Characterising Computational Thinking in Mathematics Education:

A literature-informed Delphi study

Anonymised for review

Abstract

Recently, computational thinking has attracted much research attention, especially within primary and secondary education settings. However, incorporating computational thinking (CT) in mathematics or other disciplines is not a straightforward process and introduces many challenges concerning the way disciplines are organised and taught in school. The aim of this paper is to identify what characterises CT in mathematics education and which CT aspects can be addressed within mathematics education. First, we present a systematic literature review that identifies characteristics of computational thinking that have been explored in mathematics education research. Second, we present the results of a Delphi study conducted to capture the collective opinion of 25 experts in both the fields of mathematics education and computer science regarding the opportunities for addressing computational thinking in mathematics education. The results of the Delphi study, which corroborate the findings of the literature review, highlight three important aspects of computational thinking to be addressed in mathematics education: problem solving, cognitive processes, and transposition.

Keywords: computational thinking, Delphi study, mathematics education, mathematical thinking

1. Introduction

The focus on developing higher-order thinking skills in education has been emphasised in schools for many years now, primarily as a means of advancing educational standards and preparing students for lifetime learning by promoting and setting their intellectual and cognitive growth at the centre of the educational process (Fisher, 1999). Even in the first half of the 20th century, Dewey already underlined thinking as the goal of classroom instruction, and he specifically emphasised that the educational process should mirror scientific inquiry (Lipman, 2003,). In his book *How we think*, Dewey (1933) posits that the origin of thinking is uncertainty or doubt, confusion or bewilderment, and he encourages educators to engage students in how to think though framing hypotheses and testing these in practice. However, both the process of thinking and the skills involved in this process are not firmly defined or described. This, as Lipman (2014) states, is because a list with thinking skills "consists of nothing less than an inventory of the intellectual powers of [hu]mankind" (p. 83).

According to Paul (1990, p. 2), all disciplines, from mathematics and physics to sociology etc., are "modes of thought", and, as such, people "know mathematics" when they can "think mathematically" and "understand science" when they "think scientifically". Each discipline engenders a system of thinking which mirrors the epistemology of the discipline. For example, in mathematics, students engage with mathematical thinking, and in sciences, students additionally engage with scientific inquiry. The last decade, much attention has concentrated on a process of thinking named *computational thinking* (Wing, 2006). The reason for this attention refers mostly to the widely accepted view that computational thinking is not only applicable in computer science, but also across a diversity of disciplines, such as mathematics, science, and humanities (Barr & Stephenson, 2011). Wing (2006) specifically highlighted that “to reading, writing, and arithmetic, we should add computational thinking to every child’s analytical ability” (p. 33).

However, the incorporation of computational thinking into mathematics or any other discipline is not a smooth process; among other things, it necessitates a move in education towards what are called *thickly authentic* practices (Perez, 2018) that echo Dewey’s vision of seeing schools and classrooms replicating real-life scenarios in which students engage in activities in multiple social settings, and problem-solving within a community (Dewey, 1938, as cited in Williams, 2017). The question that arises then is how these compelling and engaging classrooms can be constructed when computational thinking is embedded in school subjects.

In this study, we focus on the integration of computational thinking in mathematics education and we consider computational thinking as a process of thinking “associated with but not limited to problem-solving” (Selby & Woollard, 2013, p.5) and applicable to all disciplines. The main goal of our research is to identify what characterises computational thinking in mathematics education and which aspects of computational thinking can be addressed in mathematics education. The research study endeavours to provide answers to the following main research question:

What characterises computational thinking in mathematics education and what aspects of computational thinking can be addressed in mathematics education?

In this paper, we explore what computational thinking is and how it can be addressed in mathematics education by employing a methodological approach that brings together research findings from current literature through a systematic literature review and the perspectives of experts in these fields through a Delphi study.

2. Conceptual Framework: Mathematical and Computational Thinking

Characterizing thinking processes within disciplines touches upon the so-called *epistemic frames* (Shaffer, 2005) of those disciplines. Indeed, mathematical thinking is considered the central epistemic frame of mathematics (Perez, 2018).

Both mathematical thinking and computational thinking have been characterized in several ways that often appear to fluctuate depending on one's perspectives about the nature of mathematics and computer science, respectively. A common aspect in many definitions, however, is the emphasis on the *contextualization* of the discipline; that is, the connection between real-world situations and mathematical and computational concepts. In this view, four categories of cognitive activities can be distinguished: (1) translating a situation into mathematical or computational model, drawing on, e.g., modeling, abstraction and pattern recognition; (2) reasoning and working within mathematics and computer science; and (3) translating the result back into the context, involving, e.g., generalization, and (4) verifying if this really solves the real-world problem adequately (evaluation). These activities are depicted in Figure 1. In mathematics education, this cycle is also referred to as the mathematical modelling cycle that begins with a real-life problem which is then described and solved using a mathematical model (e.g., Blum and Leiβ, 2007).

[figure 1 here]

In the following, we explore our understanding of mathematical thinking and computational thinking, respectively, and propose an integrated view combining mathematical and computational thinking.

