Coupled electro-elastic deformation and instabilities of a toroidal membrane Supplementary material: Appendix

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Appendix A. Pressure term in the variational formulation

The pressure term in the total potential energy functional

$$\int_{V_0}^{V_0+\Delta V} \widetilde{P}\mathrm{d}V \tag{A.1}$$

should be such that upon taking its first variation, the virtual work (δV) obtained is of the form [63]

$$\delta V = \int_{\Gamma} \left[\widetilde{P}\mathbf{n}\mathrm{d}s_0\right] \cdot \delta\mathbf{x} \tag{A.2}$$

where Γ is the domain comprising the deformed mid-surface of the membrane and $\delta\mathbf{x}$ is a virtual displacement of a point on the mid-surface. The normal vector is given by

$$\mathbf{n} = \frac{1}{\sqrt{g}}\left[\widetilde{\varrho}\,\widetilde{\eta}_\theta \cos\phi\mathbf{E}_1 + \widetilde{\varrho}\,\widetilde{\eta}_\theta \sin\phi\mathbf{E}_2 - \widetilde{\varrho}\,\widetilde{\varrho}_\theta\,\mathbf{E}_3\right] \quad \text{where} \quad \sqrt{g} = \widetilde{\varrho}\left[\widetilde{\varrho}_\theta^2 + \widetilde{\eta}_\theta^2\right]^{1/2}. \tag{A.3}$$

From equations (3) and (A.3), one obtains

$$\delta V = \int_0^{2\pi}\int_0^{2\pi} \widetilde{P}\left[\widetilde{\varrho}\,\widetilde{\eta}_\theta\,\delta\,\widetilde{\varrho} - \widetilde{\varrho}\,\widetilde{\varrho}_\theta\,\delta\,\widetilde{\eta}\right]\mathrm{d}\theta\,\mathrm{d}\phi. \tag{A.4}$$

It can be shown that this is the first variation of the functional following

$$\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \widetilde{P}\,\widetilde{\varrho}^2\,\widetilde{\eta}_\theta\;\mathrm{d}\theta\,\mathrm{d}\phi, \tag{A.5}$$

after using the condition $\delta\eta|_{\theta=0} = \delta\eta|_{\theta=2\pi}$.

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Further note that total volume of the deformed torus is given by

$$V(\widetilde{\varrho},\widetilde{\eta}) = 2\int_{\widetilde{\varrho}(\pi)}^{\widetilde{\varrho}(0)} 2\pi\,\widetilde{\varrho}\,\widetilde{\eta}\,\mathrm{d}\,\widetilde{\varrho} = -4\pi\int_{0}^{\pi}\widetilde{\varrho}\,\widetilde{\varrho}_{\theta}\,\widetilde{\eta}\,\mathrm{d}\theta. \tag{A.6}$$

In the domain $\theta \in [0, 2\pi]$, it can be shown that the function $\widetilde{\varrho}$ is even while $\widetilde{\varrho}_{\theta}$ and $\widetilde{\eta}$ are odd with respect to the point $\theta = \pi$. Thus the product $\widetilde{\varrho}\,\widetilde{\varrho}_{\theta}\,\widetilde{\eta}$ is an even function and the above integrals can be written as

$$V(\widetilde{\varrho},\widetilde{\eta}) = -2\pi\int_{0}^{2\pi}\widetilde{\varrho}\,\widetilde{\varrho}_{\theta}\,\widetilde{\eta}\,\mathrm{d}\theta = -\int_{0}^{2\pi}\int_{0}^{2\pi}\widetilde{\varrho}\,\widetilde{\varrho}_{\theta}\,\widetilde{\eta}\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{A.7}$$

Upon using the identity

$$\int_{0}^{2\pi}\left[\widetilde{\varrho}^{2}\,\widetilde{\eta}\right]_{\theta}\mathrm{d}\theta = 0 \quad \Rightarrow \int_{0}^{2\pi}\widetilde{\varrho}^{2}\,\widetilde{\eta}_{\theta}\,\mathrm{d}\theta = -2\int_{0}^{2\pi}\widetilde{\varrho}\,\widetilde{\varrho}_{\theta}\,\widetilde{\eta}\,\mathrm{d}\theta, \tag{A.8}$$

we can write the above volume integral as

$$V(\widetilde{\varrho},\widetilde{\eta}) = \frac{1}{2}\int_{0}^{2\pi}\int_{0}^{2\pi}\widetilde{\varrho}^{2}\,\widetilde{\eta}_{\theta}\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{A.9}$$

If the torus deforms from the reference configuration with volume V_0 to the current configuration with volume V at constant pressure $\widetilde{P}$ then the total work done can be written as

$$\frac{1}{2}\int_{0}^{2\pi}\int_{0}^{2\pi}\widetilde{P}\,\widetilde{\varrho}^{2}\,\widetilde{\eta}_{\theta}\,\mathrm{d}\theta\,\mathrm{d}\phi - \widetilde{P}V_0, \tag{A.10}$$

which is the same expression as (A.5) barring the constant term.

Appendix B. Derivatives of the energy density function

The first derivatives of the right Cauchy–Green deformation tensor $\boldsymbol{C}$ as expressed in Equation (8) with respect to ϱ, ϱ_{θ}, η and η_{θ} are given by

$$\begin{aligned}
\boldsymbol{C}_{\varrho} &= \mathrm{diag}\left(0, \frac{2\lambda_2}{1+\gamma\cos\theta}, \frac{-2}{\lambda_1^2\lambda_2^3[1+\gamma\cos\theta]}\right),\\
\boldsymbol{C}_{\varrho_\theta} &= \mathrm{diag}\left(\frac{2\,\varrho_{\theta}}{\gamma^2}, 0, \frac{-2\,\varrho_{\theta}}{\lambda_1^4\lambda_2^2\gamma^2}\right),\\
\boldsymbol{C}_{\eta} &= \boldsymbol{0},\\
\boldsymbol{C}_{\eta_\theta} &= \mathrm{diag}\left(\frac{2\eta_{\theta}}{\gamma^2}, 0, \frac{-2\eta_{\theta}}{\lambda_1^4\lambda_2^2\gamma^2}\right).
\end{aligned} \tag{B.1}$$

Then non-zero second derivatives of $\boldsymbol{C}$ are

$$
\begin{aligned}
\boldsymbol{C}_{\varrho\,\varrho} &= \mathrm{diag}\left(0, \frac{2}{[1+\gamma\cos\theta]^2}, \frac{6}{\lambda_1^2\lambda_2^4[1+\gamma\cos\theta]^2}\right),\\
\boldsymbol{C}_{\varrho\,\varrho_\theta} &= \mathrm{diag}\left(0, 0, \frac{4\,\varrho_\theta}{\gamma^2\lambda_1^4\lambda_2^3[1+\gamma\cos\theta]}\right),\\
\boldsymbol{C}_{\varrho\,\eta_\theta} &= \mathrm{diag}\left(0, 0, \frac{4\eta_\theta}{\gamma^2\lambda_1^4\lambda_2^3[1+\gamma\cos\theta]}\right),\\
\boldsymbol{C}_{\varrho_\theta\,\varrho_\theta} &= \mathrm{diag}\left(\frac{2}{\gamma^2}, 0, \frac{8\,\varrho_\theta^2}{\gamma^4\lambda_1^6\lambda_2^2} - \frac{2}{\gamma^2\lambda_1^4\lambda_2^2}\right),\\
\boldsymbol{C}_{\eta_\theta\eta_\theta} &= \mathrm{diag}\left(\frac{2}{\gamma^2}, 0, \frac{8\eta_\theta^2}{\gamma^4\lambda_1^6\lambda_2^2} - \frac{2}{\gamma^2\lambda_1^4\lambda_2^2}\right),\\
\boldsymbol{C}_{\varrho_\theta\,\eta_\theta} &= \mathrm{diag}\left(0, 0, \frac{8\,\varrho_\theta\,\eta_\theta}{\gamma^4\lambda_1^6\lambda_2^2}\right).
\end{aligned}
\tag{B.2}
$$

