A note on representations of welded braid groups

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Abstract

In this note, we adapt the procedure of the Long-Moody procedure to construct linear representations of *welded* braid groups. We exhibit the natural setting in this context and compute the first examples of representations we obtain thanks to this method. We take this way also the opportunity to review the few known linear representations of welded braid groups.

Introduction

The theory of linear representations of the braid group $\mathbf{B}_n$ is a very large topic. One of the most famous representations is the Burau representation [10], which is non faithful for $n \geq 5$. For a long period, it was an open problem whether $\mathbf{B}_n$ was linear until the independent works of Bigelow [6], Krammer [16] and Lawrence [17] showing a faithful representation. Since the braid group $\mathbf{B}_n$ is an ubiquitous object in mathematics it is natural to ask whether other generalizations are linear too, but except some cases (for instance Artin groups of type B and D, braid groups of the sphere and of the projective plane) this question remains widely open.

In this work note we focus on welded braid groups, which, as braid groups, admit several different definitions, for instance in terms of configuration spaces of (euclidean) circles, as automorphisms of free groups, or as *tubes* in $\mathbb{R}^4$.

The representation theory for these groups is just at the beginning: Burau representation extends, in terms of Fox derivatives and Magnus expansion, to welded braid groups [3] and few other results are known on representations arising from braided vector spaces [14] and on extensions of representations of $\mathbf{B}_n$ to particular subgroups of welded braid groups [9].

The main idea of this work is to extend Long-Moody procedure [7, 20, 18, 19, 24, 23] to $\mathbf{wB}_n$: in §1 we recall briefly the interpretation of welded braid groups in terms of fundamental groups of configuration spaces of circles and as automorphisms of free groups through Artin homomophism; this latter interpretation will be extended to other possible representations, extending Wada representations of the classical braid group $\mathbf{B}_n$. In §1.3 we recall and compare different Burau representations for $\mathbf{wB}_n$, the reducible one (Proposition 1.1), the reduced one (Proposition 1.2), and the dual version (Proposition 1.4). Then we introduce Tong-Yang-Ma representations of $\mathbf{wB}_n$ (Proposition 1.6) and extending the heuristic approach proposed in [25] we show that Burau and Tong-Yang-Ma representations are the only representations allowing a certain diagram to commute (Proposition 1.8). The main results are given in §2, where we adapt to welded braid groups the Long-Moody procedure for obtaining iterated linear representations. At the first step we obtain the Burau representation (Theorem 2.4) as in the case of $\mathbf{B}_n$. Surprisingly at the second iteration we do not obtain any new information, since we get the tensor product of two Burau representations (Theorem 2.6), while in the case of $\mathbf{B}_n$ we recover this way Lawrence-Krammer representation (see [24, Section 2.3.1]). This result can be also compared with the fact that the "trivial" extension of

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Bigelow representation to $\mathbf{wB}_n$ (associating to "braid" generators corresponding Bigelow matrices and to "permutation" generators the corresponding permutations matrices) is not well defined (see [3, 14]). Finally, once more contrarily to the case of $\mathbf{B}_n$, we show in Theorem 2.7 that it is not possible to recover the Tong-Yang-Ma representation for $\mathbf{wB}_n$ by a Long-Moody construction and we conclude with some further possible directions on the study of linear representations for $\mathbf{wB}_n$. In a further paper, we intend to study the Long-Moody iteration on the Tong-Yang-Ma representation, but here we focus on the two above interesting facts.

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Notations and conventions. Throughout this work, for a group defined by a (finite) presentation, we take the convention to read from the right to the left for the group operation.

1 State of the art of welded braid groups representations

1.1 Recollections on welded braid groups

We refer the reader to [11] for a complete and unified presentation of the various definitions of welded braid groups, which correspond to (unextended) loop braid groups in this reference. In the following we will use essentially the interpretation of welded braid groups in terms of automorphisms of free groups and fundamental groups of configuration spaces of circles.

In this work, we focus on a 3-dimensional analogue of $\mathbf{B}_n$: it is the fundamental group of all configurations of n unlinked Euclidean circles lying on planes that are parallel to a fixed one (called *untwisted rings* in [8]). According to [8] we will denote by $\mathcal{UR}_n$ the space of configurations of n unlinked Euclidean circles being all parallel to a fixed plane and by UR_n its fundamental group (called *group of rings* in [8]). The group UR_n is generated by 2 types of moves (see Figure 1.1).

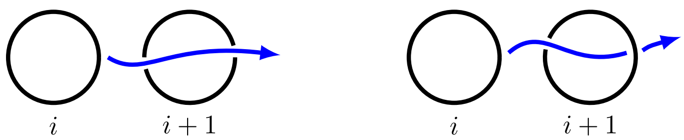

Figure 1.1 The moves τ_i and σ_i.

The move τ_i is the path permuting the i-th and the $i+1$-th circles by passing over (or around) while σ_i permutes them by passing the i-th circle through the $i+1$-th (let us remark that our notation is different from the one of [8], where τ_i was denoted by σ_i and σ_i by ρ_i; here we change the notation because σ_i's generate a subgroup isomorphic to $\mathbf{B}_n$).

The fundamental group UR_n is here denoted by $\mathbf{wB}_n$, and it is called welded braid group. Note that the convention of reading from the right to the left for the group operation is coherent with the interpretation of $\mathbf{wB}_n$ as the fundamental group of the space of configurations of n unlinked Euclidean circles being all parallel to a fixed plane: this convention corresponds to the composition of morphisms. We abuse the notation throughout this work, identifying $\lambda \circ \lambda' = \lambda\lambda'$ for all elements λ and λ' of $\mathbf{wB}_n$.

In [8] was proven that the welded braid group on n generators $\mathbf{wB}_n$ admits a presentation with generators $\{\sigma_i, \tau_i \mid i \in \{1, \ldots, n-1\}\}$ together with relations:

$$\begin{cases} \sigma_i\sigma_k = \sigma_k\sigma_i & \text{if } \mid i-k \mid \geqslant 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{if } i \in \{1,\ldots,n-2\}, \\ \tau_i\tau_k = \tau_k\tau_i & \text{if } \mid i-k \mid \geqslant 2, \\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} & \text{if } i \in \{1,\ldots,n-2\}, \\ \tau_i^2 = 1 & \text{if } i \in \{1,\ldots,n-1\}, \\ \sigma_i\tau_k = \tau_k\sigma_i & \text{if } \mid i-k \mid \geqslant 2, \\ \tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1} & \text{if } i \in \{1,\ldots,n-2\}, \\ \sigma_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\sigma_{i+1} & \text{if } i \in \{1,\ldots,n-2\}. \end{cases} \tag{1.1}$$

The map $\mathcal{UR}_n \to \mathcal{UR}_{n+1}$ which adds a circle on the left induces an (injective) homomorphism from $\mathbf{wB}_n$ to $\mathbf{wB}_{n+1}$.

Let $\mathcal{PUR}_n$ be the space of configurations of n ordered unlinked Euclidean circles being all parallel to a fixed plane; the fundamental group of $\mathcal{PUR}_n$ is usually denoted by $\mathbf{wP}_n$ and called welded pure braid group.

The map $\mathcal{PUR}_{n+1} \to \mathcal{PUR}_n$ which forgets the first circle is a fibration and the long exact sequence in homotopy provides the following (splitting) sequence:

$$1 \longrightarrow \mathbf{D}_n \longrightarrow \mathbf{wP}_{n+1} \longrightarrow \mathbf{wP}_n \longrightarrow 1,$$

where $\mathbf{D}_n$ is the fundamental group of configurations with n circles in a fixed position and the first circle varying [8].