2.1 Mathematical Thinking

For many years, there has been a shift from behaviourist to constructivist approaches to teaching and learning. Irrespectively of whether these two philosophies are considered as opposing views, behaviourism and constructivism have influenced the way mathematics is being taught in schools. The behaviourist approach is sometimes considered as a 'traditional' approach to teaching and learning. Kimble's (1961) view of learning "*as a relatively permanent change in behavioral potentiality occurs as a result of reinforced practice*" highlights the behaviourist focus on learning being manifested in changes in behaviour shaped by reinforcement through practice and a reward or punishment system (Kimble, 1961 cited in Lessani et al., 2016, p.166). The teacher is responsible for transferring skills and knowledge to students by focusing mostly on making them do something. Orton (2004, p.29) points out that "*exposition by the teacher followed by practise of skills and techniques is a feature which most people remember when they think of how they learned mathematics*". The constructivist approach, in contrast, discards the dominant view of the teacher as the ultimate source of knowledge and invites students to actively participate in lessons and thereby construct their new knowledge and understanding. Social constructivism, in particular, places a major role in the social interactions for the construction of knowledge, and thus, knowledge is socially constructed (Cobb, 1994). As such, in sociocultural organised classrooms, activities are connected to participation in culturally organised practices (Cobb, 1994, p.14) and the teacher is responsible for creating contexts where the students participate in social interactions and culturally organised activities for constructing their new knowledge.

Drawing from the significant amount of literature that positions mathematical learning in the social and constructivist gestalts that consider mathematical learning as an innately social and constructive activity, Schoenfield advocates developing a ‘mathematical view’ - *seeing the world through the lens of the mathematician* - as a central component of thinking mathematically, thus stressing the connection between the world and mathematical objects; i.e., activity (1) above. He argues that thinking mathematically involves the following elements: growing a mathematical point of view; appreciating the process of abstraction and mathematization; having an inclination and affinity to apply them; and being able to use tools for structuring understanding and mathematical sense-making. He posits that “core knowledge, problem-solving strategies, effective use of one’s resources, having a mathematical perspective, and engagement in mathematical practices” are central parts of thinking mathematically and places a particular emphasis on the social side of mathematics and on creating communities of practice for fostering mathematical thinking – “microcosms of mathematical practice,” as he called them(Schoenfeld, 1992, p. 335).

The social and cultural aspect of mathematical thinking is also recognized by Tall (1991), who postulates that mathematical thinking must be considered in the context of “human mental and cultural activity” (p. 6). As such, it is not search for an absolute and true way of thinking about mathematics but instead a search for various ways of thinking that are socially and culturally established, and in which different aspects are related to the specific contexts.

Burton (1984) emphasizes the role of cognitive activity (2) mentioned above. She stresses that mathematical thinking does not refer to mathematics as a subject, but it refers to mathematical operations, processes, and dynamics applicable to every content and, thus, it can generally be applied to any field. These processes, according to Mason, Burton and Stacey (1991), are the following: *specialising, conjecturing, generalising* and *convincing*. Mason and Johnston-Wilder (2004 as cited in Breen & O’Shea, 2010) give a detailed list of words that they perceive as processes and actions that mathematicians use when they confront mathematical problems*: "exemplifying, specialising, completing, deleting, correcting, comparing, sorting, organising, changing, varying, reversing, altering, generalising, conjecturing explaining, justifying, verifying, convincing,* and *refuting"* (p. 109). They argue that students could practice aspects of mathematical thinking if questions administered to them stem from these words (as cited in Breen & O'Shea, 2010).

In the view of Freudenthal (1973), mathematics is the human action of mathematically organising and structuring the world, a process known as *mathematizing*. In fact, Treffers (1978) recognises two forms of mathematization: horizontal and vertical mathematization. The former corresponds to activities (1) and (3) in Figure 1 above, and "leads from the world of life to the world of symbols. In the world of life, one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly; this is vertical mathematization" (Freudenthal, 1991, pp. 41–42). Vertical mathematizing corresponds to activity (2) and refers to delegating a real problem situation to mathematical analysis (Treffers, 1987) and includes activities like "experimenting, pattern snooping, classifying, conjecturing, organising, and identifying" (Rasmussen et al., 2005, p. 54). Vertical mathematization builds on horizontal activities, creates new mathematical realities and includes activities like reasoning about abstract structures, generalising and formalising (Rasmussen et al., 2005, p. 54-55). In other words, horizontal mathematization refers to the process of "translating contextual problems into mathematical problems" (Gravemeijer & Cobb, 2013, p. 90), while vertical mathematization refers to the process of "reorganising and constructing within the world of symbols" (Jupri & Drijvers, 2016, p. 2483). Mathematization, horizontal as well as vertical, is an essential activity in doing mathematics and in thinking mathematically.

When done consecutively, cognitive activities (1), (2) and (3) can be interpreted as problem solving steps. This problem-solving aspect is recognized by Drijvers (2015), who distinguished three core aspects in mathematical thinking: problem-solving, modelling and abstraction. The modelling aspect corresponds to horizontal mathematization, whereas Drijvers (2015) positions abstraction within vertical mathematization.