The full derivatives of some of the second derivatives with respect to θ are computed as

$$
\begin{aligned}
\frac{\mathrm{d}\,\boldsymbol{C}_{\varrho\,\varrho_\theta}}{\mathrm{d}\theta} &= \mathrm{diag}\left(0, 0, \frac{4}{\gamma^2}\left[\frac{\varrho_{\theta\theta}}{\lambda_1^4\lambda_2^3[1+\gamma\cos\theta]} + \frac{\varrho_\theta\,\gamma\sin\theta}{\lambda_1^4\lambda_2^3[1+\gamma\cos\theta]^2} - \frac{3\,\varrho_\theta\,\lambda_{2\theta}}{\lambda_1^4\lambda_2^4[1+\gamma\cos\theta]} - \frac{4\,\varrho_\theta\,\lambda_{1\theta}}{\lambda_1^5\lambda_2^3[1+\gamma\cos\theta]}\right]\right),\\
\frac{\mathrm{d}\,\boldsymbol{C}_{\varrho\,\eta_\theta}}{\mathrm{d}\theta} &= \mathrm{diag}\left(0, 0, \frac{4}{\gamma^2}\left[\frac{\eta_{\theta\theta}}{\lambda_1^4\lambda_2^3[1+\gamma\cos\theta]} + \frac{\eta_\theta\gamma\sin\theta}{\lambda_1^4\lambda_2^3[1+\gamma\cos\theta]^2} - \frac{3\eta_\theta\lambda_{2\theta}}{\lambda_1^4\lambda_2^4[1+\gamma\cos\theta]} - \frac{4\eta_\theta\lambda_{1\theta}}{\lambda_1^5\lambda_2^3[1+\gamma\cos\theta]}\right]\right),\\
\frac{\mathrm{d}\,\boldsymbol{C}_{\varrho_\theta\,\eta_\theta}}{\mathrm{d}\theta} &= \mathrm{diag}\left(0, 0, \frac{8\,\varrho_{\theta\theta}\,\eta_\theta}{\gamma^4\lambda_1^6\lambda_2^2} + \frac{8\,\varrho_\theta\,\eta_{\theta\theta}}{\gamma^4\lambda_1^6\lambda_2^2} - \frac{16\,\varrho_\theta\,\eta_\theta\lambda_{2\theta}}{\gamma^4\lambda_1^6\lambda_2^3} - \frac{48\,\varrho_\theta\,\eta_\theta\lambda_{1\theta}}{\gamma^4\lambda_1^7\lambda_2^2}\right),\\
\frac{\mathrm{d}\,\boldsymbol{C}_{\eta_\theta\eta_\theta}}{\mathrm{d}\theta} &= \mathrm{diag}\left(0, 0, \frac{8}{\gamma^4}\left[\frac{2\eta_\theta\eta_{\theta\theta}}{\lambda_1^6\lambda_2^2} - \frac{6\eta_\theta^2}{\lambda_1^7\lambda_2^2}\lambda_{1\theta} - \frac{2\eta_\theta^2}{\lambda_1^6\lambda_2^3}\lambda_{2\theta}\right] + \frac{2}{\gamma^2}\left[\frac{4}{\lambda_1^5\lambda_2^2}\lambda_{1\theta} + \frac{2}{\lambda_1^4\lambda_2^3}\lambda_{2\theta}\right]\right).
\end{aligned}
\tag{B.3}
$$

Given the energy density function in equation (29), the first derivatives of the energy density with respect to ϱ is computed as

$$
\begin{aligned}
\Omega_\varrho =& C_1\frac{\partial I_1}{\partial\,\varrho} + C_2\frac{\partial I_2}{\partial\,\varrho} + \beta[\boldsymbol{C}_\varrho\,\mathbb{D}]\cdot\mathbb{D}\\
=& C_1\frac{2\lambda_2}{[1+\gamma\cos\theta]}\left[\,[1+\alpha\lambda_1^2]\left[1-\frac{1}{\lambda_1^2\lambda_2^4}\right]\right] - \frac{\Phi_0^2\lambda_2\lambda_1^2}{2\beta H^2[1+\gamma\cos\theta]}.
\end{aligned}
\tag{B.4}
$$

Similarly, other first derivatives of energy density function Ω are given by

$$
\begin{aligned}
\Omega_{\mathbb{D}} &= 2\beta\,\boldsymbol{C}\,\mathbb{D},\\
\Omega_{\varrho_\theta} &= C_1\frac{2\varrho_\theta}{\gamma^2}\left[\,[1+\alpha\lambda_2^2]\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]\right] - \frac{\varrho_\theta\,\Phi_0^2\lambda_2^2}{2\beta H^2\gamma^2},\\
\Omega_{\eta_\theta} &= C_1\frac{2\eta_\theta}{\gamma^2}\left[\,[1+\alpha\lambda_2^2]\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]\right] - \frac{\eta_\theta\Phi_0^2\lambda_2^2}{2\beta H^2\gamma^2},\\
\Omega_\eta &= 0.
\end{aligned}
\tag{B.5a}
$$

The non-zero second derivatives of the energy density are given by

$$
\begin{aligned}
\Omega_{\varrho\,\varrho} &= 2C_1\frac{[1+\alpha\lambda_1^2]}{[1+\gamma\cos\theta]^2}\left[1+\frac{3}{\lambda_1^2\lambda_2^4}\right]+\frac{3\Phi_0^2\lambda_1^2}{2\beta H^2[1+\gamma\cos\theta]^2},\\
\Omega_{\varrho\,\varrho_\theta} &= C_1\frac{4\lambda_2}{[1+\gamma\cos\theta]}\frac{\varrho_\theta}{\gamma^2}\left[\alpha+\frac{1}{\lambda_1^4\lambda_2^4}\right]+\frac{\varrho_\theta}{\gamma^2}\frac{\Phi_0^2\lambda_2}{\beta H^2[1+\gamma\cos\theta]},\\
\Omega_{\varrho\,\eta_\theta} &= C_1\frac{4\lambda_2}{[1+\gamma\cos\theta]}\frac{\eta_\theta}{\gamma^2}\left[\alpha+\frac{1}{\lambda_1^4\lambda_2^4}\right]+\frac{\eta_\theta}{\gamma^2}\frac{\Phi_0^2\lambda_2}{\beta H^2[1+\gamma\cos\theta]},\\
\Omega_{\varrho_\theta\,\varrho_\theta} &= \frac{2C_1}{\gamma^2}[1+\alpha\lambda_2^2]\left[\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]+\frac{4\,\varrho_\theta^2}{\gamma^2\lambda_1^6\lambda_2^2}\right]+\frac{2\Phi_0^2\,\varrho_\theta^2\,\lambda_2^2}{\beta H^2\gamma^4\lambda_1^2}-\frac{\Phi_0^2\lambda_2^2}{2\beta H^2\gamma^2},\\
\Omega_{\varrho_\theta\,\eta_\theta} &= 8C_1\frac{\varrho_\theta\,\eta_\theta}{\gamma^4}\,[1+\alpha\lambda_2^2]\,\frac{1}{\lambda_1^6\lambda_2^2}+\frac{\varrho_\theta\,\eta_\theta}{\gamma^4}\frac{2\Phi_0^2\lambda_2^2}{\beta H^2\lambda_1^2},\\
\Omega_{\eta_\theta\eta_\theta} &= \frac{2C_1}{\gamma^2}[1+\alpha\lambda_2^2]\left[\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]+\frac{4\eta_\theta^2}{\gamma^2\lambda_1^6\lambda_2^2}\right]+\frac{2\Phi_0^2\eta_\theta^2\lambda_2^2}{\beta H^2\gamma^4\lambda_1^2}-\frac{\Phi_0^2\lambda_2^2}{2\beta H^2\gamma^2}.
\end{aligned}
\tag{B.6}
$$

Full derivatives of some second derivatives of the energy density function with respect to θ are computed as

