i=4}^{n} r_i(\underline{xaxP(1,i-4)axa} + \underline{x^2aP(1,i-4)axa}) + \sum_{i=5}^{n} r_i(\underline{x^3P(2,i-5)axa} + x^2aP(2,i-5)x^2a$$

$$+\underline{xaxP(2,i-5)x^2a}) + \sum_{i=6}^{n} r_i \underline{x^3P(3,i-6)x^2a}) + r_3\underline{x^2a^{n+1}}.$$

All the words in (14) and (15) are irreducible except $x^2a^3x^{n-3}a \in x^2aP(2,n-5)x^2a$. But $x^2a^3x^{n-3}a$ appears with opposite sign to the same reducible word in (9) so they cancel out, thus dealing with (**II**). So the reduction process ends here and substituting (9), (10), (11), (12), (13), (14), (15) into (8) and simplifying gives

$$(16)\quad a^3\omega_1 \to (r_2\underline{xa^4} + r_2\underline{axa^{n+1}} + \sum_{i=3}^{n-1} r_i(\underline{(x^2a+xax)P(0,i-3)a^3} + \sum_{i=4}^{n} r_i(\underline{x^3P(1,i-4)a^3} +$$

$$\underline{a^2xP(1,i-4)(ax^2+x^2a+xax)} + \underline{axaP(1,i-4)(ax^2+xax)} + \underline{xa^2P(1,i-4)ax^2}) +$$

$$\sum_{i=5}^{n} r_i(\underline{(ax^2+x^2a+xax)P(2,i-5)ax^2} + \underline{ax^2P(2,i-5)xax}) + \sum_{i=6}^{n} r_i \underline{x^3P(3,i-6)ax^2}$$

$$-(r_2\underline{a^3P(1,1)} + r_3\underline{ax^2a^n} + \sum_{i=3}^{n} r_i(\underline{xa^2P(0,i-3)(a^2x+axa)} + \underline{axaP(0,i-3)(axa+xa^2)}) +$$

$$\sum_{i=4}^{n} r_i(\underline{(xax+ax^2+x^2a)P(1,i-4)axa} + \underline{(xax+x^2a)P(1,i-4)a^2x} + \underline{ax^2P(1,i-4)xa^2})$$

$$\sum_{i=4}^{n-1} r_i a^3P(1,i-4)x^3 + \underline{a^3Q(1,n-4)x^3} + \sum_{i=5}^{n} r_i \underline{x^3P(2,i-5)(a^2x+axa)}.$$

Moving now to $\omega_4 x^3$, use (5) to write σ_4 as

$$\omega_4 \to -(\sum_{i=4}^{n-1} r_i \underline{a^3P(1,i-4)} + \underline{a^3Q(1,n-4)} + \sum_{i=5}^{n-1} r_i \underline{(a^2x+axa+xa^2)P(2,i-5)} +$$

$$\underline{(a^2x+axa+xa^2)P(2,n-5)} + \sum_{i=6}^{n-1} r_i \underline{(ax^2+xax+x^2a)P(3,i-6)} +$$

$$\underline{(ax^2+xax+x^2a)P(3,n-6)} + \sum_{i=7}^{n-1} r_i \underline{x^3P(4,i-7)} + \underline{x^3P(4,n-7)}) + r_4\underline{a^n}$$

$$= -(\sum_{i=4}^{n-1} r_i \underline{a^3P(1,i-4)} + \underline{a^3Q(1,n-4)} + \sum_{i=5}^{n} r_i \underline{(a^2x+axa+xa^2)P(2,i-5)}$$

$$+ \sum_{i=6}^{n} r_i \underline{(ax^2+xax+x^2a)P(3,i-6)} + \sum_{i=7}^{n} r_i \underline{x^3P(4,i-7)}) + r_4\underline{a^n}.$$

Post multiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$(17)\quad \omega_4 x^3 \to -(\sum_{i=4}^{n-1} r_i a^3P(1,i-4)x^3 + \underline{a^3Q(1,n-4)x^3} + \sum_{i=5}^{n} r_i (a^2x+axa+xa^2)P(2,i-5)x^3$$

$$+ \sum_{i=6}^{n} r_i (ax^2+xax+x^2a)P(3,i-6)x^3 + \sum_{i=7}^{n} r_i x^3P(4,i-7)x^3) + r_4\underline{x^3a^n}.$$

We find the following reducible words of length $n+3$ in (17):

(I) $xa^2(a^2x^{n-2}) \in xa^2P(2,n-5)x^3$,
(II) $axa(a^2x^{n-2}) \in axaP(2,n-5)x^3$,
(III) $a^2x(a^2x^{n-2}) \in a^2xP(2,n-5)x^3$,
(IV) $x^2a(a^3x^{n-3}) \in x^2aP(3,n-6)x^3$,
(V) $ax^2(a^3x^{n-3}) \in ax^2P(3,n-6)x^3$,
(VI) $xax(a^3x^{n-3}) \in xaxP(3,n-6)x^3$,
(VII) $x^3(a^4x^{n-4}) \in x^3P(4,n-7)x^3$.

Use (4) to expand σ_2 as

$$a^2x^{n-2} \to -(\sum_{i=3}^{n-1} r_i \underline{P(0,i-3)(xa^2+axa+a^2x)} + \underline{P(0,n-3)(xa^2+axa+a^2x)} +$$

$$\sum_{i=4}^{n-1} r_i \underline{P(1,i-4)(ax^2+xax+x^2a)} + \underline{P(1,n-4)(ax^2+xax+x^2a)} +$$

$$\sum_{i=5}^{n-1} r_i \underline{P(2,i-5)x^3} + \underline{Q(2,n-5)x^3} + r_2\underline{a^2}) + r_2\underline{a^n}.$$

$$= -(\sum_{i=3}^{n} r_i \underline{P(0,i-3)(xa^2+axa+a^2x)} + \sum_{i=4}^{n} r_i \underline{P(1,i-4)(ax^2+xax+x^2a)} +$$

$$\sum_{i=5}^{n-1} r_i \underline{P(2,i-5)x^3} + \underline{Q(2,n-5)x^3} + r_2\underline{a^2}) + r_2\underline{a^n}.$$

Premultiplying this by xa^2, axa and a^2x, using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields respectively

(18)

$$-xa^2(a^2x^{n-2}) \to (\sum_{i=3}^{n} r_i xa^2 P(0,i-3)(xa^2+axa+a^2x)+\sum_{i=4}^{n} r_i xa^2 P(1,i-4)(ax^2+xax+x^2a)+ \sum_{i=5}^{n-1} r_i xa^2 P(2,i-5)x^3 + \underline{xa^2Q(2,n-5)x^3} + r_2\underline{xa^4}) - r_2\underline{xa^{n+2}},$$

(19)

$$-axa(a^2x^{n-2}) \to (\sum_{i=3}^{n} r_i \underline{axaP(0,i-3)(xa^2+axa+a^2x)}+\sum_{i=4}^{n} r_i axaP(1,i-4)(ax^2+xax+x^2a)+ \sum_{i=5}^{n-1} r_i \underline{axaP(2,i-5)x^3} + \underline{axaQ(2,n-5)x^3} + r_2\underline{axa^3}) - r_2\underline{axa^{n+1}},$$

and

(20)

$$-a^2x(a^2x^{n-2}) \to (\sum_{i=3}^{n} r_i a^2xP(0,i-3)(xa^2+axa+a^2x)+\sum_{i=4}^{n} r_i \underline{a^2xP(1,i-4)(ax^2+xax+x^2a)}+ \sum_{i=5}^{n-1} r_i \underline{a^2xP(2,i-5)x^3} + \underline{a^2xQ(2,n-5)x^3} + r_2\underline{a^2xa^2}) - r_2\underline{a^2xa^n},$$

so dealing with (**I**), (**II**) and (**III**). Also, use (4) to expand σ_3 as

$$\begin{aligned} a^3x^{n-3} \to -(&\sum_{i=3}^{n-1} r_i \underline{P(0,i-3)a^3} + \underline{P(0,n-3)a^3} + \sum_{i=4}^{n-1} r_i \underline{P(1,i-4)(a^2x+axa+xa^2)}+ \\ &\underline{P(1,n-4)(a^2x+axa+xa^2)} + \sum_{i=5}^{n-1} r_i \underline{P(2,i-5)(ax^2+xax+x^2a)}+ \\ &\underline{P(2,n-5)(ax^2+xax+x^2a)} + \sum_{i=6}^{n-1} r_i \underline{P(3,i-6)x^3} + \underline{Q(3,n-6)x^3}) + r_3\underline{a^n} \\ = -(&\sum_{i=3}^{n} r_i \underline{P(0,i-3)a^3} + \sum_{i=4}^{n} r_i \underline{P(1,i-4)(a^2x+axa+xa^2)}+ \\ &\sum_{i=5}^{n} r_i \underline{P(2,i-5)(ax^2+xax+x^2a)} + \sum_{i=6}^{n-1} r_i \underline{P(3,i-6)x^3} + \underline{Q(3,n-6)x^3}) + r_3\underline{a^n}. \end{aligned}$$

Premultiplying this by x^2a, ax^2 and xax, using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields respectively

$$(21)\qquad -x^2a(a^3x^{n-3}) \to (\sum_{i=3}^{n} r_i \underline{x^2aP(0,i-3)a^3} + \sum_{i=4}^{n} r_i \underline{x^2aP(1,i-4)(a^2x+axa+xa^2)}+ \sum_{i=5}^{n} r_i x^2aP(2,i-5)(ax^2+xax+x^2a) + \sum_{i=6}^{n-1} r_i \underline{x^2aP(3,i-6)x^3} + \underline{x^2aQ(3,n-6)x^3}) - r_3\underline{x^2a^{n+1}},$$

$$(22)\qquad -ax^2(a^3x^{n-3}) \to (\sum_{i=3}^{n} r_i ax^2P(0,i-3)a^3 + \sum_{i=4}^{n} r_i \underline{ax^2P(1,i-4)(a^2x+axa+xa^2)}+$$

$$+\sum_{i=5}^{n} r_i \underline{ax^2P(2,i-5)(ax^2+xax+x^2a)} + \sum_{i=6}^{n-1} r_i \underline{ax^2P(3,i-6)x^3} + \underline{ax^2Q(3,n-6)x^3}) - r_3\underline{ax^2a^n},$$

and

$$(23) \qquad -xax(a^3x^{n-3}) \to (\sum_{i=3}^{n} r_i \underline{xaxP(0,i-3)a^3} + \sum_{i=4}^{n} r_i \underline{xaxP(1,i-4)(a^2x+axa+xa^2)}+$$

$$+\sum_{i=5}^{n} r_i \underline{xaxP(2,i-5)(ax^2+xax+x^2a)} + \sum_{i=6}^{n-1} r_i \underline{xaxP(3,i-6)x^3} + \underline{xaxQ(3,n-6)x^3}) - r_3\underline{xaxa^n},$$

dealing with (**IV**), (**V**) and (**VI**). Moreover, use (4) to expand σ_4 as

$$a^4x^{n-4} \to -(\sum_{i=4}^{n-1} r_i \underline{P(1,i-4)a^3} + \underline{P(1,n-4)a^3} + \sum_{i=5}^{n-1} r_i \underline{P(2,i-5)(a^2x+axa+xa^2)}+$$

$$\underline{P(2,n-5)(a^2x+axa+xa^2)} + \sum_{i=6}^{n-1} r_i \underline{P(3,i-6)(ax^2+xax+x^2a)}$$

$$\underline{P(3,n-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i \underline{P(4,i-7)x^3} + \underline{Q(4,n-7)x^3}) + r_4\underline{a^n}$$

$$= -(\sum_{i=4}^{n} r_i \underline{P(1,i-4)a^3} + \sum_{i=5}^{n} r_i \underline{P(2,i-5)(a^2x+axa+xa^2)}+$$

$$\sum_{i=6}^{n} r_i \underline{P(3,i-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i \underline{P(4,i-7)x^3} + \underline{Q(4,n-7)x^3}) + r_4\underline{a^n}.$$

Premultiplying this by x^3, using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields, to handle (**VII**),

$$(24) \qquad -x^3(a^4x^{n-4}) \to (\sum_{i=4}^{n} r_i \underline{x^3P(1,i-4)a^3} + \sum_{i=5}^{n} r_i \underline{x^3P(2,i-5)(a^2x+axa+xa^2)}+$$

$$\sum_{i=6}^{n} r_i \underline{x^3P(3,i-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i \underline{x^3P(4,i-7)x^3} + \underline{x^3Q(4,n-7)x^3}) - r_4\underline{x^3a^n}.$$

We are now left with the following reducible words of length $n+3$ from (18)-(24):

(**I**) $xa^3x^{n-2}a \in xa^2P(1,n-4)x^2a$ in (18);
(**II**) $xa^3x^{n-3}ax \in xa^2P(1,n-4)xax$ in (18);
(**III**) $axa^2x^{n-2}a \in axaP(1,n-4)x^2a$ in (19);
(**IV**) $a^2x^{n-1}a^2 \in a^2xP(0,n-3)xa^2$ in (20);
(**V**) $a^2x^{n-2}a^2x \in a^2xP(0,n-3)a^2x$ in (20);
(**VI**) $a^2x^{n-2}axa \in a^2xP(0,n-3)axa$ in (20);
(**VII**) $x^2a^3x^{n-3}a \in x^2aP(2,n-5)x^2a$ in (21,
(**VIII**) $ax^{n-1}a^3 \in ax^2P(0,n-3)a^3$, in (22);
(**IX**) $xa^2x^{n-2}a^2 \in xa^2P(0,n-3)xa^2$ in (18).

Recall from (15) that

$$xa^3x^{n-3}xa \to -(\sum_{i=3}^{n} r_i \underline{xa^2P(0,i-3)axa} + \sum_{i=4}^{n-1} r_i \underline{xa^2P(1,i-4)x^2a} + \underline{xa^2Q(1,n-4)x^2a}+$$

$$\sum_{i=4}^{n} r_i(\underline{xaxP(1,i-4)axa} + \underline{x^2aP(1,i-4)axa}) + \sum_{i=5}^{n} r_i(\underline{x^3P(2,i-5)axa} + x^2aP(2,i-5)x^2a$$

$$+\underline{xaxP(2,i-5)x^2a}) + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)x^2a}) + r_3\underline{x^2a^{n+1}}.$$

All the words in (15) are irreducible except $-x^2a^3x^{n-3}a \in -x^2aP(2,n-5)x^2a$. But this term cancels out with (**VII**) in the list above since they appear with opposite signs.

