

A note on spherical functors

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ABSTRACT

We provide a new and very short proof of the fact that a spherical functor between certain triangulated categories induces an auto-equivalence.

Introduction

Let $\mathcal{D}(X)$ denote the bounded derived category of coherent sheaves on a smooth projective variety X . If Y is another smooth projective variety, then any object $\mathcal{P} \in \mathcal{D}(X \times Y)$ gives rise to a Fourier–Mukai functor $F = \Phi_{\mathcal{P}} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$, and we refer to the object \mathcal{P} as the Fourier–Mukai kernel of F . Similarly, if Z is a third smooth projective variety and $\mathcal{Q} \in \mathcal{D}(Y \times Z)$ is any object, then the composition $\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}}$ is induced by the convolution $\mathcal{Q} * \mathcal{P} := \pi_{13*}(\pi_{12}^* \mathcal{P} \otimes \pi_{23}^* \mathcal{Q}) \in \mathcal{D}(X \times Z)$.

Now, since such an $F = \Phi_{\mathcal{P}} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ has a left adjoint $L = \Phi_{\mathcal{P}_L}$ and a right adjoint $R = \Phi_{\mathcal{P}_R}$, we can use the unit and counit of adjunction to define new kernels via the following triangles:

$$\mathcal{C} \rightarrow \Delta_* \mathcal{O}_X \xrightarrow{\eta} \mathcal{P}_R * \mathcal{P} \quad \text{and} \quad \mathcal{P} * \mathcal{P}_R \xrightarrow{\varepsilon} \Delta_* \mathcal{O}_Y \rightarrow \mathcal{T}.$$

The induced functors $C = \Phi_{\mathcal{C}} : \mathcal{D}(X) \rightarrow \mathcal{D}(X)$ and $T = \Phi_{\mathcal{T}} : \mathcal{D}(Y) \rightarrow \mathcal{D}(Y)$ are called the cotwist and twist of F , respectively.

In this brief note, we give a short and simple proof, relying solely on the structure of adjunctions, and a classical result found in [5], of the following theorem.

THEOREM [2.3 and 3.2]. *Let $F = \Phi_{\mathcal{P}} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ be a Fourier–Mukai functor between the bounded derived categories of two smooth projective varieties X and Y , with left adjoint $L = \Phi_{\mathcal{P}_L}$ and right adjoint $R = \Phi_{\mathcal{P}_R}$. Suppose that the cotwist $C = \Phi_{\mathcal{C}}$ is an auto-equivalence of $\mathcal{D}(X)$ and $\mathcal{P}_R \simeq \mathcal{C} * \mathcal{P}_L[1]$ is any isomorphism. Then the canonical map $\mathcal{P}_R \rightarrow \mathcal{P}_R * \mathcal{P} * \mathcal{P}_L \rightarrow \mathcal{C} * \mathcal{P}_L[1]$ is an isomorphism and the twist $T = \Phi_{\mathcal{T}}$ is an auto-equivalence of $\mathcal{D}(Y)$.*

The observation that spherical functors give rise to interesting auto-equivalences is well documented. It is hard to underestimate the importance of the paper [10]. Their ideas were further developed in [2, 8] and the foundations were finally completed in [3]. Other notable works include [1, 6, 9]. Our proof is different from all of these and so we feel it is worthy of mention.

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1. Preliminaries

DEFINITION 1.1. If $F = \Phi_{\mathcal{P}} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ is a Fourier–Mukai functor with left adjoint $L = \Phi_{\mathcal{P}_L}$ and right adjoint $R = \Phi_{\mathcal{P}_R}$, then we distinguish the units and counits with subscripts as follows:

$$\Delta_* \mathcal{O}_X \xrightarrow{\eta_R} \mathcal{P}_R * \mathcal{P} \quad \mathcal{P} * \mathcal{P}_R \xrightarrow{\varepsilon_R} \Delta_* \mathcal{O}_Y \quad \Delta_* \mathcal{O}_Y \xrightarrow{\eta_L} \mathcal{P} * \mathcal{P}_L \quad \mathcal{P}_L * \mathcal{P} \xrightarrow{\varepsilon_L} \Delta_* \mathcal{O}_X.$$

If an argument only deals with one adjoint pair, then we shall drop the subscripts.