$$
\begin{aligned}
\frac{\mathrm{d}\Omega_{\eta_\theta\eta_\theta}}{\mathrm{d}\theta} =& \frac{2C_1}{\gamma^2}\Bigg[2\alpha\lambda_2\lambda_{2\theta}\left[\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]+\frac{4\eta_\theta^2}{\gamma^2\lambda_1^6\lambda_2^2}\right] \\
&+[1+\alpha\lambda_2^2]\left[\frac{4\lambda_{1\theta}}{\lambda_1^5\lambda_2^2}+\frac{2\lambda_{2\theta}}{\lambda_1^4\lambda_2^3}+\frac{4}{\gamma^2}\left[\frac{2\eta_\theta\eta_{\theta\theta}}{\lambda_1^6\lambda_2^2}-\frac{6\eta_\theta^2\lambda_{1\theta}}{\lambda_1^7\lambda_2^2}-\frac{2\eta_\theta^2\lambda_{2\theta}}{\lambda_1^6\lambda_2^3}\right]\right]\Bigg] \\
&+\frac{\Phi_0^2}{\beta H^2}\left[\frac{2}{\gamma^4}\left[\frac{2\eta_\theta\eta_{\theta\theta}\lambda_2^2}{\lambda_1^2}-\frac{6\eta_\theta^2\lambda_2^2}{\lambda_1^3}\lambda_{1\theta}-\frac{2\eta_\theta^2\lambda_2}{\lambda_1^2}\lambda_{2\theta}\right]+\frac{1}{2\gamma^2}\left[\frac{4\lambda_2^2}{\lambda_1}\lambda_{1\theta}+2\lambda_2\lambda_{2\theta}\right]\right] \\
\frac{\mathrm{d}\Omega_{\varrho\,\varrho_\theta}}{\mathrm{d}\theta} =& \frac{4C_1}{\gamma^2}\Bigg[\left[\frac{\varrho_\theta\,\gamma\sin\theta}{[1+\gamma\cos\theta]^2}+\frac{\varrho_{\theta\theta}}{[1+\gamma\cos\theta]}\right]\left[\alpha\lambda_2+\frac{1}{\lambda_1^4\lambda_2^3}\right] \\
&+\frac{\varrho_\theta}{[1+\gamma\cos\theta]}\left[\alpha\lambda_{2\theta}-\frac{4}{\lambda_1^5\lambda_2^3}\lambda_{1\theta}-\frac{3}{\lambda_1^4\lambda_2^4}\lambda_{2\theta}\right]\Bigg] \\
&-\frac{\Phi_0^2}{\beta H^2\gamma^2}\left[\frac{\varrho_{\theta\theta}\,\lambda_2}{[1+\gamma\cos\theta]}+\frac{\varrho_\theta\,\gamma\sin\theta\lambda_2}{[1+\gamma\cos\theta]^2}-\frac{3\,\varrho_\theta\,\lambda_{2\theta}}{[1+\gamma\cos\theta]}-\frac{4\,\varrho_\theta\,\lambda_2\lambda_{1\theta}}{\lambda_1[1+\gamma\cos\theta]}\right] \\
\frac{\mathrm{d}\Omega_{\varrho\,\eta_\theta}}{\mathrm{d}\theta} =& \frac{4C_1}{\gamma^2}\Bigg[\left[\frac{\eta_\theta\gamma\sin\theta}{[1+\gamma\cos\theta]^2}+\frac{\eta_{\theta\theta}}{[1+\gamma\cos\theta]}\right]\left[\alpha\lambda_2+\frac{1}{\lambda_1^4\lambda_2^3}\right] \\
&+\frac{\eta_\theta}{[1+\gamma\cos\theta]}\left[\alpha\lambda_{2\theta}-\frac{4}{\lambda_1^5\lambda_2^3}\lambda_{1\theta}-\frac{3}{\lambda_1^4\lambda_2^4}\lambda_{2\theta}\right]\Bigg] \\
&-\frac{\Phi_0^2}{\beta H^2\gamma^2}\left[\frac{\eta_{\theta\theta}\lambda_2}{[1+\gamma\cos\theta]}+\frac{\eta_\theta\gamma\sin\theta\lambda_2}{[1+\gamma\cos\theta]^2}-\frac{3\eta_\theta\lambda_{2\theta}}{[1+\gamma\cos\theta]}-\frac{4\eta_\theta\lambda_2\lambda_{1\theta}}{\lambda_1[1+\gamma\cos\theta]}\right] \\
\frac{\mathrm{d}\Omega_{\varrho_\theta\,\eta_\theta}}{\mathrm{d}\theta} =& \frac{8C_1}{\gamma^4}\Bigg[[\varrho_{\theta\theta}\,\eta_\theta+\varrho_\theta\,\eta_{\theta\theta}]\left[1+\alpha\lambda_2^2\right]\frac{1}{\lambda_1^6\lambda_2^2}+\varrho_\theta\,\eta_\theta\,[2\alpha\lambda_2\lambda_{2\theta}]\,\frac{1}{\lambda_1^6\lambda_2^2} \\
&-\varrho_\theta\,\eta_\theta\gamma^4\left[1+\alpha\lambda_2^2\right]\left[\frac{6\lambda_{1\theta}}{\lambda_1^7\lambda_2^2}+\frac{2\lambda_{2\theta}}{\lambda_1^6\lambda_2^3}\right]\Bigg] \\
&+\frac{\Phi_0^2}{\beta H^2}\left[\frac{2\,\varrho_{\theta\theta}\,\eta_\theta\lambda_2^2}{\gamma^4\lambda_1^2}+\frac{2\,\varrho_\theta\,\eta_{\theta\theta}\lambda_2^2}{\gamma^4\lambda_1^2}-\frac{4\,\varrho_\theta\,\eta_\theta\lambda_{2\theta}\lambda_2}{\gamma^4\lambda_1^2}-\frac{12\,\varrho_\theta\,\eta_\theta\lambda_{1\theta}\lambda_2^2}{\gamma^4\lambda_1^3}\right]
\end{aligned} \tag{B.7}
$$

where

$$
\lambda_{1\theta}=\frac{\varrho_\theta\,\varrho_{\theta\theta}+\eta_\theta\eta_{\theta\theta}}{\gamma\,[\varrho_\theta^2+\eta_\theta^2]^{1/2}}=\frac{\varrho_\theta\,\varrho_{\theta\theta}+\eta_\theta\eta_{\theta\theta}}{\gamma^2\lambda_1},\quad \lambda_{2\theta}=\frac{\varrho_\theta}{1+\gamma\cos\theta}+\frac{\varrho\,\gamma\sin\theta}{[1+\gamma\cos\theta]^2}. \tag{B.8}
$$

The second various derivatives of Ω with respect to $\mathbb{D}$ are computed as

$$\begin{aligned}
\Omega_{\mathbb{D}\,\mathbb{D}} &= 2\beta\,\boldsymbol{C},\\
\Omega_{\mathbb{D}\,\varrho} &= 2\beta\,\boldsymbol{C}_{\varrho}\,\mathbb{D} = \frac{\Phi_0}{H}\,\boldsymbol{C}_{\varrho}\,\boldsymbol{C}^{-1}\,\mathbf{N},\\
\Omega_{\mathbb{D}\,\varrho_\theta} &= 2\beta\,\boldsymbol{C}_{\varrho_\theta}\,\mathbb{D} = \frac{\Phi_0}{H}\,\boldsymbol{C}_{\varrho_\theta}\,\boldsymbol{C}^{-1}\,\mathbf{N},\\
\Omega_{\mathbb{D}\,\eta_\theta} &= 2\beta\,\boldsymbol{C}_{\eta_\theta}\,\mathbb{D} = \frac{\Phi_0}{H}\,\boldsymbol{C}_{\eta_\theta}\,\boldsymbol{C}^{-1}\,\mathbf{N},\\
\frac{\mathrm{d}\Omega_{\mathbb{D}\,\varrho_\theta}}{\mathrm{d}\theta} &= 2\beta\,\boldsymbol{C}_{\varrho_\theta\,\theta}\,\mathbb{D} = \frac{\Phi_0}{H}\frac{\mathrm{d}\,\boldsymbol{C}_{\varrho_\theta}}{\mathrm{d}\theta}\,\boldsymbol{C}^{-1}\,\mathbf{N},\\
\frac{\mathrm{d}\Omega_{\mathbb{D}\,\eta_\theta}}{\mathrm{d}\theta} &= 2\beta\,\boldsymbol{C}_{\eta_\theta\theta}\,\mathbb{D} = \frac{\Phi_0}{H}\frac{\mathrm{d}\,\boldsymbol{C}_{\eta_\theta}}{\mathrm{d}\theta}\,\boldsymbol{C}^{-1}\,\mathbf{N},
\end{aligned} \tag{B.9a}$$

where

$$\frac{\mathrm{d}\,\boldsymbol{C}_{\varrho_\theta}}{\mathrm{d}\theta} = \mathrm{diag}\Bigg(-\frac{2\,\varrho_{\theta\theta}}{\gamma^2}, 0, -\frac{2}{\gamma^2}\left(\frac{\varrho_{\theta\theta}}{\lambda_1^4\lambda_2^2} - \frac{4\,\varrho_\theta}{\lambda_1^5\lambda_2^2}\lambda_{1\theta} - \frac{2\,\varrho_\theta}{\lambda_1^4\lambda_2^3}\lambda_{2\theta}\right)\Bigg), \tag{B.10a}$$

$$\frac{\mathrm{d}\,\boldsymbol{C}_{\eta_\theta}}{\mathrm{d}\theta} = \mathrm{diag}\Bigg(-\frac{2\eta_{\theta\theta}}{\gamma^2}, 0, -\frac{2}{\gamma^2}\left(\frac{\eta_{\theta\theta}}{\lambda_1^4\lambda_2^2} - \frac{4\eta_\theta}{\lambda_1^5\lambda_2^2}\lambda_{1\theta} - \frac{2\eta_\theta}{\lambda_1^4\lambda_2^3}\lambda_{2\theta}\right)\Bigg). \tag{B.10b}$$

Upon rewriting, we get

$$\Omega_{\mathbb{D}\,\mathbb{D}} = 2\beta\mathrm{diag}\left(\lambda_1^2, \lambda_2^2, \lambda_3^2\right), \tag{B.11a}$$

$$\Omega_{\mathbb{D}\,\varrho} = \frac{\Phi_0}{H}\mathrm{diag}\Bigg(0, \frac{2}{\lambda_2(1+\gamma\cos(\theta))}, -\frac{2}{\lambda_2(1+\gamma\cos(\theta))}\Bigg)\mathbf{N}, \tag{B.11b}$$

$$\Omega_{\mathbb{D}\,\varrho_\theta} = \frac{\Phi_0}{H}\mathrm{diag}\Bigg(\frac{2\,\varrho_\theta}{\lambda_1^2\gamma^2}, 0, \frac{-2\,\varrho_\theta}{\lambda_1^2\gamma^2}\Bigg)\mathbf{N}, \tag{B.11c}$$

$$\Omega_{\mathbb{D}\,\eta_\theta} = \frac{\Phi_0}{H}\mathrm{diag}\Bigg(\frac{2\eta_\theta}{\lambda_1^2\gamma^2}, 0, \frac{-2\eta_\theta}{\lambda_1^2\gamma^2}\Bigg)\mathbf{N}, \tag{B.11d}$$

$$\frac{\mathrm{d}\Omega_{\mathbb{D}\,\varrho_\theta}}{\mathrm{d}\theta} = \frac{\Phi_0}{H}\mathrm{diag}\Bigg(-\frac{2\,\varrho_{\theta\theta}}{\gamma^2\lambda_1^2}, 0, -\frac{2}{\gamma^2}\left(\frac{\varrho_{\theta\theta}}{\lambda_1^2} - \frac{4\,\varrho_\theta}{\lambda_1^3}\lambda_{1\theta} - \frac{2\,\varrho_\theta}{\lambda_1^2\lambda_2}\lambda_{2\theta}\right)\Bigg)\mathbf{N}, \tag{B.11e}$$

$$\frac{\mathrm{d}\Omega_{\mathbb{D}\,\eta_\theta}}{\mathrm{d}\theta} = \frac{\Phi_0}{H}\mathrm{diag}\Bigg(-\frac{2\eta_{\theta\theta}}{\gamma^2\lambda_1^2}, 0, -\frac{2}{\gamma^2}\left(\frac{\eta_{\theta\theta}}{\lambda_1^2} - \frac{4\eta_\theta}{\lambda_1^3}\lambda_{1\theta} - \frac{2\eta_\theta}{\lambda_1^2\lambda_2}\lambda_{2\theta}\right)\Bigg)\mathbf{N}. \tag{B.11f}$$