Let now $\mathcal{UR}_{1,n}$ be the orbit space $\mathcal{PUR}_{n+1}/\mathfrak{S}_n$ where the symmetric group $\mathfrak{S}_n$ acts by permutation on last n circles. We will denote by $\mathbf{wB}_{1,n}$ its fundamental group. As in previous case we have a splitting sequence:

$$1 \longrightarrow \mathbf{D}_n \longrightarrow \mathbf{wB}_{1,n} \longrightarrow \mathbf{wB}_n \longrightarrow 1.$$

1.2 Artin homomorphism

In the following we will use another interpretation of $\mathbf{wB}_n$, this time in terms of automorphisms of free groups. Let $\mathbf{F}_n$ be the free group on n generators $\langle x_1, \ldots, x_n \rangle$. We call *Artin homomorphism* the map $a_n \colon \mathbf{wB}_n \to \mathrm{Aut}\,(\mathbf{F}_n)$ defined as follows:

$$\sigma_i \longmapsto \begin{cases} x_i \longmapsto x_{i+1} \\ x_{i+1} \longmapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j \longmapsto x_j \quad j \notin \{i, i+1\} \end{cases}$$

$$\tau_i \longmapsto \begin{cases} x_i \longmapsto x_{i+1} \\ x_{i+1} \longmapsto x_i \\ x_j \longmapsto x_j \quad j \notin \{i, i+1\} \end{cases}$$

This map is well defined: the relations involving only generators $\{\sigma_i \mid i \in \{1, \ldots, n-1\}\}$ are verified since it is the usual Artin representation of $\mathbf{B}_n$ in $\mathrm{Aut}\,(\mathbf{F}_n)$. The same remark holds for relations involving only generators $\{\tau_i \mid i \in \{1, \ldots, n-1\}\}$ (permutation automorphisms), therefore the only relations that we have to verify are relations involving generators of both type σ_i and τ_i. We check here relations $\tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1}$ and $\tau_{i+1}\tau_i\sigma_{i+1} = \sigma_i\tau_{i+1}\tau_i$; commutation relations are evidently verified (see also [13]).

$$a_n\left(\sigma_{i+1}\sigma_i\tau_{i+1}\right)=\begin{cases}x_i\longmapsto x_{i+2}\\ x_{i+1}\longmapsto x_{i+2}^{-1}x_{i+1}x_{i+2}\\ x_{i+2}\longmapsto x_{i+1}^{-1}x_ix_{i+1}\\ x_j\longmapsto x_j\quad j\notin\{i,i+1,i+2\}\end{cases}=a_n\left(\tau_i\sigma_{i+1}\sigma_i\right)$$

$$a_n\left(\sigma_i\tau_{i+1}\tau_i\right)=\begin{cases}x_i\longmapsto x_{i+2}\\ x_{i+1}\longmapsto x_{i+1}\\ x_{i+2}\longmapsto x_{i+1}^{-1}x_ix_{i+1}\\ x_j\longmapsto x_j\quad j\notin\{i,i+1,i+2\}\end{cases}=a_n\left(\tau_{i+1}\tau_i\sigma_{i+1}\right)$$

Geometrically, we are associating to any generator of $\mathbf{wB}_n$ the corresponding action on the fundamental group of the ball B^3 less n trivial circles, which is a free group on n generators. We refer to [11] for a rigorous proof of this construction and of the fact that Artin homomorphism is injective. Moreover, the image of the group $\mathbf{wB}_n$ in $\mathrm{Aut}\left(\mathbf{F}_n\right)$ is the subgroup of those automorphisms of $\mathbf{F}_n$ that send each generator of $\mathbf{F}_n$ to a conjugate of some generator [13].

Notice also that the homomorphism from $\mathbf{wB}_n$ to $\mathbf{wB}_{n+1}$ previously defined ("add a circle on the left") becomes here the restriction of the map $id_1 * - : \mathrm{Aut}\left(\mathbf{F}_n\right)\hookrightarrow\mathrm{Aut}\left(\mathbf{F}_{n+1}\right)$. In other words, $id_1*\sigma_i=\sigma_{i+1}$ and $id_1*\tau_i=\tau_{i+1}$.

The group $\mathbf{D}_n$ previously defined is isomorphic to the group generated by automorphisms $\{\epsilon_{i,1},\epsilon_{1,i}\mid i\in\{2,\ldots,n+1\}\}$ [1], where

$$\begin{aligned}\epsilon_{1,i}&\longmapsto\begin{cases}x_i\longmapsto x_1^{-1}x_ix_1\\ x_j\longmapsto x_j\quad j\neq i\end{cases}\\ \epsilon_{i,1}&\longmapsto\begin{cases}x_1\longmapsto x_i^{-1}x_1x_i\\ x_j\longmapsto x_j\quad j\neq 1\end{cases}\end{aligned}$$

Note that $\{\epsilon_{i,1}\}$ generate a free group of rank n and $\{\epsilon_{1,i}\}$ a free abelian group of rank n [1], but $\mathbf{D}_n$ is not finitely presented when $n\geq 3$ [22].

Alternative to Artin homomorphism. Wada [27] found several *local* representations of $\mathbf{B}_n$ in $\mathrm{Aut}\left(\mathbf{F}_n\right)$ of the following form: any generator (and therefore its inverse) of $\mathbf{B}_n$ acts trivially on generators of $\mathbf{F}_n$ except a pair of generators:

$$\begin{aligned}\sigma_i\cdot x_i&=u(x_i,x_{i+1})\,,\\ \sigma_i\cdot x_{i+1}&=v(x_i,x_{i+1})\,,\\ \sigma_i\cdot x_j&=x_j\qquad j\neq i,i+1\,,\end{aligned}$$

where u and v are now words in the generators x_i,x_{i+1}, with $\langle x_i,x_{i+1}\rangle\simeq\mathbf{F}_2$. Wada found seven families of representations of local type (see Section 4 of [5] for a short survey on these representations), up to the dual equivalence (corresponding to the involution of $\mathbf{F}_n$ given by simultaneously replacing all x_i with x_i^{-1}) and inverse equivalence (derived from the involution of $\mathbf{B}_n$ defined by sending σ_i to σ_i^{-1}):[1]

- Type 1, ψ_1: $u(x_i,x_{i+1})=x_i$ and $v(x_i,x_{i+1})=x_{i+1}$;
- Type 2, ψ_2: $u(x_i,x_{i+1})=x_{i+1}^{-1}$ and $v(x_i,x_{i+1})=x_i$;

[1] In [5] and [27], the action described here is the one for σ_i^{-1}. However, we choose to write the inverse symmetry equivalent here to be consistent with respect to the above Artin representation a_n and with the fact that we compose elements from the right to the left.

- Type 3, ψ_3: $u(x_i, x_{i+1}) = x_{i+1}^{-1}$ and $v(x_i, x_{i+1}) = x_i^{-1}$;
- Type 4, $\psi_{4,h}$: $u(x_i, x_{i+1}) = x_{i+1}$ and $v(x_i, x_{i+1}) = x_{i+1}^{-h} x_i x_{i+1}^h$;
- Type 5, ψ_5: $u(x_i, x_{i+1}) = x_{i+1}$ and $v(x_i, x_{i+1}) = x_{i+1} x_i^{-1} x_{i+1}$;
- Type 6, ψ_6: $u(x_i, x_{i+1}) = x_{i+1}^{-1}$ and $v(x_i, x_{i+1}) = x_{i+1} x_i x_{i+1}$;
- Type 7, ψ_7: $u(x_i, x_{i+1}) = x_i x_{i+1}^{-1} x_i^{-1}$ and $v(x_i, x_{i+1}) = x_i x_{i+1}^2$.

