ON THE DUAL OF A P-ALGEBRA AND ITS COMODULES, WITH APPLICATIONS TO COMPARISON OF SOME BOUSFIELD CLASSES

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Abstract. In his seminal work on localisation of spectra, Ravenel initiated the study of Bousfield classes of spectra related to the chromatic perspective. In particular he showed that there were infinitely many distinct Bousfield classes between $\langle M\mathrm{U}\rangle$ and $\langle S^0\rangle$. The main topological goal of this paper is investigate how these Bousfield classes are related to that of another classical Thom spectrum $M\mathrm{Sp}$, and in particular how $\langle M\mathrm{Sp}\rangle$ is related to $\langle M\mathrm{U}\rangle$.

We follow the approach of Ravenel, but adapt it using the theory of P-algebras to give vanishing results for cohomology. Our work involves dualising and considering comodules over duals of P-algebras; these ideas are then applied to the mod 2 Steenrod algebra and certain subHopf algebras.

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Introduction

In his seminal paper on localisation [Rav84], Doug Ravenel initiated the study of Bousfield classes of spectra related to the chromatic perspective. In particular he showed that there were infinitely many distinct Bousfield classes between $\langle M\mathrm{U}\rangle$ and $\langle S^0\rangle$. The main topological goal

Date: last updated 01/03/2021 – version 1.

2020 *Mathematics Subject Classification.* Primary 55P42; Secondary 55S10, 57T05.

Key words and phrases. Homotopy theory, Steenrod algebra, Bousfield class.

I would like to thank the following: The Max-Planck-Institut für Mathematik in Bonn for supporting my visit in January 2020; Tobias Barthel for raising the question that motivated this work; Doug Ravenel for support and encouragement and teaching me so much over many decades; Ken Brown and Nige Ray for numerous helpful conversations on algebra and topology.

I would like to dedicate this paper to the memory of **Pete Bousfield** (1941–2020) whose work on localisation has become a central feature of modern homotopy theory.

of this paper is investigate how these Bousfield classes are related to that of another classical Thom spectrum $M\mathrm{Sp}$, and in particular how $\langle M\mathrm{Sp}\rangle$ is related to $\langle M\mathrm{U}\rangle$. Of course, there is a canonical map of ring spectra $M\mathrm{Sp} \to M\mathrm{U}$ so $\langle M\mathrm{Sp}\rangle \geqslant \langle M\mathrm{U}\rangle$. This issue is mentioned by Hovey & Palmieri [HP99], who also say that Ravenel had outlined an argument that $\langle M\mathrm{Sp}\rangle > \langle M\mathrm{U}\rangle$ but this has not appeared in print.

To achieve this we rework the algebraic machinery in Ravenel's approach making use of the theory of P-algebras of Moore & Peterson [MP73], subsequently developed further by Margolis [Mar83]. We first consider the dual of a P-algebra and its comodules; in order to set up some Cartan-Eilenberg spectral sequences we discuss the homological algebra required to make a bivariant derived functor of the homomorphism for comodules with the aim of obtaining three such spectral sequences, one of which involves dualising to modules over the P-algebra itself because comodules do not always have resolutions by projective objects. After reviewing some properties of the mod 2 Steenrod algebra and its dual, we discuss doubling and then describe some families of subHopf algebras and their duals. Moving on to topology, we introduce a sequence of loop spaces whose colimit is equivalent to $B\mathrm{Sp}$ and which give rise to a family of Thom spectra whose colimit is equivalent to $M\mathrm{Sp}$. After describing the homology of these as comodule algebras we move on to our main result which says that 2-locally, $\langle M\mathrm{Sp}\rangle > \langle BP\rangle$ and there is an infinite sequence of distinct Bousfield classes between $\langle S^0\rangle$ and $\langle M\mathrm{Sp}\rangle$.

1. P-algebras and their duals

We will make use material on P-algebras and their modules contained in [Mar83, chapter 13]. Here is a summary of assumptions and conventions.

- We will often suppress explicit mention of internal grading in cohomology and just write M for M^* when discussing a module over $A = A^*$. When working in homology we will usually write M_* or A_*.
- We will work with graded vector spaces over a field $\Bbbk$ and in particular, $\Bbbk$-algebras and their (co)modules. In our topological applications, $\Bbbk = \mathbb{F}_2$.

  For a finite type graded vector space V_* we think of V_n as dual to $V^n = \mathrm{Hom}_{\Bbbk}(V_n, \Bbbk)$, so V^* is the cohomologically graded degree-wise dual, and bounded below means that V^* is bounded below. We can also start with a finite type graded vector space V^* and form V_* where $V_n = \mathrm{Hom}_{\Bbbk}(V^n, \Bbbk)$; the double dual of V_* is canonically isomorphic to V_*, and vice versa.

  A graded vector space which is finite dimensional will be referred to as a *finite*; when $\Bbbk$ is finite this terminology agrees with Margolis' use of finite module over a P-algebra.
- Recall that a P-algebra A is a union of connected cocommutative Hopf algebras $A(n) \subseteq A(n+1) \subseteq A$. Each $A(n)$ is a Poincaré duality algebra and we will denote its highest non-trivial degree by $\mathrm{pd}(n)$, which is also its Poincaré duality degree; this satisfies $\mathrm{pd}(n) < \mathrm{pd}(n+1)$. In general, P-algebras are not required to be of finite type, but all the examples we consider have that property and it is required in some of our homological results so we will assume it holds. We also stress that each $A(n)$ is a local graded ring and $A(n)^{\mathrm{pd}(n)} = A(n)^0 = \Bbbk$. Other important properties are that A is flat as a left or right $A(n)$-module and a coherent $\Bbbk$-algebra.

  Of course our main examples are the Steenrod algebra $\mathcal{A}$ and infinite dimensional subHopf algebras.

We will use the following basic result stated on page 195 of [Mar83] but left as an exercise.

Proposition 1.1. *Suppose that A is a P-algebra. Let M be a finite A-module and F a bounded below free A-module. Then*

$$\mathrm{Ext}_A^*(M, F) = 0.$$

Proof. By [Mar83, theorem 13.12], bounded below projective A-modules are also injective, so $\mathrm{Ext}_A^s(M, F) = 0$ when $s > 0$, therefore we only have to show that $\mathrm{Hom}_A(M, F) = 0$. It suffices to prove this for the case $F = A$.

Suppose that $0 \neq \theta \in \mathrm{Hom}_A(M, A)$. The image of θ contains a simple submodule in its top degree, so let $\theta(x) \neq 0$ be in this submodule; then $a\theta(x) = 0$ for every positive degree element $a \in A$. Now for some n, $\theta(x) \in A(n)^k$ where $k < \mathrm{pd}(n)$, and by Poincaré duality for $A(n)$ there exists $z \in A(n)$ for which $0 \neq z\theta(x) \in A(n)^{\mathrm{pd}(n)}$. This gives a contradiction, hence no such θ can exist. $\square$

A particular concern for us will be the situation where we have a *pair* of P-algebras $B \subseteq A$ (so B is a subHopf algebra of A).

Proposition 1.2. *For a pair of P-algebras $B \subseteq A$,*

$$\mathrm{Ext}_A^*(A \otimes_B \Bbbk, A) \cong \mathrm{Ext}_B^*(\Bbbk, A) = \mathrm{Hom}_B(\Bbbk, A) = 0.$$

Proof. First we recall a classic result of Milnor & Moore [MM65, proposition 4.9]: A is free as a left or right B-module. This guarantees the change of rings isomorphism and the second isomorphism follows from Proposition 1.1. $\square$

It is crucial that B is it self a P-algebra, for example the vanishing result does not hold if $B = A(n)$.