Similarly, recall from (14) that

$$xa^3x^{n-3}ax \to -(\sum_{i=3}^{n} r_i\underline{xa^2P(0,i-3)a^2x} + \sum_{i=4}^{n-1} r_i\underline{xa^2P(1,i-4)xax} + \underline{xa^2Q(1,n-4)xax}+$$

$$\sum_{i=4}^{n} r_i(\underline{xaxP(1,i-4)a^2x} + \underline{x^2aP(1,i-4)a^2x}) + \sum_{i=5}^{n} r_i(\underline{x^3P(2,i-5)a^2x} + \underline{x^2aP(2,i-5)xax}$$

$$+\underline{xaxP(2,i-5)xax}) + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)xax}) + r_3\underline{xaxa^n}.$$

Furthermore, use (5) to expand σ_2 as

$$a^2x^{n-2} \to -(\sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)} + \sum_{i=3}^{n-1} r_i\underline{(xa^2+axa)P(0,i-3)} + \underline{(xa^2+axa)P(0,n-3)}+$$

$$\sum_{i=4}^{n-1} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)} + \underline{(ax^2+xax+x^2a)P(1,n-4)}+$$

$$\sum_{i=5}^{n-1} r_i\underline{x^3P(2,i-5)} + \underline{x^3P(2,n-5)} + r_2\underline{a^2}) + r_2\underline{a^n}$$

$$= -(\sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)} + \sum_{i=3}^{n} r_i\underline{(xa^2+axa)P(0,i-3)} + \sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)}$$

$$+\sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)} + r_2\underline{a^2}) + r_2\underline{a^n}.$$

Postmultiply this by a^2x, xa^2 and axa and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words to get, respectively,

$$a^2x^{n-2}a^2x \to -(\sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)a^2x} + \sum_{i=3}^{n} r_i\underline{(xa^2+axa)P(0,i-3)a^2x}+ \tag{25}$$

$$\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)a^2x} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)a^2x} + r_2\underline{a^4x}) + r_2\underline{a^2xa^n},$$

$$a^2x^{n-2}xa^2 \to -(\sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)xa^2} + \sum_{i=3}^{n} r_i(xa^2+axa)P(0,i-3)xa^2+ \tag{26}$$

$$\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)xa^2} + \underline{x^3P(2,n-5)xa^2} + xa^2x^{n-2}a^2 + \underline{axax^{n-2}a^2})$$

and

$$(27)\qquad a^2x^{n-2}axa \to -(\sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)axa} + \sum_{i=3}^{n} r_i\underline{(xa^2+axa)P(0,i-3)axa} +$$

$$\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)axa} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)axa} + r_2\underline{a^3xa}) + r_2\underline{axa^{n+1}}.$$

The reducible word $(-xa^2x^{n-2}a^2)$ in (26) cancels out with $xa^2x^{n-2}a^2$ which is (**IX**) in the list of reducible words above.

Recall from (12) that

$$axa^2x^{n-2}a \to -(\sum_{i=3}^{n} r_i(\underline{axaP(0,i-3)axa} + \underline{axaP(0,i-3)xa^2} + ax^2P(0,i-3)a^3) +$$

$$\sum_{i=4}^{n-1} r_i\underline{axaP(1,i-4)x^2a} + \underline{axaQ(1,n-4)x^2a} + \sum_{i=4}^{n} r_i(\underline{ax^2P(1,i-4)axa} +$$

$$\underline{ax^2P(1,i-4)xa^2}) + \sum_{i=5}^{n} r_i\underline{ax^2P(2,i-5)x^2a} + r_2\underline{axa^3}) + r_2\underline{axa^{n+1}}.$$

All the words in (12) are irreducible except $-ax^2P(0,i-3)a^3$ which can be written as $-ax^{n-1}a^3$. But this cancels out with $ax^{n-1}a^3$ which is (**VIII**) in the list of reducible words above.

Thus, the reduction process ends here and when we substitute (12), (14), (15), (18), (19), (20), (21), (22), (23), (24), (25), (26) and (27) into (17), we get

$$(28)\qquad \omega_4x^3 \to (r_2\underline{xa^4} + r_2\underline{axa^{n+1}} + \sum_{i=3}^{n-1} r_i\underline{(x^2a+xax)P(0,i-3)a^3} + \sum_{i=4}^{n} r_i(\underline{x^3P(1,i-4)a^3} +$$

$$\underline{a^2xP(1,i-4)(ax^2+x^2a+xax)} + \underline{axaP(1,i-4)(ax^2+xax)} + \underline{xa^2P(1,i-4)ax^2}) +$$

$$\sum_{i=5}^{n} r_i(\underline{(ax^2+x^2a+xax)P(2,i-5)ax^2} + \underline{ax^2P(2,i-5)xax}) + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)ax^2}$$

$$-(r_2\underline{a^3P(1,1)} + r_3\underline{ax^2a^n} + \sum_{i=3}^{n} r_i(\underline{xa^2P(0,i-3)(a^2x+axa)} + \underline{axaP(0,i-3)(axa+xa^2)}) +$$

$$\sum_{i=4}^{n} r_i(\underline{(xax+ax^2+x^2a)P(1,i-4)axa} + \underline{(xax+x^2a)P(1,i-4)a^2x} + \underline{ax^2P(1,i-4)xa^2})$$

$$\sum_{i=4}^{n-1} r_ia^3P(1,i-4)x^3 + \underline{a^3Q(1,n-4)x^3} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)(a^2x+axa)}.$$

Comparing (16) and (28), we conclude that the overlap ambiguity $(a^4x^{n-4}, ax^{n-1}, a^3, ax^{n-4}, x^3)$ is resolvable.

3.2. **Case (b):** $j=2$: We consider the overlap ambiguity $(\sigma_5, \sigma_2, a^3, a^2x^{n-5}, x^3)$. Use (4) to expand σ_2 as

$$a^2x^{n-2} \to -\big(\sum_{i=3}^{n-1} r_i\underline{P(0,i-3)(a^2x+axa+xa^2)} + \underline{P(0,n-3)(a^2x+axa+xa^2)}$$

$$\sum_{i=4}^{n-1} r_i\underline{P(1,i-4)(ax^2+xax+x^2a)} + \underline{P(1,n-4)(ax^2+xax+x^2a)}$$

$$+\sum_{i=5}^{n-1} r_i\underline{P(2,i-5)x^3} + \underline{Q(2,n-5)x^3} + r_2\underline{a^2}\big) + r_2\underline{a^n}$$

$$= -\big(\sum_{i=3}^{n} r_i\underline{P(0,i-3)(a^2x+axa+xa^2)} + \sum_{i=4}^{n} r_i\underline{P(1,i-4)(ax^2+xax+x^2a)}$$

$$+\sum_{i=5}^{n-1} r_i\underline{P(2,i-5)x^3} + \underline{Q(2,n-5)x^3} + r_2\underline{a^2}\big) + r_2\underline{a^n}.$$

Premultiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, we obtain

$$(29)\quad a^3\omega_2 \to -\big(\sum_{i=3}^{n} r_i a^3P(0,i-3)(a^2x+axa+xa^2) + \sum_{i=4}^{n} r_i a^3P(1,i-4)(ax^2+xax+x^2a)+$$

$$\sum_{i=5}^{n-1} r_i a^3P(2,i-5)x^3 + \underline{a^3Q(2,n-5)x^3} + r_2\underline{a^5}\big) + r_2\underline{a^{n+3}}.$$

The following words of length $n+3$ in (29) are reducible:

(a) $(a^4x^{n-4})x^2a \in a^3P(1,n-4)x^2a$
(b) $(a^4x^{n-4})xax \in a^3P(1,n-4)xax$
(c) $(a^4x^{n-4})ax^2 \in a^3P(1,n-4)ax^2$
(d) $a^3x^{n-3}axa$
(e) $a^3x^{n-3}a^2x$
(f) $a^3x^{n-2}a^2$.

To handle (a), (b) and (c), use (5) to expand σ_4 as

$$a^4x^{n-4} \to -\big(\sum_{i=4}^{n-1} r_i\underline{a^3P(1,i-4)} + \underline{a^3Q(1,n-4)} + \sum_{i=5}^{n-1} r_i\underline{(a^2x+axa+xa^2)P(2,i-5)}$$

$$+\underline{(a^2x+axa+xa^2)P(2,n-5)} + \sum_{i=6}^{n-1} r_i\underline{(ax^2+xax+x^2a)P(3,i-6)} + \underline{(ax^2+xax+x^2a)P(3,n-6)}$$

$$+\sum_{i=7}^{n-1} r_i\underline{x^3P(4,i-7)} + \underline{x^3P(4,n-7)}\big) + r_4\underline{a^n}$$

$$= -\big(\sum_{i=4}^{n-1} r_i\underline{a^3P(1,i-4)} + \underline{a^3Q(1,n-4)} + \sum_{i=5}^{n} r_i\underline{(a^2x+axa+xa^2)P(2,i-5)}$$

$$+\sum_{i=6}^{n} r_i\underline{(ax^2+xax+x^2a)P(3,i-6)} + \sum_{i=7}^{n-1} r_i\underline{x^3P(4,i-7)}\big) + r_4\underline{a^n}.$$

Thus, postmultiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields for (a), (b) and (c) respectively

$$(30)\qquad -(a^4x^{n-4})x^2a \to \big(\sum_{i=4}^{n-1} r_i a^3P(1,i-4)x^2a + \sum_{i=5}^{n} r_i(a^2x+axa+xa^2)P(2,i-5)x^2a$$

$$+\sum_{i=6}^{n} r_i\underline{(ax^2+xax+x^2a)P(3,i-6)x^2a}+\sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)x^2a}+\underline{a^3Q(1,n-4)x^2a}\big)-r_4\underline{x^2a^{n+1}};$$

$$(31)\qquad -(a^4x^{n-4})xax \to \big(\sum_{i=4}^{n-1} r_i \underline{a^3P(1,i-4)xax} + \sum_{i=5}^{n} r_i(a^2x+axa+xa^2)P(2,i-5)xax$$

$$+\sum_{i=6}^{n} r_i\underline{(ax^2+xax+x^2a)P(3,i-6)xax}+\sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)xax}+\underline{a^3Q(1,n-4)xax}\big)-r_4\underline{xaxa^n};$$

and

$$(32)\qquad -\big(a^4x^{n-4})ax^2 \to \big(\sum_{i=4}^{n-1} r_i \underline{a^3P(1,i-4)ax^2} + \sum_{i=5}^{n} r_i\underline{(a^2x+axa+xa^2)P(2,i-5)ax^2}$$

$$+\sum_{i=6}^{n} r_i\underline{(ax^2+xax+x^2a)P(3,i-6)ax^2}+\sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)ax^2}+\underline{a^3Q(1,n-4)ax^2}\big)-r_4\underline{ax^2a^n}.$$

Similarly, for (d), (e) and (f) use (5) to expand σ_3 as

$$a^3x^{n-3} \to -\big(\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)}+\sum_{i=4}^{n-1} r_i\underline{(a^2x+axa+xa^2)P(1,i-4)}+\underline{(a^2x+axa+xa^2)P(1,n-4)}$$

$$+\sum_{i=5}^{n-1} r_i\underline{(ax^2+xax+x^2a)P(2,i-5))}+\underline{(ax^2+xax+x^2a)P(2,n-5)}+\sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)}\big)+r_3\underline{a^n}$$

$$= -\big(\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)}+\sum_{i=4}^{n} r_i\underline{(a^2x+axa+xa^2)P(1,i-4)}$$

$$+\sum_{i=5}^{n} r_i\underline{(ax^2+xax+x^2a)P(2,i-5)}+\sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)}\big)+r_3\underline{a^n}.$$

Postmultiply this by axa, a^2x and xa^2, and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words to get

$$(33)\qquad -a^3x^{n-3}axa \to \big(\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)axa}+\sum_{i=4}^{n} r_i\underline{(a^2x+axa+xa^2)P(1,i-4)axa}$$

$$+\sum_{i=5}^{n} r_i\underline{(ax^2+xax+x^2a)P(2,i-5)axa}+\sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)axa}\big)-r_3\underline{axa^{n+1}};$$

$$(34)\qquad -a^3x^{n-3}a^2x \to \big(\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)a^2x}+\sum_{i=4}^{n} r_i\underline{(a^2x+axa+xa^2)P(1,i-4)a^2x}$$

$$+\sum_{i=5}^{n} r_i\underline{(ax^2+xax+x^2a)P(2,i-5)a^2x}+\sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)a^2x}\big)-r_3\underline{a^2xa^n};$$

and

$$(35)\qquad -a^3x^{n-3}xa^2 \to \Big(\sum_{i=3}^{n-1} r_i \underline{a^3P(0,i-3)xa^2} + \sum_{i=4}^{n} r_i(a^2x+axa+xa^2)P(1,i-4)xa^2$$

$$+\sum_{i=5}^{n} r_i\underline{(ax^2+xax+x^2a)P(2,i-5)xa^2} + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)xa^2}\Big) - r_3\underline{xa^{n+2}}.$$

The following words of length $n+3$ from (30), (31), (32), (33), (34) and (35) are reducible:

(g) $x^2a^4x^{n-4}a \in x^2aP(3,n-6)x^2a$ in (30),
(h) $x(a^4x^{n-4})xa \in xa^2P(2,n-5)x^2a$ in (30),
(i) $axa^3x^{n-3}a \in axaP(2,n-5)x^2a$ in(30),
(j) $xa^4x^{n-4}ax \in xa^2P(2,n-5)xax$ in (31),
(k) $xa^3x^{n-3}a^2 \in xa^2P(1,n-4)xa^2$ in (35).

We deal with these as follows. First, use (6) to expand σ_4 as

$$a^4x^{n-4} \to -\Big(\sum_{i=4}^{n-1} r_i\underline{a^2P(1,i-4)a} + \underline{a^2P(1,n-4)a} + \sum_{i=5}^{n-1} r_i\underline{a^2P(2,i-5)x} + \underline{a^2Q(2,n-5)x}$$

$$+\sum_{i=5}^{n-1} r_i\big(\underline{axP(2,i-5)a} + \underline{xaP(2,i-5)a}\big) + \underline{axP(2,n-5)a} + \underline{xaP(2,n-5)a}$$

$$+\sum_{i=6}^{n-1} r_i\big(\underline{x^2P(3,i-6)a} + \underline{axP(3,i-6)x} + \underline{xaP(3,i-6)x}\big) + \underline{x^2P(3,n-6)a}$$

$$+\underline{axP(3,n-6)x} + \underline{xaP(3,n-6)x} + \sum_{i=7}^{n-1} r_i\underline{x^2P(4,i-7)x} + \underline{x^2P(4,n-7)x}\Big) + r_4\underline{a^n}$$

$$= -\Big(\sum_{i=4}^{n} r_i\underline{a^2P(1,i-4)a} + \sum_{i=5}^{n-1} r_i\underline{a^2P(2,i-5)x} + \sum_{i=5}^{n} r_i\big(\underline{axP(2,i-5)a} + \underline{xaP(2,i-5)a}\big)$$

$$+\sum_{i=6}^{n} r_i\big(\underline{x^2P(3,i-6)a} + \underline{axP(3,i-6)x} + \underline{xaP(3,i-6)x}\big) + \sum_{i=7}^{n} r_i\underline{x^2P(4,i-7)x}$$

$$+\underline{a^2Q(2,n-5)x}\Big) + r_4\underline{a^n}.$$

To handle (h) and (j) we premultiply the above by x and postmultiply it by xa and also by ax, using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, so obtaining respectively

$$(36)\quad xa^4x^{n-4}xa \to -\Big(\sum_{i=4}^{n} r_i\underline{xa^2P(1,i-4)axa} + \sum_{i=5}^{n-1} r_i\underline{xa^2P(2,i-5)x^2a} + \underline{xa^2Q(2,n-5)x^2a}$$

$$+\sum_{i=5}^{n} r_i\big(\underline{xaxP(2,i-5)axa} + \underline{x^2aP(2,i-5)axa}\big) + \sum_{i=6}^{n} r_i\big(\underline{x^3P(3,i-6)axa}$$

$$+\underline{xaxP(3,i-6)x^2a} + x^2aP(3,i-6)x^2a\big) + \sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)x^2a}\Big) + r_4\underline{x^2a^{n+1}};$$

and

$$(37)\quad xa^4x^{n-4}ax \to -\Big(\sum_{i=4}^{n} r_i\underline{xa^2P(1,i-4)a^2x} + \sum_{i=5}^{n-1} r_i\underline{xa^2P(2,i-5)xax} + \underline{xa^2Q(2,n-5)xax}$$

$$+\sum_{i=5}^{n} r_i(\underline{xaxP(2,i-5)a^2x}+\underline{x^2aP(2,i-5)a^2x})+\sum_{i=6}^{n} r_i(\underline{x^3P(3,i-6)a^2x}$$
$$+\underline{xaxP(3,i-6)xax}+\underline{x^2aP(3,i-6)xax})+\sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)xax})+r_4\underline{xaxa^n}.$$

The reducible word $-x^2a^4x^{n-4}a$ in (36) has an opposite sign to $x^2a^4x^{n-4}a$, the term (g) in the list above, so they cancel.