We can try to extend Wada representations to welded braid groups associating to any generator σ_i the Wada representation of type k and to any generator τ_i the corresponding permutation automorphism. We will say that a Wada representation extends to $\mathbf{wB}_n$ if the map defined as above on generators of $\mathbf{wB}_n$ is actually a homomorphism. Type 1 does not extend, while Type 2 and Type 3 extend to $\mathbf{wB}_n$ but these extensions are not interesting since the image of the group generated by σ_i's in $\mathrm{Aut}(\mathbf{F}_n)$ is trivial or isomorphic to $\mathfrak{S}_n$. Between the other four representations it is easy to check (see [4]) that only Type 4 and Type 5 representations extend this way to a representation; let denote these two representations respectively by χ_1 and χ_2. These two representations are not equivalent meaning that there are no automorphisms $\phi \in \mathrm{Aut}(\mathbf{F}_n)$ and $\mu : \mathbf{wB}_n \to \mathbf{wB}_n$ such that

$$\phi^{-1} \chi_1(\beta) \phi = \chi_2(\mu(\beta)),$$

for any $\beta \in \mathbf{wB}_n$ (see [4]).

1.3 Known linear representations for welded braid groups

We present in this section the linear representations of welded braid groups which can be straightforwardly derived from those of braid groups. In particular, we extend heuristic procedure on matrices. For the remainder of §1.3, fix a natural number $n \geqslant 2$.

Beforehand we indicate that we study here in the "specific" representations of welded braid groups, not those which factor through symmetric groups. Namely, recall that sending both σ_i and τ_i in the transposition $(i, i+1)$ for $i \in \{1, \ldots, n-1\}$ we obtain the short exact sequence:

$$1 \longrightarrow \mathbf{wP}_n \longrightarrow \mathbf{wB}_n \longrightarrow \mathfrak{S}_n \longrightarrow 1.$$

Therefore, all representations of the symmetric group $\mathfrak{S}_n$ lift to $\mathbf{wB}_n$, but we will not consider them since we loose too many informations on welded braid groups.

1.3.1 The Burau representations

The family of *Burau* representations for braid groups were first introduced in [10] and has been intensively studied. We refer the reader to [15] for a complete presentation.

These representations can be extended to welded braid groups in the following way.

Proposition 1.1 *The following assignment defines a representation $Bur : \mathbf{wB}_n \to GL_n\left(\mathbb{Z}[t^{\pm 1}]\right)$ called the Burau representation of the welded braid group $\mathbf{wB}_n$:*

$$\sigma_i \mapsto Id_{i-1} \oplus \begin{bmatrix} 0 & t \\ 1 & 1-t \end{bmatrix} \oplus Id_{n-i-1} \quad \text{and} \quad \tau_i \mapsto Id_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus Id_{n-i-1}$$

for all natural numbers $i \in \{1, \ldots, n-1\}$.

Proof. The matrices for σ_i and τ_i define representations of braid groups and symmetric groups respectively. It follows from the relations of $\mathbf{wB}_n$ of (1.1) that we just have to check the mixed relations between the braid and symmetric generators: this is done by straightforward computations. □

Reduced version. As for the case of braid groups, the Burau representation is reducible and we can define an irreducible version.

Proposition 1.2 *The following assignment defines a representation* $\overline{Bur} : \mathbf{wB}_n \to GL_{n-1}\left(\mathbb{Z}[t^{\pm1}]\right)$ *called the reduced Burau representation of the welded braid group* $\mathbf{wB}_n$*:*

$$\sigma_1 \mapsto \begin{bmatrix} -t & t \\ 0 & 1 \end{bmatrix} \oplus Id_{n-3} \text{ and } \tau_1 \mapsto \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \oplus Id_{n-3};$$

$$\sigma_{n-1} \mapsto Id_{n-3} \oplus \begin{bmatrix} 1 & 0 \\ 1 & -t \end{bmatrix} \text{ and } \tau_{n-1} \mapsto Id_{n-3} \oplus \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix};$$

$$\sigma_i \mapsto Id_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & -t & t \\ 0 & 0 & 1 \end{bmatrix} \oplus Id_{n-i-2} \text{ and } \tau_i \mapsto Id_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus Id_{n-i-2}$$

for all natural numbers $i \in \{2, \dots, n-2\}$*. Moreover we have a short exact sequence of* $\mathbf{wB}_n$*-representations*

$$0 \longrightarrow \overline{Bur} \longrightarrow Bur \longrightarrow \mathbb{Z}\left[t^{\pm1}\right] \longrightarrow 0 \tag{1.2}$$

where $\mathbf{wB}_n$ *acts trivially on the kernel* $\mathbb{Z}\left[t^{\pm1}\right]$*. In particular, the map* $Bur \to \mathbb{Z}\left[t^{\pm1}\right]$ *is defined by the matrix* $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$*.*

Proof. The matrices for σ_i define the reduced Burau representations of braid groups (see [15, Section 3.3]) and those for τ_i define the standard representation of symmetric groups. Again the compatibility with respect to the mixed relations of $\mathbf{wB}_n$ are checked by straightforward computations.

Let r_n be the $n \times n$-matrix with coefficients $r_{i,j} = 1$ if $j \leqslant i$ and $r_{i,j} = 0$ otherwise. Then

$$r_n \circ Bur\left(\sigma_i\right) \circ r_n^{-1} = \begin{bmatrix} \overline{Bur}\left(\sigma_i\right) & C_i \\ 0 & 1 \end{bmatrix}$$

where $C_i = \begin{bmatrix} 0 & \cdots & 0 & t \cdot \delta_{i,n-1} \end{bmatrix}^T$ (where the label ^{T} means the transpose matrix) and $\delta_{i,n-1}$ denotes the Kronecker delta: this defines the short exact sequence (1.2). □

Remark 1.3 Since Burau representation for $\mathbf{B}_n$ is not faithful when $n \geq 5$, it follows that also its extension is not faithful; indeed, the Burau representation for $\mathbf{wB}_n$ has non trivial kernel even for $n = 2$ (see Lemma 6 of [2]).

Dual versions. Actually, there are two non-equivalent versions of the Burau representation for braid groups: the one which matrices are given in Proposition 1.1, and its dual which matrices are the transpose of the inverse of these matrices. This dual version also lifts to the welded braid group:

Proposition 1.4 *Assigning* $Bur^*\left(\sigma_i\right) = Bur^T\left(\sigma_i^{-1}\right)$ *and* $Bur^*\left(\tau_i\right) = Bur^T\left(\tau_i^{-1}\right)$ *for all natural numbers* $i \in \{1, \dots, n-1\}$ *defines a representation* $Bur^* : \mathbf{wB}_n \to GL_n\left(\mathbb{Z}[t^{\pm1}]\right)$ *called the dual Burau representation.*

It induces a dual reduced Burau representation $\overline{Bur}^* : \mathbf{wB}_n \to GL_{n-1}\left(\mathbb{Z}[t^{\pm1}]\right)$ *which is defined by* $\overline{Bur}^*\left(\sigma_i\right) = \overline{Bur}^T\left(\sigma_i^{-1}\right)$ *and* $\overline{Bur}^*\left(\tau_i\right) = \overline{Bur}^T\left(\tau_i^{-1}\right)$ *for all natural numbers* $i \in \{1, \dots, n-1\}$*, and defines the short exact sequence of* $\mathbf{wB}_n$*-representations*

$$0 \longrightarrow \mathbb{Z}\left[t^{\pm1}\right] \longrightarrow Bur^* \longrightarrow \overline{Bur}^* \longrightarrow 0 \tag{1.3}$$

where $\mathbf{wB}_n$ *acts trivially on the kernel* $\mathbb{Z}\left[t^{\pm1}\right]$*.*

Proof. Taking the transpose of the inverse of a multiplication of matrices keeps the order of the multiplication. Therefore all the relations of (1.1) are therefore straightforwardly satisfied and proves that Bur^* is a representation. The reduced version $\overline{Bur}^*$ is induced by taking the transpose of the inverse of the short exact sequence (1.2): taking the inverse keeps the direction of the arrows, whereas the transpose reverses this direction (since it exchanges lines and columns of matrices) and we obtain (1.3). □

The dual Burau representation was already introduced in [26, Section 4].

