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Asymptotics of the k-free diffraction measure via discretisation

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Abstract

We determine the diffraction intensity of the k-free integers near the origin.

1. *Introduction*

Point sets in Euclidean space exhibiting pure point diffraction play an important rôle in the theory of aperiodic order as mathematical models of quasicrystals. The growth of the diffraction intensity $Z(\varepsilon)$ as $\varepsilon \to 0^+$ demonstrates how stable the structure of the point set is. For instance, a homogeneous Poisson process displays growth $Z(\varepsilon) = \varepsilon$, whereas for a lattice one has $Z(\varepsilon) = 0$. Power laws are typical of aperiodically ordered sets, cf. [3].

Recently, sets of number theoretic origin, such as the k-free integers, have gained attention as they are conjectured to be weak model sets with extremal density. Baake and Coons [1] studied the fluctuation of the density of this set by considering the scaling behaviour of the diffraction measure ν_k, given by $Z_k(\varepsilon) = \nu_k((0, \varepsilon])/\nu_k(\{0\})$, as $\varepsilon \to 0^+$. They used a sieving argument to show

$$\lim_{\varepsilon \to 0} \frac{\log Z_k(\varepsilon)}{\log \varepsilon} = 2 - 1/k.$$

We prove that a power law holds for k-free integers, thus confirming the conjectured behavior:

Theorem 1.1. *For all $k > 1$, as $\varepsilon \to 0^+$ we have*

$$Z_k(\varepsilon) = \frac{c_k}{2k} \varepsilon^{2-1/k} \Big(1 + o(\varepsilon^{1/k})\Big),$$

where c_k is an explicit positive constant.

The constant c_k is given in (2.4). It stabilises quite rapidly, specifically,

$$c_k = 1 + O(1/k).$$

Our proof gives a more specific error term, namely, $o(\varepsilon^{1/k})$ can be replaced by

$$\varepsilon^{1/k} \exp\{-\gamma k^{-1} (\log 1/\varepsilon)^{3/5} (\log \log 1/\varepsilon)^{-1/5}\}$$

for some positive absolute constant γ. The improvements over previous works stem from using a discretisation approach, which is new in this problem and allows the use of number theory estimates.

We shall see that the Riemann hypothesis implies a much stronger approximation of the diffraction intensity by a power law; we are not aware of a previous connection between the Riemann hypothesis and aperiodic structures.

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THEOREM 1.2. *Assume the Riemann Hypothesis. For every $k > 1$ and $\delta > 0$, as $\varepsilon \to 0^+$, we have*

$$Z_k(\varepsilon) = \frac{c_k}{2k}\varepsilon^{2-1/k} + O(\varepsilon^{2-11/35k-\delta}).$$

1.1. *The discretisation approach*

Our method is entirely different from the one used by Baake and Coons. Before explaining its steps, we must note that the crucial reason behind our improvements over the work of Baake and Coons is our discretisation trick and not the use of analytic number theory estimates. Indeed, our discretisation trick followed by a sieving argument that is similar to the one of Baake and Coons, would produce an error term $O(\varepsilon^{1/k})$. This is plainly weaker than our Theorem 1.1 but still an ample improvement over what was known before.

Our proof has three steps.

(1) (*Discretisation*) We approximate $Z_k(\varepsilon)$ by $Z_k(1/N)$ for a certain integer N in Lemma 2.1. We make a slightly unusual use of the auxiliary variable in Lemma 2.2: noting that it divides certain integers allows expressing $Z_k(1/N)$ as a sum of certain quantities $z_k(c)$. These objects are closer to number theory than the diffraction measure.

(2) (*Analysing $z_k(c)$*) In Lemma 2.3 and Proposition 2.4, we study $z_k(c)$. By taking the validity of Proposition 2.4 for granted, we prove Theorem 1.1 at the end of § 2.2.