Central to mathematical thinking is the concept of a problem. The word ‘problem’, although widely used, has acquired meanings that sometimes are interpreted differently. A problem may indicate “a routine exercise for the practice and consolidation of newly learned mathematical techniques”or “tasks whose difficulty or complexity makes them genuinely problematic or non-routine” (Xenofontos & Andrews, 2014, p.2). A characteristic example is Webster’s (1979, p. 1434 cited in Schoenfeld, 2017, p.4) definition of a problem: a. “In mathematics, anything required to be done, or requiring the doing of something” b. “A question…that is perplexing or difficult”. The first definition highlights mathematical tasks as routine exercises used for practicing and acquiring skills, but has nothing to do with the notion of problems as defined in the second definition above. More aligned with Webster’s second definition is Lester’s (1980) view who postulates that “a problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution” (Lester, 1980, p. 287). Along the same lines, McLeod (1988) describes problems as “those tasks where the solution or goal is not immediately attainable and there is no obvious algorithm for the student to use” (p. 135). He argues that students’ preliminary reactions to the problem are that there is not an obvious solution.

2.1. Computational Thinking

Wing's vision of computational thinking as an attitude and skillset for everyone, and not only for computer scientists, has put computational thinking at the centre of educational researchers' attention. Subsequently, in the last decade, research on computational thinking has been growing in many directions, including the formation of a representative definition (e.g., CSTA & ISTE, 2011; Selby & Woollard, 2010; Cuny, Snyder &Wing, 2010), the development of frameworks that incorporate computational thinking in sciences and mathematics (e.g., Weintrop et al. 2016; Barrand & Stephenson, 2011), empirical studies concentrating on the effects on learning when computational thinking is embedded in different disciplines (Orton et al., 2016; Voskoglou, & Buckley 2012; Costa et al., 2017; Van Dyne & Braun, 2014), studies of teachers' perspectives, capacity and confidence (Yadav et al., 2014; Jenkin et al., 2012; Grgurina et al., 2014), the assessment of computational thinking skills (e.g., Roman-Gonzalez et al., 2017), as well as papers that adopt a more critical stance questioning its authenticity (e.g., Tedre & Denning, 2016).

Examining the variety of definitions attributed to computational thinking, Roman-Gonzalez et al. (2017, p. 679) grouped them into three broad categories: generic definitions (e.g., Wing, 2006), operational definitions (e.g., CSTA & ISTE, 2011), and educational and curricular definitions (e.g., Barefoot, 2014; Brennan & Resnick, 2012).

After reviewing papers since 2006, Selby and Woollard (2010, p.5), refer to computational thinking as "an activity, often product-oriented, associated with, but not limited to, problem-solving". Part of this process are capabilities such as abstracting, decomposing, algorithmic thinking, evaluating, and generalising, in which the cognitive activities (1), (2) and (3) mentioned earlier can be recognized, with an emphasis on the first (abstracting, decomposing) and third (evaluating, generalising) categories. Similarly, Kalelioglu et al. (2016), having conducted a systematic literature review, developed a framework for computational thinking that includes the following components:

a. identifying the problem: abstraction, decomposition
b. gathering, representing and analysing data: data collection, analysis, pattern recognition, conceptualising, data representation
c. generating, selecting and planning solutions: mathematical reasoning, building algorithms and procedures, parallelisation
d. implementing solutions: Automation, modelling and simulations
e. assessing solutions and continue for improvement: testing, debugging and generalisation

A characteristic example elaborating on various 'technical' computational skills and reasoning (activity (2) is the definition proposed by the Computer Science Teachers Association and the International Society for Technology in Education (CSTA & ISTE, 2011) which refers to computational thinking as a problem-solving process with the following characteristics:

a. formulating problems in a way that enables us to use a computer and other tools to help solve them
b. logically organizing and analysing data
c. representing data through abstractions such as models and simulations
d. automating solutions through algorithmic thinking (a series of ordered steps)
e. identifying, analysing, and implementing possible solutions with the goal of achieving the most efficient and effective combination of steps and resources

f. generalizing and transferring this problem-solving process to a wide variety of problems. (p. 1)

Brennan and Resnick (2012) regard computational concepts, practices and perspectives as part of computational thinking. They highlight the computational thinkers' need not only for consuming but for *expressing* and implementing ideas, *connecting* by creating, sharing and learning in social learning environments, and *questioning* as they try to understand the world. The ISTE and CSTA (2011) definition also emphasises computational thinking attitudes, which include confidence in dealing with complexity, persistence in working with difficult problems, tolerance for ambiguity, the competence to deal with open-ended problems, and the capability to communicate and work with others to achieve a common goal or solution (p. 1).

From this perspective, the learner's dispositions or attitudes are highlighted as central to their engagement in computational thinking practices. Kafai (2016) also recognises the social aspect of computational thinking and calls for a change from computational thinking to computational participation. She argues that both computational thinking and programming are social practices and as such, should be practised within a community of collaboration and sharing. In line with this, computational thinking can effectively create interdisciplinary connections and support students' participation in different communities of practice.