Since a P-algebra A is coherent, its finitely generated modules are its coherent modules, and they form an abelian category with finite limits and colimits (in particular it has kernels, cokernels and images). A coherent A-module M admits a finite presentation

$$A^{\oplus k} \xrightarrow{\pi} A^{\oplus \ell} \longrightarrow M \longrightarrow 0$$

which can be defined over some $A(n)$, i.e., there is a finite presentation

$$A(n)^{\oplus k} \xrightarrow{\pi'} A(n)^{\oplus \ell} \longrightarrow M' \longrightarrow 0$$

of $A(n)$-modules and a commutative diagram of A-modules

$$\begin{tikzcd}
A \otimes_{A(n)} A(n)^{\oplus k} \arrow[r, "\mathrm{Id} \otimes \pi'"] \arrow[d] & A \otimes_{A(n)} A(n)^{\oplus \ell} \arrow[r] \arrow[d, leftrightarrow, "\cong"] & A \otimes_{A(n)} M' \arrow[r] \arrow[d, leftrightarrow, "\cong"] & 0 \\
A^{\oplus k} \arrow[r, "\pi"] & A^{\oplus \ell} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

with exact rows. It is standard that every coherent A-module admits a resolution by finitely generated free modules. But for a P-algebra we also have injective resolutions by finitely generated free modules.

Proposition 1.3. *Let M be a coherent module over a P-algebra A. Then M admits an injective resolution by finitely generated free modules.*

Proof. By [Mar83, theorem 13.12], bounded below projective A-modules are injective.

For some n, $M \cong A \otimes_{A(n)} M'$ where M' is a finitely generated $A(n)$-module. It is standard that M' admits a monomorphism $M' \to J'$ into a finitely generated free module (since $A(n)$ is a Poincaré duality algebra this is obvious for a simple module and can be proved by induction on dimension). By flatness we can find a monomorphism of A-modules

$$M \xrightarrow{\cong} A \otimes_{A(n)} M' \to A \otimes_{A(n)} J'$$

into a finitely generated free module with coherent cokernel. Iterating this we can build an injection resolution of the stated form. □

Proposition 1.4. *Let M be a finite A-module and N a coherent A-module. Then*

$$\mathrm{Ext}^*_A(M, N) = 0.$$

Proof. By Proposition 1.3 there is an injective resolution $N \to J^*$ where each J^s is a finitely generated free module. Also, by Proposition 1.1 $\mathrm{Hom}_A(M, J^s) = 0$ and so $\mathrm{Ext}^s_A(M, N) = 0$. □

For example, every left A-module of the form $A//A(n) = A \otimes_{A(n)} \Bbbk$ is coherent so it admits such an injective resolution and $\mathrm{Ext}^s_A(\Bbbk, A//A(n)) = 0$.

The dual of a P-algebra and its comodules. The theory of P-algebras can be dualised: given a finite type P-algebra A, we define its dual A_* by setting $A_n = \mathrm{Hom}_\Bbbk(A^n, \Bbbk)$ and making this a commutative Hopf algebra. We will refer to the dual Hopf algebra of a finite type P-algebra as a *P_*-algebra*; as far as we know this is not standard terminology but it seems appropriate.

When working with comodules over a P_*-algebra A_* we will use homological grading. For left A_*-comodules which are bounded below and of finite type there is no significant difference between working with them or their (degree-wise) duals as A-modules. In particular,

$$\mathrm{Cohom}_{A_*}(M_*, N_*) \cong \mathrm{Hom}_A(N^*, M^*),$$

where $M^n = \mathrm{Hom}_{\mathbb{F}_2}(M_n, \mathbb{F}_2)$ and M^* is made into a left A-module using the antipode. More generally,

$$\mathrm{Coext}^{s,*}_{A_*}(M_*, N_*) \cong \mathrm{Ext}^{s,*}_A(N^*, M^*),$$

where $\mathrm{Coext}^{s,*}_{A_*}(M_*, -)$ denotes the right derived functor of $\mathrm{Cohom}_{A_*}(M_*, -)$, which can be computed using extended comodules which are injective comodules here since we are working over a field.

Here are the dual versions of Propositions 1.1 and 1.2. Recall that a cofree A_*-comodule is one of the form $A_* \otimes W_*$.

Proposition 1.5. *Suppose that A_* is a P_*-algebra. Let M_* be a finite A_*-comodule and C_* a bounded below finite type cofree A_*-comodule. Then*

$$\mathrm{Coext}^*_{A_*}(C_*, M_*) = 0.$$

Proposition 1.6. *For a surjective morphism of Hopf algebras $A_* \to B_*$ dual to a pair of P-algebras $B \subseteq A$,*

$$\mathrm{Coext}^{*,*}_{A_*}(A_*, A_* \Box_{B_*} \Bbbk) \cong \mathrm{Coext}^{*,*}_{B_*}(A_*, \Bbbk) = \mathrm{Cohom}_{B_*}(A_*, \Bbbk) = 0.$$

If A_* is a P_*-algebra then we will call an A_*-comodule M_* *coherent* if its dual M^* is a coherent A-module. Here is a dual version of Proposition 1.3.

Proposition 1.7. *Let M_* be a coherent comodule over a P_*-algebra A_*. Then M_* admits a resolution by finitely generated cofree comodules.*

Proof. Take an injective resolution of M^* as in Proposition 1.3 and then take duals to obtain a projective resolution. □

Notice that since A_* is an injective comodule we have

$$\mathrm{Coext}^*_{A_*}(A_*, A_*) \cong \mathrm{Hom}_{\Bbbk}(A_*, \Bbbk).$$

Proposition 1.8. *Let M_* be a coherent comodule over a P_*-algebra A_* and let N_* be a finite P_*-comodule. Then*

$$\mathrm{Coext}^*_{A_*}(M_*, N_*) = 0.$$

Proof. Let $P_{\bullet,*} \to M_* \to 0$ be a resolution of M_* by cofree comodules. Then by Proposition 1.5, for each $s \geqslant 0$ we have

$$\mathrm{Cohom}_{A_*}(P_{s,*}, N_*) = 0,$$

and the result follows. □

Remark 1.9. For bounded below finite type comodules over a P_*-algebra A_* dual to a P-algebra A, taking degree-wise duals defines an equivalence of categories

$$\begin{tikzcd} \mathbf{Comod}^{\flat,\,\mathrm{f.t.}}_{A_*} \arrow[r, shift left, "(-)^*"] & (\mathbf{Mod}^{\flat,\,\mathrm{f.t.}}_{A})^{\mathrm{op}} \arrow[l, shift left, "(-)_*"] \end{tikzcd}$$

between the A_*-comodule and the A-module categories. Moreover, these functors are exact, so this equivalence identifies injective comodules (which are retracts of extended comodules) with projective modules. By [Mar83, theorem 13.12], projective A-modules are injective so it also identifies injective comodules as projective objects (this is not true in general).

In fact this equivalence fits into a bigger diagram

(1.1)
$$\begin{tikzcd} & \mathbf{Mod}^{\sharp,\,\mathrm{f.t.}}_{A} \arrow[d, shift left, "(-)^*"] \\ \mathbf{Comod}^{\flat,\,\mathrm{f.t.}}_{A_*} \arrow[ur, bend left] \arrow[dr, bend right] \arrow[r, shift left, "(-)^*"] & (\mathbf{Mod}^{\flat,\,\mathrm{f.t.}}_{A})^{\mathrm{op}} \arrow[l, shift left, "(-)_*"] \arrow[u, shift left, "(-)_*"] \arrow[d] \\ & \mathbf{Mod}^{\mathrm{op}}_{A} \end{tikzcd}$$

where $\mathbf{Mod}^{\natural,\,\mathrm{f.t.}}_{A}$ denotes the category of finite type bounded below homologically graded A-modules (with A acting by decreasing degree), $\mathbf{Mod}^{\flat,\,\mathrm{f.t.}}_{A}$ denotes the category of finite type bounded below cohomologically graded A-modules and $\mathbf{Mod}_A$ denoting the category of all A-modules. All of the functors here are exact.