To deal with (i) we use (7) to expand σ_3 as

$$a^3x^{n-3}\to -\big(\sum_{i=3}^{n-1} r_i\underline{aP(0,i-3)a^2}+\underline{aP(0,n-3)a^2}+\sum_{i=4}^{n-1} r_i(\underline{aP(1,i-4)ax}+\underline{aP(1,i-4)xa}$$
$$+\underline{xP(1,i-4)a^2})+\underline{aP(1,n-4)ax}+\underline{aP(1,n-4)xa}+\underline{xP(1,n-4)a^2}$$
$$+\sum_{i=5}^{n-1} r_i\underline{aP(2,i-5)x^2}+\underline{aQ(2,n-5)x^2}+\sum_{i=5}^{n-1} r_i\big(\underline{xP(2,i-5)ax}+\underline{xP(2,i-5)xa}\big)$$
$$+\underline{xP(2,n-5)ax}+\underline{xP(2,n-5)xa}+\sum_{i=6}^{n-1} r_i\underline{xP(3,i-6)x^2}+\underline{xP(3,n-6)x^2}\big)+r_3\underline{a^n}$$
$$=-\big(\sum_{i=3}^{n} r_i\underline{aP(0,i-3)a^2}+\sum_{i=4}^{n} r_i(\underline{aP(1,i-4)ax}+\underline{aP(1,i-4)xa}+\underline{xP(1,i-4)a^2})$$
$$+\sum_{i=5}^{n-1} r_i\underline{aP(2,i-5)x^2}+\sum_{i=5}^{n} r_i\big(\underline{xP(2,i-5)ax}+\underline{xP(2,i-5)xa}\big)+\sum_{i=6}^{n} r_i\underline{xP(3,i-6)x^2}$$
$$+\underline{aQ(2,n-5)x^2}\big)+r_3\underline{a^n}.$$

Premultiply the above by ax and postmultiply it by a, and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words, so as to obtain

$$(38)\quad axa^3x^{n-3}a\to -\big(\sum_{i=3}^{n} r_i\underline{axaP(0,i-3)a^3}+\sum_{i=4}^{n} r_i(\underline{axaP(1,i-4)axa}+\underline{axaP(1,i-4)xa^2}$$
$$+\underline{ax^2P(1,i-4)a^3})+\sum_{i=5}^{n-1} r_i\underline{axaP(2,i-5)x^2a}+\sum_{i=5}^{n} r_i\big(\underline{ax^2P(2,i-5)axa}$$
$$+\underline{ax^2P(2,i-5)xa^2}\big)+\sum_{i=6}^{n} r_i\underline{ax^2P(3,i-6)x^2a}+\underline{axaQ(2,n-5)x^2a}\big)+r_3\underline{axa^{n+1}}.$$

For (k), that is $xa^3x^{n-3}a^2$, use (6) to expand σ_3 as

$$a^3x^{n-3}\to -\big(\sum_{i=3}^{n-1} r_i\underline{a^2P(0,i-3)a}+\underline{a^2P(0,n-3)a}+\sum_{i=4}^{n-1} r_i\underline{a^2P(1,i-4)x}+\underline{a^2Q(1,n-4)x}$$
$$+\sum_{i=4}^{n-1} r_i\big(\underline{axP(1,i-4)a}+\underline{xaP(1,i-4)a}\big)+\underline{axP(1,n-4)a}+\underline{xaP(1,n-4)a}$$
$$+\sum_{i=5}^{n-1} r_i\big(\underline{x^2P(2,i-5)a}+\underline{axP(2,i-5)x}+\underline{xaP(2,i-5)x}\big)$$
$$+\underline{x^2P(2,n-5)a}+\underline{axP(2,n-5)x}+\underline{xaP(2,n-5)x}\big)+r_3\underline{a^n}$$

$$= -\Big(\sum_{i=3}^{n} r_i \underline{a^2P(0,i-3)a} + \sum_{i=4}^{n-1} r_i \underline{a^2P(1,i-4)x} + \sum_{i=4}^{n} r_i \big(\underline{axP(1,i-4)a} + \underline{xaP(1,i-4)a}\big)$$

$$+ \sum_{i=5}^{n} r_i \big(\underline{x^2P(2,i-5)a} + \underline{axP(2,i-5)x} + \underline{xaP(2,i-5)x}\big) + \underline{a^2Q(1,n-4)x}\Big) + r_3\underline{a^n}.$$

Premultiplying this by x and postmultiplying by a^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, we obtain

$$(39)\quad xa^3x^{n-3}a^2 \to -\Big(\sum_{i=3}^{n} r_i \underline{xa^2P(0,i-3)a^3} + \sum_{i=4}^{n-1} r_i \underline{xa^2P(1,i-4)xa^2} + \sum_{i=4}^{n} r_i \big(\underline{xaxP(1,i-4)a^3}$$

$$+\underline{x^2aP(1,i-4)a^3}\big) + \sum_{i=5}^{n} r_i \big(\underline{x^3P(2,i-5)a^3} + \underline{xaxP(2,i-5)xa^2} + \underline{x^2aP(2,i-5)xa^2}\big)$$

$$+\underline{xa^2Q(1,n-4)xa^2}\Big) + r_3\underline{xa^{n+2}}.$$

Thus, the reduction process ends here and when we substitute (30), (31), (32), (33), (34), (35), (36), (37), (38) and (39) into (29), we obtain

$$(40)\qquad a^3\omega_2 \to r_2\underline{a^{n+3}} + \sum_{i=4}^{n} r_i \big(\underline{a^2xP(1,i-4)(a^2x+axa+xa^2)} + \underline{axaP(1,i-4)a^2x}$$

$$+ \sum_{i=5}^{n} r_i \big(\underline{a^2xP(2,i-5)(ax^2+xax+x^2a)} + \underline{axaP(2,i-5)(ax^2+xax)} + \underline{xa^2P(2,i-5)ax^2}$$

$$+\underline{ax^2P(2,i-5)a^2x}\big) + \sum_{i=6}^{n} r_i \big(\underline{ax^2P(3,i-6)xax} + \underline{(ax^2+xax+x^2a)P(3,i-6)ax^2}\big)$$

$$+ \sum_{i=7}^{n} r_i \underline{x^3P(4,i-7)ax^2}\big) - \big(r_2\underline{a^5} + r_3\underline{a^2xa^n} + r_4\underline{ax^2a^n} + \sum_{i=3}^{n} r_i \underline{(axa+xa^2)P(0,i-3)a^3}$$

$$+ \sum_{i=4}^{n} r_i \underline{(ax^2+xax+x^2a)P(1,i-4)a^3} + \sum_{i=5}^{n} r_i \underline{x^3P(2,i-5)a^3}$$

$$+ \sum_{i=5}^{n-1} r_i a^3P(2,i-5)x^3 + \underline{a^3Q(2,n-5)x^3}\big).$$

Turning now to the reduction of ω_5x^3, use (5) to expand σ_5 as

$$\omega_5 \to -\Big(\sum_{i=5}^{n-1} r_i \underline{a^3P(2,i-5)} + \underline{a^3Q(2,n-5)} + \sum_{i=6}^{n-1} r_i \underline{(a^2x+axa+xa^2)P(3,i-6)}$$

$$+\underline{(a^2x+axa+xa^2)P(3,n-6)} + \sum_{i=7}^{n-1} r_i \underline{(ax^2+xax+xa^2)P(4,i-7)}$$

$$+\underline{(ax^2+xax+xa^2)P(4,n-7)} + \sum_{i=8}^{n-1} r_i \underline{x^3P(5,i-8)} + \underline{x^3P(5,n-8)}\Big) + r_5\underline{a^n}$$

$$= -\Big(\sum_{i=5}^{n-1} r_i \underline{a^3P(2,i-5)} + \underline{a^3Q(2,n-5)} + \sum_{i=6}^{n} r_i \underline{(a^2x+axa+xa^2)P(3,i-6)}$$

$$+ \sum_{i=7}^{n} r_i \underline{(ax^2+xax+xa^2)P(4,i-7)} + \sum_{i=8}^{n} r_i \underline{x^3P(5,i-8)}\Big) + r_5\underline{a^n}.$$

Postmultiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$(41)\quad \omega_5 x^3 \to -\big(\sum_{i=5}^{n-1} r_i a^3 P(2,i-5)x^3 + \underline{a^3 Q(2,n-5)x^3} + \sum_{i=6}^{n} r_i(a^2x+axa+xa^2)P(3,i-6)x^3 + \sum_{i=7}^{n} r_i(ax^2+xax+xa^2)P(4,i-7)x^3 + \sum_{i=8}^{n} r_i x^3 P(5,i-8)x^3\big) + r_5\underline{a^{n+3}}.$$

The following words in (41) of length $n+3$ are reducible:

(l) $x^3a^5x^{n-5} \in x^3P(5,n-8)x^3$
(m) $x^2a(a^4x^{n-4}) \in x^2aP(4,n-7)x^3$
(n) $xaxa^4x^{n-4} \in xaxP(4,n-7)x^3$
(o) $ax^2a^4x^{n-4} \in ax^2P(4,n-7)x^3$
(p) $xa^2(a^3x^{n-3}) \in xa^2P(3,n-6)x^3$
(q) $a^2xa^3x^{n-3} \in a^2xP(3,n-6)x^3$
(r) $axa(a^3x^{n-3}) \in axaP(3,n-6)x^3$.

In order to reduce (m), (n) and (o) in the list above, use (4) to expand σ_4 as

$$\begin{aligned} a^4x^{n-4} \to -\big(&\sum_{i=4}^{n-1} r_i\underline{P(1,i-4)a^3} + \underline{P(1,n-4)a^3} + \sum_{i=5}^{n-1} r_i\underline{P(2,i-5)(a^2x+axa+xa^2)} \\ &+\underline{P(2,n-5)(a^2x+axa+xa^2)} + \sum_{i=6}^{n-1} r_i\underline{P(3,i-6)(ax^2+xax+x^2a)} \\ &+\underline{P(3,n-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i\underline{P(4,i-7)x^3} + \underline{Q(4,n-7)x^3}\big) + r_4\underline{a^n} \\ = -\big(&\sum_{i=4}^{n} r_i\underline{P(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{P(2,i-5)(a^2x+axa+xa^2)} \\ &+ \sum_{i=6}^{n} r_i\underline{P(3,i-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i\underline{P(4,i-7)x^3} + \underline{Q(4,n-7)x^3}\big) + r_4\underline{a^n}. \end{aligned}$$

Premultiply this by x^2a, xax and ax^2 and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words, to obtain, respectively for (m), (n) and (o),

$$(42)\quad \begin{aligned} -x^2a(a^4x^{n-4}) \to \big(&\sum_{i=4}^{n} r_i\underline{x^2aP(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{x^2aP(2,i-5)(a^2x+axa+xa^2)} \\ &+ \sum_{i=6}^{n} r_i x^2aP(3,i-6)(ax^2+xax+x^2a) + \sum_{i=7}^{n-1} r_i\underline{x^2aP(4,i-7)x^3} \\ &+\underline{x^2aQ(4,n-7)x^3}\big) - r_4\underline{x^2a^{n+1}}; \end{aligned}$$

$$(43)\quad \begin{aligned} -xax(a^4x^{n-4}) \to \big(&\sum_{i=4}^{n} r_i\underline{xaxP(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{xaxP(2,i-5)(a^2x+axa+xa^2)} \\ &+ \sum_{i=6}^{n} r_i\underline{xaxP(3,i-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i\underline{xaxP(4,i-7)x^3} \end{aligned}$$

$$+\underline{xaxQ(4,n-7)x^3}\big) - r_4\underline{xaxa^n};$$

and

(44) $$-ax^2(a^4x^{n-4}) \to \big(\sum_{i=4}^{n} r_i\underline{ax^2P(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{ax^2P(2,i-5)(a^2x+axa+xa^2)}$$
$$+\sum_{i=6}^{n} r_i\underline{ax^2P(3,i-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i\underline{ax^2P(4,i-7)x^3}$$
$$+\underline{ax^2Q(4,n-7)x^3}\big) - r_4\underline{ax^2a^n}.$$

To deal with (l), use (4) to expand σ_5 as

$$a^5x^{n-5} \to -\big(\sum_{i=5}^{n-1} r_i\underline{P(2,i-5)a^3} + \underline{P(2,n-5)a^3} + \sum_{i=6}^{n-1} r_i\underline{P(3,i-6)(a^2x+axa+xa^2)}$$
$$+\underline{P(3,n-6)(a^2x+axa+xa^2)} + \sum_{i=7}^{n-1} r_i\underline{P(4,i-7)(ax^2+xax+ax^2)}$$
$$+\underline{P(4,n-7)(ax^2+xax+ax^2)} + \sum_{i=8}^{n-1} r_i\underline{P(5,i-8)x^3} + \underline{Q(5,n-8)x^3}\big) + r_5\underline{a^n}$$
$$= -\big(\sum_{i=5}^{n} r_i\underline{P(2,i-5)a^3} + \sum_{i=6}^{n} r_i\underline{P(3,i-6)(a^2x+axa+xa^2)}$$
$$+\sum_{i=7}^{n} r_i\underline{P(4,i-7)(ax^2+xax+ax^2)} + \sum_{i=8}^{n-1} r_i\underline{P(5,i-8)x^3} + \underline{Q(5,n-8)x^3}\big) + r_5\underline{a^n}.$$

Premultiply this by x^3 and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words to get, for (l),

(45) $$-x^3a^5x^{n-5} \to \big(\sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)a^3} + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)(a^2x+axa+xa^2)}$$
$$+\sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)(ax^2+xax+ax^2)} + \sum_{i=8}^{n-1} r_i\underline{x^3P(5,i-8)x^3}$$
$$+\underline{x^3Q(5,n-8)x^3}\big) - r_5\underline{x^3a^n}.$$

Similarly, expand σ_3 using (4) to get

$$a^3x^{n-3} \to -\big(\sum_{i=3}^{n-1} r_i\underline{P(0,i-3)a^3} + \underline{P(0,n-3)a^3} + \sum_{i=4}^{n-1} r_i\underline{P(1,i-4)(a^2x+axa+xa^2)}$$
$$+\underline{P(1,n-4)(a^2x+axa+xa^2)}+\sum_{i=5}^{n-1} r_i\underline{P(2,i-5)(ax^2+xax+x^2a)}+\underline{P(2,n-5)(ax^2+xax+x^2a)}$$
$$+\sum_{i=6}^{n-1} r_i\underline{P(3,i-6)x^3} + \underline{Q(3,n-6)x^3}\big) + r_3\underline{a^n}$$
$$= -\big(\sum_{i=3}^{n} r_i\underline{P(0,i-3)a^3} + \sum_{i=4}^{n} r_i\underline{P(1,i-4)(a^2x+axa+xa^2)}$$
$$+\sum_{i=5}^{n} r_i\underline{P(2,i-5)(ax^2+xax+x^2a)} + \sum_{i=6}^{n-1} r_i\underline{P(3,i-6)x^3} + \underline{Q(3,n-6)x^3}\big) + r_3\underline{a^n}.$$

Premultiplying this by xa^2, a^2x and axa, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields, respectively for (p), (q) and (r),

$$(46)\qquad -xa^2(a^3x^{n-3}) \to \Big(\sum_{i=3}^{n} r_i\underline{xa^2P(0,i-3)a^3} + \sum_{i=4}^{n} r_i xa^2P(1,i-4)(a^2x+axa+xa^2) + \sum_{i=5}^{n} r_i xa^2P(2,i-5)(ax^2+xax+x^2a) + \sum_{i=6}^{n-1} r_i\underline{xa^2P(3,i-6)x^3} + \underline{xa^2Q(3,n-6)x^3}\Big) - r_3\underline{xa^{n+2}};$$

$$(47)\qquad -a^2x(a^3x^{n-3}) \to \Big(\sum_{i=3}^{n} r_i a^2xP(0,i-3)a^3 + \sum_{i=4}^{n} r_i\underline{a^2xP(1,i-4)(a^2x+axa+xa^2)} + \sum_{i=5}^{n} r_i\underline{a^2xP(2,i-5)(ax^2+xax+x^2a)} + \sum_{i=6}^{n-1} r_i\underline{a^2xP(3,i-6)x^3} + \underline{a^2xQ(3,n-6)x^3}\Big) - r_3\underline{a^2xa^n};$$

and

$$(48)\qquad -axa(a^3x^{n-3}) \to \Big(\sum_{i=3}^{n} r_i\underline{axaP(0,i-3)a^3} + \sum_{i=4}^{n} r_i\underline{axaP(1,i-4)(a^2x+axa+xa^2)} + \sum_{i=5}^{n} r_i axaP(2,i-5)(ax^2+xax+x^2a) + \sum_{i=6}^{n-1} r_i\underline{axaP(3,i-6)x^3} + \underline{axaQ(3,n-6)x^3}\Big) - r_3\underline{axa^{n+1}}.$$

The following words in (42)-(48) of length $n+3$ are irreducible:

(s) $x^2a^4x^{n-4}a \in x^2aP(3,n-6)x^2a$ in (42)
(t) $x(a^4x^{n-4})xa \in xa^2P(2,n-5)x^2a$ in (46)
(u) $xa^4x^{n-4}ax \in xa^2P(2,n-5)xax$ in (46)
(v) $xa^3x^{n-3}a^2 \in xa^2P(1,n-4)xa^2$ in (46)
(w) $a^2x^{n-2}a^3$ in (47)
(x) $axa^3x^{n-3}a \in axaP(2,n-5)x^2a$ in (48)

Now observe that (36), (37), (38) and (39) respectively provide resolutions for (t), (u), (x) and (v).