Although research in the area of computational thinking is growing, there are those who adopt a more sceptical or modest stance regarding computational thinking and the idea behind the term (Denning 2009; Tedre & Denning, 2016; Hemmendinger, 2010). Indeed, long before the introduction of digital tools, computation was a fundamental part of mathematics and other scientific disciplines, manifested in computations performed by humans. Denning (2010) explores the definitions of computation since the early 1930s. He points out that originally “computation meant the mechanical steps followed to evaluate mathematical functions while computers were people who did calculations” (Denning, 2010, p.3). Law (2011) provides a comprehensive discussion on the definition of the word (human) computation. She points out that human computation is “computation carried out by humans and human computation systems can be defined as intelligent systems that explicitly organizes human efforts to carry out the process of computation” (Law, 2011, p.2). In the mid and late 1980s, use of the word computation tightened with digital computers as its main tool. However, Denning (2010) highlights that computation did not only reflect an activity of machines but a new way of thinking; the computer is not only regarded as a set of tools but “it plays a role in communicating the actions, sharing and re-negotiating mathematical expression and facilitating the (co-)construction of mathematical meanings” (Noss & Hoyles, 1996, p. 228).

Tedre and Denning (2016) provide a comprehensive article that demonstrates the historical development of CT and the intellectual ideas that guided its development, while Hemmendinger (2010) questions the exclusiveness of the reported aspects of computational thinking and purports that most of them are typical parts of problem-solving in different disciplines. Revisiting research in computer programming, one cannot fail to notice the connection between computational thinking and the considerable attention that was given to the cognitive aspects of computer programming from the early ‘80s. An illustrative example is Casey’s (1997) work seeing computer programming, and the skills involved, as a means of practising problem-solving and criticising the practice of problem-solving being housed only in the mathematics curriculum. Along the same lines, Jansson et al. (1987) advocate the cognitive benefits of practising programming and the cognitive functions evident in this practice and Linn (1985) sees computer programming as ideal for encouraging problem-solving. In her paper, she addresses the question of how links between problem-solving in programming and problem-solving in other disciplines can be formulated.

2.2. Combining Mathematical and Computational Thinking

Computational thinking is characterized by aspects that are central to computer science but also to other scientific disciplines like mathematics. Considering both computational and mathematical thinking, it is evident that both approach thinking by employing concepts of cognition, metacognition, and dispositions central to problem solving. Moreover, both recognise and promote social-cultural learning opportunities that mould ways of thinking and practising that reflect those of the real world. However, the growing speed with which computational advancements inundate and reshape our society, positions computational thinking at the centre of disciplinary practices.

Considering that computational thinking can provide new opportunities for designing disciplinary content and context that facilitate learners to explore ideas and ways of thinking instilled in the corresponding disciplines, the question that arises is how computational thinking can be embedded in mathematics education to acquaint learners with the way mathematics is practised in the real world, to enhance learning of mathematics content (Weintrop et al., 2016), and to build students' capacity to attain and apply knowledge to new situations. Bower and Falkner (2015) highlight the need for educational systems to provide computational thinking opportunities that enhance students' understanding of and experiences with computational practices apparent in different scientific fields.

Therefore, computational thinking could extend the processes central to mathematics by restructuring both how problems are formulated and how they are solved. In terms of contextualization, this point of view puts mathematics in the position of context for computational thinking. In this way, both mathematical objects (resulting from horizontal mathematization) and mathematical activities (in the process of vertical mathematization) can be starting points for computational thinking.

The relationship between computational thinking and mathematics has been previously examined by researchers who mostly examine the interplay between computational and mathematical thinking in a mathematics context. Weintrop et al. (2016) presented a taxonomy of computational thinking in mathematics and science that includes practices in data, modelling and simulation, computational problem-solving, and systems thinking. Along the same lines, to support the integration of computational thinking in multiple disciplines, Barr and Stephenson (2011) proposed a structured model that includes core CT skills and examples of how they can be incorporated in different school subjects. Costa et al. (2017) listed specific guidelines for effectively incorporating computational thinking in mathematics questions, while Perez (2018) presents a framework that facilitates the incorporation of computational thinking in mathematics learning based on computational thinking dispositions.

Kotsoupoulos et al. (2017) also developed a pedagogical framework for computational thinking that includes four pedagogical experiences: unplugged, tinkering, making and remixing. Considering the relationship between computational and mathematical thinking, Sneider et al. (2014) highlight this association in view of capabilities related to mathematical thinking, capabilities related to computational thinking and capabilities related to both. Barcelos and Silvera (2012) identified three groups of skills that can be developed when both mathematical and computational thinking are considered. The first one is mathematical representations and their semiotic relationship to algorithms, the second refers to establishing relationships and identifying pattern regularities, and the final skill identified is the descriptive and representative models, which refers to defining and interpreting mathematical models to analyse and explain situations.

[figure 2 here]

We suggest that the relationship between computational and mathematical thinking can be studied in two ways: first, by comparing them as contextualizing activities as in Figure 1, and second, by investigating their interplay in a mathematical context as in Figure 2. In our study, we employ the latter view to explore the characteristics of learning opportunities that consider computational thinking in mathematics education.