For a fixed A_*-comodule M_*, the functor

$$\mathrm{Cohom}_{A_*}(M_*, -) = \mathbf{Comod}^{\flat,\,\mathrm{f.t.}}_{A_*}(M_*, -) \to \mathbf{Mod}^{\flat,\,\mathrm{f.t.}}_{\Bbbk}$$

is left exact and has right derived functors $\mathrm{Coext}^*_{A_*}(M_*,-)$. Since

$$\mathrm{Cohom}_{A_*}(M_*,-) \cong \mathrm{Hom}_A((-)^*,M^*) = \mathbf{Mod}_A^{\flat,\,\mathrm{f.t.}}((-)^*,M^*) = \mathbf{Mod}_A((-)^*,M^*)$$

and injective comodules are sent to projective modules, we also have

$$\mathrm{Coext}^*_{A_*}(M_*,-) \cong \mathrm{Ext}^*_A((-)^*,M^*). \tag{1.2}$$

The contravariant functor $\mathbf{Comod}_{A_*}^{\flat,\,\mathrm{f.t.}} \to \mathbf{Mod}_A^{\mathrm{op}}$ allows us to define cohomological invariants of comodules using injective resolutions in $\mathbf{Mod}_A$ as a substitute for projective resolutions in $\mathbf{Comod}_{A_*}^{\flat,\,\mathrm{f.t.}}$. In effect for a comodule N_* we define

$$\mathrm{Coext}^*_{A_*}(-,N_*) = \mathrm{Ext}_A(N^*,(-)^*).$$

Of course this is calculated using injective resolutions of A-modules; since $\mathrm{Ext}_A(-,-)$ is a balanced functor, (1.2) implies that $\mathrm{Coext}^*_{A_*}(-,-)$ is too, whenever we can use projective comodule resolutions in the first variable. For example, if we restrict to the subcategory of coherent comodules we obtain balanced bifunctors

$$\mathrm{Coext}^s_{A_*}(-,-)\colon (\mathbf{Comod}^{\mathrm{coh}}_{A_*}(-,-))^{\mathrm{op}} \otimes \mathbf{Comod}^{\mathrm{coh}}_{A_*}(-,-) \to \mathbf{Mod}_{\Bbbk}^{\flat,\,\mathrm{f.t.}}.$$

Given a surjection of P_*-algebras $A_* \to B_*$ there are adjunction isomorphisms of the form

$$\mathrm{Cohom}_{A_*}(-,-) \cong \mathrm{Cohom}_{B_*\backslash\backslash A_*}((B_*\backslash\backslash A_*)\square_{A_*}(-),-), \tag{1.3}$$

$$\mathrm{Cohom}_{B_*}(-,-) \cong \mathrm{Cohom}_{A_*}(-,A_*\square_{B_*}(-)), \tag{1.4}$$

which we will use to construct Grothendieck spectral sequences.

2. Some homological algebra for comodules

In this section we describe some Cartan-Eilenberg spectral sequences for comodules over a commutative Hopf algebra over a field. Some of these are similar to other examples in the literature such as that for computing Cotor for Hopf algebroids in [Rav86]. However, the spectral sequence that plays the biggest rôle in our calculations is not defined in the same way but requires dualising and is a standard Cartan-Eilenberg spectral sequence for Ext of modules over a ring.

To ease notation, in this section we suppress internal gradings and assume that all our objects are connective and of finite type over a field $\Bbbk$. We refer to the classic [MM65] as well as the more recent [MP12] for notation and basic ideas about graded Hopf algebras.

Suppose we have a sequence of homomorphisms of connected commutative graded Hopf algebras over $\Bbbk$,

$$K\backslash\backslash H \rightarrowtail H \twoheadrightarrow K,$$

where in the notation of [MM65, definition 3.5],

$$K\backslash\backslash H = \Bbbk\square_K H = H\square_K \Bbbk \subseteq H.$$

We also assume given a left $K\backslash\backslash H$-comodule M and a left H-comodule N. Of course M and N inherit structures of H-comodule and K-comodule respectively, where M is trivial as a K-comodule. Our aim is to calculate $\mathrm{Coext}^*_H(M,N)$, the right derived functor of

$$\mathit{Cohom}_H(M,-)\colon \mathbf{Comod}_{K\backslash\backslash H} \to \mathbf{Mod}_{\Bbbk}; \quad N \mapsto \mathrm{Cohom}_H(M,N).$$

Following Hovey [Hov04], we will write $U \overset{H}{\wedge} V$ to indicate the tensor product of two H-comodules $U \otimes V = U \otimes_{\Bbbk} V$ with the diagonal coaction given by the composition

$$\begin{tikzcd} U \otimes V \arrow[r, "\mu\otimes\mu"'] \arrow[rrr, bend left] & (H \otimes U) \otimes (H \otimes V) \arrow[r, "\cong"'] & (H \otimes H) \otimes (U \otimes V) \arrow[r, "\varphi\otimes\mathrm{Id}"'] & H \otimes (U \otimes V). \end{tikzcd}$$

For a vector space W, the notation $U \otimes W$ will be used to denote H-comodule with coaction

$$\begin{tikzcd} U \otimes W \arrow[r, "\mu\otimes\mathrm{Id}"'] \arrow[rr, bend left] & (H \otimes U) \otimes W \arrow[r, "\cong"'] & H \otimes (U \otimes W) \end{tikzcd}$$

carried on the first factor alone.

If L is a left H-comodule, then there is a well-known isomorphism of left H-comodules

$$K\backslash\backslash H \overset{H}{\wedge} L = (H \square_K \Bbbk) \overset{H}{\wedge} L \cong H \square_K L, \tag{2.1}$$

We can also regard $K\backslash\backslash H = \Bbbk \square_K H$ as a right H-comodule to form the left $K\backslash\backslash H$-comodule

$$K\backslash\backslash H \square_H L = (\Bbbk \square_K H) \square_H L \cong \Bbbk \square_K L; \tag{2.2}$$

in particular, if L is a trivial K-comodule then as left $K\backslash\backslash H$-comodules,

$$K\backslash\backslash H \square_H L \cong L. \tag{2.3}$$

We will use two more functors

$$\mathbf{Comod}_H \to \mathbf{Comod}_{K\backslash\backslash H}; \quad N \mapsto K\backslash\backslash H \square_H N = (\Bbbk \square_K H) \square_H N \cong \Bbbk \square_K N$$

and

$$\mathbf{Comod}_{K\backslash\backslash H} \to \mathbf{Mod}_{\Bbbk}; \quad N \mapsto \mathrm{Cohom}_{K\backslash\backslash H}(M, N).$$

Notice that there is a natural isomorphism

$$\mathrm{Cohom}_{K\backslash\backslash H}(M, K\backslash\backslash H \square_H (-)) \cong \mathrm{Cohom}_H(M, -)$$

and for an injective H-comodule J, $K\backslash\backslash H \square_H J$ is an injective $K\backslash\backslash H$-comodule. This means we are in a situation where we have a Grothendieck composite functor spectral sequence which in this case is a form of Cartan-Eilenberg spectral sequence; for details see [Wei94, section 5.8] for example.