LEMMA 1.2. *Units and counits are exchanged under the adjunction isomorphisms:*

$$\mathrm{Hom}(\Delta_* \mathcal{O}_X, \mathcal{P}_R * \mathcal{P}) \simeq \mathrm{Hom}(\mathcal{P}_L * \mathcal{P}, \Delta_* \mathcal{O}_X) ; \eta_R \mapsto \varepsilon_L$$

$$\mathrm{Hom}(\mathcal{P} * \mathcal{P}_R, \Delta_* \mathcal{O}_Y) \simeq \mathrm{Hom}(\Delta_* \mathcal{O}_Y, \mathcal{P} * \mathcal{P}_L) ; \varepsilon_R \mapsto \eta_L.$$

Proof. For the first one, recall from [7, Chapter IV] that an adjunction is a bijection which assigns arrows according to a specific recipe. In particular, we have:

$$\begin{array}{ccc} \Delta_* \mathcal{O}_X & & \mathcal{P}_L * \mathcal{P} \xrightarrow{\mathcal{P}_L * \mathcal{P} * \eta_R} \mathcal{P}_L * \mathcal{P} * \mathcal{P}_R * \mathcal{P} \\ \eta_R \downarrow & \mapsto & \searrow \varepsilon_L \quad \downarrow \varepsilon \simeq \varepsilon_L \circ (\mathcal{P}_L * \varepsilon_R * \mathcal{P}) \\ \mathcal{P}_R * \mathcal{P} & & \Delta_* \mathcal{O}_X, \end{array}$$

where ε is the counit associated to the adjoint pair $(\mathcal{P}_L * \mathcal{P}, \mathcal{P}_R * \mathcal{P})$. Using naturality of counits, we can rewrite this universal arrow as $\varepsilon \simeq \varepsilon_L \circ (\mathcal{P}_L * \varepsilon_R * \mathcal{P})$. Finally, the composition of the arrows being ε_L follows from convolving the triangular identity $(\varepsilon_R * \mathcal{P}) \circ (\mathcal{P} * \eta_R) \simeq \mathrm{id}_{\mathcal{P}}$ on the left with \mathcal{P}_L ; see [7, Theorem IV.1.1.(ii)]. The second statement follows from a similar argument. \square

DEFINITION 1.3. Let $F = \Phi_{\mathcal{P}} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ be a Fourier–Mukai functor between the derived categories of two smooth projective varieties X and Y with left adjoint $L = \Phi_{\mathcal{P}_L}$ and right adjoint $R = \Phi_{\mathcal{P}_R}$. Using the units and counits above, we can define the *twist* $T = \Phi_{\mathcal{T}}$ and *cotwist* $C = \Phi_{\mathcal{C}}$ of F by the following exact triangles:

$$\mathcal{P} * \mathcal{P}_R \xrightarrow{\varepsilon_R} \Delta_* \mathcal{O}_Y \xrightarrow{\alpha_R} \mathcal{T} \xrightarrow{\beta_R} \mathcal{P} * \mathcal{P}_R[1] \quad \text{and} \quad \mathcal{C} \xrightarrow{\delta_R} \Delta_* \mathcal{O}_X \xrightarrow{\eta_R} \mathcal{P}_R * \mathcal{P} \xrightarrow{\gamma_R} \mathcal{C}[1].$$

Similarly, we define the *dual twist* $T' = \Phi_{\mathcal{T}'}$ and *dual cotwist* $C' = \Phi_{\mathcal{C}'}$ of F by

$$\mathcal{T}' \xrightarrow{\delta'_L} \Delta_* \mathcal{O}_Y \xrightarrow{\eta'_L} \mathcal{P} * \mathcal{P}_L \xrightarrow{\gamma'_L} \mathcal{T}'[1] \quad \text{and} \quad \mathcal{P}_L * \mathcal{P} \xrightarrow{\varepsilon'_L} \Delta_* \mathcal{O}_X \xrightarrow{\alpha'_L} \mathcal{C}' \xrightarrow{\beta'_L} \mathcal{P}_L * \mathcal{P}[1].$$

LEMMA 1.4. *$C' = \Phi_{\mathcal{C}'}$ and $T' = \Phi_{\mathcal{T}'}$ are left adjoint to $C = \Phi_{\mathcal{C}}$ and $T = \Phi_{\mathcal{T}}$, respectively.*