3. Methods

For the research methods of our study we adopted a synthesis of a systematic literature review (Kitchenham, 2004) and a Delphi study (Vernon, 2009). The starting point of this exploration was a systematic literature review that aimed to identify core aspects of computational thinking that are addressed in mathematics education. Once the literature review was complete, the next step was to conduct a Delphi study. The main reason for employing this technique was because we deemed it critical to reach consensus among experts as to what characterises computational thinking in the mathematics classroom and what aspects of computational thinking can be incorporated into mathematics education. In the following subsections, we describe in detail these two methodological approaches and the way we employed them to carry out our research.

3.1Systematic literature review

As mentioned above, the first step of our investigation was to conduct a systematic literature review regarding computational thinking in mathematics education. The purpose of the systematic literature review was to investigate the opportunities for addressing computational thinking in mathematics education, inform the Delphi questions correspondingly and compare the findings from the two approaches.

For conducting and reporting the systematic literature review, we followed the guidelines suggested by Kitchenham (2004)of three broad steps: planning the review, conducting the review and reporting the review.

3.1.1 Planning the review

We focused on finding research papers from six online repositories: ACM, IEEE, Web of Sciences, ERIC, Scopus, and PsycINFO. The search term used for searching the repositories was the following: (‘computational thinking’ and ‘mathematics’) in title, abstract and keywords.

To evaluate the relevance of each research paper returned from the search, we set inclusion and exclusion criteria. Specifically, the inclusion criteria we determined were the following:

1. the study empirically investigates computational thinking in the mathematics curriculum or
2. the study should consider, describe and discuss how computational thinking can be intertwined with mathematics education

Correspondingly, the criteria for excluding papers retrieved were the following:

1. the study considers how computational thinking can be embedded in sciences other than mathematics
2. the study investigates programming in the mathematics curriculum without considering computational thinking skills as the focus of the investigation
3. the study considers the impact of digital tools in learning mathematics without mentioning computational thinking.

3.1.2 Conducting the review

The study was performed in March 2019, and a total of 496 documents were retrieved and then classified as being relevant to the research purpose or not. Two researchers were involved in this process, and for each retrieved paper, they both indicated whether it should be "included", "excluded", or whether it was "unclear". The inter-rater reliability on the selection of studies was substantial with Kappa=.77 and percentage of agreement 94.98%. Disagreement between the researchers was resolved by a discussion. The process followed is depicted in the following figure, and the statistics for this analysis are presented in Table 1.

[figure 3 here]

[table 1 here]

In total, 56 papers were found to match the inclusion criteria defined above. To identify the computational thinking aspects evident in each paper, the researchers read the papers carefully, and for each paper, they each listed each of the aspects that were mentioned by the authors as being practised considering a mathematics setting. The two researchers worked individually, and the inter-rater reliability was calculated again for each aspect;see Table 2. Any disagreement between the two researchers was again resolved with an extensive discussion.

[table 2 here]

3.2 The Delphi study

Having completed the literature review, we moved on to explore the topic under study empirically. We employed the Delphi method (Vernon, 2009) with the aim to investigate experts’ perspectives on the characteristics of learning opportunities that considers computational thinking in mathematics education and the aspects of computational thinking that can be addressed in mathematics education.

The Delphi method is a consensus technique and, thus, it is particularly useful when the researchers seek to reach expert agreement on a topic with insufficient evidence. Considering that there is limited research regarding our research questions, the Delphi method and, thus, the experts' collective opinion were deemed appropriate.

When the Delphi method is considered, a panel of experts is formed by the researchers to exchange and suggest their opinions on a specific question; in our case by email. A vital characteristic of this method is anonymity, which indicates that the participants keep their identities hidden, and none of the participants is aware of who is taking part in the research or the participants' capacity. Equally important for this method is the feedback that the participants receive after each round and, therefore, it is the researcher's role to generate an accurate reflection of the participants' ideas without favouring some over others and without revealing the identity of the participants. Finally, a Delphi study is iterative, which means that in each round, the participants are asked the same questions having first considered the feedback of the previous round. This iterative process is repeated until consensus is reached.

In the literature, consensus within Delphi studies is not well defined, and researchers often employ their own criteria. In this study, we adopted Giannarou and Zerva’s (2014) criteria:

1. The percentage of those who selected “agree” and “strongly agree” and correspondingly “disagree” and “strongly disagree” should be more than 51%. We consider this down limit to be low for our sample of 25 participants, and therefore, we increase this limit to 70%, which has been used in other studies as well (Green, 1982; Vernon, 2009).

2. The standard deviation should not exceed 1.5
3. The interquartile range should not exceed 1

Apart from determining consensus, it is important that stability is reached in the participants' responses before terminating the Delphi study (Dajani et al., 1979). This suggests that the researchers should explore whether or not there is a statistically significant difference in the way the participants replied between two successive rounds. To this end, Seagle and Iverson (2002) suggest the use of the Wilcoxon signed ranks-test, which we employed to determine if the Delphi study should be terminated or not.

3.2.1 Participants

In a Delphi study, it is important that the panel is formed by people that are knowledgeable about the topic as the Delphi's outcome is based on their knowledge capacity and experience (Habibi et al., 2014).