Remark 1.5 For a group G and a commutative ring R, composing a linear representation $\phi : G \to GL_n(R)$ by an automorphism $\alpha \in Aut(GL_n(R))$ defines another representation $\alpha \circ \phi : G \to GL_n(R)$. Hence, considering the contragredient automorphism sending $M \in GL_n(R)$ to $(M^{-1})^T$, we reconstruct the dual Burau representation of Proposition 1.4 from the original one of Proposition 1.1.

1.3.2 The Tong-Yang-Ma procedure and associated representations.

In 1996, Tong, Yang and Ma [25] investigated the representations of $\mathbf{B}_n$ where the i-th generator is sent to a non-singular matrix of the form

$$Id_{i-1} \oplus \begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus Id_{n-i-1}.$$

In particular, they proved that there exist (up to equivalence and dual) only two non trivial representations of this type: the unreduced Burau representations and a new irreducible representation, called the *Tong-Yang-Ma* representation. This last family lifts to define two families of linear representations of the welded braid group $\mathbf{wB}_n$.

Proposition 1.6 *Let α be an invertible element of $\mathbb{Z}\left[t^{\pm 1}\right]$. The following assignment defines a representation $TYM_\alpha : \mathbf{wB}_n \to GL_n\left(\mathbb{Z}[t^{\pm 1}]\right)$ called the Tong-Yang-Ma representation (with parameter α) of the welded braid group $\mathbf{wB}_n$:*

$$\sigma_i \mapsto Id_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} \oplus Id_{n-i-1} \text{ and } \tau_i \mapsto Id_{i-1} \oplus \begin{bmatrix} 0 & \alpha^{-1} \\ \alpha & 0 \end{bmatrix} \oplus Id_{n-i-1}$$

for all natural numbers $i \in \{1, \ldots, n-1\}$. Taking the transpose of the inverse of the above matrices defines the dual Tong-Yang-Ma representation $TYM^*_\alpha : \mathbf{wB}_n \to GL_n\left(\mathbb{Z}[t^{\pm 1}]\right)$ (which is obviously not equivalent to TYM_α).

Proof. As above for the Burau representations, we just have to check the compatibility with respect to the mixed relations (1.1) between the braid and symmetric generators of $\mathbf{wB}_n$: this is done by straightforward computations. □

Remark 1.7 For $n \geqslant 3$, note that $TYM_\alpha(\sigma_1^2) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \oplus Id_{n-2}$ and $TYM_\alpha(\sigma_2^2) = Id_1 \oplus \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \oplus Id_{n-3}$. Hence $TYM_\alpha(\sigma_1^2\sigma_2^2\sigma_1^{-2}\sigma_2^{-2}) = Id_n$ and therefore the Tong-Yang-Ma representation is not faithful.

In addition, we can carry out the analogous heuristic approach to [25] for the welded braid groups. For all $i \in \{1, \ldots, n-1\}$, we denote by $\mathrm{incl}_i^n : \mathbf{wB}_2 \cong \mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbf{wB}_n$ the inclusion morphism induced by $\mathrm{incl}_i^n(\sigma_1) = \sigma_i$ and $\mathrm{incl}_i^n(\tau_1) = \tau_i$.

Proposition 1.8 *Let $\eta_n : \mathbf{wB}_n \longrightarrow GL_n\left(\mathbb{Z}\left[t^{\pm 1}\right]\right)$ be a representation. Assume that for all $i \in \{1, \ldots, n-1\}$ the following diagram is commutative:*

$$\begin{tikzcd} \mathbf{wB}_n \arrow[r, "\eta_n"] & GL_n\left(\mathbb{Z}\left[t^{\pm 1}\right]\right) \\ \mathbf{wB}_2 \arrow[u, "incl_i^n"] \arrow[r, "\eta_2"'] & GL_2\left(\mathbb{Z}\left[t^{\pm 1}\right]\right). \arrow[u, "id_{i-1}\oplus - \oplus id_{n-i-1}"'] \end{tikzcd}$$

Then η_n is equivalent or dual to the trivial representation, or to the (unreduced) Burau representation Bur, or to a Tong-Yang-Ma representation TYM_α, or else to the specialisation at $t = 1$ of a Tong-Yang-Ma representation.

Proof. Restricting along the natural inclusion $\mathbf{B}_n \hookrightarrow \mathbf{wB}_n$, it follows from [25, Part II] that three only possible matrices (up to equivalence and dual) on which the braid generators $\{\sigma_i\}_{i\in\{1,\ldots,n-1\}}$ can be sent to are the trivial, Burau, Tong-Yang-Ma and the specialisation at $t = 1$ of a Tong-Yang-Ma matrix.

Note that the definition of η_n on the the symmetric generators $\{\tau_i\}_{i\in\{1,\ldots,n-1\}}$ amounts to considering a representation $\mathbb{Z}/2\mathbb{Z} \to GL_2\left(\mathbb{Z}\left[t^{\pm 1}\right]\right)$. We denote it by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in \mathbb{Z}[t^{\pm 1}]$. For the representation of $\mathbf{wB}_3$, it follows from the assignments that the image of τ_1 is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus Id_1$ and the image of τ_2 is $Id_1 \oplus \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since the square of this matrix must be trivial, we deduce that $a^2 + bc = 1$, $d^2 + bc = 1$, $ab = -bd$ and $ac = -cd$. There are two types of situations.

Case 1 : Either $b = 0$ or $c = 0$, and therefore $a^2 = d^2 = 1$. Let us assume that $b = 0$, the other case being similar. The relation $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$ of (1.1) implies that $a = da^-1 = d^-1 = 1$, hence $a = d = 1$. Finally, it follows from the relation $\tau_1\sigma_2\sigma_1 = \sigma_2\sigma_1\tau_1$ of (1.1) that the braid generators must be sent to the identity matrix.