In order to reduce (w), that is $a^2x^{n-2}a^3$, use (5) to expand σ_2 as

$$\begin{aligned} a^2x^{n-2} \to -\Big(&\sum_{i=3}^{n-1} r_i\underline{(axa+xa^2)P(0,i-3)} + \underline{(axa+xa^2)P(0,n-3)} + \sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)} \\ &+ \sum_{i=4}^{n-1} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)} + \underline{(ax^2+xax+x^2a)P(1,n-4)} \\ &+ \sum_{i=5}^{n-1} r_i\underline{x^3P(2,i-5)} + \underline{x^3P(2,n-5)} + r_2\underline{a^2}\Big) + r_2\underline{a^n} \\ = -\Big(&\sum_{i=3}^{n} r_i\underline{(axa+xa^2)P(0,i-3)} + \sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)} \end{aligned}$$

$$+\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)} + r_2\underline{a^2}\big) + r_2\underline{a^n}.$$

Postmultiply this by a^3 and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words to get

$$a^2x^{n-2}a^3 \to -\big(\sum_{i=3}^{n} r_i\underline{(axa+xa^2)P(0,i-3)a^3} + \sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)a^3} \tag{49}$$

$$+\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)a^3} + r_2\underline{a^5}\big) + r_2\underline{a^{n+3}}.$$

The reducible word $-x^2a^4x^{n-4}a$ in (36) has an opposite sign to $x^2a^4x^{n-4}a$, the term (s) in the above list, so they cancel. Hence, the reduction process ends here and when we substitute (42)-(49) and (36)-(39) into (31), we obtain

$$\omega_5x^3 \to \big(r_2\underline{a^{n+3}} + \sum_{i=4}^{n} r_i(\underline{a^2xP(1,i-4)(a^2x+axa+xa^2)} + \underline{axaP(1,i-4)a^2x}) \tag{50}$$

$$+\sum_{i=5}^{n} r_i\big(\underline{a^2xP(2,i-5)(ax^2+xax+x^2a)} + \underline{axaP(2,i-5)(ax^2+xax)} + \underline{xa^2P(2,i-5)ax^2}$$

$$+\underline{ax^2P(2,i-5)a^2x}\big) + \sum_{i=6}^{n} r_i\big(\underline{ax^2P(3,i-6)xax} + \underline{(ax^2+xax+x^2a)P(3,i-6)ax^2}\big)$$

$$+\sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)ax^2}\big) - \big(r_2\underline{a^5} + r_3\underline{a^2xa^n} + r_4\underline{ax^2a^n} + \sum_{i=3}^{n} r_i\underline{(axa+xa^2)P(0,i-3)a^3}$$

$$+\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)a^3}$$

$$+\sum_{i=5}^{n-1} r_ia^3P(2,i-5)x^3 + \underline{a^3Q(2,n-5)x^3}\big).$$

Comparing (40) and (50), we conclude that the overlap ambiguity $(\sigma_5, \sigma_2, a^3, a^2x^{n-5}, x^3)$ is resolvable.

3.3. **Case (c):** $3 \le j < n-4$. Consider the overlap ambiguity $(\sigma_{j+3}, \sigma_j, a^3, a^jx^{n-j-3}, x^3)$. Using (4), expand σ_j to get

$$a^jx^{n-j} \to -\big(\sum_{i=j}^{n-1} r_i\underline{P(j-3,i-j)a^3} + \underline{P(j-3,n-j)a^3} + \sum_{i=j+1}^{n-1} r_i\underline{P(j-2,i-j-1)(a^2x+axa+xa^2)}$$

$$+\underline{P(j-2,n-j-1)(a^2x+axa+xa^2)} + \sum_{i=j+2}^{n-1} r_i\underline{P(j-1,i-j-2)(ax^2+xax+x^2a)}$$

$$+\underline{P(j-1,n-j-2)(ax^2+xax+x^2a)} + \sum_{i=j+3}^{n-1} r_i\underline{P(j,i-j-3)x^3} + \underline{Q(j,n-j-3)x^3}\big) + r_j\underline{a^n}$$

$$= -\big(\sum_{i=j}^{n} r_i\underline{P(j-3,i-j)a^3} + \sum_{i=j+1}^{n} r_i\underline{P(j-2,i-j-1)(a^2x+axa+xa^2)}$$

$$+\sum_{i=j+2}^{n} r_i \underline{P(j-1,i-j-2)(ax^2+xax+x^2a)}+\sum_{i=j+3}^{n-1} r_i \underline{P(j,i-j-3)x^3}+\underline{Q(j,n-j-3)x^3}\big)+r_j\underline{a^n}.$$

Premultiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$(51)\qquad a^3\omega_j \to -\big(\sum_{i=j}^{n} r_i a^3 P(j-3,i-j)a^3 + \sum_{i=j+1}^{n} r_i a^3 P(j-2,i-j-1)(a^2x+axa+xa^2)$$
$$+\sum_{i=j+2}^{n} r_i a^3 P(j-1,i-j-2)(ax^2+xax+x^2a)+\sum_{i=j+3}^{n-1} r_i a^3 P(j,i-j-3)x^3$$
$$+\underline{a^3Q(j,n-j-3)x^3}\big)+r_j\underline{a^{n+3}}.$$

The following words in (51) of length $n+3$ are reducible:

(a) $a^jx^{n-j}a^3 \in a^3P(j-3,n-j)a^3$
(b) $(a^{j+2}x^{n-j-2})x^2a \in a^3P(j-1,n-j-2)x^2a$
(c) $(a^{j+2}x^{n-j-2})xax \in a^3P(j-1,n-j-2)xax$
(d) $(a^{j+2}x^{n-j-2})ax^2 \in a^3P(j-1,n-j-2)ax^2$
(e) $(a^{j+1}x^{n-j-1})xa^2 \in a^3P(j-2,n-j-1)xa^2$
(f) $(a^{j+1}x^{n-j-1})axa \in a^3P(j-2,n-j-1)axa$
(g) $(a^{j+1}x^{n-j-1})a^2x \in a^3P(j-2,n-j-1)a^2x.$

To handle (a), first expand σ_j using (5) to obtain:

$$a^jx^{n-j} \to -\big(\sum_{i=j}^{n-1} r_i\underline{a^3P(j-3,i-j)}+\underline{a^3Q(j-3,n-j)}+\sum_{i=j+1}^{n-1} r_i\underline{(a^2x+axa+xa^2)P(j-2,i-j-1)}$$
$$+\underline{(a^2x+axa+xa^2)P(j-2,n-j-1)}+\sum_{i=j+2}^{n-1} r_i\underline{(ax^2+xax+x^2a)P(j-1,i-j-2)}$$
$$+\underline{(ax^2+xax+x^2a)P(j-1,n-j-2)}+\sum_{i=j+3}^{n-1} r_i\underline{x^3P(j,i-j-3)}+\underline{x^3P(j,n-j-3)}\big)+r_j\underline{a^n}$$
$$=-\big(\sum_{i=j}^{n-1} r_i\underline{a^3P(j-3,i-j)}+\underline{a^3Q(j-3,n-j)}+\sum_{i=j+1}^{n} r_i\underline{(a^2x+axa+xa^2)P(j-2,i-j-1)}$$
$$+\sum_{i=j+2}^{n} r_i\underline{(ax^2+xax+x^2a)P(j-1,i-j-2)}+\sum_{i=j+3}^{n} r_i\underline{x^3P(j,i-j-3)}\big)+r_j\underline{a^n}.$$

Postmultiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields

$$(52)\qquad -a^jx^{n-j}a^3 \to \big(\sum_{i=j}^{n-1} r_i\underline{a^3P(j-3,i-j)a^3}+\underline{a^3Q(j-3,n-j)a^3}$$
$$+\sum_{i=j+1}^{n} r_i\underline{(a^2x+axa+xa^2)P(j-2,i-j-1)a^3}$$
$$+\sum_{i=j+2}^{n} r_i\underline{(ax^2+xax+x^2a)P(j-1,i-j-2)a^3}+\sum_{i=j+3}^{n} r_i\underline{x^3P(j,i-j-3)a^3}\big)-r_j\underline{a^{n+3}}.$$

For (b), (c) and (d), expand σ_{j+2} using (5) to obtain

$$a^{j+2}x^{n-j-2} \to -\Big(\sum_{i=j+2}^{n-1} r_i \underline{a^3P(j-1,i-j-2)} + \underline{a^3Q(j-1,n-j-2)}$$

$$+ \sum_{i=j+3}^{n-1} r_i \underline{(a^2x+axa+xa^2)P(j,i-j-3)} + \underline{(a^2x+axa+xa^2)P(j,n-j-3)}$$

$$+ \sum_{i=j+4}^{n-1} r_i \underline{(ax^2+xax+x^2a)P(j+1,i-j-4)} + \underline{(ax^2+xax+x^2a)P(j+1,n-j-4)}$$

$$+ \sum_{i=j+5}^{n-1} r_i \underline{x^3P(j+2,i-j-5)} + \underline{x^3P(j+2,n-j-5)}\Big) + r_{j+2}\underline{a^n}$$

$$= -\Big(\sum_{i=j+2}^{n-1} r_i \underline{a^3P(j-1,i-j-2)} + \underline{a^3Q(j-1,n-j-2)}$$

$$+ \sum_{i=j+3}^{n} r_i \underline{(a^2x+axa+xa^2)P(j,i-j-3)} + \sum_{i=j+4}^{n} r_i \underline{(ax^2+xax+x^2a)P(j+1,i-j-4)}$$

$$+ \sum_{i=j+5}^{n} r_i \underline{x^3P(j+2,i-j-5)}\Big) + r_{j+2}\underline{a^n}$$

Postmultiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields, for (b), (c) and (d) respectively,

$$-(a^{j+2}x^{n-j-2})x^2a \to \Big(\sum_{i=j+2}^{n-1} r_i a^3P(j-1,i-j-2)x^2a + \underline{a^3Q(j-1,n-j-2)x^2a} \tag{53}$$

$$+ \sum_{i=j+3}^{n} r_i (a^2x+axa+xa^2)P(j,i-j-3)x^2a + \sum_{i=j+4}^{n} r_i (ax^2+xax+x^2a)P(j+1,i-j-4)x^2a$$

$$+ \sum_{i=j+5}^{n} r_i \underline{x^3P(j+2,i-j-5)x^2a}\Big) - r_{j+2}\underline{x^2a^{n+1}};$$

$$-(a^{j+2}x^{n-j-2})xax \to \Big(\sum_{i=j+2}^{n-1} r_i \underline{a^3P(j-1,i-j-2)xax} + \underline{a^3Q(j-1,n-j-2)xax} \tag{54}$$

$$+ \sum_{i=j+3}^{n} r_i (a^2x+axa+xa^2)P(j,i-j-3)xax + \sum_{i=j+4}^{n} r_i \underline{(ax^2+xax+x^2a)P(j+1,i-j-4)xax}$$

$$+ \sum_{i=j+5}^{n} r_i \underline{x^3P(j+2,i-j-5)xax}\Big) - r_{j+2}\underline{xaxa^n};$$

and

$$-(a^{j+2}x^{n-j-2})ax^2 \to \Big(\sum_{i=j+2}^{n-1} r_i \underline{a^3P(j-1,i-j-2)ax^2} + \underline{a^3Q(j-1,n-j-2)ax^2} \tag{55}$$

$$+\sum_{i=j+3}^{n} r_i\underline{(a^2x+axa+xa^2)P(j,i-j-3)ax^2}+\sum_{i=j+4}^{n} r_i\underline{(ax^2+xax+x^2a)P(j+1,i-j-4)ax^2}$$

$$+\sum_{i=j+5}^{n} r_i\underline{x^3P(j+2,i-j-5)ax^2}\big)-r_{j+2}\underline{ax^2a^n}.$$

Now, to deal with (e), (f) and (g), expand σ_{j+1} using (5) as

$$a^{j+1}x^{n-j-1}\to-\big(\sum_{i=j+1}^{n-1} r_i\underline{a^3P(j-2,i-j-1)}+\underline{a^3Q(j-2,n-j-1)}$$

$$+\sum_{i=j+2}^{n-1} r_i\underline{(a^2x+axa+xa^2)P(j-1,i-j-2)}+\underline{(a^2x+axa+xa^2)P(j-1,n-j-2)}$$

$$+\sum_{i=j+3}^{n-1} r_i\underline{(ax^2+xax+x^2a)P(j,i-j-3)}+\underline{(ax^2+xax+x^2a)P(j,n-j-3)}$$

$$+\sum_{i=j+4}^{n-1} r_i\underline{x^3P(j+1,i-j-4)}+\underline{x^3P(j+1,n-j-4)}\big)+r_{j+1}\underline{a^n}.$$

$$=-\big(\sum_{i=j+1}^{n-1} r_i\underline{a^3P(j-2,i-j-1)}+\underline{a^3Q(j-2,n-j-1)}$$

$$+\sum_{i=j+2}^{n} r_i\underline{(a^2x+axa+xa^2)P(j-1,i-j-2)}+\sum_{i=j+3}^{n} r_i\underline{(ax^2+xax+x^2a)P(j,i-j-3)}$$

$$+\sum_{i=j+4}^{n} r_i\underline{x^3P(j+1,i-j-4)}\big)+r_{j+1}\underline{a^n}.$$