Proof. This can be found in [6, Remark 2.10]. For a direct argument, first take left adjoints of the triangle $\mathcal{C} \rightarrow \Delta_* \mathcal{O}_X \xrightarrow{\eta_R} \mathcal{P}_R * \mathcal{P}$ defining the kernel of the cotwist $C = \Phi_{\mathcal{C}}$ to get an exact triangle $\mathcal{P}_L * \mathcal{P} \rightarrow \Delta_* \mathcal{O}_X \rightarrow \mathcal{C}'$ and then use Lemma 1.2 to see that the $\mathrm{Hom}(\Delta_* \mathcal{O}_X, \mathcal{P}_R * \mathcal{P}) \simeq \mathrm{Hom}(\mathcal{P}_L * \mathcal{P}, \Delta_* \mathcal{O}_X)$ maps the unit η_R to the counit ε_L . This shows that $C' \dashv C$. Similarly, we can show $T' \dashv T$. \square

LEMMA 1.5. *Let $F = \Phi_{\mathcal{P}} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ be a Fourier–Mukai functor with right adjoint $R = \Phi_{\mathcal{P}_R}$. Then we have the following natural isomorphisms:*

$$\begin{aligned} \mathcal{T} * \mathcal{P}[-1] &\xrightarrow{\beta_R * \mathcal{P}[-1]} \mathcal{P} * \mathcal{P}_R * \mathcal{P} \xrightarrow{\mathcal{P} * \gamma_R} \mathcal{P} * \mathcal{C}[1], \quad \text{and} \\ \mathcal{P}_R * \mathcal{T}[-1] &\xrightarrow{\mathcal{P}_R * \beta_R[-1]} \mathcal{P}_R * \mathcal{P} * \mathcal{P}_R \xrightarrow{\gamma_R * \mathcal{P}_R} \mathcal{C} * \mathcal{P}_R[1]. \end{aligned}$$

Similarly, if $F = \Phi_{\mathcal{P}} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ is a Fourier–Mukai functor with left adjoint $L = \Phi_{\mathcal{P}_L}$, then we have the following natural isomorphisms:

$$\begin{aligned} \mathcal{P} * \mathcal{C}'[-1] &\xrightarrow{\mathcal{P} * \beta_L[-1]} \mathcal{P} * \mathcal{P}_L * \mathcal{P} \xrightarrow{\gamma_L * \mathcal{P}} \mathcal{T}' * \mathcal{P}[1] \quad \text{and} \\ \mathcal{C}' * \mathcal{P}_L[-1] &\xrightarrow{\beta_L * \mathcal{P}_L[-1]} \mathcal{P}_L * \mathcal{P} * \mathcal{P}_L \xrightarrow{\mathcal{P}_L * \gamma_L} \mathcal{P}_L * \mathcal{T}'[1]. \end{aligned}$$

Proof. These identities are standard, and all stem from the triangular identities associated to the adjoint pairs. For example, [7, Theorem IV.1.1(ii)] tells us that the composition $\mathcal{P}_R \xrightarrow{\eta_R * \mathcal{P}_R} \mathcal{P}_R * \mathcal{P} * \mathcal{P}_R \xrightarrow{\mathcal{P}_R * \varepsilon_R} \mathcal{P}_R$ is the identity on \mathcal{P}_R . This provides a splitting of $\mathcal{P}_R * \mathcal{P}_R$ and allows us to complete the following diagram:

$$\begin{array}{ccccc} & & \mathcal{P}_R * \mathcal{T}[-1] & & \\ & & \downarrow \mathcal{P}_R * \beta_R[-1] & \searrow \sim & \\ \mathcal{P}_R & \xrightarrow{\eta_R * \mathcal{P}_R} & \mathcal{P}_R * \mathcal{P} * \mathcal{P}_R & \xrightarrow{\gamma_R * \mathcal{P}_R} & \mathcal{C} * \mathcal{P}_R[1] \\ & \searrow & \downarrow \mathcal{P}_R * \varepsilon_R & & \\ & & \mathcal{P}_R & & \end{array}$$

using the octahedral axiom to get a functorial isomorphism:

$$\mathcal{P}_R * \mathcal{T}[-1] \xrightarrow{\mathcal{P}_R * \beta_R[-1]} \mathcal{P}_R * \mathcal{P} * \mathcal{P}_R \xrightarrow{\gamma_R * \mathcal{P}_R} \mathcal{C} * \mathcal{P}_R[1]. \quad \square$$

The following result is the key technical lemma which will allow us to easily deduce that the twist associated to a spherical functor is an auto-equivalence.