Case 2 : Either $b \neq 0$ and $c \neq 0$. Since $\mathbb{Z}[t^{\pm 1}]$ is an integral domain, we deduce that $a = -d$. Writing the matrices defined by the relation $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$ of (1.1), we deduce that $2a^6 + a^4bc = 0$. Assuming that $a \neq 0$ leads to the contradiction that $a = bc = 1$ and $a^2 + bc = 1$. Hence $a = d = 0$ and therefore $bc = 1$. Let us now investigate the compatibility with the braid generators $\{\sigma_i\}_{i\in\{1,\ldots,n-1\}}$. There are three possibilities:

- The braid generators are sent to the identity matrices, but the relation $\tau_1\sigma_2\sigma_1 = \sigma_2\sigma_1\tau_2$ leads to the fact that $c = 0$ which is impossible.
- The braid generators are sent to the Burau matrices (or its dual). We then deduce from the relation $\tau_1\sigma_2\sigma_1 = \sigma_2\sigma_1\tau_2$ that $(1 - c)(1 - t) = 0$ and therefore that $c = 1$ since $\mathbb{Z}[t^{\pm 1}]$ is an integral domain. We thus recover the Burau representations of the welded braid groups.
- The braid generators are sent to one of the Tong-Yang-Ma matrices (or a dual, or possibly with the specialisation at $t = 1$) and then we recover the Tong-Yang-Ma representation TYM_c (or its dual, or its specialisation at $t = 1$).

This analysis thus ends the proof. □

2 The Long-Moody construction for welded braid groups

In 1994, Long and Moody [20] gave a method to construct a new linear representation of $\mathbf{B}_n$ from a representation of $\mathbf{B}_{n+1}$, complexifying in a sense the initial representation. For instance, it reconstructs the unreduced Burau representation from a one dimensional representation. It was studied from a functorial point of view and extended in [24] and then generalised to other families of groups [23]. In particular, the underlying framework of this method, called the *Long-Moody construction*, naturally arises considering representations of *welded* braid groups: the aim of this section is the study of this construction in this case. We fix a natural number $n \geqslant 3$ all along §2.

2.1 The theoretical setting of the Long-Moody construction

We detail here the required tool and present the abstract definition of the Long-Moody construction.

Tool. Recall that $\mathbf{F}_n = \langle x_1, \ldots, x_n \rangle$ is the free group on n generators. The key ingredient to define the Long-Moody construction for welded braid groups is to find a group morphisms $\alpha_n \colon \mathbf{wB}_n \to \mathrm{Aut}(\mathbf{F}_n)$ and $\xi_n \colon \mathbf{F}_n \to \mathbf{wB}_{n+1}$ such that the morphism $\mathbf{F}_n * \mathbf{wB}_n \to \mathbf{wB}_{n+1}$ given by the free product of ξ_n and $id_1 * -$ factors across the canonical surjection to the semidirect product $\mathbf{F}_n \underset{\alpha_n}{\rtimes} \mathbf{wB}_n$. In other words, the following diagram is commutative

$$\begin{tikzcd} \mathbf{F}_n \arrow[r, hook] \arrow[dr, "\xi_n"'] & \mathbf{F}_n \underset{\alpha_n}{\rtimes} \mathbf{wB}_n \arrow[d] & \mathbf{wB}_n \arrow[l, hook'] \arrow[dl, "id_1 * -"] \\ & \mathbf{wB}_{n+1}, & \end{tikzcd} \tag{2.1}$$

where the vertical morphism $\mathbf{F}_n \underset{\alpha_n}{\rtimes} \mathbf{wB}_n \to \mathbf{wB}_{n+1}$ is induced by the free product of ξ_n and $id_1 * -$ morphism $\mathbf{F}_n * \mathbf{wB}_n \to \mathbf{wB}_{n+1}$. Equivalently, we require that for all elements $\lambda \in \mathbf{wB}_n$ and $x \in \mathbf{F}_n$ the morphism ξ_n satisfies the following equality in $\mathbf{wB}_{n+1}$:

$$(id_1 * \lambda) \cdot \xi_n(x) = \xi_n(\alpha_n(\lambda)(x)) \cdot (id_1 * \lambda). \tag{2.2}$$

Definition of the Long-Moody construction. The Long-Moody construction is defined as follows. We fix an abelian group V. We denote by $\mathcal{I}_{\mathbf{F}_n}$ the augmentation ideal of the group ring $\mathbb{Z}[\mathbf{F}_n]$. Note that the action α_n canonically induces an action of $\mathbf{wB}_n$ on $\mathcal{I}_{\mathbf{F}_n}$ (that we denote in the same way for convenience).

Let $\rho : \mathbf{wB}_{n+1} \to Aut_{\mathbb{Z}}(V)$ be a linear representation. Precomposing by the morphism ξ_n, ρ gives the module V a $\mathbf{F}_n$-module structure. Then the Long-Moody construction

$$\mathbf{LM}(\rho) : \mathbf{wB}_n \to Aut_{\mathbb{Z}}\left(\mathcal{I}_{\mathbf{F}_n} \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} V\right)$$

is the map defined by:

$$\mathbf{LM}(\rho)(\lambda)\left(i \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} v\right) = \alpha_n(\lambda)(i) \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \rho(id_1 * \lambda)(v)$$

for all $\lambda \in \mathbf{wB}_n$, $i \in \mathcal{I}_{\mathbf{F}_n}$ and $v \in V$. For sake of completeness, we detail that:

Lemma 2.1 *[23, Section 2.2.4] The representation* $\mathbf{LM}(\rho)$ *is well-defined.*

Proof. We consider elements $\lambda \in \mathbf{wB}_n$, $x \in \mathbf{F}_n$, $v \in V$ and $i \in \mathcal{I}_{\mathbf{F}_n}$. First, since ρ is a morphism, we have that

$$\begin{aligned} \mathbf{LM}(\rho)(\lambda)\left(i \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \rho(\xi_n(x))(v)\right) &= \alpha_n(\lambda)(i) \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \rho(id_1 * \lambda)(\rho(\xi_n(x))(v)) \\ &= \alpha_n(\lambda)(i) \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \rho((id_1 * \lambda)\xi_n(x))(v). \end{aligned}.$$

Furthermore, since $\alpha_n(\lambda)$ is an automorphism of $\mathbf{F}_n$ and the $\mathbb{Z}[\mathbf{F}_n]$-module structure of V is induced by the composite $\rho \circ \xi_n$, we compute that

$$\begin{aligned} \mathbf{LM}(\rho)(\lambda)\left(i \cdot x \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} v\right) &= \alpha_n(\lambda)(i) \cdot \alpha_n(\lambda)(x) \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \rho(id_1 * \lambda)(v) \\ &= \alpha_n(\lambda)(i) \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \rho(\xi_n(\alpha_n(\lambda)(x))) \circ \rho((id_1 * \lambda))(v) \\ &= \alpha_n(\lambda)(i) \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \rho(\xi_n(\alpha_n(\lambda)(x)) \cdot (id_1 * \lambda))(v) \end{aligned}$$

Then it follows from (2.2) that

$$\mathbf{LM}(\rho)(\lambda)\left(i \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \rho(\xi_n(x))(v)\right) = \mathbf{LM}(\rho)(\lambda)\left(i \cdot x \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} v\right),$$

This gives the compatibility of the assignment $\mathbf{LM}(\rho)$ with respect to the tensor product over $\mathbb{Z}[\mathbf{F}_n]$. Then this assignment $\mathbf{LM}(\rho)$ defines a morphism on $\mathbf{wB}_n$ follows from the fact that α_n and ρ are themselves morphisms. $\square$

The natural candidate for the morphism α_n is the Artin homomorphism a_n recalled in §1.2 and we fix the assignment $\alpha_n = a_n$ from now on. We could use another Wada representation for a_n (recalled in §1.2), but we prove in §2.7 that this is actually not relevant for welded braid groups.