Hence, postmultiplying this by xa^2, axa and a^2x, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields respectively for (e), (f) and (g)

$$(56)\qquad -(a^{j+1}x^{n-j-1})xa^2\to\big(\sum_{i=j+1}^{n-1} r_i\underline{a^3P(j-2,i-j-1)xa^2}+\underline{a^3Q(j-2,n-j-1)xa^2}$$

$$+\sum_{i=j+2}^{n} r_i(a^2x+axa+xa^2)P(j-1,i-j-2)xa^2+\sum_{i=j+3}^{n} r_i\underline{(ax^2+xax+x^2a)P(j,i-j-3)xa^2}$$

$$+\sum_{i=j+4}^{n} r_i\underline{x^3P(j+1,i-j-4)xa^2}\big)-r_{j+1}\underline{xa^{n+2}};$$

$$(57)\qquad -(a^{j+1}x^{n-j-1})axa\to\big(\sum_{i=j+1}^{n-1} r_i\underline{a^3P(j-2,i-j-1)axa}+\underline{a^3Q(j-2,n-j-1)axa}$$

$$+\sum_{i=j+2}^{n} r_i\underline{(a^2x+axa+xa^2)P(j-1,i-j-2)axa}+\sum_{i=j+3}^{n} r_i\underline{(ax^2+xax+x^2a)P(j,i-j-3)axa}$$

$$+\sum_{i=j+4}^{n} r_i\underline{x^3P(j+1,i-j-4)axa}\big)-r_{j+1}\underline{axa^{n+1}};$$

and

$$(58)\qquad -(a^{j+1}x^{n-j-1})a^2x \to \Big(\sum_{i=j+1}^{n-1} r_i\underline{a^3P(j-2,i-j-1)a^2x} + \underline{a^3Q(j-2,n-j-1)a^2x}$$

$$+\sum_{i=j+2}^{n} r_i\underline{(a^2x+axa+xa^2)P(j-1,i-j-2)a^2x} + \sum_{i=j+3}^{n} r_i\underline{(ax^2+xax+x^2a)P(j,i-j-3)a^2x}$$

$$+\sum_{i=j+4}^{n} r_i\underline{x^3P(j+1,i-j-4)a^2x}\Big) - r_{j+1}\underline{a^2xa^n}$$

respectively. The words in (52)-(58) of length $n+3$ which are reducible are as follows:

(h) $x^2a^{j+2}x^{n-j-2}a \in x^2aP(j+1,n-j-4)x^2a$ in (53),
(i) $x(a^{j+2}x^{n-j-2})xa \in xa^2P(j,n-j-3)x^2a$ in (53),
(j) $x(a^{j+2}x^{n-j-2})ax \in xa^2P(j,n-j-3)xax$ in (54),
(k) $xa^{j+1}x^{n-j-1}a^2 \in xa^2P(j-1,n-j-2)xa^2$ in (56),
(l) $axa^{j+1}x^{n-j-1}a \in axaP(j,n-j-3)x^2a$ in (53).

To handle (i) and (j) expand σ_{j+2} using (6), to obtain

$$a^{j+2}x^{n-j-2} \to -\Big(\sum_{i=j+2}^{n-1} r_i\underline{a^2P(j-1,i-j-2)a} + \underline{a^2P(j-1,n-j-2)a}$$

$$+\sum_{i=j+3}^{n-1} r_i\underline{a^2P(j,i-j-3)x} + \underline{a^2Q(j,n-j-3)x} + \sum_{i=j+3}^{n-1} r_i\big(\underline{axP(j,i-j-3)a}$$

$$+\underline{xaP(j,i-j-3)a}\big) + \underline{axP(j,n-j-3)a} + \underline{xaP(j,n-j-3)a}$$

$$+\sum_{i=j+4}^{n-1} r_i\big(\underline{axP(j+1,i-j-4)x} + \underline{xaP(j+1,i-j-4)x} + \underline{x^2P(j+1,i-j-4)a}\big)$$

$$+\underline{axP(j+1,n-j-4)x} + \underline{xaP(j+1,n-j-4)x} + \underline{x^2P(j+1,n-j-4)a}$$

$$+\sum_{i=j+5}^{n-1} r_i\underline{x^2P(j+2,i-j-5)x} + \underline{x^2P(j+2,n-j-5)x}\big) + r_{j+2}\underline{a^n}$$

$$= -\Big(\sum_{i=j+2}^{n} r_i\underline{a^2P(j-1,i-j-2)a} + \sum_{i=j+3}^{n-1} r_i\underline{a^2P(j,i-j-3)x} + \underline{a^2Q(j,n-j-3)x}$$

$$+\sum_{i=j+3}^{n} r_i\big(\underline{axP(j,i-j-3)a} + \underline{xaP(j,i-j-3)a}\big) + \sum_{i=j+4}^{n} r_i\big(\underline{axP(j+1,i-j-4)x}$$

$$+\underline{xaP(j+1,i-j-4)x} + \underline{x^2P(j+1,i-j-4)a}\big) + \sum_{i=j+5}^{n} r_i\underline{x^2P(j+2,i-j-5)x}\big) + r_{j+2}\underline{a^n}.$$

Premultiplying this by x and postmultiplying by xa and ax, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields, for (i) and (j) respectively,

$$(59)\quad x(a^{j+2}x^{n-j-2})xa \to -\Big(\sum_{i=j+2}^{n} r_i\underline{xa^2P(j-1,i-j-2)axa} + \sum_{i=j+3}^{n-1} r_i\underline{xa^2P(j,i-j-3)x^2a}$$

$$+\underline{xa^2Q(j,n-j-3)x^2a}+\sum_{i=j+3}^{n} r_i\big(\underline{xaxP(j,i-j-3)axa}+\underline{x^2aP(j,i-j-3)axa}\big)$$

$$+\sum_{i=j+4}^{n} r_i\big(\underline{xaxP(j+1,i-j-4)x^2a}+x^2aP(j+1,i-j-4)x^2a$$

$$+\underline{x^3P(j+1,i-j-4)axa}\big)+\sum_{i=j+5}^{n} r_i\underline{x^3P(j+2,i-j-5)x^2a}\big)+r_{j+2}\underline{x^2a^{n+1}};$$

and

$$x(a^{j+2}x^{n-j-2})ax \to -\big(\sum_{i=j+2}^{n} r_i\underline{xa^2P(j-1,i-j-2)a^2x}+\sum_{i=j+3}^{n-1} r_i\underline{xa^2P(j,i-j-3)xax} \tag{60}$$

$$+\underline{xa^2Q(j,n-j-3)xax}+\sum_{i=j+3}^{n} r_i\big(\underline{xaxP(j,i-j-3)a^2x}+\underline{x^2aP(j,i-j-3)a^2x}\big)$$

$$+\sum_{i=j+4}^{n} r_i\big(\underline{xaxP(j+1,i-j-4)xax}+\underline{x^2aP(j+1,i-j-4)xax}$$

$$+\underline{x^3P(j+1,i-j-4)a^2x}\big)+\sum_{i=j+5}^{n} r_i\underline{x^3P(j+2,i-j-5)xax}\big)+r_{j+2}\underline{xaxa^n}.$$

For (k), expand σ_{j+1} using (6) to get

$$a^{j+1}x^{n-j-1} \to -\big(\sum_{i=j+1}^{n-1} r_i\underline{a^2P(j-2,i-j-1)a}+\underline{a^2P(j-2,n-j-1)a}+$$

$$\sum_{i=j+2}^{n-1} r_i\underline{a^2P(j-1,i-j-2)x}+\underline{a^2Q(j-1,n-j-2)x}$$

$$+\sum_{i=j+2}^{n-1} r_i\big(\underline{axP(j-1,i-j-2)a}+\underline{xaP(j-1,i-j-2)a}\big)$$

$$+\underline{axP(j-1,n-j-2)a}+\underline{xaP(j-1,n-j-2)a}$$

$$\sum_{i=j+3}^{n-1} r_i\big(\underline{axP(j,i-j-3)x}+\underline{xaP(j,i-j-3)x}+\underline{x^2P(j,i-j-3)a}\big)$$

$$+\underline{axP(j,n-j-3)x}+\underline{xaP(j,n-j-3)x}+\underline{x^2P(j,n-j-3)a}$$

$$+\sum_{i=j+4}^{n-1} r_i\underline{x^2P(j+1,i-j-4)x}+\underline{x^2P(j+1,n-j-4)x}\big)+r_{j+1}\underline{a^n}$$

$$=-\big(\sum_{i=j+1}^{n} r_i\underline{a^2P(j-2,i-j-1)a}+\sum_{i=j+2}^{n-1} r_i\underline{a^2P(j-1,i-j-2)x}$$

$$+\underline{a^2Q(j-1,n-j-2)x}+\sum_{i=j+2}^{n} r_i\big(\underline{axP(j-1,i-j-2)a}+\underline{xaP(j-1,i-j-2)a}\big)$$

$$+\sum_{i=j+3}^{n} r_i\big(\underline{axP(j,i-j-3)x}+\underline{xaP(j,i-j-3)x}+\underline{x^2P(j,i-j-3)a}\big)$$

$$+\sum_{i=j+4}^{n} r_i \underline{x^2P(j+1,i-j-4)x}\big) + r_{j+1}\underline{a^n}.$$

Premultiplying this by x and postmultiplying by a^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields, for (k),

$$xa^{j+1}x^{n-j-1}a^2 \to -\big(\sum_{i=j+1}^{n} r_i\underline{xa^2P(j-2,i-j-1)a^3} + \sum_{i=j+2}^{n-1} r_i\underline{xa^2P(j-1,i-j-2)xa^2} \tag{61}$$

$$+\underline{xa^2Q(j-1,n-j-2)xa^2} + \sum_{i=j+2}^{n} r_i\big(\underline{xaxP(j-1,i-j-2)a^3} + \underline{x^2aP(j-1,i-j-2)a^3}\big)$$

$$+\sum_{i=j+3}^{n} r_i\big(\underline{xaxP(j,i-j-3)xa^2} + \underline{x^2aP(j,i-j-3)xa^2} + \underline{x^3P(j,i-j-3)a^3}\big)$$

$$+\sum_{i=j+4}^{n} r_i\underline{x^3P(j+1,i-j-4)xa^2}\big) + r_{j+1}\underline{xa^{n+2}}.$$

To deal with (l) expand σ_{j+1} using (7) to get

$$a^{j+1}x^{n-j-1} \to -\big(\sum_{i=j+1}^{n-1} r_i\underline{aP(j-2,i-j-1)a^2} + \underline{aP(j-2,n-j-1)a^2}$$

$$+\sum_{i=j+2}^{n-1} r_i\big(\underline{xP(j-1,i-j-2)a^2} + \underline{aP(j-1,i-j-2)ax} + \underline{aP(j-1,i-j-2)xa}\big)$$

$$\underline{xP(j-1,n-j-2)a^2} + \underline{aP(j-1,n-j-2)ax} + \underline{aP(j-1,n-j-2)xa}$$

$$+\sum_{i=j+3}^{n-1} r_i\big(\underline{xP(j,i-j-3)ax} + \underline{xP(j,i-j-3)xa}\big) + \underline{xP(j,n-j-3)ax}$$

$$+\underline{xP(j,n-j-3)xa} + \sum_{i=j+3}^{n-1} r_i\underline{aP(j,i-j-3)x^2} + \underline{aQ(j,n-j-3)x^2}$$

$$+\sum_{i=j+4}^{n-1} r_i\underline{xP(j+1,i-j-4)x^2} + \underline{xP(j+1,n-j-4)x^2}\big) + r_{j+1}\underline{a^n}$$

$$= -\big(\sum_{i=j+1}^{n} r_i\underline{aP(j-2,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i(\underline{xP(j-1,i-j-2)a^2}$$

$$+\underline{aP(j-1,i-j-2)ax} + \underline{aP(j-1,i-j-2)xa}\big) + \sum_{i=j+3}^{n} r_i\big(\underline{xP(j,i-j-3)ax}$$

$$+\underline{xP(j,i-j-3)xa}\big) + \sum_{i=j+3}^{n-1} r_i\underline{aP(j,i-j-3)x^2} + \underline{aQ(j,n-j-3)x^2}$$

$$+\sum_{i=j+4}^{n} r_i\underline{xP(j+1,i-j-4)x^2}\big) + r_{j+1}\underline{a^n}.$$

Premultiplying this by ax and postmultiplying by a, using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields, for (l),

$$(62)\quad axa^{j+1}x^{n-j-1}a \to -\Big(\sum_{i=j+1}^{n} r_i \underline{axaP(j-2,i-j-1)a^3} + \sum_{i=j+2}^{n} r_i\big(\underline{ax^2P(j-1,i-j-2)a^3}$$
$$+\underline{axaP(j-1,i-j-2)axa} + \underline{axaP(j-1,i-j-2)xa^2}\big) + \sum_{i=j+3}^{n} r_i\big(\underline{ax^2P(j,i-j-3)axa}$$
$$+\underline{ax^2P(j,i-j-3)xa^2}\big) + \sum_{i=j+3}^{n-1} r_i \underline{axaP(j,i-j-3)x^2a} + \underline{axaQ(j,n-j-3)x^2a}$$
$$+\sum_{i=j+4}^{n} r_i \underline{ax^2P(j+1,i-j-4)x^2a}\Big) + r_{j+1}\underline{axa^{n+1}}.$$

The reducible word $-x^2a^{j+2}x^{n-j-2}a$ in (59) has an opposite sign with $x^2a^{j+2}x^{n-j-2}a$, which appears as (h) in the list above, so they cancel. Thus, the reduction process ends here and when we substitute (52)-(62) into (51), we get

$$(63)\quad a^3\omega_j \to \Big(\sum_{i=j+1}^{n} r_i \underline{a^2xP(j-2,i-j-1)a^3} + \sum_{i=j+2}^{n} r_i\big(\underline{a^2xP(j,i-j-2)(a^2x+axa+xa^2)}$$
$$+\underline{axaP(j,i-j-2)a^2x}\big) + \sum_{i=j+3}^{n} r_i\big(\underline{a^2xP(j,i-j-3)(ax^2+xax+x^2a)} + \underline{axaP(j,i-j-3)(ax^2+xax)}$$
$$+\underline{xa^2P(j,i-j-3)ax^2} + \underline{ax^2P(j,i-j-3)a^2x}\big) + \sum_{i=j+4}^{n} r_i\big(\underline{ax^2P(j+1,i-j-4)(ax^2+xax)}$$
$$+\underline{(xax+x^2a)P(j+1,i-j-4)ax^2}\big) + \sum_{i=j+5}^{n} r_i \underline{x^3P(j+2,i-j-5)ax^2}\Big)$$
$$-\big(r_{j+1}\underline{a^2xa^n} + r_{j+2}\underline{ax^2a^n} + \sum_{i=j+3}^{n-1} r_i a^3P(j,i-j)x^3 + \underline{a^3Q(j,n-j)x^3}\big).$$

We turn now to the alternative face of the ambiguity. Begin by using (5) to expand σ_{j+3} as

$$a^{j+3}x^{n-j-3} \to -\Big(\sum_{i=j+3}^{n-1} r_i \underline{a^3P(j,i-j-3)} + \underline{a^3Q(j,n-j-3)} + \sum_{i=j+4}^{n-1} r_i \underline{(a^2x+axa+xa^2)P(j+1,i-j-4)}$$
$$+\underline{(a^2x+axa+xa^2)P(j+1,n-j-4)} + \sum_{i=j+5}^{n} r_i \underline{(ax^2+xax+x^2a)P(j+2,i-j-5)}$$
$$+\sum_{i=j+6}^{n-1} r_i \underline{x^3P(j+3,i-j-6))} + \underline{x^3P(j+3,n-j-6))}\Big) + r_{j+3}\underline{a^n}$$
$$= -\Big(\sum_{i=j+3}^{n-1} r_i \underline{a^3P(j,i-j-3)} + \underline{a^3Q(j,n-j-3)} + \sum_{i=j+4}^{n} r_i \underline{(a^2x+axa+xa^2)P(j+1,i-j-4)}$$
$$+\sum_{i=j+5}^{n} r_i \underline{(ax^2+xax+x^2a)P(j+2,i-j-5)} + \sum_{i=j+6}^{n} r_i \underline{x^3P(j+3,i-j-6))}\Big) + r_{j+3}\underline{a^n}.$$

Postmultiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

(64)
$$\omega_{j+3}x^3 \to -\Big(\sum_{i=j+3}^{n-1} r_i a^3 P(j,i-j-3)x^3 + \underline{a^3 Q(j,n-j-3)x^3}$$
$$+\sum_{i=j+4}^{n} r_i(a^2x+axa+xa^2)P(j+1,i-j-4)x^3$$
$$+\sum_{i=j+5}^{n} r_i(ax^2+xax+x^2a)P(j+2,i-j-5)x^3$$
$$+\sum_{i=j+6}^{n} r_i x^3 P(j+3,i-j-6)x^3\Big) + r_{j+3}\underline{a^{n+3}}.$$

The reducible words in (64) of length $n+3$ are:

(a) $x^2a(a^{j+2}x^{n-j-2}) \in x^2aP(j+2,n-j-5)x^3$
(b) $xax(a^{j+2}x^{n-j-2}) \in xaxP(j+2,n-j-5)x^3$
(c) $ax^2(a^{j+2}x^{n-j-2}) \in ax^2P(j+2,n-j-5)x^3$
(d) $xa^2(a^{j+1}x^{n-j-1}) \in xa^2P(j+1,n-j-4)x^3$
(e) $axa(a^{j+1}x^{n-j-1}) \in axaP(j+1,n-j-4)x^3$
(f) $a^2x(a^{j+1}x^{n-j-1}) \in a^2xP(j+1,n-j-4)x^3$
(g) $x^3a^{j+3}x^{n-j-3} \in x^3P(j+3,n-j-6)x^3$.