(3) (*Zero-free region information*) The proof of Proposition 2.4 is given in § 2.3. It uses a result of Walfisz on the distribution of square-free numbers, whose proof hinges upon the zero-free region of the Riemann zeta function.

The leading constant of Theorem 1.1 is analysed in Section 3. Section 4 gives the implications of the Riemann hypothesis about the diffraction measure, namely Theorem 1.2.

NOTATION. All implied constants in the Landau/Vinogradov O-big notation $O(), \ll$ are absolute. Any further dependence on a further quantity h will be recorded by the use of a subscript $O_h(), \ll_h$. The number of positive integer divisors of an integer n is denoted by $\tau(n)$, the Möbius function by $\mu(n)$ and the indicator function of the k-free integers n by $\mu_k(n)$.

2. *The proof of Theorem 1.1*

2.1. *Discretisation*

For any $k, N \in \mathbb{N}$, we let

$$\widetilde{Z}_k(N) := \sum_{q\in\mathbb{N}} \mu_{k+1}(q)\left(\prod_{p \text{ prime}, p|q} \frac{1}{(p^k-1)^2}\right)\sharp\left\{m \in \mathbb{N}\cap\left[1, \frac{q}{N}\right] : \gcd(m,q) = 1\right\}. \quad (2.1)$$

The function $\widetilde{Z}_k(N)$ is well defined because its modulus is at most

$$\sum_{q\in\mathbb{N}} \mu_{k+1}(q)\left(\prod_{p \text{ prime}, p|q} \frac{1}{(p^k-1)^2}\right) q \leqslant \prod_p \left(1 + \sum_{n=1}^{k} \frac{p^n}{(p^k-1)^2}\right) \leqslant \prod_p \left(1 + \frac{k}{p^k-2}\right) < \infty. \quad (2.2)$$

LEMMA 2.1. *For any $\varepsilon \in (0,1)$, let N be the integer part of $1/\varepsilon$. Then $\widetilde{Z}_k(N+1) \leqslant Z_k(\varepsilon) \leqslant \widetilde{Z}_k(N)$.*

Proof. The work of Baake, Moody and Pleasants [5] gives

$$Z_k(\varepsilon) = \sum_{q \geqslant 1/\varepsilon} \mu_{k+1}(q) \sum_{\substack{1 \leqslant m \leqslant q\varepsilon \\ \gcd(m,q)=1}} \prod_{p \mid q} \frac{1}{(p^k-1)^2}.$$

The condition $q \geqslant 1/\varepsilon$ is implied by the presence of the sum over m and it can therefore be omitted. The inequality $N \leqslant \frac{1}{\varepsilon} < N+1$ shows that $\widetilde{Z}_k(N+1)$ equals

$$\sum_{q=1}^{\infty} \mu_{k+1}(q) \sum_{\substack{1 \leqslant m \leqslant q/(N+1) \\ \gcd(m,q)=1}} \prod_{p \mid q} \frac{1}{(p^k-1)^2} \leqslant Z_k(\varepsilon) \leqslant \sum_{q=1}^{\infty} \mu_{k+1}(q) \sum_{\substack{1 \leqslant m \leqslant q/N \\ \gcd(m,q)=1}} \prod_{p \mid q} \frac{1}{(p^k-1)^2} = \widetilde{Z}_k(N).$$

□

Lemma 2.2. *For any positive integer N, we have*

$$\widetilde{Z}_k(N) = \sum_{\substack{c \in \mathbb{N} \\ N \mid c}} z_k(c), \tag{2.3}$$

where

$$z_k(c) := \sum_{\substack{r \in \mathbb{N} \\ r \geqslant c}} \sum_{d \in \mathbb{N}} \mu(d)\mu_{k+1}(dr) \prod_{p \mid dr} \frac{1}{(p^k-1)^2}.$$