There is always the choice of the trivial morphism $\mathbf{F}_n \to 0 \to \mathbf{wB}_{n+1}$ as ξ_n so that relation (2.2) is satisfied. However the construction is much more interesting using a non-trivial morphism for this parameter. Indeed applying the Long-Moody construction with the trivial ξ_n to a one-dimensional representation provides the permutation representation of $\mathbf{wB}_n$ (sending both the symmetric and braid generators on the permutation matrix). Moreover, the iteration of this Long-Moody construction gives the tensor powers of that permutation representation. We refer the reader to [23, Section 2.2.5] for further details on that point. For that reason, the main point consists in finding non-trivial ξ_n such that the diagram (2.1) is commutative, which is the aim the following section.

2.2 The natural example for welded braids

We give in this section a first example of a (non-trivial) Long-Moody construction for welded braid groups. It relies on the following natural candidate for the choice of the required morphism ξ_n.

Let $\xi_{n,1} : \mathbf{F}_n \hookrightarrow \mathbf{wB}_{n+1}$ be the injective morphism defined by

$$x_i \longmapsto (\tau_{i-1} \cdots \tau_2 \tau_1)^{-1} (\sigma_i \tau_i) (\tau_{i-1} \cdots \tau_2 \tau_1).$$

It is natural in the sense that it identifies the free group $\mathbf{F}_n$ with the free subgroup of rank n of $\mathbf{D}_n$ (see §1.2) generated by the elements $\{\epsilon_{j,1}\}_{2 \leqslant j \leqslant n+1}$. They are also similar to the analogous parameter for the original construction for braid groups (see [24, Example 2.7]). Moreover they satisfy the key condition to define a Long-Moody construction:

Lemma 2.2 *The morphism $\xi_{n,1}$ satisfies the equality (2.2).*

Proof. Since

$$\sigma_{i+1} \sigma_i \tau_i \sigma_{i+1}^{-1} = \tau_i \sigma_{i+1} \sigma_i \tau_{i+1} \tau_i \sigma_{i+1}^{-1} = \tau_i \sigma_{i+1} \tau_{i+1} \tau_i$$

then

$$\xi_{n,1} (a_n (\sigma_i) (x_i)) = \sigma_{i+1} \xi_{n,1} (x_i) \sigma_{i+1}^{-1}.$$

Also, note that

$$\tau_i \sigma_{i+1} \tau_{i+1} \tau_i \sigma_{i+1} \tau_i \sigma_{i+1} \tau_{i+1} \tau_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tau_{i+1} \tau_i = \sigma_i \sigma_{i+1} \tau_{i+1} \tau_i \sigma_{i+1}$$

then

$$\tau_i \tau_{i+1} \sigma_{i+1}^{-1} \tau_i \sigma_i \sigma_{i+1} \tau_{i+1} \tau_i = \sigma_{i+1} \tau_i \sigma_{i+1} \tau_{i+1} \tau_i \sigma_{i+1}^{-1}$$

hence

$$\xi_{n,1} (a_n (\sigma_i) (x_{i+1})) = \sigma_{i+1} \xi_{n,1} (x_{i+1}) \sigma_{i+1}^{-1}.$$

□

We thus define a Long-Moody construction associated with this parameter $\xi_{n,1}$ and denote it by $\mathbf{LM}_1$. The condition (2.2) is however quite restrictive and the current example follows the spirit of the original Long-Moody construction [20]: we therefore focus on the study of $\mathbf{LM}_1$ in this paper. By the way we do not claim to be exhaustive here concerning the study of other possible Long-Moody constructions defined by some other compatible parameters α_n and ξ_n.

2.3 Applications

In addition to recovering the unreduced Burau representation, the second iteration of the original Long-Moody construction recovers the Lawrence-Krammer representation [6, 16, 17] as a subrepresentation (see [20, Corollary 2.10] or [24, Section 2.3.1]). The linearity of the welded braid groups being an open problem, there was a hope to construct an analogue of the Lawrence-Krammer representation for these groups. Unfortunately, if we can reconstruct the unreduced Burau representation (see §2.3.1), the construction $\mathbf{LM}_1$ does not produce something new at the second iteration (see §2.3.2).

On another note, the variants of the Long-Moody construction introduced in [24] allow to recover the Tong-Yang-Ma representation for braid groups (see [24, Section 2.3.2]). However, we prove in §2.3.2 that it is impossible to reconstruct the Tong-Yang-Ma representation for welded braid groups using *any* Long-Moody construction.

Notation 2.3 Let R be a ring and r an invertible element of R. We denote by $r : \mathbf{wB}_n \to R^\times$ the one-dimensional representation defined by $r(\sigma_i)$ is the multiplication by r and $r(\tau_i)$ is the identity for all $i \in \{1, \ldots, n-1\}$. Also for $\rho : \mathbf{wB}_n \to GL_R(V)$ a representation, we denote by $r\rho : \mathbf{wB}_n \to GL_R(V)$ the tensor representation $r \underset{R}{\otimes} \rho$.

2.3.1 Recovering the Burau representation

We start from the one-dimensional representation $t : \mathbf{wB}_n \to \mathbb{Z}\left[t^{\pm 1}\right]^\times$. We obtain that:

Theorem 2.4 *The representation $t^{-1}\mathbf{LM}_1(t)$ is equivalent to Bur.*

Proof. The key point to determine the form of the matrices of $t^{-1}\mathbf{LM}_1(t)$ is to understand the Artin homomorphism on the augmentation ideal. We compute that for all $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, n\}$, $a_n(\sigma_i)$ sends $x_j - 1$ to

$$\begin{cases} x_{i+1} - 1 & \text{if } j = i \\ (x_i - 1)\, x_{i+1} + (x_{i+1} - 1)\left(1 - x_{i+1}^{-1} x_i x_{i+1}\right) & \text{if } j = i+1 \\ x_j - 1 & \text{if } j \notin \{i, i+1\} \end{cases}$$

and $a_n(\tau_i)$ sends $x_j - 1$ to

$$\begin{cases} x_{i+1} - 1 & \text{if } j = i \\ x_i - 1 & \text{if } j = i+1 \\ x_j - 1 & \text{if } j \notin \{i, i+1\}. \end{cases}$$

Moreover, for all $k \in \{1, \ldots, n\}$, we denote $\mathbb{Z}\left[t^{\pm 1}\right]_k = \mathbb{Z}\left[(x_k - 1)\right] \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \mathbb{Z}\left[t^{\pm 1}\right]$ associated to the generator x_k of $\mathbf{F}_n$. Since the augmentation ideal $\mathcal{I}_{\mathbf{F}_n}$ is a free $\mathbf{F}_n$-module on the set $\{x_k - 1, k \in \{1, \ldots, n\}\}$, we thus define an isomorphism

$$\begin{array}{cccc} \varLambda : & \mathcal{I}_{\mathbf{F}_n} \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \mathbb{Z}\left[t^{\pm 1}\right] & \longrightarrow & \bigoplus_{k=1}^{n} \mathbb{Z}\left[t^{\pm 1}\right]_k \\ & (x_k - 1) \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} v & \longmapsto & \left(0, \ldots, 0, \overbrace{v}^{k\text{-}th}, 0, \ldots, 0\right). \end{array}$$

Note that $t(\xi_{n,1}(x_i)) = t$ for all $i \in \{1, \ldots, n\}$. Then result then directly follows from writing the obtained matrix through $\varLambda$. $\square$

2.3.2 Iteration on the Burau representations

We follow here the iteration procedure of [20]. In particular, an attempt to define an analogue of the Lawrence-Krammer requires to consider one more variable compared to the Burau representation, i.e. to work on the ring of Laurent polynomials in two variables $\mathbb{Z}\left[t^{\pm 1}, q^{\pm 1}\right]$. For that purpose, we iterate $\mathbf{LM}_1$ on the tensor product of the Burau representation with a one-dimensional representation in the new variable.