LEMMA 1.6. *Let $F = \Phi_{\mathcal{P}} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ be a Fourier–Mukai functor with a right adjoint $R = \Phi_{\mathcal{P}_R}$. If there is any natural isomorphism (not necessarily the unit of adjunction) between $\mathcal{P}_R * \mathcal{P}$ and $\Delta_* \mathcal{O}_X$, then the unit $\eta : \Delta_* \mathcal{O}_X \xrightarrow{\sim} \mathcal{P}_R * \mathcal{P}$ of adjunction is an isomorphism. That is, $F = \Phi_{\mathcal{P}}$ is fully faithful.*

Proof. This statement is the dual of [5, Lemma 1.1.1] translated into the setting of Fourier–Mukai functors. □

2. Spherical functors

DEFINITION 2.1. We say that a Fourier–Mukai functor $F = \Phi_{\mathcal{P}} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ with left adjoint $L = \Phi_{\mathcal{P}_L}$ and right adjoint $R = \Phi_{\mathcal{P}_R}$ is *spherical* if the cotwist $C = \Phi_{\mathcal{C}}$ is an auto-equivalence of $\mathcal{D}(X)$ and the canonical map:

$$(\gamma_R * \mathcal{P}_L) \circ (\mathcal{P}_R * \eta_L) : \mathcal{P}_R \rightarrow \mathcal{P}_R * \mathcal{P} * \mathcal{P}_L \rightarrow \mathcal{C} * \mathcal{P}_L[1],$$

is a functorial isomorphism.

REMARK 2.2. Proposition 3.2 shows that if $C = \Phi_{\mathcal{C}}$ is an auto-equivalence, then any isomorphism $\mathcal{P}_R \simeq \mathcal{C} * \mathcal{P}_L[1]$ ensures that $(\gamma_R * \mathcal{P}_L) \circ (\mathcal{P}_R * \eta_L)$ is an isomorphism.

THEOREM 2.3. *Let $F = \Phi_{\mathcal{P}} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ be a Fourier–Mukai functor between the bounded derived categories of two smooth projective varieties X and Y , with left adjoint $L = \Phi_{\mathcal{P}_L}$ and right adjoint $R = \Phi_{\mathcal{P}_R}$.*

- (i) If the canonical map $(\gamma_R * \mathcal{P}_L) \circ (\mathcal{P}_R * \eta_L) : \mathcal{P}_R \rightarrow \mathcal{P}_R * \mathcal{P} * \mathcal{P}_L \rightarrow \mathcal{C} * \mathcal{P}_L[1]$ is an isomorphism, then the unit $\eta_T : \Delta_* \mathcal{O}_Y \xrightarrow{\sim} \mathcal{T} * \mathcal{T}'$ of adjunction is an isomorphism.
- (ii) If the canonical map $(\gamma_R * \mathcal{P}_L) \circ (\mathcal{P}_R * \eta_L) : \mathcal{P}_R \rightarrow \mathcal{P}_R * \mathcal{P} * \mathcal{P}_L \rightarrow \mathcal{C} * \mathcal{P}_L[1]$ is an isomorphism and $C = \Phi_C$ is an auto-equivalence of $\mathcal{D}(X)$, then $T = \Phi_T$ is an auto-equivalence of $\mathcal{D}(Y)$.