To this end, the participants were selected based on their experience with computational thinking and mathematical thinking. An invitation was sent by email to mathematics and computer science teachers that are part of our research project as well as to researchers, lecturers, and education officers who had already experience with computational and mathematical thinking research projects. We employed expert sampling because we were interested in the consent of people that are experts in the area of the investigation (Etikan et al., 2015). Table 3 depicts in detail the participants' characteristics in the Delphi study. In total, 25 teachers, academics and education officers took part in the Delphi study.

[table 3 here]

3.2.2 Data Collection and analysis

In total, the Delphi study spanned for two months, and three rounds were needed before consensus and stability were reached. The study started on 5 April 2019 and lasted until the end of May.

The participants were individually invited to take part in the study, which secured that anonymity was kept. The invitations included some information about our project and asked the participants to participate in our research by answering our questionnaire in an online platform. In each round, the participants were given two weeks in total to respond to the questionnaire.

In the first round, the questionnaire comprised two sections. The first section included questions regarding the educational and professional background of the participants. The second section consisted of four open-ended questions (in this paper we report the findings that are relevant to two of these) and one semi-open question that included a five-point Likert-scale option and a "comment" field for the participants to include their own opinion. Specifically, the questions included in the questionnaire and relevant to this paper were the following:

1. What characterises computational thinking in mathematics education?
2. What are the common aspects of computational and mathematical thinking?
3. Which aspects of computational thinking can be addressed in mathematics instruction?

The semi-open question was the third one in the list above. The options given to the participants were generated by our systematic literature review. However, since the aspects that are involved in computational thinking are still negotiable, we provided the participants with the option to add aspects in the comment field.

As soon as we collected all the responses, a qualitative analysis was performed by two researchers. The analysis aimed to generate a list of arguments for each question based on the participants' responses. To this end, one of the researchers generated the argument list (themes of arguments) for each question by combining the participants' responses that addressed the same opinion. The second researcher then reviewed the argument list for each question and identified the participant or participants' responses that fit in the corresponding theme. The correspondence was absolute for all the questions and, thus, the reliability of the analysis was insured. Apart from that, this procedure was necessary to secure that all the participants' opinions would be reflected in the list.

In the second round, we administered the questionnaire with the same questions as the first round, but this time, the participants were provided with the theme-arguments produced by their suggestions in the previous round. The participants were asked to indicate their level of agreement with each argument. In each question, we also included a "comment" field for the participants to express an additional opinion. The participants were again given two weeks to respond, but some of them requested more time and, thus, there was some delay in collecting all the responses. Having collected all the participants' responses, we proceeded with a quantitative analysis of the data collected. For each item (argument) in a question, the following statistical information was calculated: Mean, Median, Standard deviation, Interquartile range, percentage of agreeing and strongly agreeing as well as the percentage of disagreeing and strongly disagreeing. This statistical data was used as feedback for the participants to commence the third round of the Delphi study.

In the third and final round, the participants were given the same questionnaire as in the second round and were asked to review the feedback and reconsider their opinion by retaking the questionnaire only if they would like to change their previous level of agreement. Having collected all the responses of the third round, a statistical analysis of the data was conducted again calculating the same metrics as in the second round. Additionally, to determine whether the Delphi study should terminate, we employed the Wilcoxon signed ranks test, to examine if stability was reached for all the items provided in the questionnaire.

In the third round, consensus was achieved for some arguments in each question as well as stability for all the arguments. Therefore, the Delphi study was successfully terminated after the third round.

4. Results

We report the results in two sub-sections. The first sub-section reports the results of the systematic literature review, and the second reports the results of the Delphi study.

4.1 Literature review results

4.1.1 Aspects of computational thinking most frequently examined in mathematics

Two researchers were involved in investigating the aspects of computational thinking that were most frequently addressed in papers that explore CT in mathematics courses. Figure 4 depicts the aspects that are most frequently addressed in empirical research as well as those that are theoretically considered.

From Figure 4 it is evident that there are similarities and differences between the practice of computational thinking in mathematics education and the theoretical frameworks that have been developed. In the leading positions in both the theory and the practice are the aspects of automation, abstraction, algorithmic thinking and modelling, which highlights the significance of these aspects in both computational thinking and mathematics education. Many empirical papers also highlight visualisation and decomposition, whereas in the theoretical papers visualisation is rarely mentioned.

With respect to visualisation, there is a difference in the context in which the aspect is mentioned between the theoretical and the empirical papers. From a theoretical perspective, visualisation is acknowledged by Weintrop et al. (2016) as part of scientific practice in the STEM fields, i.e., in using simulations and communicating results. Niemela et al. (2017) mention visualisation as a part of abstraction. In the empirical papers, visualisation is viewed in the light of classroom practice, occurring in many different ways; for example in using a spreadsheet tool such as Excel (e.g., Sanford & Naidu, 2016), in geometry software (e.g., Pei et al., 2018), in visual programming tools such as Scratch (e.g., Gadanidis et al., 2018; Grover & Pea, 2013), in visualizing data with programming languages such as R or Python (e.g., Benakli et al., 2017; Landau et al., 2013), and in graphical representations of phenomena (e.g., Perez, 2016; Shodiev, 2015).