To reduce (a), (b) and (c), first expand σ_{j+2} using (4) as

$$a^{j+2}x^{n-j-2} \to -\Big(\sum_{i=j+2}^{n-1} r_i\underline{P(j-1,i-j-2)a^3} + \underline{P(j-1,n-j-2)a^3}$$
$$+\sum_{i=j+3}^{n-1} r_i\underline{P(j,i-j-3)(a^2x+axa+xa^2)} + \underline{P(j,n-j-3)(a^2x+axa+xa^2)}$$
$$+\sum_{i=j+4}^{n-1} r_i\underline{P(j+1,i-j-4)(ax^2+xax+x^2a)} + \underline{P(j+1,n-j-4)(ax^2+xax+x^2a)}$$
$$+\sum_{i=j+5}^{n-1} r_i\underline{P(j+2,i-j-5)x^3} + \underline{Q(j+2,n-j-5)x^3}\Big) + r_{j+2}\underline{a^n}$$
$$= -\Big(\sum_{i=j+2}^{n} r_i\underline{P(j-1,i-j-2)a^3} + \sum_{i=j+3}^{n} r_i\underline{P(j,i-j-3)(a^2x+axa+xa^2)}$$
$$+\sum_{i=j+4}^{n} r_i\underline{P(j+1,i-j-4)(ax^2+xax+x^2a)} + \sum_{i=j+5}^{n-1} r_i\underline{P(j+2,i-j-5)x^3}$$
$$+\underline{Q(j+2,n-j-5)x^3}\Big) + r_{j+2}\underline{a^n}.$$

Premultiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields respectively, for (a), (b) and (c),

(65)
$$-x^2a(a^{j+2}x^{n-j-2}) \to \Big(\sum_{i=j+2}^{n} r_i\underline{x^2aP(j-1,i-j-2)a^3} + \sum_{i=j+3}^{n} r_i\underline{x^2aP(j,i-j-3)(a^2x+axa+xa^2)}$$

$$+\sum_{i=j+4}^{n} r_i x^2aP(j+1,i-j-4)(ax^2+xax+x^2a)+\sum_{i=j+5}^{n-1} r_i\underline{x^2aP(j+2,i-j-5)x^3}$$
$$+\underline{x^2aQ(j+2,n-j-5)x^3}\big)-r_{j+2}\underline{x^2a^{n+1}};$$

(66)
$$-xax(a^{j+2}x^{n-j-2})\to\big(\sum_{i=j+2}^{n} r_i\underline{xaxP(j-1,i-j-2)a^3}+\sum_{i=j+3}^{n} r_i\underline{xaxP(j,i-j-3)(a^2x+axa+xa^2)}$$
$$+\sum_{i=j+4}^{n} r_i\underline{xaxP(j+1,i-j-4)(ax^2+xax+x^2a)}+\sum_{i=j+5}^{n-1} r_i\underline{xaxP(j+2,i-j-5)x^3}$$
$$+\underline{xaxQ(j+2,n-j-5)x^3}\big)-r_{j+2}\underline{xaxa^n};$$

and
(67)
$$-ax^2(a^{j+2}x^{n-j-2})\to\big(\sum_{i=j+2}^{n} r_i\underline{ax^2P(j-1,i-j-2)a^3}+\sum_{i=j+3}^{n} r_i\underline{ax^2P(j,i-j-3)(a^2x+axa+xa^2)}$$
$$+\sum_{i=j+4}^{n} r_i\underline{ax^2P(j+1,i-j-4)(ax^2+xax+x^2a)}+\sum_{i=j+5}^{n-1} r_i\underline{ax^2P(j+2,i-j-5)x^3}$$
$$+\underline{ax^2Q(j+2,n-j-5)x^3}\big)-r_{j+2}\underline{ax^2a^n}.$$

For (d), (e) and (f) first expand σ_{j+1} using (4) as

$$a^{j+1}x^{n-j-1}\to-\big(\sum_{i=j+1}^{n-1} r_i\underline{P(j-2,i-j-1)a^3}+\underline{P(j-2,n-j-1)a^3}$$
$$+\sum_{i=j+2}^{n-1} r_i\underline{P(j-1,i-j-2)(a^2x+axa+xa^2)}+\underline{P(j-1,n-j-2)(a^2x+axa+xa^2)}$$
$$+\sum_{i=j+3}^{n-1} r_i\underline{P(j,i-j-3)(ax^2+xax+x^2a)}+\underline{P(j,n-j-3)(ax^2+xax+x^2a)}$$
$$+\sum_{i=j+4}^{n-1} r_i\underline{P(j+1,i-j-4)x^3}+\underline{Q(j+1,n-j-4)x^3}\big)+r_{j+1}\underline{a^n}.$$
$$=-\big(\sum_{i=j+1}^{n} r_i\underline{P(j-2,i-j-1)a^3}+\sum_{i=j+2}^{n} r_i\underline{P(j-1,i-j-2)(a^2x+axa+xa^2)}$$
$$+\sum_{i=j+3}^{n} r_i\underline{P(j,i-j-3)(ax^2+xax+x^2a)}+\sum_{i=j+4}^{n-1} r_i\underline{P(j+1,i-j-4)x^3}$$
$$+\underline{Q(j+1,n-j-4)x^3}\big)+r_{j+1}\underline{a^n}.$$

Premultiplying this by xa^2, axa and a^2x, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields respectively for (d), (e) and (f),
(68)
$$-xa^2(a^{j+1}x^{n-j-1})\to\big(\sum_{i=j+1}^{n} r_i\underline{xa^2P(j-2,i-j-1)a^3}+\sum_{i=j+2}^{n} r_i xa^2P(j-1,i-j-2)(a^2x+axa+xa^2)$$

$$+\sum_{i=j+3}^{n} r_i xa^2P(j,i-j-3)(ax^2+xax+x^2a)+\sum_{i=j+4}^{n-1} r_i \underline{xa^2P(j+1,i-j-4)x^3}$$
$$+\underline{xa^2Q(j+1,n-j-4)x^3}\big)-r_{j+1}\underline{xa^{n+2}};$$

(69)

$$-axa(a^{j+1}x^{n-j-1})\to\big(\sum_{i=j+1}^{n} r_i \underline{axaP(j-2,i-j-1)a^3}+\sum_{i=j+2}^{n} r_i \underline{axaP(j-1,i-j-2)(a^2x+axa+xa^2)}$$
$$+\sum_{i=j+3}^{n} r_i axaP(j,i-j-3)(ax^2+xax+x^2a)+\sum_{i=j+4}^{n-1} r_i \underline{axaP(j+1,i-j-4)x^3}$$
$$+\underline{axaQ(j+1,n-j-4)x^3}\big)-r_{j+1}\underline{axa^{n+1}};$$

and

(70)

$$-a^2x(a^{j+1}x^{n-j-1})\to\big(\sum_{i=j+1}^{n} r_i \underline{a^2xP(j-2,i-j-1)a^3}+\sum_{i=j+2}^{n} r_i \underline{a^2xP(j-1,i-j-2)(a^2x+axa+xa^2)}$$
$$+\sum_{i=j+3}^{n} r_i \underline{a^2xP(j,i-j-3)(ax^2+xax+x^2a)}+\sum_{i=j+4}^{n-1} r_i \underline{a^2xP(j+1,i-j-4)x^3}$$
$$+\underline{a^2xQ(j+1,n-j-4)x^3}\big)-r_{j+1}\underline{a^2xa^n}.$$

To deal with (g), use (4) to expand σ_{j+3} as

$$a^{j+3}x^{n-j-3}\to-\big(\sum_{i=j+3}^{n-1} r_i \underline{P(j,i-j-3)a^3}+\underline{P(j,n-j-3)a^3}$$
$$+\sum_{i=j+4}^{n-1} r_i \underline{P(j+1,i-j-4)(a^2x+axa+xa^2)}+\underline{P(j+1,n-j-4)(a^2x+axa+xa^2)}$$
$$+\sum_{i=j+5}^{n-1} r_i \underline{P(j+2,i-j-5)(ax^2+xax+x^2a)}+\underline{P(j+2,n-j-5)(ax^2+xax+x^2a)}$$
$$+\sum_{i=j+6}^{n-1} r_i \underline{P(j+3,i-j-6)x^3}+\underline{Q(j+3,n-j-6)x^3}\big)+r_{j+3}\underline{a^n}$$
$$=-\big(\sum_{i=j+3}^{n} r_i \underline{P(j,i-j-3)a^3}+\sum_{i=j+4}^{n} r_i \underline{P(j+1,i-j-4)(a^2x+axa+xa^2)}$$
$$\sum_{i=j+5}^{n} r_i \underline{P(j+2,i-j-5)(ax^2+xax+x^2a)}+\sum_{i=j+6}^{n-1} r_i \underline{P(j+3,i-j-6)x^3}$$
$$+\underline{Q(j+3,n-j-6)x^3}\big)+r_{j+3}\underline{a^n}$$

Premultiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

(71)

$$-x^3a^{j+3}x^{n-j-3}\to\big(\sum_{i=j+3}^{n} r_i \underline{x^3P(j,i-j-3)a^3}+\sum_{i=j+4}^{n} r_i \underline{x^3P(j+1,i-j-4)(a^2x+axa+xa^2)}$$

$$+\sum_{i=j+5}^{n} r_i \underline{x^3P(j+2,i-j-5)(ax^2+xax+x^2a)} + \sum_{i=j+6}^{n-1} r_i \underline{x^3P(j+3,i-j-6)x^3}$$
$$+\underline{x^3Q(j+3,n-j-6)x^3}\big) - r_{j+3}\underline{x^3a^n}.$$

The words in (65)-(71) of length $n+3$ which are reducible are as follows:

(h) $x^2a^{j+2}x^{n-j-2}a \in x^2aP(j+1,n-j-4)x^2a$ in (65),
(i) $x(a^{j+2}x^{n-j-2})xa \in xa^2P(j,n-j-3)x^2a$ in (68),
(j) $x(a^{j+2}x^{n-j-2})ax \in xa^2P(j,n-j-3)xax$ in (68),
(k) $xa^{j+1}x^{n-j-1}a^2 \in xa^2P(j,n-j-2)xa^2$ in (68),
(l) $axa^{j+1}x^{n-j-1}a \in axaP(j,n-j-3)x^2a$ in (69).

Resolutions of (i), (j), (k) and (l) are provided by, respectively, (59), (60), (61) and (62). As for (h), namely $-x^2a^{j+2}x^{n-j-2}a$, it occurs in (59) with opposite sign to its occurrence in (65), so they cancel each other.

Hence, the reduction process ends here and when we substitute (59)-(61) and (65)-(71) into (64), we get
(72)

$$\omega_{j+3}x^3 \to \Big(\sum_{i=j+1}^{n} r_i\underline{a^2xP(j-2,i-j-1)a^3} + \sum_{i=j+2}^{n} r_i(\underline{a^2xP(j,i-j-3)(a^2x+axa+xa^2)}+$$
$$\underline{axaP(j,i-j-3)a^2x}) + \sum_{i=j+3}^{n} r_i(\underline{a^2xP(j,i-j-3)(ax^2+xax+x^2a)} + \underline{axaP(j,i-j-3)(ax^2+xax)}$$
$$+\underline{xa^2P(j,i-j-3)ax^2} + \underline{ax^2P(j,i-j-3)a^2x}) + \sum_{i=j+4}^{n} r_i(\underline{ax^2P(j+1,i-j-4)(ax^2+xax)}$$
$$+\underline{(xax+x^2a)P(j+1,i-j-4)ax^2}) + \sum_{i=j+5}^{n} r_i\underline{x^3P(j+2,i-j-5)ax^2}\Big)$$
$$-\big(r_{j+1}\underline{a^2xa^n} + r_{j+2}\underline{ax^2a^n} + \sum_{i=j+3}^{n-1} r_ia^3P(j,i-j)x^3 + \underline{a^3Q(j,n-j)x^3}\big).$$

Comparing (64) and (72), we conclude that the overlap ambiguity $(\sigma_{j+3}, \sigma_j, a^3, a^j, x^{n-j-3}, x^3)$ is resolvable for $3 \le j < n-4$

3.4. **Case (d):** $j = n-4$: Consider the overlap ambiguity $\sigma_{n-1}, \sigma_{n-4}, a^3, a^{n-4}x, x^3)$.

Using (4), expand σ_{n-4} as

$$a^{n-4}x^4 \to -\Big(\sum_{i=n-4}^{n-1} r_i\underline{P(n-7,i-n+4)a^3} + \underline{P(n-7,4)a^3}$$
$$+\sum_{i=n-3}^{n-1} r_i\underline{P(n-6,i-n+3)(a^2x+axa+xa^2)} + \underline{P(n-6,3)(a^2x+axa+xa^2)}$$
$$+\sum_{i=n-2}^{n-1} r_i\underline{P(n-5,i-n+2)(ax^2+xax+x^2a)} + \underline{P(n-5,2)(ax^2+xax+x^2a)}$$
$$+r_{n-1}\underline{P(n-4,0)x^3} + \underline{Q(n-4,1)x^3}\Big) + r_{n-4}\underline{a^n}$$

$$= -\Big(\sum_{i=n-4}^{n} r_i \underline{P(n-7, i-n+4)a^3} + \sum_{i=n-3}^{n} r_i \underline{P(n-6, i-n+3)(a^2x + axa + xa^2)}$$

$$+ \sum_{i=n-2}^{n} r_i \underline{P(n-5, i-n+2)(ax^2 + xax + x^2a)} + r_{n-1} \underline{P(n-4, 0)x^3} + \underline{Q(n-4, 1)x^3}\Big) + r_{n-4}\underline{a^n}.$$

Premultiply this by a^3 and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words, to obtain

(73)

$$a^3(a^{n-4}x^4) \to -\Big(\sum_{i=n-4}^{n} r_i a^3 P(n-7, i-n+4)a^3 + \sum_{i=n-3}^{n} r_i a^3 P(n-6, i-n+3)(a^2x + axa + xa^2)$$

$$+ \sum_{i=n-2}^{n} r_i a^3 P(n-5, i-n+2)(ax^2 + xax + x^2a) + r_{n-1} a^3 P(n-4, 0)x^3$$

$$+ \underline{a^3 Q(n-4, 1)x^3}\Big) + r_{n-4}\underline{a^{n+3}}.$$

The reducible words of length $n+3$ in the right side of (73) are as follows:

(a) $a^{n-4}x^4a^3 \in a^3P(n-7, 4)a^3$
(b) $(a^{n-2}x^2)x^2a \in a^3P(n-5, 2)x^2a$
(c) $(a^{n-2}x^2)xax \in a^3P(n-5, 2)xax$
(d) $(a^{n-2}x^2)ax^2 \in a^3P(n-5, 2)ax^2$
(e) $(a^{n-3}x^3)xa^2 \in a^3P(n-6, 3)xa^2$
(f) $(a^{n-3}x^3)axa \in a^3P(n-6, 3)axa$
(g) $(a^{n-3}x^3)a^2x \in a^3P(n-6, 3)a^2x$.