Proposition 2.1. *Let M be a left $K\backslash\backslash H$-comodule and N a left H-comodule. Then there is a first quadrant cohomologically indexed spectral sequence with*

$$\mathrm{E}_2^{s,t} = \mathrm{Coext}^s_{K\backslash\backslash H}(M, \mathrm{Cotor}^t_K(\Bbbk, N)) \Longrightarrow \mathrm{Coext}^{s+t}_H(M, N).$$

If N is a trivial K-comodule then

$$\mathrm{E}_2^{s,t} \cong \mathrm{Coext}^s_{K\backslash\backslash H}(M, \mathrm{Cotor}^t_K(\Bbbk, \Bbbk) \overset{K\backslash\backslash H}{\wedge} N).$$

There is another spectral sequence that we will use whose construction requires that one of the Hopf algebras involved is a P_*-algebra. The reason for this is discussed in Remark 1.9: in the category of finite type connected comodules, extended comodules are projective objects.

Proposition 2.2. *Assume that H and $K\backslash\backslash H$ are P_*-algebras. Let M be a left H-comodule which admits a projective resolution and let N be a left $K\backslash\backslash H$-comodule. Then there is a first quadrant cohomologically indexed spectral sequence with*

$$\mathrm{E}_2^{s,t} = \mathrm{Coext}^s_{K\backslash\backslash H}(\mathrm{Cotor}^t_K(\Bbbk, M), N) \Longrightarrow \mathrm{Coext}^{s+t}_H(M, N).$$

If M is a trivial K-comodule then

$$\mathrm{E}_2^{s,t} \cong \mathrm{Coext}^s_{K\backslash\backslash H}(\mathrm{Cotor}^t_K(\Bbbk, \Bbbk) \overset{K\backslash\backslash H}{\wedge} M, N).$$

Proof. The construction is similar to the other one, and involves expressing $\mathrm{Cohom}_H(-, N)$ as a composition

$$\mathrm{Cohom}_{K\backslash\backslash H}(-, N) \circ (K\backslash\backslash H\square_H(-)) = \mathrm{Cohom}_{K\backslash\backslash H}(K\backslash\backslash H\square_H(-), N) \cong \mathrm{Cohom}_H(-, N).$$

The functor $K\backslash\backslash H\square_H(-)$ sends injective H-comodules to projective objects in $\mathbf{Comod}_{K\backslash\backslash H}$ (see Remark 1.9). Therefore the standard construction can be applied. $\square$

Of course the assumption that M admits a projective resolution is crucial; in the case of P_*-algebras this amounts to working with coherent comodules.

Unfortunately, these spectral sequences are not sufficient for our purposes and we also need to dualise and use a classical Cartan-Eilenberg spectral sequence [CE99, theorem 6.1(1)] for a normal sequence of $\Bbbk$-algebras

$$R \to S \to S//R \tag{2.4}$$

together with a left $S//R$-module L and a left S-module M. This spectral sequence has the form

$$\mathrm{E}_2^{s,t} = \mathrm{Ext}^s_{S//R}(L, \mathrm{Ext}^t_R(\Bbbk, M)) \Longrightarrow \mathrm{Ext}^{s+t}_S(L, M), \tag{2.5}$$

where we have suppressed internal gradings. In our applications (2.4) will be a sequence of cocommutative Hopf algebras.

3. The Steenrod algebra and its dual

The theory of P-algebras applies to many situations involving subHopf algebras of the Steenrod algebra for a prime. Of course the change of rings isomorphism of Proposition 1.2 holds for any subalgebra where A is flat over B, but the vanishing will not be true in general (for example, when $B = A(n)$).

To illustrate this, here is a simple application involving the mod 2 Steenrod algebra; this result appears in [Rav84, corollary 4.10]. We denote the mod 2 Eilenberg-Mac Lane spectrum by $H = H\mathbb{F}_2$.

Proposition 3.1. *The 2-completed BP-cohomology of H is trivial, i.e., $(BP\hat{_2})^*(H) = 0$.*

Proof. There is an Adams spectral sequence of form

$$\mathrm{E}_2^{s,t} = \mathrm{Coext}^{s,t}_{\mathcal{A}_*}(H_*(H), H_*(BP)) \Longrightarrow (BP\hat{_2})^{s-t}(H).$$

Now $H_*(H) = \mathcal{A}_*$ and it is well-known that as $\mathcal{A}_*$-comodule algebras,

$$H_*(BP) \cong \mathcal{A}_*\square_{\mathcal{E}_*}\mathbb{F}_2 = \mathcal{A}_*^{(1)} = \mathbb{F}_2[\zeta_1^2, \ldots, \zeta_n^2, \ldots] \tag{3.1}$$

where $\mathcal{E}_* = \mathcal{A}_*//\mathcal{A}_*^{(1)}$ which is an infinitely generated exterior Hopf algebra so it is a P_*-algebra. Now by a change of rings isomorphism and Proposition 1.2,

$$\mathrm{E}_2^{s,t} = \mathrm{Coext}_{\mathcal{A}_*}^{s,t}(\mathcal{A}_*, \mathcal{A}_*\square_{\mathcal{E}_*}\mathbb{F}_2) \cong \mathrm{Coext}_{\mathcal{E}_*}^{s,t}(\mathcal{A}_*, \mathbb{F}_2) = 0. \qquad \square$$

Doubling. The operation of *doubling* has been used frequently in studying $\mathcal{A}$-modules. The reader is referred to the account of Margolis [Mar83, section 15.3] which we will use as background.

Since the dual $\mathcal{A}_*$ is a commutative Hopf algebra, it admits a Frobenius endomorphism $\mathcal{A}_* \to \mathcal{A}_*$ which doubles degrees and has Hopf algebra cokernel

$$\mathcal{E}_* = \mathcal{A}_*//\mathcal{A}_*^{(1)} = \Lambda_{\mathbb{F}_2}(\overline{\zeta}_s : s \geqslant 1),$$

where $\mathcal{A}_*^{(1)} = \mathbb{F}_2[\zeta_s^2 : s \geqslant 1]$. Dually, there is a Verschiebung $\mathcal{A} \to \mathcal{A}$ which halves degrees and satisfies

$$\mathrm{Sq}^r \mapsto \begin{cases} \mathrm{Sq}^{r/2} & \text{if } r \text{ is even,} \\ 0 & \text{if } r \text{ is odd.} \end{cases}$$

The kernel of this Verschiebung is the ideal generated by the Milnor primitives P_t^0 $(t \geqslant 1)$, hence there is a grade-halving isomorphism of Hopf algebras $\mathcal{A}//\mathcal{E} \xrightarrow{\cong} \mathcal{A}$, where $\mathcal{E} \subseteq \mathcal{A}$ is the subHopf algebra generated by the primitives P_t^0 and dual to the exterior quotient Hopf algebra $\mathcal{E}_*$.