Also, from now on, we specify by an index in the notation which parameter we consider (i.e. t or q) and the dimension of the welded braid group we consider for the Burau representation.

Finally, for convenience of computations reasons, we prefer to use the dual version of Burau representation $Bur^*_{n+1,t}$ as input representation and consider $q^{-1}\mathbf{LM}_1\left(qBur^*_{n+1,t}\right)$. We explain below why this choice does not impact the results presented here. We denote by $\{e_i\}_{i\in\{1,\ldots,n+1\}}$ the basis for the matrices of the representation $Bur^*_{n+1,t}$.

First we prove that this iterate of the Long-Moody construction automatically admits the Burau representation as subrepresentation.

Proposition 2.5 *The submodule of* $\mathcal{I}_{\mathbf{F}_n} \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \mathbb{Z}\left[t^{\pm1},q^{\pm1}\right]^{\oplus n}$ *generated by the elements* $\{(x_k-1)\otimes e_1\}_{k\in\{1,\ldots,n\}}$ *is closed under the action of* $\mathbf{wB}_n$. *The induced subrepresentation is isomorphic to* $Bur_{n,qt}$.

Proof. First of all, note that $qBur^*_{n+1,t}(\sigma_{i+1})(e_1) = qBur^*_{n+1,t}(\tau_{i+1})(e_1) = e_1$ for all $i\in\{1,\ldots,n\}$. Therefore the action of Artin homomorphism gives that $q^{-1}\mathbf{LM}_1\left(qBur^*_{n+1,t}\right)(\sigma_i)((x_k-1)\otimes e_1)$ is equal to

$$\begin{cases}(x_{i+1}-1)\otimes e_1 & \text{if } j=i\\ (x_i-1)\otimes qBur^*_{n+1,t}(\xi_n(x_{i+1}))\,e_1 & \\ +(x_{i+1}-1)\otimes qBur^*_{n+1,t}\left(\xi_n\left(1-x_{i+1}^{-1}x_ix_{i+1}\right)\right)e_1 & \text{if } j=i+1\\ (x_j-1)\otimes e_1 & \text{otherwise}\end{cases}$$

and

$$q^{-1}\mathbf{LM}_1\left(qBur^*_{n+1,t}\right)(\tau_i)((x_k-1)\otimes e_1) = \begin{cases}(x_{i+1}-1)\otimes e_1 & \text{if } j=i\\ (x_i-1)\otimes e_1 & \text{if } j=i+1\\ (x_j-1)\otimes e_1 & \text{if } j\notin\{i,i+1\}.\end{cases}$$

The result thus follows from the fact that $qBur^*_{n+1,t}(\xi_n(x_j)) = qt$ for all $j\in\{1,\ldots,n\}$ and the use of the canonical isomorphism Λ (see the proof of Theorem 2.4). $\square$

It remains to identify the quotient of $q^{-1}\mathbf{LM}_1\left(qBur^*_{n+1,t}\right)$ by the subrepresentation $Bur_{n,qt}$. For that purpose, let us first study the case of $n=3$. The matrix of the morphism $q^{-1}\mathbf{LM}_1\left(qBur^*_{4,t}\right)(\sigma_1)$ is

$$\begin{bmatrix}
0 & 0 & 0 & 0 & qt & q(1-t) & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & q(1-t) & qt & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1-qt & qt^{-1}\left(1-t-t^2+t^3\right) & qt(1-t) & 0 & 0 & 0 & 0 & 0\\
0 & 1-t & t & 0 & 0 & (1-q)(1-t) & t(1-q) & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 1-q & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1-q & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-t & t & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

and the one of $q^{-1}\mathbf{LM}_1\left(qBur^*_{4,t}\right)(\sigma_2)$ is

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1-t & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & qt & 0 & q(1-t) & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q(1-t) & qt\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1-qt & 0 & qt^{-1}\left(1-t-t^2+t^3\right) & qt(1-t)\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1-q & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1-t & t & 0 & 0 & (1-q)(1-t) & t(1-q)\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1-q & 0
\end{bmatrix}.$$

Those for τ_1 and τ_2 are the analogues with $t = q = 1$. Then the quotient by the subspace $\{e_1, e_5, e_9\}$ gives the matrices

$$\begin{bmatrix} 0 & 0 & 0 & q(1-t) & qt & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\ 1-t & t & 0 & (1-q)(1-t) & t(1-q) & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1-q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1-q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-t & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q(1-t) & qt \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-t & t & 0 & (1-q)(1-t) & t(1-q) \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1-q & 0 \end{bmatrix}.$$

Writing down the matrices of the tensor product of representations

$$Bur^*_{3,t} \underset{\mathbb{Z}[t^{\pm1},q^{\pm1}]}{\otimes} Bur_{3,q}$$

for the generators of the welded braid group, we then recognise the above matrices: we thus identify the quotient of the Long-Moody construction to a tensor product of two Burau representations.

The situation for any $n \geqslant 3$ works in the same way: indeed consecutive generators (σ_i, τ_i) and $(\sigma_{i+1}, \tau_{i+1})$ in $\mathbf{wB}_n$ can be identified to $\mathbf{wB}_3$. Then the matrices for the quotient of $q^{-1}\mathbf{LM}_1\left(qBur^*_{n+1,t}\right)$ are similar to the above ones, except that there are more diagonal matrix blocks given by $Bur_{3,t}$. Hence we proved that the iteration of the Long-Moody construction does not construct any new representation of $\mathbf{wB}_n$; more precisely we have that:

Theorem 2.6 *Applying the Long-Moody construction* $\mathbf{LM}_1$ *to the representation* $Bur^*_{n+1,t}$ *gives the short exact sequence of* $\mathbf{wB}_n$*-representations:*

$$0 \longrightarrow Bur_{n,qt} \longrightarrow q^{-1}\mathbf{LM}_1\left(qBur^*_{n+1,t}\right) \longrightarrow Bur_{3,t} \underset{\mathbb{Z}[t^{\pm1},q^{\pm1}]}{\otimes} Bur^*_{3,q} \longrightarrow 0\,. \tag{2.3}$$

Cases of the dual and reduced versions. The main conclusion on the iteration of the Long-Moody construction of Theorem 2.6 remains true if we apply $\mathbf{LM}_1$ to the other versions $Bur_{n+1,t}$ and $\overline{Bur}_{n+1,t}$ of the Burau representation.