Proof. The changes in the order of summation in the following arguments are justified by the absolute convergence of the sum in (2.1), which is proved in (2.2). The expression

$$\sum_{\substack{d \in \mathbb{N} \\ d \mid m, d \mid q}} \mu(d)$$

is the indicator function of the event $\gcd(m,q)=1$. Injecting it into (2.1) yields

$$\widetilde{Z}_k(N) = \sum_{d \in \mathbb{N}} \mu(d) \sum_{\substack{q \in \mathbb{N} \\ d \mid q}} \mu_{k+1}(q) \left[\frac{q}{dN}\right] \prod_{p \mid q} \frac{1}{(p^k-1)^2},$$

where $[x]$ denotes the integer part of a real number x. The integers q appearing above are of the form dr for some $r \in \mathbb{N}$, hence,

$$\widetilde{Z}_k(N) = \sum_{d \in \mathbb{N}} \mu(d) \sum_{r \in \mathbb{N}} \mu_{k+1}(dr) \left[\frac{r}{N}\right] \prod_{p \mid dr} \frac{1}{(p^k-1)^2}.$$

We now replace the term $[r/N]$ by $\sharp\{c \in \mathbb{N} \cap [1,r] : c \equiv 0 (\bmod N)\}$, thus obtaining

$$\widetilde{Z}_k(N) = \sum_{\substack{c \in \mathbb{N} \\ N \mid c}} \sum_{\substack{r \in \mathbb{N} \\ r \geqslant c}} \sum_{d \in \mathbb{N}} \mu(d)\mu_{k+1}(dr) \prod_{p \mid dr} \frac{1}{(p^k-1)^2}.$$

□

2.2. *Analysing $z_k(c)$*

We express $z_k(c)$ via the tail of a convergent series.

Lemma 2.3. *For any positive integer c, we have*

$$z_k(c) = \xi_k \sum_{t \in \mathbb{N} \cap [c^{1/k}, \infty)} \frac{|\mu(t)|}{t^{2k}} \prod_{p \mid t} \frac{1}{1 - \frac{2}{p^k}},$$

where $\xi_k := \prod_p (1 - (p^k-1)^{-2})$.

Proof. The integers r in the definition of $z_k(c)$ are $(k+1)$-free, hence can be written uniquely as $r = \prod_{i=1}^{k} r_i^i$, where $r_i \in \mathbb{N}$ are square-free and coprime in pairs. The integer d in the definition of $z_k(c)$ is square-free and therefore coprime to r_k. Therefore, letting $\delta_i := \gcd(r_i, d)$, we infer that there are unique positive integers $\delta_i, s_i, (0 < i < k)$, d_0 such that

$$r_i = \delta_i s_i \quad (0 < i < k), \quad d = d_0 \prod_{i=1}^{k-1} \delta_i.$$

Writing $m = r_k^k \prod_{i=1}^{k-1} (\delta_i s_i)^i$ transforms $z_k(c)$ into

$$\sum_{m \geqslant c} \left(\prod_{p|m} (p^k - 1)^{-2} \right) \left(\sum_{\substack{d_0 \in \mathbb{N} \\ \gcd(d_0, m)=1}} \mu(d_0) \prod_{p|d_0} \frac{1}{(p^k-1)^2} \right) \sum_{\substack{r_k \in \mathbb{N}, \boldsymbol{\delta} \in \mathbb{N}^{k-1}, \mathbf{s} \in \mathbb{N}^{k-1} \\ m = r_k^k \prod_{i=1}^{k-1} (s_i \delta_i)^i}} \mu(\delta_1) \cdots \mu(\delta_{k-1}),$$

where the sum over $r_k, \boldsymbol{\delta}, \mathbf{s}$ is subject to the conditions $\gcd(s_i\delta_i, s_j\delta_j) = 1$ for all $i \neq j$ and $\gcd(r_k\delta_i, s_i) = 1$ for all $i \neq k$. One can see that the sum over $r_k, \boldsymbol{\delta}, \mathbf{s}$ forms a multiplicative function of m and looking at its values at prime powers makes clear that it is the indicator function of integers of the form $m = t^k$ with t square-free. Indeed, for $1 \leqslant j < k$, we have