Given a left (graded) $\mathcal{A}$-module M, we can induce an $\mathcal{A}//\mathcal{E}$-module $M_{(1)}$ where

$$M_{(1)}^n = \begin{cases} M^{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

and we write $x_{(1)}$ to indicate the element $x \in M$ regarded as an element of $M_{(1)}$; the module structure is given by

$$\mathrm{Sq}^r(x_{(1)}) = \begin{cases} (\mathrm{Sq}^{r/2} x)_{(1)} & \text{if } r \text{ is even,} \\ 0 & \text{if } r \text{ is odd,} \end{cases}$$

Using this construction, the category of left $\mathcal{A}$-modules $\mathbf{Mod}_{\mathcal{A}}$ admits an additive functor to the category of evenly graded $\mathcal{A}//\mathcal{E}$-modules,

$$\Phi \colon \mathbf{Mod}_{\mathcal{A}} \to \mathbf{Mod}_{\mathcal{A}//\mathcal{E}}^{\mathrm{ev}}; \quad M \mapsto M_{(1)}$$

which is an isomorphism of categories. The quotient homomorphism $\rho \colon \mathcal{A} \to \mathcal{A}//\mathcal{E}$ also induces an additive isomorphism of categories $\rho^* \colon \mathbf{Mod}_{\mathcal{A}//\mathcal{E}}^{\mathrm{ev}} \to \mathbf{Mod}_{\mathcal{A}}^{\mathrm{ev}}$ and it is often useful to consider the composition $\rho^* \circ \Phi \colon \mathbf{Mod}_{\mathcal{A}} \to \mathbf{Mod}_{\mathcal{A}}^{\mathrm{ev}}$.

By iterating $\Phi^{(1)} = \Phi$ we obtain isomorphisms

$$\Phi^{(s)} = \Phi \circ \Phi^{(s-1)} \colon \mathbf{Mod}_{\mathcal{A}} \to \mathbf{Mod}_{\mathcal{A}//\mathcal{E}^{(s)}}^{(s)}; \quad M \mapsto M_{(s)}$$

where the codomain is the category of $\mathcal{A}//\mathcal{E}^{(s)}$-modules concentrated in degrees divisible by 2^s and $\mathcal{E}^{(s)} \subseteq \mathcal{A}$ is the subHopf algebra multiplicatively generated by the elements

$$\mathrm{P}_b^a \quad (s \geqslant a \geqslant 0,\ b \geqslant 1).$$

By doubling all three of the variables involved the following homological result is immediate for $e \geqslant 1$ and two $\mathcal{A}$-modules M, N:

$$\mathrm{Ext}_{\mathcal{A}_{(e)}}^{s,2^e t}(M_{(e)}, N_{(e)}) \cong \mathrm{Ext}_{\mathcal{A}}^{s,t}(M, N). \tag{3.2}$$

Because doubling is induced using a grade changing Hopf algebra endomorphism, the double $\mathcal{A}_{(1)}$ is also a Hopf algebra isomorphic to the quotient Hopf algebra $\mathcal{A}//\mathcal{E}$ and dual to the subHopf algebra of squares $\mathcal{A}_*^{(1)} \subseteq \mathcal{A}_*$ which is also given by

$$\mathcal{A}_*^{(1)} = \mathcal{A}_* \Box_{\mathcal{A}_*//\mathcal{A}_*^{(1)}} \mathbb{F}_2 = \mathbb{F}_2 \Box_{\mathcal{A}_*//\mathcal{A}_*^{(1)}} \mathcal{A}_* = (\mathcal{A}_*//\mathcal{A}_*^{(1)})\backslash\backslash\mathcal{A}_*.$$

More generally, for any $s \geqslant 1$, $\mathcal{A}_{(s)}$ is isomorphic to the quotient Hopf algebra of $\mathcal{A}//\mathcal{E}^{(s)}$ dual to the subalgebra of 2^s-th powers

$$\mathcal{A}_*^{(s)} = (\mathcal{A}_*//\mathcal{A}_*^{(s)})\backslash\backslash\mathcal{A}_* \subseteq \mathcal{A}_*.$$

In many ways, doubling is more transparent when viewed in terms of comodules. For an $\mathcal{A}_*$-comodule M_*, we can define a $\mathcal{A}_*^{(1)}$-coaction $\mu^{(1)} \colon M_*^{(1)} \to \mathcal{A}_*^{(1)} \otimes M_*^{(1)}$ where $M_*^{(1)}$ denotes M_* with its degrees doubled; this is given on elements by the composition

$$\begin{tikzcd} M_* \arrow[r, "\mu"'] \arrow[rr, bend left, "\mu^{(1)}"] & \mathcal{A}_* \otimes M_* \arrow[r, "(-)^2\otimes\mathrm{Id}"'] & \mathcal{A}_*^{(1)} \otimes M_*. \end{tikzcd}$$

By iterating we also obtain a $\mathcal{A}_*^{(s)}$-coaction $\mu^{(s)} \colon M_*^{(s)} \to \mathcal{A}_*^{(s)} \otimes M_*^{(s)}$.

Then the comodule analogue of (3.2) is

$$\mathrm{Coext}^{s,2^e t}_{\mathcal{A}_*^{(e)}}(M_*^{(e)}, N_*^{(e)}) \cong \mathrm{Coext}^{s,t}_{\mathcal{A}_*}(M_*, N_*). \tag{3.3}$$

We can use iterated doubling combined with Proposition 1.2 to show that for any $d \geqslant 1$,

$$\mathrm{Coext}^{s,t}_{\mathcal{A}_*}(\mathcal{A}_*, \mathcal{A}_*^{(d)}) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathcal{A}_{(d)}, \mathcal{A}) = 0. \tag{3.4}$$

By doubling all three of the variables involved here we can also prove that for $e \geqslant 0$,

$$\mathrm{Coext}^{s,2^e t}_{\mathcal{A}_*^{(e)}}(\mathcal{A}_*^{(e)}, \mathcal{A}_*^{(d+e)}) \cong \mathrm{Coext}^{s,t}_{\mathcal{A}_*}(\mathcal{A}_*, \mathcal{A}_*^{(d)}) = 0. \tag{3.5}$$

We will use this to show that $(M\mathrm{Sp}\widehat{_2})^*(BP) = 0$ and $(M\mathrm{Sp}\widehat{_2})^*(H) = 0$ for example.

Some families of quotient P_*-algebras of $\mathcal{A}_*$. We will begin by describing some quotients of the dual Steenrod algebra $\mathcal{A}_*$. For any $n \geqslant 1$, $(\zeta_1, \ldots, \zeta_n) \lhd \mathcal{A}_*$ is a Hopf ideal so there is a quotient Hopf algebra $\mathcal{A}_*/(\zeta_1, \ldots, \zeta_n)$ together with the subHopf algebra

$$\mathcal{P}(n)_* = \mathcal{A}_* \Box_{\mathcal{A}_*/(\zeta_1,\ldots,\zeta_n)} \mathbb{F}_2 = \mathbb{F}_2[\zeta_1, \ldots, \zeta_n] \subseteq \mathcal{A}_*$$

and in fact

$$\mathcal{A}_*//\mathcal{P}(n)_* = \mathcal{A}_*/(\zeta_1, \ldots, \zeta_n).$$

Similarly, for any $s \geqslant 0$, the ideal $(\zeta_1^{2^s}, \ldots, \zeta_n^{2^s}) \lhd \mathcal{A}_*$ is a Hopf ideal and there is a quotient Hopf algebra

$$\mathcal{A}_*//\mathcal{P}(n)_*^{(s)} = \mathcal{A}_*/(\zeta_1^{2^s}, \ldots, \zeta_n^{2^s})$$

with associated subHopf algebra

$$\mathcal{P}(n)_*^{(s)} = \mathcal{A}_* \Box_{\mathcal{A}_*//\mathcal{P}(n)_*^{(s)}} \mathbb{F}_2 = \mathbb{F}_2[\zeta_1^{2^s}, \ldots, \zeta_n^{2^s}] \subseteq \mathcal{A}_*.$$