For the dual Burau representation $Bur_{n+1,t}$, it is defined at the level of the matrices by the transpose of the inverse of $Bur^*_{n+1,t}$. Therefore the $(n+1)\times(n+1)$-blocks of $q^{-1}\mathbf{LM}_1\left(qBur_{n+1,t}\right)$ are the transpose of those of $q^{-1}\mathbf{LM}_1\left(qBur^*_{n+1,t}\right)$. Recall that $\{e_i\}_{i\in\{1,\dots,n+1\}}$ denotes the basis for the matrices of the representation $Bur^*_{n+1,t}$. Then the analogue of Proposition 2.5 shows that the submodule of $\mathcal{I}_{\mathbf{F}_n} \underset{\mathbb{Z}[\mathbf{F}_n]}{\otimes} \mathbb{Z}\left[t^{\pm1}, q^{\pm1}\right]^{\oplus n}$ generated by the elements $\{(x_k - 1)\otimes e_j\}_{(k,j)\in\{1,\dots,n\}\times\{2,\dots,n\}}$ is closed under the action of $\mathbf{wB}_n$ by $q^{-1}\mathbf{LM}_1\left(qBur_{n+1,t}\right)$. Moreover repeating mutatis mutandis the proof of Theorem 2.6, we prove that there is a short exact sequence of $\mathbf{wB}_n$-representations:

$$0 \longrightarrow Bur^*_{3,t} \underset{\mathbb{Z}[t^{\pm1},q^{\pm1}]}{\otimes} Bur_{3,q} \longrightarrow q^{-1}\mathbf{LM}_1\left(qBur_{n+1,t}\right) \longrightarrow Bur^*_{n,qt} \longrightarrow 0\,.$$

For the reduced Burau representation $\overline{Bur}_{n+1,t}$, note that we have for all $i \in \{1, \ldots, n-1\}$

$$r_{n+1} \circ Bur^{*}_{n+1,t}(\sigma_i) \circ r^{-1}_{n+1} = \begin{bmatrix} \overline{Bur}^{*}_{n+1,t}(\sigma_i) & 0 \\ L_i & 1 \end{bmatrix}$$

where $L_i = \begin{bmatrix} 0 & \cdots & 0 & \delta_{i,n} & 1 \end{bmatrix}$ and r_{n+1} is the $n \times n$-matrix with coefficients $r_{i,j} = 1$ if $j \leqslant i$ and $r_{i,j} = 0$. Therefore, along the injection $id_1 * - : \mathbf{wB}_n \hookrightarrow \mathbf{wB}_{n+1}$, the short exact sequence (1.3) splits as a short exact sequence of $\mathbf{wB}_n$-representations. Then it follows from the freeness (and a fortiori flatness) of the augmentation ideal $\mathcal{I}_{\mathbf{F}_n}$ as a $\mathbb{Z}[\mathbf{F}_n]$-module that we have an isomorphism of $\mathbf{wB}_n$-representations:

$$q^{-1}\mathbf{LM}_1\left(qBur^{*}_{n+1,t}\right) \cong q^{-1}\mathbf{LM}_1\left(q\overline{Bur}^{*}_{n+1,t}\right) \oplus q^{-1}\mathbf{LM}_1\left(q\mathbb{Z}\left[t^{\pm 1}\right]\right).$$

The representation $q^{-1}\mathbf{LM}_1\left(q\overline{Bur}^{*}_{n+1,t}\right)$ is thus determined by the short exact sequence (2.3).

2.3.3 On the impossibility to recover the Tong-Yang-Ma representation

We detail in §2.1 that a Long-Moody construction is equivalent to the setting of two parameters $a_n : \mathbf{wB}_n \to Aut(\mathbf{F}_n)$ and $\xi_n : \mathbf{F}_n \to \mathbf{wB}_{n+1}$ satisfying the compatibility condition (2.1). So far, we have used the Artin homomorphism for the action a_n and this parameter determines the shape of the obtained matrices (see §2.3.1 and §2.3.2).

In [24, Section 2.3.2], the Tong-Yang-Ma representation for braid groups is recovered by playing on the choice of this morphism. However, we prove that it is not the case for welded braid groups in the following result. In particular, we call the representation which space is $\mathbb{Z}\left[t^{\pm 1}\right]^{\oplus n}$ and on which $\mathbf{wB}_n$ acts trivially the *trivial n-dimensional representation* of $\mathbf{wB}_n$.

Theorem 2.7 *Let* $\mathbf{LM}$ *be the Long-Moody construction associated with a Wada representation for a welded braid groups and an abstract compatible* ξ_n*. Then* $t^{-1}\mathbf{LM}(t)$ *is equivalent either to the Burau representation* Bur *(or its dual* Bur^{*}*), or to the permutation representation extended to* $\mathbf{wB}_n$*, or to the trivial n-dimensional representation of* $\mathbf{wB}_n$*.*

In particular, all the Tong-Yang-Ma representations (or their dual) cannot be recovered by any Long-Moody construction for welded braid groups.

Proof. We recall from §1.2 that we can only consider four Wada representations (Types 2, 3, 4 and 5). First of all, restricting to $\mathbf{B}_n$, [24, Section 2.3.2] automatically implies that:

- with the Type 3 Wada representation, $t^{-1}\mathbf{LM}(t)$ is equivalent to the permutation representation extended to $\mathbf{wB}_n$;
- with the Type 4 or 5 Wada representation, $t^{-1}\mathbf{LM}(t)$ is equivalent the Burau representation.

[24, Section 2.3.2] uses the Type 2 Wada representation to obtain the Tong-Yang-Ma representation for braid groups. Nevertheless, with the extension to $\mathbf{wB}_n$, it follows from the compatibility condition (2.2) that $\tau_2\xi_n(x_1)\tau_2 = \xi_n(x_2) = \sigma_2^{-1}\xi_n(x_1)\sigma_2$ and that $\xi_n\left(x_2^{-1}\right) = \sigma_2\xi_n(x_1)\sigma_2^{-1}$. We deduce that $\xi_n\left(x_2^{-1}\right) = \sigma_2^{-2}\xi_n(x_1)\sigma_2^{2}$ and that $\xi_n(x_1) = (\tau_2\sigma_2)^{-1}\xi_n(x_1)\tau_2\sigma_2$. Hence we obtain that:

$$\xi_n(x_1) = \sigma_2^{-1}\tau_2^{2}\sigma_2^{-1}\xi_n(x_1)\sigma_2\tau_2^{2}\sigma_2 = \xi_n\left(x_1^{-1}\right).$$

Then $(t \circ \xi_n)^2 = 1$ since t and ξ_n are morphisms. The only order 2 elements of $\mathbb{Z}\left[t^{\pm 1}\right]$ are 1 and -1 and it follows from the definition of t that -1 is not in its image. A fortiori the composite $t \circ \xi_n$ is the trivial morphism $\mathbf{F}_n \to \mathbf{wB}_{n+1}$. Then it follows from a straightforward computation of the matrices that $t^{-1}\mathbf{LM}(t)$ is equivalent to the permutation representation extended to $\mathbf{wB}_n$.

Note that a Long-Moody construction multiplies by n the dimension of an input representation. Therefore the only way to construct a Tong-Yang-Ma representation would be to start from a one-dimensional representation. Also, the form of the matrix

$$Id_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} \oplus Id_{n-i-1}$$

for each braid generator σ_i implies that the parameter α_n of the Long-Moody construction which would recover a Tong-Yang-Ma representation has to be a Wada representation. Therefore the above study of $t^{-1}\mathbf{LM}(t)$ proves that no Long-Moody construction can recover any Tong-Yang-Ma representation. □

Therefore, Artin and, more generally, Wada representations do not seem to be a useful tool to obtain interesting linear representations for welded braid groups, at least in the framework of Long-Moody procedure; a possible lead should be to consider other free groups embedded in $\mathbf{wB}_n$ or other actions. For instance, recently in [12] was constructed a lift of Artin representation to an action at the π_2-level (remind that B^3 less n trivial circles is not aspherical) and it seems interesting to adapt Long-Moody procedure in this framework. It seems also clear that a deeper understanding of $\mathbf{wB}_n$ and of its subgroup $\mathbf{D}_{n-1}$ in terms of *motion groups* of circles [11] could provide new perspectives. On another hand, an homological approaches to construct linear representations of welded braid groups is set in [21]: the connection with the representations of the present paper would deserve to be explored.

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