Proof. (i) We use the triangles $\mathcal{P} * \mathcal{P}_R \rightarrow \Delta_* \mathcal{O}_Y \rightarrow \mathcal{T}$ and $\mathcal{T}' \rightarrow \Delta_* \mathcal{O}_Y \rightarrow \mathcal{P} * \mathcal{P}_L$ to construct a commutative diagram:

$$\begin{array}{ccccc}
 \mathcal{P} * \mathcal{P}_R & \xrightarrow{\mathcal{P} * \mathcal{P}_R * \eta_L} & \mathcal{P} * \mathcal{P}_R * \mathcal{P} * \mathcal{P}_L & \xrightarrow{\mathcal{P} * \mathcal{P}_R * \gamma_L} & \mathcal{P} * \mathcal{P}_R * \mathcal{T}'[1] \\
 \varepsilon_R \downarrow & & \downarrow \varepsilon_R * \mathcal{P} * \mathcal{P}_L & & \downarrow \varepsilon_R * \mathcal{T}'[1] \\
 \Delta_* \mathcal{O}_Y & \xrightarrow{\eta_L} & \mathcal{P} * \mathcal{P}_L & \xrightarrow{\gamma_L} & \mathcal{T}'[1] \\
 \alpha_R \downarrow & & \downarrow \alpha_R * \mathcal{P} * \mathcal{P}_L & & \downarrow \alpha_R * \mathcal{T}'[1] \\
 \mathcal{T} & \xrightarrow{\mathcal{T} * \eta_L} & \mathcal{T} * \mathcal{P} * \mathcal{P}_L[1] & \xrightarrow{\mathcal{T} * \gamma_L} & \mathcal{T} * \mathcal{T}'[1].
 \end{array}$$

If we consider the top right square of the previous diagram together with the commutative square:

$$\begin{array}{ccc}
 \mathcal{P} * \mathcal{P}_L & \xlongequal{\quad} & \mathcal{P} * \mathcal{P}_L \\
 \parallel & & \parallel \\
 \mathcal{P} * \mathcal{P}_L & \xlongequal{\quad} & \mathcal{P} * \mathcal{P}_L,
 \end{array}$$

consisting of four copies of $\mathcal{P} * \mathcal{P}_L$ and all maps being the identity, then we can use the natural map $\mathcal{P} * \eta_R * \mathcal{P}_L : \mathcal{P} * \mathcal{P}_L \rightarrow \mathcal{P} * \mathcal{P}_R * \mathcal{P} * \mathcal{P}_L$ to form a commutative diagram:

$$\begin{array}{ccc}
 & \mathcal{P} * \mathcal{P}_R * \mathcal{P} * \mathcal{P}_L & \xrightarrow{\mathcal{P} * \mathcal{P}_R * \gamma_L} & \mathcal{P} * \mathcal{P}_R * \mathcal{T}'[1] \\
 \mathcal{P} * \eta_R * \mathcal{P}_L \nearrow & \downarrow & \nearrow (\mathcal{P} * \mathcal{P}_R * \gamma_L) \circ (\mathcal{P} * \eta_R * \mathcal{P}_L) & \downarrow \varepsilon_R * \mathcal{T}'[1] \\
 \mathcal{P} * \mathcal{P}_L & \xlongequal{\quad} & \mathcal{P} * \mathcal{P}_L & \\
 \parallel & \downarrow \varepsilon_R * \mathcal{P} * \mathcal{P}_L & \parallel & \\
 \mathcal{P} * \mathcal{P}_L & \xrightarrow{\quad} & \mathcal{P} * \mathcal{P}_L & \xrightarrow{\gamma_L} & \mathcal{T}'[1]. \\
 \parallel & \nearrow & \parallel & \nearrow \gamma_L \\
 \mathcal{P} * \mathcal{P}_L & \xlongequal{\quad} & \mathcal{P} * \mathcal{P}_L & &
 \end{array}$$

Indeed, the left face is just the triangular identity convolved with \mathcal{P}_L on the right; the top and bottom faces are clearly commutative and the commutativity of the right face follows from the commutativity of the other faces of the cube.

Applying the octahedral axiom to the top and right faces of this commutative diagram produces the following commutative diagram of triangles (figure 1).

The canonical map $(\gamma_R * \mathcal{P}_L) \circ (\mathcal{P}_R * \eta_L) : \mathcal{P}_R \rightarrow \mathcal{P}_R * \mathcal{P} * \mathcal{P}_L \rightarrow \mathcal{C} * \mathcal{P}_L[1]$ is an isomorphism by assumption. Therefore, convolving the canonical map with \mathcal{P} on the left, to get $(\mathcal{P} * \gamma_R * \mathcal{P}_L) \circ (\mathcal{P} * \mathcal{P}_R * \eta_L)$, must also be an isomorphism. This implies $\mathcal{P} * \mathcal{Q}[1] \simeq 0$ which