Appendix C. Second derivatives for computing the second variation of the potential energy

In Section 5, the loss of symmetry in the ϕ direction was considered. Bifurcation occurs when the second variation of the potential energy function becomes zero. However, the symmetric assumption in solutions is no longer valid and λ_1, λ_2 and λ_3 in the right Cauchy–Green deformation tensor are computed using the full expressions (6) and (7). The second derivatives of the energy density function with respect to ϱ is expressed as

$$\Omega_{\varrho\,\varrho} = C_1[I_{1\,\varrho\,\varrho} + \alpha I_{2\,\varrho\,\varrho}] + \beta[\boldsymbol{C}_{\varrho\,\varrho}\,\mathbb{D}]\cdot\mathbb{D}, \tag{C.1}$$

where

$$I_{1_{\varrho\varrho}} = \lambda_{1_{\varrho\varrho}}^{2} + \lambda_{2_{\varrho\varrho}}^{2} + \lambda_{3_{\varrho\varrho}}^{2},$$
$$I_{2_{\varrho\varrho}} = \lambda_{1_{\varrho\varrho}}^{-2} + \lambda_{2_{\varrho\varrho}}^{-2} + \lambda_{3_{\varrho\varrho}}^{-2}. \tag{C.2}$$

On substituting the expressions (6) and (7) into the previous equations, one obtains

$$I_{1_{\varrho\varrho}} = \frac{2}{[1+\gamma\cos\theta]^2} - \frac{2\gamma^2[1+\gamma\cos\theta]^2[\varrho_\theta^2+\eta_\theta^2]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2} + \frac{8\gamma^2[1+\gamma\cos\theta]^2\,\varrho^2[\varrho_\theta^2+\eta_\theta^2]^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^3}$$
$$I_{2_{\varrho\varrho}} = \frac{2\gamma^2}{[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]} - \frac{4\gamma^2\,\varrho^2[\varrho_\theta^2+\eta_\theta^2]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2}$$
$$- \frac{2[\varrho_\theta^2+\eta_\theta^2]\left[\gamma^2[\varrho_\phi^2+\eta_\phi^2+\varrho^2] + [\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2\right]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2}$$
$$+ \frac{8\,\varrho^2[\varrho_\theta^2+\eta_\theta^2]^2\left[\gamma^2[\varrho_\phi^2+\eta_\phi^2+\varrho^2] + [\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2\right]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^3} + \frac{2[\varrho_\theta^2+\eta_\theta^2]}{\gamma^2[1+\gamma\cos\theta]^2}, \tag{C.3}$$

and

$$\beta[\boldsymbol{C}_{\varrho\varrho}\,\mathbb{D}]\cdot\mathbb{D} = \frac{2\Phi_0^2\,\varrho^2[\varrho_\theta^2+\eta_\theta^2]^2}{\beta H^2\gamma^2[1+\gamma\cos\theta]^2\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]} - \frac{\Phi_0^2[\varrho_\theta^2+\eta_\theta^2]}{2\beta H^2\gamma^2[1+\gamma\cos\theta]^2}. \tag{C.4}$$

Similarly, the second derivative of Ω with respect to ϱ and ϱ_θ can be computed as

$$\Omega_{\varrho\varrho_\theta} = C_1[I_{1_{\varrho\varrho_\theta}} + \alpha I_{2_{\varrho\varrho_\theta}}] + \beta[\boldsymbol{C}_{\varrho\varrho_\theta}\,\mathbb{D}]\cdot\mathbb{D}, \tag{C.5}$$

where

$$I_{1_{\varrho\varrho_\theta}} = \frac{4\gamma^2\,\varrho[\varrho_\theta^2+\eta_\theta^2]\left[2\,\varrho^2\,\varrho_\theta + 2\eta_\phi[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\right][1+\gamma\cos\theta]^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^3} - \frac{4\gamma^2\,\varrho\,\varrho_\theta[1+\gamma\cos\theta]^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2}, \tag{C.6}$$

$$I_{2_{\varrho\varrho_\theta}} = -\frac{4\gamma^2\,\varrho\left[2\,\varrho^2\,\varrho_\theta + 2\eta_\phi[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\right]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2} - \frac{4\,\varrho\,\varrho_\theta[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2}$$
$$- \frac{4\,\varrho\,\varrho_\theta\left[[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2+\eta_\phi^2+\varrho^2]\gamma^2\right]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2}$$
$$+ \frac{4\,\varrho[\varrho_\theta^2+\eta_\theta^2]\left[2\,\varrho^2\,\varrho_\theta + 2\eta_\phi[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\right]\left[[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2+\eta_\phi^2+\varrho^2]\gamma^2\right]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^3} + \frac{4\,\varrho\,\varrho_\theta}{\gamma^2[1+\gamma\cos\theta]^2}, \tag{C.7}$$

$$\beta[\boldsymbol{C}_{\varrho\varrho_\theta}\,\mathbb{D}]\cdot\mathbb{D} = \frac{\Phi_0^2\,\varrho[\varrho_\theta^2+\eta_\theta^2][2\,\varrho^2\,\varrho_\theta + 2\eta_\phi\left[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta\right]]}{\beta H^2\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]\gamma^2[1+\gamma\cos\theta]^2} - \frac{\Phi_0^2\,\varrho\,\varrho_\theta}{\beta H^2\gamma^2[1+\gamma\cos\theta]^2}. \tag{C.8}$$

The second derivative of Ω with respect to ϱ and η_θ can be computed as

$$\Omega_{\varrho\eta_\theta} = C_1[I_{1_{\varrho\eta_\theta}} + \alpha I_{2_{\varrho\eta_\theta}}] + \beta[\boldsymbol{C}_{\varrho\eta_\theta}\,\mathbb{D}]\cdot\mathbb{D}, \tag{C.9}$$

where

$$I_{1\varrho\,\eta_\theta} = \frac{4\gamma^2\,\varrho[\varrho_\theta^2+\eta_\theta^2]\left[2\,\varrho^2\,\eta_\theta - 2\,\varrho_\phi[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\right][1+\gamma\cos\theta]^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^3} - \frac{4\gamma^2\,\varrho\,\eta_\theta[1+\gamma\cos\theta]^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2}, \tag{C.10}$$

$$\begin{aligned} I_{2\varrho\,\eta_\theta} = &-\frac{4\gamma^2\,\varrho\left[2\,\varrho^2\,\eta_\theta - 2\,\varrho_\phi[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\right]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2} - \frac{4\,\varrho\,\eta_\theta[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2} \\ &- \frac{4\,\varrho\,\eta_\theta\left[[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2+\eta_\phi^2+\varrho^2]\gamma^2\right]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2} \\ &+ \frac{4\,\varrho[\varrho_\theta^2+\eta_\theta^2]\left[2\,\varrho^2\,\eta_\theta - 2\,\varrho_\phi[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\right]\left[[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2+\eta_\phi^2+\varrho^2]\gamma^2\right]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^3} \\ &+ \frac{4\,\varrho\,\eta_\theta}{\gamma^2[1+\gamma\cos\theta]^2}, \end{aligned} \tag{C.11}$$

$$\beta[\boldsymbol{C}_{\varrho\,\eta_\theta}\,\mathbb{D}]\cdot\mathbb{D} = \frac{\Phi_0^2\,\varrho[\varrho_\theta^2+\eta_\theta^2][2\,\varrho^2\,\eta_\theta - 2\,\varrho_\phi\left[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta\right]]}{\beta H^2\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]\gamma^2[1+\gamma\cos\theta]^2} - \frac{\Phi_0^2\,\varrho\,\eta_\theta}{\beta H^2\gamma^2[1+\gamma\cos\theta]^2} \tag{C.12}$$