The aspect 'pattern recognition' also shows a difference in perspective between the theoretical papers (mentioned in four papers) on the one hand and the empirical ones (mentioned in 10 papers) on the other hand. In the theoretical papers, pattern recognition is considered a thinking skill; for example, as part of logical reasoning (Djurdjevic-Pahl et al., 2017). In the empirical papers, patterns are solely mentioned as output of the tools used; for example, a data analysis tool such as Excel (Pei et al., 2018), in geometry software (Benton et al., 2017; Hsi et al., 2012; Pei et al., 2018), or a programming tool such as Scratch (Gadanidis et al., 2017) or Logo (Kynigos & Grizioti, 2018).

Another interesting difference refers to testing/debugging and data practices (collection, analysis and representation), which in empirical papers are not frequently highlighted (only five papers considering these as part of their design).

4.2 Delphi study

This section presents the results of the Delphi study. The first section presents and discuss the results of the first question of the Delphi study, the second section presents and discuss the results of the second question, and the last one presents and discuss the results of the third round.

4.2.1 Delphi study: Question 1

The first question of the Delphi study was an open-ended question. In this question, the participants were asked to indicate what characterises computational thinking in mathematics education. Table 4 presents the arguments generated from the first Dephi question along with the percentage of participants that address each argument.

[table 4 here]

Table 5presents the most enriched and representative examples of the participants' responses and the arguments (Table 4) in which these were coded.

[table 5 here]

In this first question, most of the participants highlighted problem-solving as the key process of such an environment (Table 5). However, not all participants' responses centred on problem-solving with the use of a digital tool. For instance, one of the teachers mentioned as a characterising aspect: "*Coming up with an extensive step-by-step plan to arrive at a correct answer to a problem,*" thereby emphasising the planning and finding a solution phase rather than the "tool" for implementing the solution. A more enriched account stems from the comments of another participant:

> *The solving of complex problems by following a suite of processes could characterize a basic form of computational thinking in mathematics education. This understand-*

ing of computational thinking could be enhanced by increasingly introducing "Pseudocode" or algorithmic illustrations to solving types of problems. The role that the understanding of these processes takes in a course characterizes the computational thinking aspect of the mathematical education.

Others, however, placed an equal emphasis on the planning and solution phase by also highlighting the implementation tool: "*Analyzing a mathematical problem in order to work out (part of) the solution with the help of a computer program*."

The above observations raise questions regarding the role and necessity of a digital tool for computational thinking to be practised within the disciplines and whether 'unplugged' activities could also be an effective way in mathematics education. In fact, the unplugged approach has been explored in the literature as an alternative or supplementary to practising computational thinking through digital tools. For instance, Caeli and Yadav (2019) discuss how computational thinking is rooted in unplugged approaches to problem-solving, while Curzon et al. (2014) advocate the use of unplugged activities both for introducing students to computing concepts and for teachers.

In the second and third round, the participants were asked to indicate the levels of agreement with each of the aforementioned arguments so only the results of the third (and final) round are presented in Table 6.

[table 6 here]

4.2.2 Delphi Question 2

The second question of the Delphi study was also an open-ended question. In this question, the participants were asked to indicate what aspects do computational thinking and mathematical thinking have in common. Table 7 presents the arguments generated from the second Dephi question along with the percentage of participants that address each argument.

[table 7 here]

Table 8presents some of the most enriched and representative examples of the participants responses and the arguments (Table 8) in which they were coded.

[table 8 here]

In this question, we noticed that the participants provided more detailed answers which led to the inclusion of additional aspects and particularly that of generalisation, analytical thinking, and evaluation (Table 8). These aspects were not part of the responses of the first Delphi question, suggesting that there could be a distinction between critical characteristics that must be present in mathematics classrooms that considers computational thinking and others that may be less critical. As in the previous question, most of the participants focused on problem-solving. For example, the following response highlight problem-solving as the link between computational thinking and mathematical thinking:

> *There is a similarity between the problem-solving strategies we use in mathematics education and the process of computational thinking. We do also use many skills in mathematical thinking such as abstraction, decomposition, data collection, data analysis, pattern recognition and debugging*.

Considering that problem-solving is an important part of both computational thinking and mathematical thinking, it is not surprising that our participants identified problem-solving and thinking processes involved in this as the common ground between mathematical and computational thinking.

In the second and third round, the participants were asked to indicate the levels of agreement with each of the aforementioned arguments. Table 9 presents the arguments that reached consensus and stability in the third round.

[table 9 here]

4.2.3 Delphi Question 3

In the third question of the Delphi study, "*Which aspects of computational thinking can be addressed in mathematics courses*?", the arguments were already provided to the participants stemming from the literature review. However, the participants could also suggest their own perspectives. Table 10 presents the arguments that reached consensus and stability in the third round.

[table 10 here]

As it can be seen from Table 10, the aspects of abstraction, algorithmic thinking, decomposition, modelling and evaluation congregate the highest percentage of agreement. In this question, we also noticed the following differences between the two groups. When the groups are considered separately, one more item reaches consensus for the teachers: "*Visualisation*", which has not reached consensus when the whole and the academics group are considered. Additionally, the item "*Data Analysis*" reached consensus in the whole and the teachers' group, and "*Decomposition*" and "*Generalisation*" reached consensus in the whole and the academics group.