$$\sum_{\substack{r_k \in \mathbb{N}, \boldsymbol{\delta} \in \mathbb{N}^{k-1}, \mathbf{s} \in \mathbb{N}^{k-1} \\ p^j = r_k^k \prod_{i=1}^{k-1} (s_i \delta_i)^i}} \mu(\delta_1) \cdots \mu(\delta_{k-1}) = \sum_{\substack{s_j, \delta_j \in \mathbb{N} \\ p^j = (s_j \delta_j)^j}} \mu(\delta_j) = \sum_{\delta_j | p} \mu(\delta_j) = 0$$

since all other variables in the sum must equal 1. We thus obtain

$$z_k(c) = \sum_{t \geqslant c^{1/k}} |\mu(t)| \left(\prod_{p|t} (p^k - 1)^{-2} \right) \sum_{\substack{d_0 \in \mathbb{N} \\ \gcd(d_0, t)=1}} \mu(d_0) \prod_{p|d_0} (p^k - 1)^{-2}.$$

The proof concludes by writing the sum over d_0 as an Euler product and using $t^{-2k} = \prod_{p|t} p^{-2k}$. □

PROPOSITION 2.4. *There exists a constant $\gamma > 0$ such that for all $k > 1$ and $u \geqslant 1$, we have*

$$\sum_{1 \leqslant t \leqslant u} |\mu(t)| \prod_{p|t} \frac{1}{1 - 2p^{-k}} = \gamma_k u + O_k \left(\frac{u^{1/2}}{\exp\left(\gamma (\log u)^{3/5} (\log \log u)^{-1/5}\right)} \right),$$

where the implied constant depends at most on k and

$$\gamma_k := \frac{1}{\zeta(2)} \prod_p \left(1 + \frac{2}{(p+1)(p^k - 2)} \right).$$

We conclude this section by deducing Theorem 1.1 from Proposition 2.4. Define

$$c_k := \frac{2k}{(2k-1)} \frac{\zeta(2 - 1/k)}{\zeta(2)} \zeta(k)^2 \prod_p \left(1 - \frac{2p}{(p+1)p^k} \right). \tag{2.4}$$

Proof of Theorem 1.1. Lemma 2.3, Proposition 2.4 and Abel's summation formula give

$$\frac{z_k(c)}{\xi_k} = \frac{\gamma_k}{(2k-1)} \frac{1}{c^{2-1/k}} + O_k \left(\frac{c^{-2+1/(2k)}}{\exp\left(\gamma k^{-1} (\log c)^{3/5} (\log \log c)^{-1/5}\right)} \right).$$

Feeding this into Lemma 2.2 produces

$$\widetilde{Z}_k(N) = \sum_{b\in\mathbb{N}} z_k(Nb) = \frac{\gamma_k \xi_k}{(2k-1)} \frac{\zeta\left(2-\frac{1}{k}\right)}{N^{2-\frac{1}{k}}} + O_k\left(\frac{N^{-2+1/(2k)}}{\exp\left(\gamma k^{-1}(\log N)^{3/5}(\log\log N)^{-1/5}\right)}\right).$$

The leading constant can be turned into the form of Theorem 1.1 by noting that

$$\frac{\gamma_k \xi_k}{(2k-1)}\zeta(2-1/k) = \frac{c_k}{2k}.$$

Finally, invoking Lemma 2.1 concludes the proof because the inequality $N \leqslant \frac{1}{\varepsilon} < 1+N$ implies that both $(N+1)^{-2+\frac{1}{k}}$ and $N^{-2+\frac{1}{k}}$ are $\varepsilon^{2-\frac{1}{k}} + O_k(\varepsilon^{3-\frac{1}{k}})$. □

2.3. *Zero-free region information*

We now prove Proposition 2.4 by using the following result, which is based on the best-known zero-free region for the Riemann zeta function.