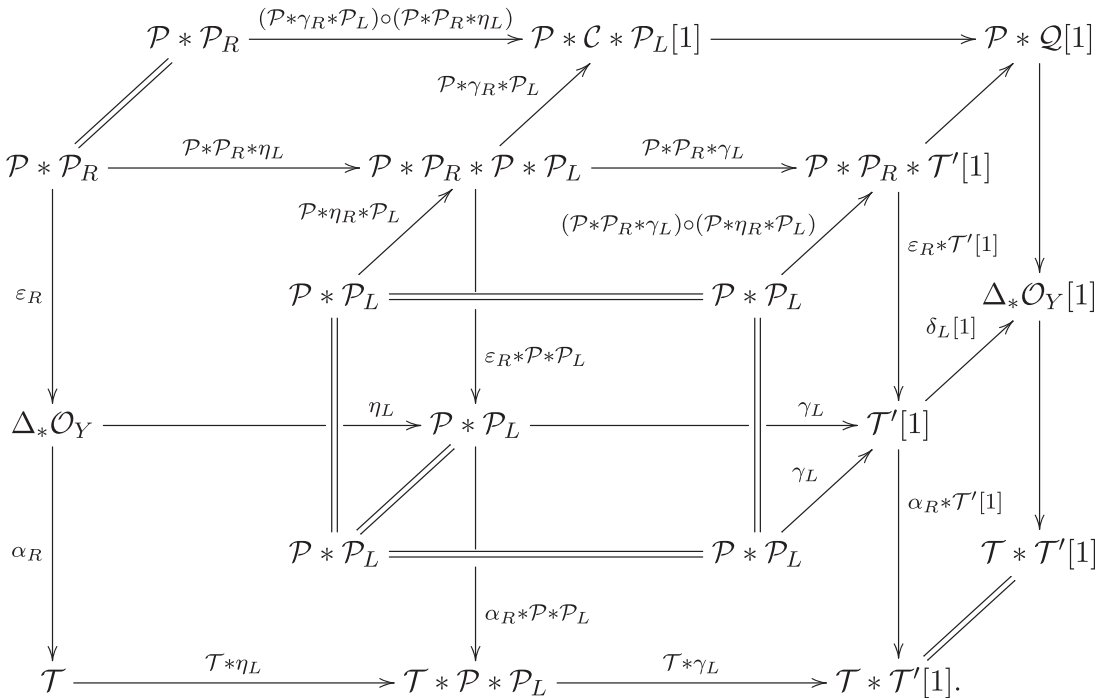


FIGURE 1. Diagram of functors associated to $\mathcal{T} * \mathcal{T}'[1]$.

in turn provides a natural isomorphism $\Delta_* \mathcal{O}_Y[1] \xrightarrow{\sim} \mathcal{T} * \mathcal{T}'[1]$. By Lemma 1.6, this implies that the unit $\eta_T : \Delta_* \mathcal{O}_Y \xrightarrow{\sim} \mathcal{T} * \mathcal{T}'$ of adjunction is an isomorphism.

(ii) Part (i) proves that $T' = \Phi_{\mathcal{T}'} : \mathcal{D}(Y) \rightarrow \mathcal{D}(Y)$ is fully faithful and so it remains to show that $T' = \Phi_{\mathcal{T}'}$ is an equivalence. By [4, Lemma 1.50], it is enough to show that $\ker \Phi_{\mathcal{T}} = 0$. To see this, suppose that $\Phi_{\mathcal{T}}(\mathcal{E}) = 0$ for some $\mathcal{E} \in \mathcal{D}(Y)$. Then, by Lemma 1.5, we have $\Phi_{\mathcal{C} * \mathcal{P}_R}(\mathcal{E}) \simeq \Phi_{\mathcal{P}_R * \mathcal{T}}(\mathcal{E})[-2] = (\Phi_{\mathcal{P}_R}(\Phi_{\mathcal{T}}(\mathcal{E})))[-2] = 0$ which implies $\Phi_{\mathcal{P}_R}(\mathcal{E}) = 0$ since the cotwist $C = \Phi_C$ is an auto-equivalence by assumption. Now, the defining triangle $\Phi_{\mathcal{P}}(\Phi_{\mathcal{P}_R}(\mathcal{E})) \rightarrow \mathcal{E} \rightarrow \Phi_{\mathcal{T}}(\mathcal{E})$ shows that $\mathcal{E} = 0$ and so $\ker \Phi_{\mathcal{T}} = 0$ as required. Finally, we know that $T = \Phi_{\mathcal{T}}$ is right adjoint to $T' = \Phi_{\mathcal{T}'}$ by Lemma 1.4 and so $T = \Phi_{\mathcal{T}}$ must be an equivalence as well. \square