5. Discussion

The aim of this investigation was to identify what characterises computational thinking in mathematics education and which aspects of computational thinking can be embedded in mathematics education. To this end, we first conducted a literature review to identify aspects of computational thinking that have already been explored in mathematics education. The results of the literature study suggest that most of the empirical papers concentrate on the following aspects: *automation, abstraction, modelling, algorithmic thinking, visualisation, decomposition, and pattern recognition*. At the same time, aspects referring to *data analysis, testing, debugging, data collection, data representation, generalisation, evaluation, and tinkering* are not often explored when mathematics settings are considered. Interestingly, these findings are also corroborated by our Delphi study findings.

Specifically, in the second phase of our research, we conducted a three-round Delphi study with 25 mathematics and computer science education experts, including teachers, academics and educational officers. The results of the Delphi study highlight that learning opportunities that consider computational thinking in *mathematics education* are characterised by:

A structured problem-solving approach in which one is able to solve and/or transfer the solution of a mathematical problem to other people or a machine by employing thinking processes that include abstraction, decomposition, pattern recognition, algorithmic thinking, modelling, logical and analytical thinking, generalisation and evaluation of solutions and strategies.

The aforementioned characterisation highlights three points:

1. problem-solving as a fundamental goal of mathematics education in which computational thinking is embedded;

2. thinking processes that include (but not limited to) *abstraction, decomposition, pattern recognition, algorithmic thinking, modelling, logical* and *analytical thinking, generalisation* and *evaluation* of solutions and strategies;

3. Phrasing the solution of a mathematical problem in such a way that it can be transferred / outsourced to another person or a machine (transposition).

The description proposed here can be extended by considering the participants' responses in the third Delphi question which enrich the previous findings by suggesting two further computational thinking aspects that can be part of mathematics education. These aspects refer to data practices and specifically, *data analysis* and *data representation*.

Comparing the above description with the definitions suggested thus far in the literature, we notice that, in essence, this characterisation is in line with most of the definitions that consider computational thinking as a thinking process. For instance, the characteristics suggested above fall under Cuny, Snyder and Wing's (2010) suggested definition: "*Computational Thinking is the thought processes involved in formulating problems and their solutions so that the solutions are represented in a form that can be effectively carried out by an information-processing agent"* (Cuny, Snyder, & Wing, 2010). It is also evident that most of the aspects mentioned above have also been reported in the literature as part of computational thinking. For example, most of the aspects apparent in Selby and Woollard's (2013) definition, are also part of this characterisation. Additionally, the aspects that were reported by our participants as part of the first Delphi question are in line with the aspects most frequently examined in empirical papers (computational thinking in mathematics education), as reported in the literature review. This observation leads us to the suggestion that maybe some aspects of computational thinking are more critical than others and learning opportunities that consider computational thinking should provide opportunities for students to practice as many aspects as possible.

One difference with the frameworks suggested so far in the literature is the aspects of logical and analytical thinking which are not usually part of theoretical or empirical papers. However, our experts considered them important aspects of computational thinking in mathematics contexts. Along the same lines, Grover and Pea (2013) include logic and logical thinking as a computational thinking concept and the Barefoot website (2014) highlights six concepts, among them logic (predicting and analysing). In contrast, Selby and Woollard (2013) excluded logical thinking from their list as they regarded it as a broad term and not well defined. Another difference refers to the testing and debugging aspects. Our participants did not consider testing and debugging as necessary aspects when computational thinking is considered in mathematics education, which contradicts some existing frameworks (e.g., Weintrop et al., 2016; Kalelioglu et al., 2016).

Our participants have not suggested any dispositions or attitudes involved when computational thinking is considered in mathematics contents. Therefore, dispositions like the ones mentioned in the CSTA & ISTA definition of computational thinking (e.g., confidence in dealing with complexity) and attitudes such as the ones mentioned by Brennan and Resnick (2012) were not part of our participants' responses. This is probably because our participants focused more on aspects that are frequently referred to in studies with an emphasis on thinking processes rather than on students' attitudes. Nevertheless, this finding highlights that future research surveys should formulate questions in a way that directs participants to consider these aspects as well.

6. Conclusion

In answer to the research question, the results of the Delphi study align with and extend the results of the literature review. The participants in the Delphi study also agreed that learning opportunities that consider computational thinking in mathematics education should highlight the following aspects: abstraction, decomposition, pattern recognition, algorithmic thinking, modelling, logical thinking and automation, followed by analytical thinking, generalisation and evaluation of solutions and strategies. Our investigation also revealed that problem-solving and therefore, the thinking processes involved in problem-solving, is regarded as the common ground between computational and mathematical thinking.

Consequently, in this paper, we identified three aspects for considering computational thinking in mathematics education, *problem solving, cognitive processes, and transposition*. These characteristics are in line with the definitions of computational thinking suggested so far and at the same time reflect a wide range of aspects. We argue that computational thinking should be considered as an "umbrella" concept that is adaptable and flexible to alterations depending on the context in which it is applied as well as on the current socio-economic trends and needs emerging from society. Computational thinking can therefore be employed in a variety of ways that reflect authentic disciplinary contexts in which students connect learning and doing inside communities of practice.

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