For each $t \geqslant 0$ there is a finite quotient Hopf algebra

$$\mathcal{P}(n)_*^{(s)}/(\zeta_1^{2^{s+t}}, \zeta_2^{2^{s+t-1}}, \ldots, \zeta_t^{2^{s+1}}, \zeta_{t+1}^{2^s}, \ldots, \zeta_n^{2^s})$$

and we have

$$\mathcal{P}(n)_*^{(s)} = \lim_t \mathcal{P}(n)_*^{(s)}/(\zeta_1^{2^{s+t}}, \zeta_2^{2^{s+t-1}}, \ldots, \zeta_t^{2^{s+1}}, \zeta_{t+1}^{2^s}, \ldots, \zeta_n^{2^s})$$

where the limit is computed degree-wise. The graded dual Hopf algebra

$$\mathcal{P}(n)_{(s)} = (\mathcal{P}(n)_*^{(s)})^* = \mathrm{Hom}(\mathcal{P}(n)_*^{(s)}, \mathbb{F}_2)$$

is the colimit of the finite dual Hopf algebras

$$\mathrm{Hom}(\mathcal{P}(n)_*^{(s)}/(\zeta_1^{2^{s+t}}, \zeta_2^{2^{s+t-1}}, \ldots, \zeta_t^{2^{s+1}}, \zeta_{t+1}^{2^s}, \ldots, \zeta_n^{2^s}), \mathbb{F}_2),$$

i.e.,

$$\mathcal{P}(n)_{(s)} = \operatorname*{colim}_t \mathrm{Hom}(\mathcal{P}(n)_*^{(s)}/(\zeta_1^{2^{s+t}}, \zeta_2^{2^{s+t-1}}, \ldots, \zeta_t^{2^{s+1}}, \zeta_{t+1}^{2^s}, \ldots, \zeta_n^{2^s}), \mathbb{F}_2).$$

Therefore $\mathcal{P}(n)_{(s)}$ is a P-algebra and $\mathcal{P}(n)_*^{(s)}$ is a P_*-algebra.

4. Recollections on Ravenel's proof

We will assume the reader is familiar with the strategy behind Ravenel's proof of [Rav84, theorem 3.9]. Since we will be working in a very similar situation with ring spectra which have torsion free homology the main steps will be applicable.

Following Ravenel [Rav84, section 3], we recall that there are compatible double loop maps $\Omega\mathrm{SU}(2^s) \to B\mathrm{U}$ defined using Bott's weak equivalence $\Omega\mathrm{SU} \to B\mathrm{U}$. The corresponding virtual complex bundle on $\Omega\mathrm{SU}(2^s)$ has Thom spectrum X_s which is an $\mathcal{E}_2$ ring spectrum. Ravenel shows that

$$\langle S^0 \rangle = \langle X_1 \rangle > \langle X_2 \rangle > \langle X_3 \rangle > \cdots > \langle X_s \rangle > \langle X_{s+1} \rangle > \cdots > \langle M\mathrm{U} \rangle.$$

Of course, locally at the prime 2, $\langle M\mathrm{U} \rangle = \langle BP \rangle$.

We are lead to consider the question: How is $\langle M\mathrm{Sp} \rangle$ related to $\langle M\mathrm{U} \rangle$, and locally at 2, to $\langle BP \rangle$? Since there are maps of ring spectra $M\mathrm{Sp} \to M\mathrm{U} \to BP$, $\langle M\mathrm{Sp} \rangle \geqslant \langle M\mathrm{U} \rangle = \langle BP \rangle$. We will show that $\langle M\mathrm{Sp} \rangle > \langle BP \rangle$ and also find an analogue of the sequence of spectra X_s for $M\mathrm{Sp}$.

Remark 4.1. Here are observations on the proof of theorem 3.1.

Rather than taking $M = M(4)$ to be the mod 4 Moore spectrum we suggest using the ring spectrum $M = F(M(2), M(2))$ (it is even an A_∞ ring spectrum in a canonical way). Then $\langle M \rangle \geqslant \langle M(2) \rangle$ since the Moore spectrum $M(2) = S^0 \cup_2 e^1$ is an M-module spectrum, while $\langle M(2) \rangle \geqslant \langle M \rangle$ since $M \sim M(2) \wedge DM(2)$. Therefore $\langle M \rangle = \langle M(2) \rangle$. Similar observations apply to the odd prime case.

Also, for (a) and (b) the following observation seems sufficient. Since

$$(X \wedge M) \wedge Z \sim * \nsim (Y \wedge M) \wedge Z \quad \Longrightarrow \quad X \wedge (Z \wedge M) \sim * \nsim Y \wedge (M \wedge Z),$$

we have

$$\langle Y \wedge M \rangle > \langle X \wedge M \rangle \quad \Longrightarrow \quad \langle Y \rangle > \langle X \rangle.$$

5. Some Thom spectra on loop spaces

For background to this discussion see Mimura & Toda [MT91, Chapter IV, §6]. The sequence of spaces making up the spectrum of connected real K-theory include

$$\underline{k\mathrm{O}}_4 \sim B\mathrm{Sp}, \quad \underline{k\mathrm{O}}_5 \sim \mathrm{SU}/\mathrm{Sp},$$

so there is a weak equivalence of infinite loop spaces $\Omega\mathrm{SU}/\mathrm{Sp} \sim B\mathrm{Sp}$. At the finite level we have compatible loop maps

$$\Omega\mathrm{SU}(2n)/\mathrm{Sp}(n) \to \Omega\mathrm{SU}(2n+2)/\mathrm{Sp}(n+1) \to \cdots \to B\mathrm{Sp}$$

which define virtual symplectic bundles on these spaces and we will denote the Thom spectrum over $\Omega\mathrm{SU}(2^{s+1})/\mathrm{Sp}(2^s)$ by Y_s. There is a map of $\mathcal{E}_1$ ring spectra $Y_s \to M\mathrm{Sp}$ which induces an injective ring homomorphism in homology.

There are also compatible loop maps $\Omega\mathrm{SU}(2n) \to \Omega\mathrm{SU}(2n)/\mathrm{Sp}(n)$, and by pulling back from $\Omega\mathrm{SU}(2n)/\mathrm{Sp}(n)$ we obtain $\mathcal{E}_1$ ring spectra Y'_s over the $\Omega\mathrm{SU}(2^{s+1})$ together with $\mathcal{E}_1$ maps $Y'_s \to Y_s$.

The mod 2 homology of Y_s is given by

$$H_*(Y_s) = \mathbb{F}_2[y_1, y_2, \dots, y_{2^s-1}],$$

where $|y_k| = 4k$. The induced homomorphism $H_*(Y_s) \to H_*(M\mathrm{Sp})$ is actually an isomorphism up to degree $2^{s+2} - 4$, so we can choose the generators y_k so that they map to polynomial generators of $H_*(M\mathrm{Sp})$. In particular, we assume that each y_{2^r-1} maps to the element corresponding to the primitive generator in $H_{2^{r+2}-4}(B\mathrm{Sp}_+)$ under the Thom isomorphism $H_*(M\mathrm{Sp}) \xrightarrow{\cong} H_*(B\mathrm{Sp}_+)$. It is well known that the coaction on these elements satisfies

$$\psi y_{2^r-1} = \sum_{0 \leqslant k \leqslant r} \zeta_k^4 \otimes y_{2^{r-k}-1}^{2^k}. \tag{5.1}$$

Consider the ideal

$$J_s = (y_1, y_3, \dots, y_{2^s-1}) \lhd H_*(Y_s).$$

Then (5.1) implies that this is a subcomodule ideal, hence the quotient ring $H_*(Y_s)/J_s$ is an $\mathcal{A}_*$-comodule algebra.