To deal with (a), use (5) to expand σ_{n-4} as follows:

$$a^{n-4}x^4 \to -\Big(\sum_{i=n-4}^{n-1} r_i \underline{a^3 P(n-7, i-n+4)} + \underline{a^3 Q(n-7, 4)}$$

$$+ \sum_{i=n-3}^{n-1} r_i \underline{(a^2x + axa + x^2a)P(n-6, i-n+3)} + \underline{(a^2x + axa + x^2a)P(n-6, 3)}$$

$$+ \sum_{i=n-2}^{n-1} r_i \underline{(ax^2 + xax + x^2a)P(n-5, i-n+2)} + \underline{(ax^2 + xax + x^2a)P(n-5, 2)}$$

$$+ \sum_{i=n-1}^{n} r_i \underline{x^3 P(n-4, i-n+1)}\Big) + r_{n-4}\underline{a^n}$$

$$= -\Big(\sum_{i=n-4}^{n-1} r_i \underline{a^3 P(n-7, i-n+4)} + \underline{a^3 Q(n-7, 4)} + \sum_{i=n-3}^{n} r_i \underline{(a^2x + axa + x^2a)P(n-6, i-n+3)}$$

$$+ \sum_{i=n-2}^{n} r_i \underline{(ax^2 + xax + x^2a)P(n-5, i-n+2)} + \sum_{i=n-1}^{n} r_i \underline{x^3 P(n-4, i-n+1)}\Big) + r_{n-4}\underline{a^n}.$$

Postmultiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-a^{n-4}x^4a^3 \to \Big(\sum_{i=n-4}^{n-1} r_i \underline{a^3 P(n-7, i-n+4)a^3} + \underline{a^3 Q(n-7, 4)a^3} \qquad (74)$$

$$+\sum_{i=n-3}^{n} r_i\underline{(a^2x+axa+x^2a)P(n-6,i-n+3)a^3}+\sum_{i=n-2}^{n} r_i\underline{(ax^2+xax+x^2a)P(n-5,i-n+2)a^3}$$

$$+\sum_{i=n-1}^{n} r_i\underline{x^3P(n-4,i-n+1)a^3}\big)+r_{n-4}\underline{a^{n+3}}.$$

For (b), (c) and (d), use (5) to expand σ_{n-2} as

$$a^{n-2}x^2\to-\big(\sum_{i=n-2}^{n-1} r_i\underline{a^3P(n-5,i-n+2)}+\underline{a^3Q(n-5,2)}+\sum_{i=n-1}^{n} r_i\underline{(a^2x+axa+xa^2)P(n-4,i-n+1)}$$

$$+\underline{(ax^2+xax+x^2a)P(n-3,0)}\big)+r_{n-2}\underline{a^n}.$$

Thus, postmultiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, we obtain, respectively,

$$-(a^{n-2}x^2)x^2a\to\big(\sum_{i=n-2}^{n-1} r_ia^3P(n-5,i-n+2)x^2a+\underline{a^3Q(n-5,2)x^2a} \tag{75}$$

$$+\sum_{i=n-1}^{n} r_i(a^2x+axa+xa^2)P(n-4,i-n+1)x^2a+(ax^2+xax+x^2a)P(n-3,0)x^2a\big)-r_{n-2}\underline{x^2a^{n+1}};$$

$$-(a^{n-2}x^2)xax\to\big(\sum_{i=n-2}^{n-1} r_ia^3P(n-5,i-n+2)xax+\underline{a^3Q(n-5,2)xax} \tag{76}$$

$$+\sum_{i=n-1}^{n} r_i(a^2x+axa+xa^2)P(n-4,i-n+1)xax+\underline{(ax^2+xax+x^2a)P(n-3,0)xax}\big)-r_{n-2}\underline{xaxa^n};$$

and

$$-(a^{n-2}x^2)ax^2\to\big(\sum_{i=n-2}^{n-1} r_i\underline{a^3P(n-5,i-n+2)ax^2}+\underline{a^3Q(n-5,2)ax^2} \tag{77}$$

$$+\sum_{i=n-1}^{n} r_i(a^2x+axa+xa^2)P(n-4,i-n+1)ax^2+(ax^2+xax+x^2a)P(n-3,0)ax^2\big)-r_{n-2}\underline{ax^2a^n}.$$

For (e), (f) and (g), expand σ_{n-3} using (5) as

$$a^{n-3}x^3\to-\big(\sum_{i=n-3}^{n-1} r_i\underline{a^3P(n-6,i-n+3)}+\underline{a^3Q(n-6,3)}+\underline{x^3P(n-3,0)}$$

$$+\sum_{i=n-2}^{n-1} r_i\underline{(a^2x+axa+xa^2)P(n-5,i-n+2)}+\underline{(a^2x+axa+xa^2)P(n-5,2)}$$

$$+\sum_{i=n-1}^{n} r_i\underline{(ax^2+xax+x^2a)P(n-4,i-n+1)}\big)+r_{n-3}\underline{a^n}.$$

$$=-\big(\sum_{i=n-3}^{n-1} r_i\underline{a^3P(n-6,i-n+3)}+\underline{a^3Q(n-6,3)}+\underline{x^3P(n-3,0)}$$

$$+\sum_{i=n-2}^{n} r_i\underline{(a^2x+axa+xa^2)P(n-5,i-n+2)}+\sum_{i=n-1}^{n} r_i\underline{(ax^2+xax+x^2a)P(n-4,i-n+1)}\big)$$

$$+r_{n-3}\underline{a^n}.$$

Hence, postmultiplying this by xa^2, axa and a^2x, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields, respectively for (e), (f) and (g),

$$(78)\quad -(a^{n-3}x^3)xa^2 \to \Big(\sum_{i=n-3}^{n-1} r_i a^3P(n-6,i-n+3)xa^2 + \underline{a^3Q(n-6,3)xa^2} + \underline{x^3P(n-3,0)xa^2}$$

$$+\sum_{i=n-2}^{n} r_i(a^2x+axa+xa^2)P(n-5,i-n+2)xa^2 + \sum_{i=n-1}^{n} r_i\underline{(ax^2+xax+x^2a)P(n-4,i-n+1)xa^2}\Big)$$

$$-r_{n-3}\underline{xa^{n+2}};$$

$$(79)\qquad -(a^{n-3}x^3)axa \to \Big(\sum_{i=n-3}^{n-1} r_i\underline{a^3P(n-6,i-n+3))axa} + \underline{a^3Q(n-6,3))axa}$$

$$+\underline{x^3P(n-3,0))axa} + \sum_{i=n-2}^{n} r_i\underline{(a^2x+axa+xa^2)P(n-5,i-n+2))axa}$$

$$+\sum_{i=n-1}^{n} r_i\underline{(ax^2+xax+x^2a)P(n-4,i-n+1))axa}\Big) - r_{n-3}\underline{axa^{n+1}};$$

and

$$(80)\qquad -(a^{n-3}x^3)a^2x \to \Big(\sum_{i=n-3}^{n-1} r_i a^3P(n-6,i-n+3)a^2x + \underline{a^3Q(n-6,3)a^2x}$$

$$+x^3P(n-3,0)a^2x + \sum_{i=n-2}^{n} r_i(a^2x+axa+xa^2)P(n-5,i-n+2)a^2x$$

$$+\sum_{i=n-1}^{n} r_i(ax^2+xax+x^2a)P(n-4,i-n+1)a^2x\Big) - r_{n-3}\underline{a^2xa^n}.$$

The following words of length $n+3$ from (74)-(80) are reducible:

(h) $x^2a^{n-2}x^2a$ in (75);
(i) $x(a^{n-2}x^2)xa \in xa^2P(n-4,1)x^2a$ in (75);
(j) $axa^{n-3}x^3a \in axaP(n-4,1)x^2a$ in (75);
(k) $xa^{n-2}x^2ax \in xa^2P(n-4,1)xax$ in (76);
(l) $xa^{n-3}x^3a^2 \in xa^2P(n-5,2)xa^2$ in (78);
(m) $x^3a^{n-1}x$ in (80);
(n) $x^2a(a^{n-2}x^2)$ in (77);
(o) $xax(a^{n-2}x^2)$ in (77);
(p) $ax^2(a^{n-2}x^2)$ (77).

To resolve (i) and (k), expand σ_{n-2} using (6), to obtain

$$a^{n-2}x^2 \to -\Big(\sum_{i=n-2}^{n-1} r_i\underline{a^2P(n-5,i-n+2)a} + \underline{a^2P(n-5,2)a}$$

$$+\underline{x^2P(n-3,0)a} + \underline{axP(n-3,0)x} + \underline{xaP(n-3,0)x} + \sum_{i=n-1}^{n} r_i\Big(\underline{axP(n-4,i-n+1)a}$$

$$+\underline{xaP(n-4,i-n+1)a}\Big) + r_{n-1}\underline{a^2P(n-4,0)x} + \underline{a^2Q(n-4,1)x}\Big) + r_{n-2}\underline{a^n}$$

$$= -\Big(\sum_{i=n-2}^{n} r_i \underline{a^2P(n-5,i-n+2)a} + \underline{x^2P(n-3,0)a} + \underline{axP(n-3,0)x} + \underline{xaP(n-3,0)x}$$

$$+ \sum_{i=n-1}^{n} r_i \big(\underline{axP(n-4,i-n+1)a} + \underline{xaP(n-4,i-n+1)a}\big)$$

$$+ r_{n-1}\underline{a^2P(n-4,0)x} + \underline{a^2Q(n-4,1)x}\Big) + r_{n-2}\underline{a^n}.$$

Thus, premultiplying this by x and postmultiplying it by xa and ax, using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields respectively for (i) and (k)

$$x(a^{n-2}x^2)xa \to -\Big(\sum_{i=n-2}^{n} r_i \underline{xa^2P(n-5,i-n+2)axa} + \underline{x^3P(n-3,0)axa} \tag{81}$$

$$+\underline{xaxP(n-3,0)x^2a} + x^2aP(n-3,0)x^2a + \sum_{i=n-1}^{n} r_i \big(\underline{xaxP(n-4,i-n+1)axa}$$

$$+\underline{x^2aP(n-4,i-n+1)axa}\big) + r_{n-1}\underline{xa^2P(n-4,0)x^2a} + \underline{xa^2Q(n-4,1)x^2a}\Big) + r_{n-2}\underline{x^2a^{n+1}};$$

and

(82)

$$x(a^{n-2}x^2)ax \to -\Big(\sum_{i=n-2}^{n} r_i \underline{xa^2P(n-5,i-n+2)a^2x} + x^3P(n-3,0)a^2x + \underline{x^2aP(n-3,0)xax}$$

$$+\underline{xaxP(n-3,0)xax} + \sum_{i=n-1}^{n} r_i \big(\underline{xaxP(n-4,i-n+1)a^2x} + \underline{x^2aP(n-4,i-n+1)a^2x}\big)$$

$$+r_{n-1}\underline{xa^2P(n-4,0)xax} + \underline{xa^2Q(n-4,1)xax}\Big) + r_{n-2}\underline{xaxa^n}.$$

To deal with (j), expand σ_{n-3} using (7) to obtain

$$a^{n-3}x^3 \to -\Big(\sum_{i=n-3}^{n-1} r_i \underline{aP(n-6,i-n+3)a^2} + \underline{aP(n-6,3)a^2}$$

$$+ \sum_{i=n-2}^{n-1} r_i \big(\underline{aP(n-5,i-n+2)ax} + \underline{aP(n-5,i-n+2)xa} + \underline{xP(n-5,i-n+2)a^2}\big)$$

$$+\underline{aP(n-5,2)ax} + \underline{aP(n-5,2)xa} + \underline{xP(n-5,2)a^2} + \sum_{i=n-1}^{n} r_i \big(\underline{xP(n-4,i-n+1)ax}$$

$$+\underline{xP(n-4,i-n+1)xa}\big) + r_{n-1}\underline{aP(n-4,0)x^2} + \underline{aQ(n-4,1)x^2} + \underline{xP(n-3,0)x^2}\Big) + r_{n-3}\underline{a^n}$$

$$= -\Big(\sum_{i=n-3}^{n} r_i \underline{aP(n-6,i-n+3)a^2} + \sum_{i=n-2}^{n} r_i \big(\underline{aP(n-5,i-n+2)ax} + \underline{aP(n-5,i-n+2)xa}$$

$$+\underline{xP(n-5,i-n+2)a^2}\big) + \sum_{i=n-1}^{n} r_i \big(\underline{xP(n-4,i-n+1)ax} + \underline{xP(n-4,i-n+1)xa}\big)$$

$$+r_{n-1}\underline{aP(n-4,0)x^2} + \underline{aQ(n-4,1)x^2} + \underline{xP(n-3,0)x^2}\Big) + r_{n-3}\underline{a^n}.$$

Premultiplying this by ax and postmultiplying it by a, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields for (j)

$$axa^{n-3}x^3a \to -\Big(\sum_{i=n-3}^{n} r_i \underline{axaP(n-6,i-n+3)a^3} + \sum_{i=n-2}^{n} r_i \big(\underline{axaP(n-5,i-n+2)axa} \tag{83}$$

$$+\underline{axaP(n-5,i-n+2)xa^2}+\underline{ax^2P(n-5,i-n+2)a^3}\big)+\sum_{i=n-1}^{n} r_i\big(\underline{ax^2P(n-4,i-n+1)axa}$$

$$+\underline{ax^2P(n-4,i-n+1)xa^2}\big)+r_{n-1}\underline{axaP(n-4,0)x^2a}+\underline{axaQ(n-4,1)x^2a}$$

$$+\underline{ax^2P(n-3,0)x^2a}\big)+r_{n-3}\underline{axa^{n+1}}.$$

For (l), expand σ_{n-3} using (6), giving

$$a^{n-3}x^3 \to -\big(\sum_{i=n-3}^{n-1} r_i\underline{a^2P(n-6,i-n+3)a}+\underline{a^2P(n-6,3)a}$$

$$+\sum_{i=n-2}^{n-1} r_i\big(\underline{axP(n-5,i-n+2)a}+\underline{xaP(n-5,i-n+2)a}\big)+\underline{axP(n-5,2)a}+\underline{xaP(n-5,2)a}$$

$$+\sum_{i=n-2}^{n-1} r_i\underline{a^2P(n-5,i-n+2)x}+\underline{a^2Q(n-5,2)x}+\sum_{i=n-1}^{n} r_i\big(\underline{x^2P(n-4,i-n+1)a}$$

$$+\underline{axP(n-4,i-n+1)x}+\underline{xaP(n-4,i-n+1)x}\big)+\underline{x^2P(n-3,0)x}\big)+r_{n-3}\underline{a^n}$$

$$=-\big(\sum_{i=n-3}^{n} r_i\underline{a^2P(n-6,i-n+3)a}+\sum_{i=n-2}^{n} r_i\big(\underline{axP(n-5,i-n+2)a}+\underline{xaP(n-5,i-n+2)a}\big)$$

$$+\sum_{i=n-1}^{n} r_i\big(\underline{x^2P(n-4,i-n+1)a}+\underline{axP(n-4,i-n+1)x}+\underline{xaP(n-4,i-n+1)x}\big)$$

$$+\sum_{i=n-2}^{n-1} r_i\underline{a^2P(n-5,i-n+2)x}+\underline{a^2Q(n-5,2)x}+\underline{x^2P(n-3,0)x}\big)+r_{n-3}\underline{a^n}.$$