COROLLARY 2.4. *The left (or right) adjoint of a spherical functor $F = \Phi_{\mathcal{P}}$ with twist $T = \Phi_{\mathcal{T}}$ and cotwist $C = \Phi_C$ is a spherical functor with twist C^{-1} and cotwist T^{-1} .*

Proof. This follows from the fact that the units and counits are exchanged under adjunction; see Lemma 1.2. \square

3. Identifying adjoints by an auto-equivalence

We work with the same notation that was introduced in Section 1. For details on adjunctions, we refer to [7, § IV.1–4].

LEMMA 3.1. *If $C = \Phi_C$ is an auto-equivalence of $\mathcal{D}(X)$, then the canonical map:*

$$\mathcal{P}_R * \mathcal{P} \xrightarrow{\mathcal{P}_R * \eta_L * \mathcal{P}} \mathcal{P}_R * \mathcal{P} * \mathcal{P}_L * \mathcal{P} \xrightarrow{\gamma_R * \mathcal{P}_L * \mathcal{P}} \mathcal{C} * \mathcal{P}_L * \mathcal{P}[1],$$

is an isomorphism.

Proof. First, let us observe that we have a commutative diagram:

$$\begin{array}{ccccc}
 \mathcal{P}_R * \mathcal{P} & \xrightarrow{\mathcal{P}_R * \eta_L * \mathcal{P}} & \mathcal{P}_R * \mathcal{P} * \mathcal{P}_L * \mathcal{P} & \xrightarrow{\gamma_R * \mathcal{P}_L * \mathcal{P}} & \mathcal{C} * \mathcal{P}_L * \mathcal{P}[1] \\
 & \searrow & \downarrow \mathcal{P}_R * \mathcal{P} * \varepsilon_L & & \downarrow \mathcal{C} * \varepsilon_L[1] \\
 & & \mathcal{P}_R * \mathcal{P} & \xrightarrow{\gamma_R} & \mathcal{C}[1].
 \end{array}$$

Indeed, the left-hand side is just the triangular identity convolved with \mathcal{P}_R on the left, and the square commutes since the arrows act on separate variables. That is,

$$\gamma_R \simeq (\mathcal{C} * \varepsilon_L[1]) \circ (\gamma_R * \mathcal{P}_L * \mathcal{P}) \circ (\mathcal{P}_R * \eta_L * \mathcal{P}). \tag{3.1}$$

Now, by Lemma 1.2 we know that ε_L is the left adjoint of η_R . Moreover, in the proof of Lemma 1.4, we observed that the dual cotwist triangle is the left adjoint of the cotwist triangle. Thus, by comparing triangles, we see that α_L must be the left adjoint of δ_R , or equivalently, δ_R is the right adjoint of α_L . That is, we have:

$$\begin{array}{ccc}
 \Delta_* \mathcal{O}_X & & \mathcal{C} \xrightarrow{\mathcal{C} * \alpha_L} \mathcal{C} * \mathcal{C}' \\
 \alpha_L \downarrow & \mapsto & \searrow \delta_R \quad \downarrow \varepsilon_{\mathcal{C}} \\
 \mathcal{C}' & & \Delta_* \mathcal{O}_X.
 \end{array} \tag{3.2}$$