We will make use of the $\mathcal{A}_*$-comodule subalgebra

$$\mathcal{P}(s)_*^{(2)} = \mathbb{F}_2[\zeta_1^4, \zeta_2^4, \dots, \zeta_s^4] \subseteq \mathcal{A}_*$$

and the quotient Hopf algebra $\mathcal{A}_*//\mathcal{P}(s)_*^{(2)}$.

Proposition 5.1. *The composition*

$$\begin{tikzcd} H_*(Y_s) \arrow[r, "\psi"] \arrow[rr, bend right] & \mathcal{A}_* \otimes H_*(Y_s) \arrow[r] & \mathcal{A}_* \square_{\mathcal{A}_*//\mathcal{P}(s)_*^{(2)}} H_*(Y_s)/J_s \end{tikzcd}$$

is an isomorphism of $\mathcal{A}_*$-comodule algebras, therefore

$$H_*(Y_s) \cong (\mathcal{A}_* \square_{\mathcal{A}_*//\mathcal{P}(s)_*^{(2)}} \mathbb{F}_2) \otimes H_*(Y_s)/J_s \cong \mathcal{P}(s)_*^{(2)} \otimes H_*(Y_s)/J_s$$

where $H_*(Y_s)/J_s$ has trivial coaction.

In the limit this coincides with well-known isomorphisms of $\mathcal{A}_*$-comodule algebras

$$\begin{aligned} H_*(M\mathrm{Sp}) &\xrightarrow{\cong} \mathcal{A}_* \square_{\mathcal{A}_*//\mathcal{A}_*^{(2)}} H_*(M\mathrm{Sp})/J \\ &\xrightarrow{\cong} (\mathcal{A}_* \square_{\mathcal{A}_*//\mathcal{A}_*^{(2)}} \mathbb{F}_2) \otimes H_*(M\mathrm{Sp})/J \xrightarrow{\cong} \mathcal{A}_*^{(2)} \otimes H_*(M\mathrm{Sp})/J \end{aligned} \tag{5.2}$$

where $J \lhd H_*(M\mathrm{Sp})$ is the comodule ideal generated by the images of all the y_{2^r-1}.

Of course the quotient Hopf algebra

$$\mathcal{A}_*//\mathcal{A}_*^{(2)} = \mathbb{F}_2[\zeta_1, \zeta_2, \dots]/(\zeta_1^4, \zeta_2^4, \dots, \zeta_s^4)$$

is also infinite dimensional and its dual is the cyclic $\mathcal{A}$-module $\mathcal{A}_{(2)} = \mathcal{A}//\mathcal{E}(2)$ where $\mathcal{E}(2) \subseteq \mathcal{A}$ is the subHopf algebra generated by the elements P_t^k ($k \geqslant 2$, $t \leqslant s$) in the notation of [Mar83, section XV.1].

6. Comparison of some Bousfield classes

Now we can give our main topological results. From now on we will work 2-locally and omit reference to this in notation, etc.

Theorem 6.1. *The following inequalities for Bousfield classes are satisfied:*

$$\langle S^0\rangle = \langle Y_1\rangle > \langle Y_2\rangle > \cdots > \langle Y_s\rangle > \langle Y_{s+1}\rangle > \cdots > \langle M\mathrm{Sp}\rangle > \langle BP\rangle.$$

Proof. We will adopt the strategy of Ravenel's proof for the analogous results on X_n and BP. In particular we will make repeated use of the Adams spectral sequence. For 2-local finite type spectra X and Y, by [Boa99, theorem 15.6] there is a strongly convergent Adams spectral sequence with

$$\mathrm{E}_2^{s,t}(X,Y) = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(Y), H^*(X)) \Longrightarrow [X, Y_2\widehat{\ }]^{s-t}. \tag{6.1}$$

We can dualise to homology and use comodules to obtain another interpretation of the Ext group:

$$\mathrm{E}_2^{s,t}(X,Y) = \mathrm{Coext}_{\mathcal{A}_*}^{s,t}(H_*(X), H_*(Y)) \Longrightarrow [X, Y_2\widehat{\ }]^{s-t}. \tag{6.2}$$

Of course we know that BP is an $M\mathrm{Sp}$-module spectrum so $\langle M\mathrm{Sp}\rangle \geqslant \langle BP\rangle$. To prove the strict inequality we follow Ravenel and reduce to showing the triviality of the E_2-term of the Adams spectral sequence

$$\mathrm{E}_2^{s,t}(BP, M\mathrm{Sp}) = \mathrm{Coext}_{\mathcal{A}_*}^{s,t}(H_*(BP), H_*(M\mathrm{Sp})) \Longrightarrow [BP, M\mathrm{Sp}_2\widehat{\ }]^{s-t}.$$

By (5.2),

$$H_*(M\mathrm{Sp}) \cong (\mathcal{A}_*\square_{\mathcal{A}_*//\mathcal{A}_*^{(2)}}\mathbb{F}_2) \otimes H_*(M\mathrm{Sp})/J$$

where $H_*(M\mathrm{Sp})/J$ has trivial $\mathcal{A}_*//\mathcal{A}_*^{(2)}$-coaction. Combining this with (3.1) we have

$$\begin{aligned}\mathrm{E}_2^{s,*}(BP, M\mathrm{Sp}) &\cong \mathrm{Coext}_{\mathcal{A}_*}^{s,*}(H_*(BP), \mathcal{A}_*\square_{\mathcal{A}_*//\mathcal{A}_*^{(2)}}\mathbb{F}_2) \otimes H_*(M\mathrm{Sp})/J \\ &\cong \mathrm{Coext}_{\mathcal{A}_*//\mathcal{A}_*^{(2)}}^{s,*}(\mathcal{A}_*^{(1)}, \mathbb{F}_2) \otimes H_*(M\mathrm{Sp})/J.\end{aligned}$$

To compute the Coext term here we will dualise and use a Cartan-Eilenberg spectral sequence of the form (2.5) for

$$\mathrm{Ext}_{(\mathcal{A}_*//\mathcal{A}_*^{(2)})^*}(\mathbb{F}_2, \mathcal{A}_{(1)}) \cong \mathrm{Coext}_{\mathcal{A}_*//\mathcal{A}_*^{(2)}}^{s,*}(\mathcal{A}_*^{(1)}, \mathbb{F}_2).$$

We will base this on the sequence of algebras

$$(\mathcal{A}_*//\mathcal{A}_*^{(1)})^* \longrightarrow (\mathcal{A}_*//\mathcal{A}_*^{(2)})^* \longrightarrow (\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)})^*$$

which is dual to

$$\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)} \to \mathcal{A}_*//\mathcal{A}_*^{(2)} \to \mathcal{A}_*//\mathcal{A}_*^{(1)}.$$