Premultiplying this by x and postmultiplying it by a^2, using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields for (l)

$$xa^{n-3}x^3a^2 \to -\big(\sum_{i=n-3}^{n} r_i\underline{xa^2P(n-6,i-n+3)a^3}+\sum_{i=n-2}^{n} r_i\big(\underline{xaxP(n-5,i-n+2)a^3} \tag{84}$$

$$+\underline{x^2aP(n-5,i-n+2)a^3}\big)+\sum_{i=n-2}^{n-1} r_i\underline{xa^2P(n-5,i-n+2)xa^2}+\underline{xa^2Q(n-5,2)xa^2}$$

$$+\sum_{i=n-1}^{n} r_i\big(\underline{x^3P(n-4,i-n+1)a^3}+\underline{xaxP(n-4,i-n+1)xa^2}+\underline{x^2aP(n-4,i-n+1)xa^2}\big)$$

$$+\underline{x^3P(n-3,0)xa^2}\big)+r_{n-3}\underline{xa^{n+2}}.$$

To resolve (n), (o) and (p) expand σ_{n-2} using (4) to obtain

$$a^{n-2}x^2 \to -\big(\sum_{i=n-2}^{n-1} r_i\underline{P(n-5,i-n+2)a^3}+\underline{P(n-5,2)a^3}$$

$$+\sum_{i=n-1}^{n} r_i\underline{P(n-4,i-n+1)(a^2x+axa+xa^2)}$$

$$+\underline{P(n-3,0)(xax+x^2a)}\big)+r_{n-3}\underline{a^n}$$

$$=-\big(\sum_{i=n-2}^{n} r_i\underline{P(n-5,i-n+2)a^3}+\sum_{i=n-1}^{n} r_i\underline{P(n-4,i-n+1)(a^2x+axa+xa^2)}$$

$$+\underline{P(n-3,0)(xax+x^2a)}\big) + r_{n-3}\underline{a^n}.$$

Premultiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields, respectively for (n), (o) and (p),

(85)

$$x^2a(a^{n-2}x^2) \to -\Big(\sum_{i=n-2}^{n} r_i\underline{x^2aP(n-5,i-n+2)a^3}+\sum_{i=n-1}^{n} r_i\underline{x^2aP(n-4,i-n+1)(a^2x+axa+xa^2)}$$
$$+x^2aP(n-3,0)(xax+x^2a)\Big) + r_{n-3}\underline{x^2a^{n+1}};$$

(86)

$$xax(a^{n-2}x^2) \to -\Big(\sum_{i=n-2}^{n} r_i\underline{xaxP(n-5,i-n+2)a^3}+\sum_{i=n-1}^{n} r_i\underline{xaxP(n-4,i-n+1)(a^2x+axa+xa^2)}$$
$$+\underline{xaxP(n-3,0)(xax+x^2a)}\Big) + r_{n-3}\underline{xaxa^n};$$

and

(87)

$$ax^2(a^{n-2}x^2) \to -\Big(\sum_{i=n-2}^{n} r_i\underline{ax^2P(n-5,i-n+2)a^3}+\sum_{i=n-1}^{n} r_i\underline{ax^2P(n-4,i-n+1)(a^2x+axa+xa^2)}$$
$$+\underline{ax^2P(n-3,0)(xax+x^2a)}\Big) + r_{n-3}\underline{ax^2a^n}.$$

The words in (81)-(87) of length $n+3$ which are reducible are as follows:

(q) $-x^2a^{n-2}x^2a \in -x^2aP(n-3,0)x^2a$ in (81);
(r) $-x^3a^{n-1}x \in -x^3P(n-3,0)^2x$ in (82);
(s) $-x^2a^{n-2}x^2a \in -x^2aP(n-3,0)x^2a$ in (85).

But terms (q) and (r) in the preceding list match, but with opposite signs, the corresponding terms (h) and (m) in the list of nine reducible words above, so they cancel. Thus, we only have to reduce the term (s). To do this, expand σ_{n-2} using (7) to obtain

$$a^{n-2}x^2 \to -\Big(\sum_{i=n-2}^{n-1} r_i\underline{aP(n-5,i-n+2)a^2}+\underline{aP(n-5,2)a^2}$$
$$+\sum_{i=n-1}^{n} r_i\big(\underline{aP(n-4,i-n+1)ax}+\underline{aP(n-4,i-n+1)xa}+\underline{xP(n-4,i-n+1)a^2}\big)$$
$$+\underline{xP(n-3,0)ax}+\underline{xP(n-3,0)xa}\Big)+r_{n-2}\underline{a^n}$$
$$=-\Big(\sum_{i=n-2}^{n} r_i\underline{aP(n-5,i-n+2)a^2}+\sum_{i=n-1}^{n} r_i\big(\underline{aP(n-4,i-n+1)ax}$$
$$+\underline{aP(n-4,i-n+1)xa}+\underline{xP(n-4,i-n+1)a^2}\big)+\underline{xP(n-3,0)ax}+\underline{xP(n-3,0)xa}\Big)+r_{n-2}\underline{a^n}.$$

Premultiplying this by x^2 and postmultiplying it by a, using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields for (s)

$$(88)\quad -x^2a^{n-2}x^2a \to \Big(\sum_{i=n-2}^{n} r_i\underline{x^2aP(n-5,i-n+2)a^3}+\sum_{i=n-1}^{n} r_i\big(\underline{x^2aP(n-4,i-n+1)axa}$$
$$+\underline{x^2aP(n-4,i-n+1)xa^2}+\underline{x^3P(n-4,i-n+1)a^3}\big)+\underline{x^3P(n-3,0)axa}$$
$$+\underline{x^3P(n-3,0)xa^2}\Big)-r_{n-2}\underline{x^2a^{n+1}}.$$

Hence, the reduction process ends here. Substituting (74)-(88) into (73), we get
(89)

$$a^3\omega_{n-4} \to \Big(\sum_{i=n-3}^{n} r_i\underline{a^2xP(n-6,i-n+3)a^3}+ \sum_{i=n-2}^{n} r_i\big(\underline{a^2xP(n-5,i-n+2)(a^2x+axa+xa^2)}$$
$$+\underline{axaP(n-5,i-n+2)a^2x}\big) + \sum_{i=n-1}^{n} r_i\big(\underline{a^2xP(n-4,i-n+1)(ax^2+xax+x^2a)}$$
$$+\underline{axaP(n-4,i-n+1)(ax^2+xax)} + \underline{xa^2P(n-4,i-n+1)ax^2} + \underline{x^3P(n-4,i-n+1)a^3}\big)$$
$$+\underline{x^3P(n-3,0)(axa+xa^2} + r_{n-2}\underline{xaxa^n}\big) - \Big(\sum_{i=n-2}^{n} r_i\underline{xaxP(n-5,i-n+2)a^3}$$
$$+ \sum_{i=n-1}^{n} r_i(\underline{ax^2P(n-4,i-n+1)(axa+xa^2)} + \underline{xaxP(n-4,i-n+1)(a^2x+axa+xa^2)}$$
$$+\underline{x^2aP(n-4,i-n+1)a^2x}\big)+r_{n-1}a^3P(n-4,0)x^3+\underline{a^3Q(n-4,1)x^3}+\underline{xaxP(n-3,0)(xax+x^2a)}$$
$$+\underline{x^2aP(n-3,0)xax} + \underline{ax^2(P(n-3,0)x^2a + P(n-5,2)a^3} + r_{n-3}\underline{a^2xa^n}\big).$$

We consider now the other side of the ambiguity, namely $\omega_{n-1}x^3$. First, using (4), expand σ_{n-1} as

$$\omega_{n-1} \to -\big(r_{n-1}\underline{a^3P(n-4,0)} + \underline{a^3Q(n-4,1)} + \underline{(a^2x+axa+xa^2)P(n-3,0)}\big) + r_{n-1}\underline{a^n}.$$

Postmultiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields
(90)

$$\omega_{n-1}x^3 \to -\big(r_{n-1}a^3P(n-4,0)x^3+\underline{a^3Q(n-4,1)x^3}+(a^2x+axa+xa^2)P(n-3,0)x^3\big)+r_{n-1}\underline{a^n x^3}.$$

The following words in (90) of length $n+3$ are reducible:

(a) $xa^{n-1}x^3$;
(b) $axa^{n-2}x^3$;
(c) $a^2xa^{n-3}x^3$.

To deal with these expand σ_{n-3} using (4), as

$$a^{n-3}x^3 \to -\Big(\sum_{i=n-3}^{n-1} r_i\underline{P(n-6,i-n+3)a^3} + \underline{P(n-6,3)a^3}$$
$$+ \sum_{i=n-2}^{n-1} r_i\underline{P(n-5,i-n+2)(a^2x+axa+xa^2)} + \underline{P(n-5,2)(a^2x+axa+xa^2)}$$
$$+ \sum_{i=n-1}^{n} r_i\underline{P(n-4,i-n+1)(ax^2+xax+x^2a)}\big) + r_{n-3}\underline{a^n}$$
$$= -\Big(\sum_{i=n-3}^{n} r_i\underline{P(n-6,i-n+3)a^3} + \sum_{i=n-2}^{n} r_i\underline{P(n-5,i-n+2)(a^2x+axa+xa^2)}$$
$$+ \sum_{i=n-1}^{n} r_i\underline{P(n-4,i-n+1)(ax^2+xax+x^2a)}\big) + r_{n-3}\underline{a^n}.$$

Premultiply this by xa^2, axa and a^2x, and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words to obtain, respectively for the reduction of (a), (b) and (c),
(91)

$$-xa^2(a^{n-3}x^3) \to \big(\sum_{i=n-3}^{n} r_i \underline{xa^2P(n-6, i-n+3)a^3} + \sum_{i=n-2}^{n} r_i xa^2P(n-5, i-n+2)(a^2x+axa+xa^2) + \sum_{i=n-1}^{n} r_i xa^2P(n-4, i-n+1)(ax^2+xax+x^2a)\big) - r_{n-3}\underline{xa^{n+2}};$$

(92)

$$-axa(a^{n-3}x^3) \to \big(\sum_{i=n-3}^{n} r_i \underline{axaP(n-6, i-n+3)a^3} + \sum_{i=n-2}^{n} r_i \underline{axaP(n-5, i-n+2)(a^2x+axa+xa^2)} + \sum_{i=n-1}^{n} r_i axaP(n-4, i-n+1)(ax^2+xax+x^2a)\big) - r_{n-3}\underline{axa^{n+1}};$$

and
(93)

$$-a^2x(a^{n-3}x^3) \to \big(\sum_{i=n-3}^{n} r_i \underline{a^2xP(n-6, i-n+3)a^3} + \sum_{i=n-2}^{n} r_i \underline{a^2xP(n-5, i-n+2)(a^2x+axa+xa^2)} + \sum_{i=n-1}^{n} r_i \underline{a^2xP(n-4, i-n+1)(ax^2+xax+x^2a)}\big) - r_{n-3}\underline{a^2xa^n}.$$

The reducible words in (91)-(93) of length $n+3$ are as follows:

(d) $xa^{n-2}x^2xa \in xa^2P(n-4,1)x^2a$ in (91);
(e) $xa^{n-2}x^2ax \in xa^2P(n-4,1)xax$ in(91);
(f) $xa^{n-3}x^3a^2 \in xa^2P(n-5,2)xa^2$ in (91);
(g) $axa^{n-3}x^3a \in axaP(n-4,1)x^2a$ in (92).

The resolutions of these four words (d), (e), (f) and (g) are provided respectively by (81), (82), (84) and (83).

The reducible words in (81)-(84) of length $n+3$ are:

(h) $-x^2a^{n-2}x^2a \in x(a^{n-2}x^2)xa$ in (81);
(i) $-x^3a^{n-1}x \in x(a^{n-2}x^2)ax$ in (82).

The first of these, $-x^2a^{n-2}x^2a$ has already been resolved in (88). To deal with the second, that is $-x^3a^{n-1}x$, first expand σ_{n-1} using (4) to obtain

$$a^{n-1}x \to -\big(\sum_{i=n-1}^{n} r_i \underline{P(n-4, i-n+1)a^3} + \underline{P(n-3,0)(axa+xa^2)}\big) + r_{n-1}\underline{a^n}.$$

Premultiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields

$$(94) \quad -x^3a^{n-1}x \to \big(\sum_{i=n-1}^{n} r_i \underline{x^3P(n-4, i-n+1)a^3} + \underline{x^3P(n-3,0)(axa+xa^2)}\big) - r_{n-1}\underline{x^3a^n}.$$

Thus, the reduction process ends here and when we substitute (81)-(84), (88), (91)-(94) into (90), we get

(95)

$$
\begin{aligned}
\omega_{n-1}x^3 \to \Big(&\sum_{i=n-3}^{n} r_i \underline{a^2xP(n-6,i-n+3)a^3} + \sum_{i=n-2}^{n} r_i \big(\underline{a^2xP(n-5,i-n+2)(a^2x+axa+xa^2)} \\
&+ \underline{axaP(n-5,i-n+2)a^2x}\big) + \sum_{i=n-1}^{n} r_i \big(\underline{a^2xP(n-4,i-n+1)(ax^2+xax+x^2a)} \\
&+ \underline{axaP(n-4,i-n+1)(ax^2+xax)} + \underline{xa^2P(n-4,i-n+1)ax^2} + \underline{x^3P(n-4,i-n+1)a^3}\big) \\
&+ \underline{x^3P(n-3,0)(axa+xa^2} + r_{n-2}\underline{xaxa^n}\Big) - \Big(\sum_{i=n-2}^{n} r_i \underline{xaxP(n-5,i-n+2)a^3} \\
&+ \sum_{i=n-1}^{n} r_i \big(\underline{ax^2P(n-4,i-n+1)(axa+xa^2)} + \underline{xaxP(n-4,i-n+1)(a^2x+axa+xa^2)} \\
&+ \underline{x^2aP(n-4,i-n+1)a^2x}\big) + r_{n-1}a^3P(n-4,0)x^3 + \underline{a^3Q(n-4,1)x^3} + \underline{xaxP(n-3,0)(xax+x^2a)} \\
&+ \underline{x^2aP(n-3,0)xax} + \underline{ax^2(P(n-3,0)x^2a + P(n-5,2)a^3)} + r_{n-3}\underline{a^2xa^n}\Big).
\end{aligned}
$$

Comparing (89) and (95), we conclude that the overlap ambiguity $(\sigma_{n-1}, \sigma_{n-4}, a^3, a^{n-4}x, x^3)$ is resolvable. This completes the proof of Case (d), and with it the proof of Proposition 2.8(iii) of[1] is complete.

References

[1] K. Brown and A.A. Tabiri, *Plane curves which are quantum homogeneous spaces*, preprint **arXiv1810.09509**.

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