The second derivative of Ω with respect to ϱ and ϱ_ϕ can be computed as

$$\Omega_{\varrho\,\varrho_\phi} = C_1[I_{1\varrho\,\varrho_\phi} + \alpha I_{2\varrho\,\varrho_\phi}] + \beta[\boldsymbol{C}_{\varrho\,\varrho_\phi}\,\mathbb{D}]\cdot\mathbb{D}, \tag{C.13}$$

where

$$I_{1\varrho\,\varrho_\phi} = -\frac{8\,\varrho\,\eta_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\varrho_\theta^2+\eta_\theta^2]\gamma^2[1+\gamma\cos\theta]^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^3}, \tag{C.14}$$

$$\begin{aligned} I_{2\varrho\,\varrho_\phi} = &\frac{4\,\varrho\,\gamma^2\eta_\theta[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2} - \frac{4\,\varrho_\phi\,\varrho[\varrho_\theta^2+\eta_\theta^2]\gamma^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^2} \\ &- \frac{8\,\varrho\,\eta_\theta[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta][\varrho_\theta^2+\eta_\theta^2]\left[[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2+\eta_\phi^2+\varrho^2]\gamma^2\right]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]^3}, \end{aligned} \tag{C.15}$$

$$\beta[\boldsymbol{C}_{\varrho\,\varrho_\phi}\,\mathbb{D}]\cdot\mathbb{D} = -\frac{2\Phi_0^2\,\varrho\,\eta_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\varrho_\theta^2+\eta_\theta^2]}{\beta H^2\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2+\eta_\theta^2]\right]\gamma^2[1+\gamma\cos\theta]^2}. \tag{C.16}$$

The second derivative of Ω with respect to ϱ and η_ϕ is given by

$$\Omega_{\varrho\,\eta_\phi} = C_1[I_{1\varrho\,\eta_\phi} + \alpha I_{2\varrho\,\eta_\phi}] + \beta[\boldsymbol{C}_{\varrho\,\eta_\phi}\,\mathbb{D}]\cdot\mathbb{D}, \tag{C.17}$$

where

$$I_{1_{\varrho\,\eta_\phi}} = \frac{8\,\varrho\,\varrho_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]\gamma^2[1+\gamma\cos\theta]^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]^3}, \tag{C.18}$$

$$I_{2_{\varrho\,\eta_\phi}} = -\frac{4\,\varrho\,\gamma^2\,\varrho_\theta[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]^2} - \frac{4\eta_\phi\,\varrho[\varrho_\theta^2 + \eta_\theta^2]\gamma^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]^2}$$
$$+ \frac{8\,\varrho\,\varrho_\theta[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]\left[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta] + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\right]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]^3}, \tag{C.19}$$

$$\beta[\boldsymbol{C}_{\varrho\,\varrho_\phi}\,\mathbb{D}]\cdot\mathbb{D} = \frac{2\Phi_0^2\,\varrho\,\varrho_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]}{\beta H^2\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]\gamma^2[1+\gamma\cos\theta]^2} \tag{C.20}$$

The second derivative of Ω with respect to ϱ_θ is given by

$$\Omega_{\varrho_\theta\,\varrho_\theta} = C_1[I_{1_{\varrho_\theta\,\varrho_\theta}} + \alpha I_{2_{\varrho_\theta\,\varrho_\theta}}] + \beta[\boldsymbol{C}_{\varrho_\theta\,\varrho_\theta}\,\mathbb{D}]\cdot\mathbb{D}, \tag{C.21}$$

where

$$I_{1_{\varrho_\theta\,\varrho_\theta}} = \frac{2}{\gamma^2} + \frac{2\gamma^2[1+\gamma\cos\theta]^2[2\eta_\phi^2\,\varrho_\theta - 2\eta_\phi\,\varrho_\phi\,\eta_\theta + 2\,\varrho^2\,\varrho_\theta]^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]^3} - \frac{\gamma^2[1+\gamma\cos\theta]^2[2\eta_\phi^2 + 2\,\varrho^2]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]^2}, \tag{C.22}$$

$$I_{2_{\varrho_\theta\,\varrho_\theta}} = \frac{2\,\varrho^2 + 2\eta_\phi^2}{\gamma^2[1+\gamma\cos\theta]^2} - \frac{4\,\varrho_\theta[1+\gamma\cos\theta]^2[2\,\varrho^2\,\varrho_\theta + 2\eta_\phi^2\,\varrho_\theta - 2\eta_\theta\,\varrho_\phi\,\eta_\phi]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]^2} + \frac{2[1+\gamma\cos\theta]^2}{[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]}$$
$$+ \frac{2\left[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\right][2\eta_\phi^2\,\varrho_\theta - 2\eta_\phi\,\varrho_\phi\,\eta_\theta + 2\,\varrho^2\,\varrho_\theta]^2}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]^3}$$
$$- \frac{\left[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\right][2\eta_\phi^2 + 2\,\varrho^2]}{\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]^2}, \tag{C.23}$$

$$\beta[\boldsymbol{C}_{\varrho_\theta\,\varrho_\theta}\,\mathbb{D}]\cdot\mathbb{D} = \frac{2\Phi_0^2[\eta_\phi^2\,\varrho_\theta - \eta_\phi\,\varrho_\phi\,\eta_\theta + \varrho^2\,\varrho_\theta]^2}{\beta H^2\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]\gamma^2[1+\gamma\cos\theta]^2} - \frac{\Phi_0^2[\eta_\phi^2 + \varrho^2]}{2\beta H^2\gamma^2[1+\gamma\cos\theta]^2}. \tag{C.24}$$

The second derivative of Ω with respect to ϱ_θ and η_θ is given by

$$\Omega_{\varrho_\theta\,\eta_\theta} = C_1[I_{1_{\varrho_\theta\,\eta_\theta}} + \alpha I_{2_{\varrho_\theta\,\eta_\theta}}] + \beta[\boldsymbol{C}_{\varrho_\theta\,\eta_\theta}\,\mathbb{D}]\cdot\mathbb{D}, \tag{C.25}$$

where

$$I_{1_{\varrho_\theta\,\eta_\theta}} = \frac{2\gamma^2[1+\gamma\cos\theta]^2[2\,\varrho_\phi^2\,\eta_\theta - 2\eta_\phi\,\varrho_\phi\,\varrho_\theta + 2\,\varrho^2\,\eta_\theta][2\,\varrho^2\,\varrho_\theta + 2\,\varrho_\theta\,\eta_\phi^2 - 2\,\varrho_\phi\,\eta_\phi\eta_\theta]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}, \qquad (C.26)$$
$$- \frac{2\,\varrho_\phi\,\eta_\phi\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}$$

$$I_{2_{\varrho_\theta\,\eta_\theta}} = -\frac{2\,\varrho_\phi\,\eta_\phi}{\gamma^2[1+\gamma\cos\theta]^2} - \frac{2\,\varrho_\theta[1+\gamma\cos\theta]^2[2\,\varrho^2\,\eta_\theta - 2\,\varrho_\phi\,\eta_\phi\,\varrho_\theta + 2\,\varrho_\phi^2\,\eta_\theta]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}$$
$$- \frac{2\eta_\theta[1+\gamma\cos\theta]^2[2\,\varrho^2\,\varrho_\theta + 2\eta_\phi^2\,\varrho_\theta - 2\eta_\phi\,\varrho_\phi\,\eta_\theta]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2} + \frac{2\,\varrho_\phi\,\eta_\phi\,\big[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}$$
$$+ \frac{2[2\,\varrho^2\,\eta_\theta - 2\,\varrho_\phi\,\eta_\phi\,\varrho_\theta + 2\,\varrho_\phi^2\,\eta_\theta][2\,\varrho^2\,\varrho_\theta + 2\eta_\phi^2\,\varrho_\theta - 2\eta_\phi\,\varrho_\phi\,\eta_\theta]\,\big[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}, \qquad (C.27)$$

$$\beta[\boldsymbol{C}_{\varrho_\theta\,\eta_\theta}\,\mathbb{D}]\cdot\mathbb{D} = \frac{2\Phi_0^2[\varrho_\phi^2\,\eta_\theta - \eta_\phi\,\varrho_\phi\,\varrho_\theta + \varrho^2\,\eta_\theta][\varrho^2\,\varrho_\theta + \varrho_\theta\,\eta_\phi^2 - \varrho_\phi\,\eta_\phi\eta_\theta]}{\beta H^2\,\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]\,\gamma^2[1+\gamma\cos\theta]^2} - \frac{\Phi_0^2\,\varrho_\phi\,\eta_\phi}{2\beta H^2\gamma^2[1+\gamma\cos\theta]^2}. \qquad (C.28)$$