LEMMA 2.5 [7]. *There exists an absolute constant* $\gamma_0 > 0$ *such that*

$$\sum_{n\in\mathbb{N}\cap[1,x]} \mu(n)^2 = \frac{x}{\zeta(2)} + O\Big(x^{\frac{1}{2}} \exp\left(-\gamma_0(\log x)^{3/5}(\log\log x)^{-1/5}\right)\Big).$$

We shall later need a stronger version of Lemma 2.5.

COROLLARY 2.6. *There exists an absolute constant* $\gamma' > 0$ *such that for every* $a \in \mathbb{N}, x \geqslant 1$, *we have*

$$\sum_{\substack{n\in\mathbb{N}\cap[1,x]\\ \gcd(n,a)=1}} \mu(n)^2 = \left(\prod_{p\mid a}\left(1+\frac{1}{p}\right)^{-1}\right)\frac{x}{\zeta(2)} + O\Big(\tau(a)^3 x^{\frac{1}{2}} \exp\left(-\gamma'(\log x)^{3/5}(\log\log x)^{-1/5}\right)\Big),$$

where the implied constant is absolute.

Proof. The Dirichlet series of $\mathbb{1}_{\gcd(a,n)=1}(n)\mu(n)^2$ is

$$\sum_{\substack{n=1\\ \gcd(n,a)=1}}^{\infty} \frac{\mu(n)^2}{n^s} = \prod_p \left(1+\frac{1}{p^s}\right)\prod_{p\mid a}\left(1+\frac{1}{p^s}\right)^{-1} = \prod_p \left(1+\frac{1}{p^s}\right)\prod_{p\mid a}\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k}{p^{ks}}\right).$$

This is the product of the Dirichlet series of $\mu(n)^2$ by the Dirichlet series of the multiplicative function $g_a(n)$, where

$$g_a(n) := \mathbb{1}_{p\mid n \Rightarrow p\mid a}(n)(-1)^{\Omega(n)}$$

and $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. We get

$$\mathbb{1}_{\gcd(a,n)=1}(n)\mu(n)^2 = (\mu^2 * g_a)(n) = \sum_{\substack{c,d\in\mathbb{N}\\ cd=n}} g_a(c)\mu(d)^2,$$

where $*$ is the Dirichlet convolution. Hence, we can write

$$\sum_{\substack{n\in\mathbb{N}\cap[1,x]\\ \gcd(n,a)=1}} \mu(n)^2 = \sum_{1\leqslant c\leqslant x} g_a(c) \sum_{1\leqslant d\leqslant x/c} \mu(d)^2.$$

Let $Y := x^{7/10}$. The terms with $Y < c \leqslant x$ contribute at most

$$x\sum_{c>Y}\frac{|g_a(c)|}{c} \leqslant \frac{x}{Y^{3/4}}\sum_{c>Y}\frac{|g_a(c)|}{c^{1/4}} \leqslant \frac{x}{Y^{3/4}}\prod_{p|a}\left(1+\sum_{k=1}^{\infty}\frac{1}{p^{k/4}}\right) \leqslant \frac{x}{Y^{3/4}}\left(\prod_{p|a}8\right) \leqslant \frac{x\tau(a)^3}{Y^{3/4}},$$

which equals $\tau(a)^3x^{\frac{19}{40}}$. By Lemma 2.5, the terms with $c \leqslant Y$ contribute

$$\frac{x}{\zeta(2)}\sum_{1\leqslant c\leqslant Y}\frac{g_a(c)}{c} + O\left(x^{\frac{1}{2}}\sum_{1\leqslant c\leqslant Y}\frac{|g_a(c)|}{c^{\frac{1}{2}}}\exp\left(-\gamma_0(\log x/c)^{3/5}(\log\log x/c)^{-1/5}\right)\right).$$