This spectral sequence is tri-graded with E_2-term

$${}^{\mathrm{CE}}\mathrm{E}_2^{s,t} = \mathrm{Ext}^s_{(\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)})^*}(\mathbb{F}_2, \mathrm{Ext}^t_{(\mathcal{A}_*//\mathcal{A}_*^{(1)})^*}(\mathbb{F}_2, \mathcal{A}_{(1)}))$$

where we have suppressed mention of the internal grading. As $(\mathcal{A}_*//\mathcal{A}_*^{(1)})^*$ acts trivially on $\mathcal{A}_{(1)}$,

$$\mathrm{Ext}^t_{(\mathcal{A}_*//\mathcal{A}_*^{(1)})^*}(\mathbb{F}_2, \mathcal{A}_{(1)}) \cong \mathrm{Ext}^t_{(\mathcal{A}_*//\mathcal{A}_*^{(1)})^*}(\mathbb{F}_2, \mathbb{F}_2) \overset{(\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)})^*}{\boxtimes} \mathcal{A}_{(1)},$$

where $\overset{(\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)})^*}{\boxtimes}$ indicates the $\Bbbk$-tensor product with the diagonal action of the cocommutative Hopf algebra $(\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)})^*$. Since there is a surjection of Hopf algebras

$$\mathcal{A}_*^{(1)} \to \mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)},$$

dually there is an injection

$$(\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)})^* \to (\mathcal{A}_*^{(1)})^* = \mathcal{A}_{(1)}$$

so the Milnor-Moore theorem implies that $\mathcal{A}_{(1)}$ is a free $(\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)})^*$-module. Hence as $(\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)})^*$-modules,

$$\mathrm{Ext}^t_{(\mathcal{A}_*//\mathcal{A}_*^{(1)})^*}(\mathbb{F}_2,\mathbb{F}_2) \overset{(\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)})^*}{\boxtimes} \mathcal{A}_{(1)} \cong \mathcal{A}_{(1)} \otimes \mathrm{Ext}^t_{(\mathcal{A}_*//\mathcal{A}_*^{(1)})^*}(\mathbb{F}_2,\mathbb{F}_2).$$

Feeding this into our E_2-term we obtain

$$\begin{aligned}
{}^{\mathrm{CE}}\mathrm{E}_2^{s,t} &\cong \mathrm{Ext}^s_{(\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)})^*}(\mathbb{F}_2,\mathcal{A}_{(1)}) \otimes \mathrm{Ext}^t_{(\mathcal{A}_*//\mathcal{A}_*^{(1)})^*}(\mathbb{F}_2,\mathbb{F}_2)\\
&\cong \mathrm{Coext}^s_{\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)}}(\mathcal{A}_*^{(1)},\mathbb{F}_2) \otimes \mathrm{Ext}^t_{(\mathcal{A}_*//\mathcal{A}_*^{(1)})^*}(\mathbb{F}_2,\mathbb{F}_2)\\
&\cong \mathrm{Coext}^s_{\mathcal{A}_*//\mathcal{A}_*^{(1)}}(\mathcal{A}_*,\mathbb{F}_2) \otimes \mathrm{Ext}^t_{(\mathcal{A}_*//\mathcal{A}_*^{(1)})^*}(\mathbb{F}_2,\mathbb{F}_2)\\
&= 0
\end{aligned}$$

since $\mathcal{A}_*$ is a cofree comodule over the P_*-algebra $\mathcal{A}_*//\mathcal{A}_*^{(1)}$.

Now we can conclude that $\mathrm{E}_2^{s,*}(BP, M\mathrm{Sp}) = 0$ and so $(M\mathrm{Sp}\hat{_2})^*(BP) = 0$.

The rest of the proof is similar and uses the spectral sequence (2.5) to reduce to calculate some Coext groups using the dual algebras. We will describe the main details required for these.

To verify that $\langle Y_n \rangle > \langle M\mathrm{Sp} \rangle$ the key step involves showing that

$$\mathrm{Coext}^{*,*}_{\mathcal{A}_*}(\mathcal{A}_*^{(2)},\mathcal{P}(n)_*^{(2)}) = \mathrm{Coext}^{*,*}_{\mathcal{A}_*}(\mathcal{A}_*^{(2)},\mathcal{A}_*\square_{\mathcal{A}_*//\mathcal{P}(n)_*^{(2)}}\mathbb{F}_2) = 0,$$

where

$$\mathrm{Coext}^{*,*}_{\mathcal{A}_*}(\mathcal{A}_*^{(2)},\mathcal{A}_*\square_{\mathcal{A}_*//\mathcal{P}(n)_*^{(2)}}\mathbb{F}_2) \cong \mathrm{Coext}^{*,*}_{\mathcal{A}_*//\mathcal{P}(n)_*^{(2)}}(\mathcal{A}_*^{(2)},\mathbb{F}_2).$$

By [MP12, corollary 21.2.5], there is a sequence of Hopf algebras

$$\mathcal{A}_*^{(2)}//\mathcal{P}(n)_*^{(2)} = (\mathcal{A}_*//\mathcal{A}_*^{(2)})\backslash\backslash(\mathcal{A}_*//\mathcal{P}(n)_*^{(2)}) \to \mathcal{A}_*//\mathcal{P}(n)_*^{(2)} \to \mathcal{A}_*//\mathcal{A}_*^{(2)}$$

whose the dual sequence of algebras gives rise to a Cartan-Eilenberg spectral sequence (2.5) with

$$\begin{aligned}
{}^{\mathrm{CE}}\mathrm{E}_2^{s,t} &\cong \mathrm{Ext}^s_{\mathcal{A}_{(2)}}(\mathbb{F}_2,\mathrm{Ext}^t_{(\mathcal{A}_*//\mathcal{A}_*^{(2)})^*}(\mathbb{F}_2,\mathcal{A}_{(2)}))\\
&\cong \mathrm{Ext}^s_{\mathcal{A}_{(2)}}(\mathbb{F}_2,\mathrm{Ext}^t_{(\mathcal{A}_*//\mathcal{A}_*^{(2)})^*}(\mathbb{F}_2,\mathbb{F}_2) \overset{\mathcal{A}_{(2)}}{\boxtimes} \mathcal{A}_{(2)})\\
&\cong \mathrm{Ext}^s_{\mathcal{A}_{(2)}}(\mathbb{F}_2,\mathcal{A}_{(2)}) \otimes \mathrm{Ext}^t_{(\mathcal{A}_*//\mathcal{A}_*^{(2)})^*}(\mathbb{F}_2,\mathbb{F}_2))\\
&= 0
\end{aligned}$$

since $\mathcal{A}_{(2)}$ is a P-algebra.

Finally, the verification that $\langle Y_n \rangle > \langle Y_{n+1} \rangle$ proceeds in this fashion with $\mathcal{P}(n+1)_*^{(2)}$ in place of $\mathcal{A}_*^{(2)}$ and involves the vanishing of

$$\mathrm{Coext}^{s,4*}_{\mathcal{P}(n+1)_*^{(2)}//\mathcal{P}(n)_*^{(2)}}(\mathcal{P}(n+1)_*^{(2)},\mathbb{F}_2) \cong \mathrm{Coext}^{s,*}_{\mathcal{P}(n+1)_*//\mathcal{P}(n)_*}(\mathcal{P}(n+1)_*,\mathbb{F}_2)$$

which follows from the facts that $\mathcal{P}(n+1)_*$ is cofree over $\mathcal{P}(n+1)_*//\mathcal{P}(n)_*$ by the Milnor-Moore theorem and the latter is a P_*-algebra. $\square$

As an exercise, the reader may like to rederive Ravenel's results for BP and the X_n's using our approach.

There are some other interesting spectra that intermediate between $M\mathrm{Sp}$ and BP. These include Pengelley's BoP [Pen82], $M\mathrm{SU}$ and the sequence of $M\mathrm{Sp}$-module spectra obtained by killing the Ray elements $\varphi_{(s)} = \varphi_s \in \pi_{2^{s+2}-3}(M\mathrm{Sp})$ as described by Botvinnik [Bot92] (both including and excluding the generator of $\pi_1(M\mathrm{Sp})$). In fact if we kill finitely many of the Ray elements this does not change the Bousfield class since they are all nilpotent, whereas if we kill all of them we get a spectrum whose Bousfield class is the same as BP; similarly, $\langle M\mathrm{SU}\rangle = \langle BP\rangle$. So there are no novel Bousfield classes between $\langle M\mathrm{Sp}\rangle$ and $\langle BP\rangle$ stemming from these.

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