The second derivative of Ω with respect to η_θ and ϱ_ϕ is computed as

$$\Omega_{\eta_\theta\,\varrho_\phi} = C_1[I_{1_{\eta_\theta\,\varrho_\phi}} + \alpha I_{2_{\eta_\theta\,\varrho_\phi}}] + \beta[\boldsymbol{C}_{\eta_\theta\,\varrho_\phi}\,\mathbb{D}]\cdot\mathbb{D}, \qquad (C.29)$$

where

$$I_{1_{\eta_\theta\,\varrho_\phi}} = -\frac{4\gamma^2[1+\gamma\cos\theta]^2[\varrho_\theta\,\eta_\theta\eta_\phi - \eta_\theta^2\,\varrho_\phi][2\,\varrho_\phi^2\,\eta_\theta - 2\eta_\phi\,\varrho_\phi\,\varrho_\theta + 2\,\varrho^2\,\eta_\theta]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}$$
$$- \frac{[2\,\varrho_\phi\,\eta_\theta - 2[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]]\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}, \qquad (C.30)$$

$$I_{2_{\eta_\theta\,\varrho_\phi}} = \frac{2\eta_\theta\,\varrho_\phi - 2[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]}{\gamma^2[1+\gamma\cos\theta]^2} + \frac{4\eta_\theta^2[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}$$
$$- \frac{2\,\varrho_\phi\,\gamma^2\,\big[2\,\varrho^2\,\eta_\theta - 2\,\varrho_\phi[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\big]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}$$
$$- \frac{\big[2\,\varrho_\phi\,\eta_\theta - 2[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\big]\,\big[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}$$
$$- \frac{4\eta_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\,\big[2\,\varrho^2\,\varrho_\theta - 2\,\varrho_\phi[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\big]\,\big[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}, \qquad (C.31)$$

$$\beta[\boldsymbol{C}_{\eta_\theta\,\varrho_\phi}\,\mathbb{D}]\cdot\mathbb{D} = -\frac{\Phi_0^2[\varrho_\theta\,\eta_\theta\eta_\phi - \eta_\theta^2\,\varrho_\phi][2\,\varrho_\phi^2\,\eta_\theta - 2\eta_\phi\,\varrho_\phi\,\varrho_\theta + 2\,\varrho^2\,\eta_\theta]}{\beta H^2\,\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]\,\gamma^2[1+\gamma\cos\theta]^2} - \frac{\Phi_0^2[\varrho_\phi\,\eta_\theta - \varrho_\theta\,\eta_\phi + \varrho_\phi\,\eta_\theta]}{2\beta H^2\gamma^2[1+\gamma\cos\theta]^2} \qquad (C.32)$$

The second derivative of Ω with respect to η_θ is computed as

$$\Omega_{\eta_\theta\eta_\theta} = C_1[I_{1_{\eta_\theta\eta_\theta}} + \alpha I_{2_{\eta_\theta\eta_\theta}}] + \beta[\boldsymbol{C}_{\eta_\theta\eta_\theta}\,\mathbb{D}]\cdot\mathbb{D}, \qquad (C.33)$$

where

$$I_{1_{\eta_\theta\eta_\theta}} = \frac{2}{\gamma^2} + \frac{2\gamma^2[1+\gamma\cos\theta]^2[2\,\varrho_\phi^2\,\eta_\theta - 2\eta_\phi\,\varrho_\phi\,\varrho_\theta + 2\,\varrho^2\,\eta_\theta]^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3} - \frac{\gamma^2[1+\gamma\cos\theta]^2[2\,\varrho_\phi^2 + 2\,\varrho^2]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}, \quad \text{(C.34)}$$

$$I_{2_{\eta_\theta\eta_\theta}} = \frac{2\,\varrho^2 + 2\,\varrho_\phi^2}{\gamma^2[1+\gamma\cos\theta]^2} - \frac{4\eta_\theta[1+\gamma\cos\theta]^2[2\,\varrho^2\,\eta_\theta + 2\,\varrho_\phi^2\,\eta_\theta - 2\,\varrho_\theta\,\varrho_\phi\,\eta_\phi]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2} + \frac{2[1+\gamma\cos\theta]^2}{[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]}$$
$$+ \frac{2\left[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\right][2\,\varrho_\phi^2\,\eta_\theta - 2\eta_\phi\,\varrho_\phi\,\varrho_\theta + 2\,\varrho^2\,\eta_\theta]^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}$$
$$- \frac{\left[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\right][2\,\varrho_\phi^2 + 2\,\varrho^2]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}, \quad \text{(C.35)}$$

$$\beta[\boldsymbol{C}_{\eta_\theta\eta_\theta}\,\mathbb{D}]\cdot\mathbb{D} = \frac{2\Phi_0^2[\varrho_\phi^2\,\eta_\theta - \eta_\phi\,\varrho_\phi\,\varrho_\theta + \varrho^2\,\eta_\theta]^2}{\beta H^2\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]\gamma^2[1+\gamma\cos\theta]^2} - \frac{\Phi_0^2[\varrho_\phi^2 + \varrho^2]}{2\beta H^2\gamma^2[1+\gamma\cos\theta]^2} \quad \text{(C.36)}$$

The second derivative of Ω with respect to η_θ and η_ϕ is given by

$$\Omega_{\eta_\theta\eta_\phi} = C_1[I_{1_{\eta_\theta\eta_\phi}} + \alpha I_{2_{\eta_\theta\eta_\phi}}] + \beta[\boldsymbol{C}_{\eta_\theta\eta_\phi}\,\mathbb{D}]\cdot\mathbb{D}, \quad \text{(C.37)}$$

where

$$I_{1_{\eta_\theta\eta_\phi}} = \frac{4\,\varrho_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\gamma^2[1+\gamma\cos\theta]^2[2\,\varrho_\phi^2\,\eta_\theta - 2\,\varrho_\phi\,\eta_\phi\,\varrho_\theta + 2\,\varrho^2\,\eta_\theta]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3} + \frac{2\,\varrho_\phi\,\varrho_\theta\,\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}, \quad \text{(C.38)}$$

$$I_{2_{\eta_\theta\eta_\phi}} = -\frac{2\,\varrho_\phi\,\varrho_\theta}{\gamma^2[1+\gamma\cos\theta]^2} - \frac{4\,\varrho_\theta\,\eta_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2} - \frac{2\eta_\phi\left[2\,\varrho^2\,\eta_\theta - 2\,\varrho_\phi[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\right]\gamma^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}$$
$$+ \frac{2\,\varrho_\phi\,\varrho_\theta\left[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\right]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}$$
$$+ \frac{4\,\varrho_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\left[2\,\varrho^2\,\eta_\theta - 2\,\varrho_\phi[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta]\right]\left[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\right]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}, \quad \text{(C.39)}$$

$$\beta[\boldsymbol{C}_{\eta_\theta\eta_\phi}\,\mathbb{D}]\cdot\mathbb{D} = \frac{\Phi_0^2\,\varrho_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][2\,\varrho_\phi^2\,\eta_\theta - 2\,\varrho_\phi\,\eta_\phi\,\varrho_\theta + 2\,\varrho^2\,\eta_\theta]}{\beta H^2\left[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\right]\gamma^2[1+\gamma\cos\theta]^2} - \frac{\Phi_0^2\,\varrho_\phi\,\varrho_\theta}{2\beta H^2\gamma^2[1+\gamma\cos\theta]^2}. \quad \text{(C.40)}$$

The second derivative of Ω with respect to ϱ_ϕ is given by

$$\Omega_{\varrho_\phi\,\varrho_\phi} = C_1[I_{1_{\varrho_\phi\,\varrho_\phi}} + \alpha I_{2_{\varrho_\phi\,\varrho_\phi}}] + \beta[\boldsymbol{C}_{\varrho_\phi\,\varrho_\phi}\,\mathbb{D}]\cdot\mathbb{D}, \quad \text{(C.41)}$$

where

$$I_{1_{\varrho_\phi\,\varrho_\phi}} = \frac{2}{[1+\gamma\cos\theta]^2} - \frac{2\eta_\theta^2\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^2} + \frac{8\eta_\theta^2[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^3}, \tag{C.42}$$

$$\begin{aligned} I_{2_{\varrho_\phi\,\varrho_\phi}} &= \frac{2\gamma^2}{[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]} + \frac{8\,\varrho_\phi\,\eta_\theta[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]\gamma^2}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^2} + \frac{2\eta_\theta^2}{\gamma^2[1+\gamma\cos\theta]^2} \\ &- \frac{2\eta_\theta^2\,\big[[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2+[\varrho_\phi^2+\eta_\phi^2+\varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^2} \\ &+ \frac{8\eta_\theta^2[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2\,\big[[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2+[\varrho_\phi^2+\eta_\phi^2+\varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^3}, \end{aligned} \tag{C.43}$$

$$\beta[\boldsymbol{C}_{\varrho_\phi\,\varrho_\phi}\,\mathbb{D}]\cdot\mathbb{D} = \frac{\Phi_0^2\eta_\theta^2}{2\beta H^2\gamma^2[1+\gamma\cos\theta]^2} + \frac{2\Phi_0^2\eta_\theta^2[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2}{\beta H^2\,\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]\,\gamma^2[1+\gamma\cos\theta]^2}. \tag{C.44}$$