Note that $x/c \geqslant x^{3/10}$, therefore,

$$(\log x/c)^{3/5}(\log\log x/c)^{-1/5} \geqslant 5^{-3/5}(\log x)^{3/5}(\log\log x/c)^{-1/5}.$$

Letting $\gamma' := 5^{-3/5}\gamma_0$, we infer that the error term contribution is

$$\ll x^{\frac{1}{2}}\exp\left(-\gamma(\log x)^{3/5}(\log\log x)^{-1/5}\right)\sum_{1\leqslant c\leqslant Y}\frac{|g_a(c)|}{c^{\frac{1}{2}}}$$

$$\ll \tau(a)^3x^{\frac{1}{2}}\exp\left(-\gamma'(\log x)^{3/5}(\log\log x)^{-1/5}\right).$$

To complete the summation over $c > Y$, we use the estimate

$$x\sum_{c>Y}\frac{|g_a(c)|}{c} \ll \tau(a)^3x^{\frac{19}{40}}$$

that was proved earlier in this proof. Finally, the proof is concluded by noting that

$$\sum_{c\in\mathbb{N}}\frac{g_a(c)}{c} = \prod_{p|a}\left(1+\frac{1}{p}\right)^{-1}. \qquad \square$$

The following result is a generalisation of Lemma 2.5 and its proof uses Corollary 2.6.

LEMMA 2.7. *Let $\delta : \mathbb{N} \to \mathbb{R}$ be a multiplicative function with $|\delta(p)| \leqslant \frac{4}{p^2}$ for every prime p. There exists a positive absolute constant γ such that for all $u \geqslant 1$, we have*

$$\sum_{1\leqslant m\leqslant u}\mu(m)^2\prod_{p|m}(1+\delta(p)) = \left(\prod_p\left(1+\frac{\delta(p)}{p+1}\right)\right)\frac{u}{\zeta(2)} + O\left(\frac{u^{1/2}}{\exp\left(\gamma(\log u)^{3/5}(\log\log u)^{-1/5}\right)}\right),$$

where the implied constant is absolute.

Proof. Switching the order of summation, the sum in the lemma becomes

$$\sum_{d\leqslant u}\delta(d)\sum_{\substack{1\leqslant m\leqslant u\\ d|m}}\mu(m)^2 = \sum_{d\leqslant u}\delta(d)\mu(d)^2\sum_{\substack{1\leqslant m'\leqslant u/d\\ \gcd(m',d)=1}}\mu(m')^2.$$

The contribution of $d > u^{3/4}$ is admissible, since it is at most

$$\ll \sum_{d>u^{3/4}}\delta(d)\mu(d)^2\frac{u}{d} \leqslant \sum_{d>u^{3/4}}\frac{\tau(d)^2}{d^2}\frac{u}{d} \ll u^{-1/4}$$

due to $\mu(d)^2 4^{\sharp\{p|d\}} \leqslant \tau(d)^2$ and the divisor bound $\tau(d) \ll_\varepsilon d^\varepsilon$ for all $\varepsilon > 0$. To the remaining range, $1 \leqslant d \leqslant u^{3/4}$, we apply Corollary 2.6 with $x = u/d$ and $a = d$. It gives

$$\sum_{d \leqslant u^{3/4}} \delta(d)\mu(d)^2 \left(\frac{u}{d\zeta(2)} \left(\prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} \right) + O\left(\frac{\tau(d)^3 u^{\frac{1}{2}}}{d^{\frac{1}{2}} \exp\left(\gamma'(\log \frac{u}{d})^{3/5} (\log\log \frac{u}{d})^{-1/5}\right)} \right) \right).$$