Combining the right-hand side of (3.2) with (3.1), we can observe that we have a commutative diagram of triangles:

$$\begin{array}{ccccc}
 \mathcal{P}_R * \mathcal{P} & \xrightarrow{\gamma_R} & \mathcal{C}[1] & \xrightarrow{\delta_R[1]} & \Delta_* \mathcal{O}_X[1] \\
 (\gamma_R * \mathcal{P}_L * \mathcal{P}) \circ (\mathcal{P}_R * \eta_L * \mathcal{P}) \downarrow & & \parallel & & \uparrow \varepsilon_{\mathcal{C}}[1] \\
 \mathcal{C} * \mathcal{P}_L * \mathcal{P}[1] & \xrightarrow{\mathcal{C} * \varepsilon_L[1]} & \mathcal{C}[1] & \xrightarrow{\mathcal{C} * \alpha_L[1]} & \mathcal{C} * \mathcal{C}'[1],
 \end{array}$$

where $\varepsilon_{\mathcal{C}} : \mathcal{C} * \mathcal{C}' \rightarrow \Delta_* \mathcal{O}_X$ is an isomorphism since $\mathcal{C} = \Phi_{\mathcal{C}}$ is an auto-equivalence by assumption. Since the second and third vertical arrows are isomorphisms, it follows that the first vertical arrow is also an isomorphism. \square

PROPOSITION 3.2. *Suppose the cotwist $\mathcal{C} = \Phi_{\mathcal{C}}$ is an auto-equivalence of $\mathcal{D}(X)$. Then any isomorphism $\varphi : \mathcal{P}_R \xrightarrow{\sim} \mathcal{C} * \mathcal{P}_L[1]$ implies the canonical map:*

$$\chi : \mathcal{P}_R \xrightarrow{\mathcal{P}_R * \eta_L} \mathcal{P}_R * \mathcal{P} * \mathcal{P}_L \xrightarrow{\gamma_R * \mathcal{P}_L} \mathcal{C} * \mathcal{P}_L[1],$$

is an isomorphism.

Proof. If we consider the triangle $\mathcal{P}_R \xrightarrow{\chi} \mathcal{C} * \mathcal{P}_L[1] \rightarrow \mathcal{Q}$, then χ is an isomorphism if and only if $\mathcal{Q} \simeq 0$. To show that $\mathcal{Q} \simeq 0$ it is sufficient to prove that the induced Fourier–Mukai functor $\Phi_{\mathcal{Q}} : \mathcal{D}(Y) \rightarrow \mathcal{D}(Y)$ is zero on a spanning class of $\mathcal{D}(Y)$. To this end, we use the spanning class $\Omega = \text{im } \Phi_{\mathcal{P}} \cup \text{ker } \Phi_{\mathcal{P}_R}$ from [1, § 2.4]. Indeed, convolving the triangle defined by χ with \mathcal{P} gives:

$$\mathcal{P}_R * \mathcal{P} \xrightarrow{\chi * \mathcal{P}} \mathcal{C} * \mathcal{P}_L * \mathcal{P}[1] \rightarrow \mathcal{Q} * \mathcal{P},$$

and Lemma 3.1 tells us that $\chi * \mathcal{P}$ is an isomorphism. That is, $\mathcal{Q} * \mathcal{P} \simeq 0$ and so we see that $\Phi_{\mathcal{Q}}$ is zero on $\text{im } \Phi_{\mathcal{P}}$. Next, we take an object $\mathcal{E} \in \text{ker } \Phi_{\mathcal{P}_R}$ and evaluate the induced triangle of Fourier–Mukai functors on it to get:

$$\Phi_{\mathcal{P}_R}(\mathcal{E}) = 0 \rightarrow \Phi_{\mathcal{C}} \circ \Phi_{\mathcal{P}_L}(\mathcal{E})[1] \rightarrow \Phi_{\mathcal{Q}}(\mathcal{E}).$$

By assumption, we have some isomorphism $\varphi : \mathcal{P}_R \xrightarrow{\sim} \mathcal{C} * \mathcal{P}_L[1]$ allowing us to conclude that $\Phi_{\mathcal{C}} \circ \Phi_{\mathcal{P}_L}(\mathcal{E}) = 0$ and hence $\Phi_{\mathcal{Q}}(\mathcal{E}) = 0$. This shows that $\Phi_{\mathcal{Q}}$ is also zero on $\ker \Phi_{\mathcal{P}_R}$ and thus $\mathcal{Q} \simeq 0$, which completes the proof. \square

Remarks 3.3. The hypotheses of the results in this section are stronger than necessary. Indeed, Lemma 3.1 and Proposition 3.2 only use the weaker statements that $\Phi_{\mathcal{C}'}$ is fully faithful and $\ker \Phi_{\mathcal{P}_R} = \ker \Phi_{\mathcal{P}_L}$, respectively.

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