The second derivative of Ω with respect to η_ϕ is given by

$$\Omega_{\eta_\phi\eta_\phi} = C_1[I_{1_{\eta_\phi\eta_\phi}} + \alpha I_{2_{\eta_\phi\eta_\phi}}] + \beta[\boldsymbol{C}_{\eta_\phi\eta_\phi}\,\mathbb{D}]\cdot\mathbb{D}, \tag{C.45}$$

where

$$I_{1_{\eta_\phi\eta_\phi}} = \frac{2}{[1+\gamma\cos\theta]^2} - \frac{2\,\varrho_\theta^2\,\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^2} + \frac{8\,\varrho_\theta^2[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^3}, \tag{C.46}$$

$$\begin{aligned} I_{2_{\eta_\phi\eta_\phi}} &= \frac{2\gamma^2}{[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]} - \frac{8\eta_\phi\,\varrho_\theta[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]\gamma^2}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^2} + \frac{2\,\varrho_\theta^2}{\gamma^2[1+\gamma\cos\theta]^2} \\ &- \frac{2\,\varrho_\theta^2\,\big[[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2+[\varrho_\phi^2+\eta_\phi^2+\varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^2} \\ &+ \frac{8\,\varrho_\theta^2[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2\,\big[[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2+[\varrho_\phi^2+\eta_\phi^2+\varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^3}, \end{aligned} \tag{C.47}$$

$$\beta[\boldsymbol{C}_{\eta_\phi\eta_\phi}\,\mathbb{D}]\cdot\mathbb{D} = \frac{\Phi_0^2\,\varrho_\theta^2}{2\beta H^2\gamma^2[1+\gamma\cos\theta]^2} + \frac{2\Phi_0^2\,\varrho_\theta^2[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2}{\beta H^2\,\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]\,\gamma^2[1+\gamma\cos\theta]^2}. \tag{C.48}$$

The second derivative of Ω with respect to ϱ_ϕ and η_ϕ is given by

$$\Omega_{\varrho_\phi\,\eta_\phi} = C_1[I_{1_{\varrho_\phi\,\eta_\phi}} + \alpha I_{2_{\varrho_\phi\,\eta_\phi}}] + \beta[\boldsymbol{C}_{\varrho_\phi\,\eta_\phi}\,\mathbb{D}]\cdot\mathbb{D}, \tag{C.49}$$

where

$$I_{1_{\varrho_\phi\,\eta_\phi}} = \frac{2\,\varrho_\theta\,\eta_\theta\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^2} - \frac{8\,\varrho_\theta\,\eta_\theta[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^3}, \tag{C.50}$$

$$\begin{aligned} I_{2_{\varrho_\phi\,\eta_\phi}} &= -\frac{2\,\varrho_\theta\,\eta_\theta}{\gamma^2[1+\gamma\cos\theta]^2} - \frac{4\,\varrho_\phi\,\varrho_\theta[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]\gamma^2}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^2} + \frac{4\eta_\phi\eta_\theta[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]\gamma^2}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^2} \\ &+ \frac{8\,\varrho_\theta\,\eta_\theta[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2\,\big[[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2+[\varrho_\phi^2+\eta_\phi^2+\varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^3} \\ &+ \frac{2\,\varrho_\theta\,\eta_\theta\,\big[[\varrho_\theta^2+\eta_\theta^2][1+\gamma\cos\theta]^2+[\varrho_\phi^2+\eta_\phi^2+\varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]^2}, \end{aligned} \tag{C.51}$$

$$\beta[\boldsymbol{C}_{\varrho_\phi\,\eta_\phi}\,\mathbb{D}]\cdot\mathbb{D} = \frac{\Phi_0^2\,\varrho_\theta\,\eta_\theta}{2\beta H^2\gamma^2[1+\gamma\cos\theta]^2} - \frac{2\Phi_0^2\,\varrho_\theta\,\eta_\theta[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2}{\beta H^2\,\big[[\varrho_\theta\,\eta_\phi-\varrho_\phi\,\eta_\theta]^2+\varrho^2[\varrho_\theta^2+\eta_\theta^2]\big]\,\gamma^2[1+\gamma\cos\theta]^2}. \tag{C.52}$$

Finally, the full derivative of $\Omega_{\varrho\,\varrho_\phi}$ with respect to ϕ is expressed as

$$\frac{\mathrm{d}\Omega_{\varrho\,\varrho_\phi}}{\mathrm{d}\phi} = C_1[\frac{\mathrm{d}I_{1_{\varrho\,\varrho_\phi}}}{\mathrm{d}\phi} + \alpha\frac{\mathrm{d}I_{2_{\varrho\,\varrho_\phi}}}{\mathrm{d}\phi}] + \beta[\frac{\mathrm{d}\,\boldsymbol{C}_{\varrho\,\varrho_\phi}}{\mathrm{d}\phi}\,\mathbb{D}]\cdot\mathbb{D}, \tag{C.53}$$

where

$$\frac{\mathrm{d}I_{1_{\varrho\,\varrho_\phi}}}{\mathrm{d}\phi} = -\frac{8\,\varrho\,\eta_\theta[\eta_{\phi\phi}\,\varrho_\theta - \varrho_{\phi\phi}\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3} - \frac{8\,\varrho_\phi\,\eta_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}$$
$$+ \frac{24\,\varrho\,\eta_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^4}\,\big[2[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\eta_{\phi\phi}\,\varrho_\theta - \varrho_{\phi\phi}\,\eta_\theta] + 2\,\varrho_\phi\,\varrho[\varrho_\theta^2 + \eta_\theta^2]\big]\,,$$

$$\frac{\mathrm{d}I_{2_{\varrho\,\varrho_\phi}}}{\mathrm{d}\phi} = \frac{4\,\varrho_\phi\,\gamma^2\eta_\theta[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2} + \frac{4\,\varrho\,\gamma^2\eta_\theta[\varrho_\theta\,\eta_{\phi\phi} - \varrho_{\phi\phi}\,\eta_\theta]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}$$
$$- \frac{8\,\varrho\,\gamma^2\eta_\theta[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}\,\big[2[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta][\varrho_\theta\,\eta_{\phi\phi} - \varrho_{\phi\phi}\,\eta_\theta] + 2\,\varrho\,\varrho_\phi[\varrho_\theta^2 + \eta_\theta^2]\big]$$
$$- \frac{4\,\varrho_{\phi\phi}\,\varrho[\varrho_\theta^2 + \eta_\theta^2]\gamma^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2} - \frac{4\,\varrho_\phi\,\varrho_\phi[\varrho_\theta^2 + \eta_\theta^2]\gamma^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2}$$
$$+ \frac{8\,\varrho_\phi\,\varrho[\varrho_\theta^2 + \eta_\theta^2]\gamma^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}\,\big[2[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta][\varrho_\theta\,\eta_{\phi\phi} - \varrho_{\phi\phi}\,\eta_\theta] + 2\,\varrho\,\varrho_\phi[\varrho_\theta^2 + \eta_\theta^2]\big]$$
$$- \frac{8\,\varrho_\phi\,\eta_\theta[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]\,\big[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}$$
$$- \frac{8\,\varrho\,\eta_\theta[\varrho_\theta\,\eta_{\phi\phi} - \varrho_{\phi\phi}\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]\,\big[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}$$
$$- \frac{8\,\varrho\,\eta_\theta[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]\,\big[2\,\varrho_\phi\,\varrho_{\phi\phi} + 2\eta_\phi\eta_{\phi\phi} + 2\,\varrho\,\varrho_\phi\big]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^3}$$
$$+ \frac{24\,\varrho\,\eta_\theta[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]\,\big[[\varrho_\theta^2 + \eta_\theta^2][1+\gamma\cos\theta]^2 + [\varrho_\phi^2 + \eta_\phi^2 + \varrho^2]\gamma^2\big]}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^4}$$
$$\cdot\,\big[2[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta][\varrho_\theta\,\eta_{\phi\phi} - \varrho_{\phi\phi}\,\eta_\theta] + 2\,\varrho\,\varrho_\phi[\varrho_\theta^2 + \eta_\theta^2]\big]$$
$$+ \frac{24\,\varrho\,\eta_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]\gamma^2[1+\gamma\cos\theta]^2}{\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^4}\,\big[2[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\eta_{\phi\phi}\,\varrho_\theta - \varrho_{\phi\phi}\,\eta_\theta] + 2\,\varrho_\phi\,\varrho[\varrho_\theta^2 + \eta_\theta^2]\big]\,,$$

$$\beta[\frac{\mathrm{d}\,\boldsymbol{C}_{\varrho\,\varrho_\phi}}{\mathrm{d}\phi}\,\mathbb{D}]\cdot\mathbb{D} = -\frac{2\Phi_0^2\,\varrho\,\eta_\theta[\eta_{\phi\phi}\,\varrho_\theta - \varrho_{\phi\phi}\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]}{\beta H^2\,\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]\,\gamma^2[1+\gamma\cos\theta]^2}$$
$$- \frac{2\Phi_0^2\,\varrho_\phi\,\eta_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]}{\beta H^2\,\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\gamma^2[1+\gamma\cos\theta]^2\big]}$$
$$+ \frac{6\Phi_0^2\,\varrho\,\eta_\theta[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]}{\beta H^2\,\big[[\varrho_\theta\,\eta_\phi - \varrho_\phi\,\eta_\theta]^2 + \varrho^2[\varrho_\theta^2 + \eta_\theta^2]\big]^2\,\gamma^2[1+\gamma\cos\theta]^2}\,\big[2[\eta_\phi\,\varrho_\theta - \varrho_\phi\,\eta_\theta][\eta_{\phi\phi}\,\varrho_\theta - \varrho_{\phi\phi}\,\eta_\theta] + 2\,\varrho_\phi\,\varrho[\varrho_\theta^2 + \eta_\theta^2]\big]\,.$$