Using $d \leqslant u^{3/4}$, we see that $\log \frac{u}{d} \geqslant \frac{1}{4} \log u$, hence the error term is

$$\frac{u^{\frac{1}{2}}}{\exp\left(\gamma(\log u)^{3/5}(\log\log u)^{-1/5}\right)} \sum_{d \leqslant u^{3/4}} \frac{\tau(d)^3}{d^{\frac{3}{2}}} \ll \frac{u^{\frac{1}{2}}}{\exp\left(\gamma(\log u)^{3/5}(\log\log u)^{-1/5}\right)}$$

for some positive absolute constant γ. Finally, we complete the summation in the main term:

$$\sum_{d > u^{3/4}} \frac{\delta(d)\mu(d)^2}{d} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} \leqslant \sum_{d > u^{3/4}} \frac{\tau(d)^2}{d^3} \ll u^{-3/4}. \qquad \square$$

Proof of Proposition 2.4. This follows from applying Lemma 2.7 with

$$\delta(p) = -1 + \frac{1}{1 - 2p^{-k}} = \frac{2}{p^k - 2} \leqslant \frac{4}{p^2}. \qquad \square$$

3. *Analysis of the leading constant*

We analyse the behaviour of the leading constant in Theorem 1.1.

THEOREM 3.1. *The following holds for all $k > 1$ and with an absolute implied constant,*

$$c_k = 1 + O\left(\frac{1}{k}\right).$$

Proof. We note that $2p/(p+1) \leqslant p^{1/2}$, hence,

$$\prod_p \left(1 - \frac{2p}{(p+1)p^k}\right) \geqslant \zeta(k - 1/2)^{-1}.$$

Using $\zeta(\sigma) = 1 + O(2^{-\sigma})$ for $\sigma > 3/2$ and (2.4) yields

$$c_k = \frac{2k}{(2k-1)} \frac{\zeta(2 - 1/k)}{\zeta(2)} (1 + O(2^{-k})).$$

We conclude the proof by using the bound $\max\{|\zeta'(\sigma)| : \frac{3}{2} \leqslant \sigma < 2\} = O(1)$ to infer that

$$\zeta\left(2 - \frac{1}{k}\right) = \zeta(2) + O\left(\frac{1}{k}\right). \qquad \square$$

4. *Approximations via the Riemann Hypothesis*

In this section, we prove Theorem 1.2. The main input is the following result.

LEMMA 4.1 [6]. *Assume the Riemann Hypothesis. Then for every fixed $\delta > 0$, we have*

$$\sum_{n \in \mathbb{N} \cap [1,x]} \mu(n)^2 = \frac{x}{\zeta(2)} + O_\delta\left(x^{\frac{11}{35} + \delta}\right).$$

This result uses van der Corput's method for estimating exponential sums.

COROLLARY 4.2. *Assume the Riemann Hypothesis and let $\delta > 0$ be arbitrary and fixed. Then for every $a \in \mathbb{N}$ and $x \geqslant 1$, we have*

$$\sum_{\substack{n\in\mathbb{N}\cap[1,x]\\ \gcd(n,a)=1}} \mu(n)^2 = \frac{x}{\zeta(2)}\left(\prod_{p\mid a}\left(1+\frac{1}{p}\right)^{-1}\right) + O_\delta\left(\tau(a)^3 x^{\frac{11}{35}+\delta}\right),$$

where the implied constant depends at most on δ.

Proof. We make use of the function $g_a(n)$ that is defined in the proof of Corollary 2.6. Thus the sum in our corollary equals

$$\sum_{1\leqslant c\leqslant x} g_a(c) \sum_{1\leqslant d\leqslant x/c} \mu(d)^2 = \frac{x}{\zeta(2)} \sum_{1\leqslant c\leqslant x} \frac{g_a(c)}{c} + O_\varepsilon\left(x^{\frac{11}{35}+\varepsilon} \sum_{1\leqslant c\leqslant x} \frac{|g_a(c)|}{c^{\frac{11}{35}+\varepsilon}}\right),$$

where a use of Lemma 4.1 has been made. The bound $|g_a(c)| \leqslant \mathbb{1}_{p\mid c\Rightarrow p\mid a}(c)$ shows that the error term is

$$\ll \prod_{p\mid a}\left(1+\sum_{k=1}^{\infty}\frac{1}{p^{\frac{11}{35}k+\varepsilon k}}\right) \leqslant \prod_{p\mid a}\left(1+\sum_{k=1}^{\infty}\frac{1}{2^{\frac{11}{35}k}}\right) \leqslant 8^{\omega(a)} \leqslant \tau(a)^3.$$

The same bound yields

$$\sum_{c>x}\frac{|g_a(c)|}{c} \leqslant \sum_{\substack{c\in\mathbb{N}\\ p\mid c\Rightarrow p\mid a}} \left(\frac{c}{x}\right)^{\frac{24}{35}}\frac{1}{c} \leqslant x^{-\frac{24}{35}} \prod_{p\mid a}\left(1+\sum_{k=1}^{\infty}\frac{1}{p^{\frac{11}{35}}}\right) \leqslant x^{-\frac{24}{35}} \prod_{p\mid a}\left(1+\sum_{k=1}^{\infty}\frac{1}{2^{\frac{11}{35}}}\right) \leqslant \frac{\tau(a)^3}{x^{\frac{24}{35}}}.$$

□

LEMMA 4.3. *Assume the Riemann Hypothesis and let $\delta > 0$ be arbitrary and fixed. Let $\delta : \mathbb{N} \to \mathbb{R}$ be a multiplicative function with $|\delta(p)| \leqslant \frac{4}{p^2}$ for every prime p. Then for all $u \geqslant 1$, we have*

$$\sum_{1\leqslant m\leqslant u} \mu(m)^2 \prod_{p\mid m}(1+\delta(p)) = \left(\prod_{p}\left(1+\frac{\delta(p)}{p+1}\right)\right)\frac{u}{\zeta(2)} + O_\delta\left(u^{\frac{11}{35}+\delta}\right),$$

where the implied constant depends at most on δ.

Proof. As in the proof of Lemma 2.7, we see that the sum in our lemma is

$$\sum_{d\leqslant u}\delta(d)\mu(d)^2 \sum_{\substack{1\leqslant m'\leqslant u/d\\ \gcd(m',d)=1}} \mu(m')^2,$$

which, by Corollary 4.2, is

$$\sum_{d\leqslant u}\delta(d)\mu(d)^2\left(\frac{u}{d\zeta(2)}\left(\prod_{p\mid d}\left(1+\frac{1}{p}\right)^{-1}\right) + O_\delta\left(\tau(d)^3\frac{u^{\frac{11}{35}+\delta}}{d^{\frac{11}{35}+\delta}}\right)\right).$$

The main term above matches the main term in our lemma up to a quantity that has modulus

$$\ll u\sum_{d>u}\frac{\delta(d)\mu(d)^2}{d} \ll u\sum_{d>u}\tau(d)^2 d^{-3} \ll 1.$$

The error term contribution is

$$\ll_\delta u^{\frac{11}{35}+\delta} \sum_{d \leqslant u} \frac{\psi(d)\mu(d)^2\tau(d)^3}{d^{\frac{11}{35}+\delta}} \ll_\delta u^{\frac{11}{35}+\delta} \prod_p \left(1+\frac{32}{p^2}\right) \ll_\delta u^{\frac{11}{35}+\delta}. \qquad \square$$

The proof of the next lemma follows directly from Lemma 4.3.

LEMMA 4.4. *Assume the Riemann Hypothesis and let $\delta > 0$ be arbitrary and fixed. Then for all $k > 1$ and $u \geqslant 1$, we have*

$$\sum_{1 \leqslant t \leqslant u} |\mu(t)| \prod_{p \mid t} \frac{1}{1-2p^{-k}} = \gamma_k u + O_{k,\delta}(u^{\frac{11}{35}+\delta}),$$

where the implied constant depends at most on δ and k.

The proof of Theorem 1.2 is now concluded as that of Theorem 1.1 by replacing the use of Proposition 2.4 by Lemma 4.4.

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