(C.54)

7 ## Appendix D. Reformulation of the coupled ODEs

The derivatives of the stretches in equations (9) with respect to θ are expressed as

$$\lambda_{1\theta} = \frac{\varrho_\theta\,\varrho_{\theta\theta} + \eta_\theta\eta_{\theta\theta}}{\gamma\,[\varrho_\theta^2 + \eta_\theta^2]^{1/2}} = \frac{\varrho_\theta\,\varrho_{\theta\theta} + \eta_\theta\eta_{\theta\theta}}{\gamma^2\lambda_1}, \quad \lambda_{2\theta} = \frac{\varrho_\theta}{1+\gamma\cos\theta} + \frac{\varrho\,\gamma\sin\theta}{[1+\gamma\cos\theta]^2}. \tag{D.1}$$

The equations (31a) and (31b) can be rewritten as

$$
\begin{aligned}
&-\gamma\sin\theta\left[\frac{2\varrho_\theta}{\gamma^2}\left[1+\alpha\lambda_2^2\right]\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]-\frac{\varrho_\theta\,\mathcal{E}\lambda_2^2}{2\gamma^2}\right]\\
&+[1+\gamma\cos\theta]\left[\frac{2\varrho_{\theta\theta}}{\gamma^2}\left[1+\alpha\lambda_2^2\right]\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]+\frac{2\varrho_\theta}{\gamma^2}\left[2\alpha\lambda_2\lambda_{2\theta}\right]\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]\right.\\
&\left.+\frac{2\varrho_\theta}{\gamma^2}\left[1+\alpha\lambda_2^2\right]\left[\frac{2\lambda_{2\theta}}{\lambda_2^3\lambda_1^4}+\frac{4\lambda_{1\theta}}{\lambda_2^2\lambda_1^5}\right]-\frac{\mathcal{E}\lambda_2^2}{2\gamma^2}\rho_{\theta\theta}-\frac{2\rho_\theta\mathcal{E}}{\gamma^2}\lambda_2\lambda_{2\theta}\right]\\
&-2\lambda_2\left[1+\alpha\lambda_1^2\right]\left[1-\frac{1}{\lambda_1^2\lambda_2^4}\right]+\frac{\mathcal{E}\lambda_2\lambda_1^2}{2}+\frac{P\,\varrho\,\eta_\theta}{\gamma}=0,
\end{aligned}
\tag{D.2}
$$

$$
\begin{aligned}
&-\gamma\sin\theta\left[\frac{2\eta_\theta}{\gamma^2}\left[1+\alpha\lambda_2^2\right]\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]-\frac{\mathcal{E}\eta_\theta\lambda_2^2}{2\gamma^2}\right]\\
&+[1+\gamma\cos\theta]\left[\frac{2\eta_{\theta\theta}}{\gamma^2}\left[1+\alpha\lambda_2^2\right]\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]+\frac{2\eta_\theta}{\gamma^2}\left[2\alpha\lambda_2\lambda_{2\theta}\right]\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]\right.\\
&\left.+\frac{2\eta_\theta}{\gamma^2}\left[1+\alpha\lambda_2^2\right]\left[\frac{2\lambda_{2\theta}}{\lambda_2^3\lambda_1^4}+\frac{4\lambda_{1\theta}}{\lambda_2^2\lambda_1^5}\right]-\frac{\mathcal{E}\lambda_2^2}{2\gamma^2}\eta_{\theta\theta}-\frac{2\eta_\theta\mathcal{E}}{\gamma^2}\lambda_2\lambda_{2\theta}\right]-\frac{P\varrho\varrho_\theta}{\gamma}=0.
\end{aligned}
\tag{D.3}
$$

From the first equation, the coefficient of $\varrho_{\theta\theta}$ is

$$
A_1=[1+\gamma\cos\theta]\left[\frac{2}{\gamma^2}\left[1+\alpha\lambda_2^2\right]\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]+\frac{8\varrho_\theta^2}{\gamma^4\lambda_1^6\lambda_2^2}\left[1+\alpha\lambda_2^2\right]-\frac{\mathcal{E}\lambda_2^2}{2\gamma^2}\right].
\tag{D.4}
$$

The coefficient of $\eta_{\theta\theta}$ is given by

$$
A_2=[1+\gamma\cos\theta]\frac{8\varrho_\theta\eta_\theta}{\gamma^4\lambda_1^6\lambda_2^2}\left[1+\alpha\lambda_2^2\right],
\tag{D.5}
$$

and the remaining term is

$$
\begin{aligned}
A_3=&-\gamma\sin\theta\left[\frac{2\,\varrho_\theta}{\gamma^2}\left[1+\alpha\lambda_2^2\right]\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]-\frac{\mathcal{E}\,\varrho_\theta\,\lambda_2^2}{2\gamma^2}\right]\\
&+[1+\gamma\cos\theta]\left[\frac{2\varrho_\theta}{\gamma^2}\left[2\alpha\lambda_2\lambda_{2\theta}\right]\left[1-\frac{1}{\lambda_1^4\lambda_2^2}\right]+\frac{4\,\varrho_\theta\,\lambda_{2\theta}}{\gamma^2\lambda_2^3\lambda_1^4}[1+\alpha\lambda_2^2]-\frac{2\rho_\theta\mathcal{E}}{\gamma^2}\lambda_2\lambda_{2\theta}\right]\\
&-2\lambda_2\left[1+\alpha\lambda_1^2\right]\left[1-\frac{1}{\lambda_1^2\lambda_2^4}\right]+\frac{\mathcal{E}\lambda_2\lambda_1^2}{2}+\frac{P\,\varrho\,\eta_\theta}{\gamma}
\end{aligned}
\tag{D.6}
$$

From the second equation, the coefficient of ϱ'' is

$$
B_1=[1+\gamma\cos\theta]\frac{8\varrho_\theta\eta_\theta}{\gamma^4\lambda_1^6\lambda_2^2}\left[1+\alpha\lambda_2^2\right]
\tag{D.7}
$$

and the coefficient of η'' is

$$
B_2=[1+\gamma\cos\theta]\left[\frac{2}{\gamma^2}\left[1+\alpha\lambda_2^2\right]\left[1-\frac{1}{\lambda_1^2\lambda_2^4}\right]+\frac{8\eta_\theta^2}{\gamma^4\lambda_1^6\lambda_2^2}\left[1+\alpha\lambda_2^2\right]-\frac{\mathcal{E}\lambda_2^2}{2\gamma^2}\right]
\tag{D.8}
$$

and the remaining term is

$$B_3 = -\gamma \sin\theta \left[\frac{2\eta_\theta}{\gamma^2} \left[1+\alpha\lambda_2^2\right] \left[1 - \frac{1}{\lambda_1^4\lambda_2^2}\right] - \frac{\mathcal{E}\lambda_2^2\eta_\theta}{2\gamma^2} \right]$$
$$+ \left[1+\gamma\cos\theta\right] \left[\frac{2\eta'}{\gamma^2} \left[2\alpha\lambda_2\lambda_{2\theta}\right] \left[1 - \frac{1}{\lambda_1^4\lambda_2^2}\right] + \left[1+\alpha\lambda_2^2\right] \frac{4\eta_\theta\lambda_{2\theta}}{\gamma^2\lambda_1^4\lambda_2^3} - \frac{2\eta_\theta\mathcal{E}}{\gamma^2}\lambda_2\lambda_{2\theta} \right] - \frac{P\varrho\varrho_\theta}{\gamma}. \tag{D.9}$$

Therefore the above set of coupled ODEs can be rewritten as

$$A_1\, \varrho_{\theta\theta} + A_2\eta_{\theta\theta} + A_3 = 0, \tag{D.10}$$

$$B_1\, \varrho_{\theta\theta} + B_2\eta_{\theta\theta} + B_3 = 0. \tag{D.11}$$

8 Appendix E. Reformulation of the ODEs arising from the relaxed energy

The governing equations (31) are now modified as in equations (50) where the modified energy density function is expressed as:

$$\Omega^*(\lambda_1, \mathcal{E}) = C_1[I_1^* - 3] + C_2[I_2^* - 3] + \beta[\boldsymbol{C}\,\mathbb{D}] \cdot \mathbb{D}\,. \tag{E.1}$$

As Ω^* is not a function of λ_2, $\frac{\partial\Omega^*}{\partial\,\varrho}$ is vanished. We first use the chain rule to compute the derivatives of Ω^* with respect to ϱ_θ and η_θ as

$$\frac{\partial\Omega^*}{\partial\,\varrho_\theta} = \frac{\partial\Omega^*}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial\,\varrho_\theta}, \qquad \frac{\partial\Omega^*}{\partial\eta_\theta} = \frac{\partial\Omega^*}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial\eta_\theta}. \tag{E.2}$$

The derivatives of λ_1 with respect to ϱ_θ and η_θ are easy to be computed as

$$\frac{\partial\lambda_1}{\partial\,\varrho_\theta} = \frac{\varrho_\theta}{\gamma}[\varrho_\theta^2 + \eta_\theta^2]^{-\frac{1}{2}}, \qquad \frac{\partial\lambda_1}{\partial\eta_\theta} = \frac{\eta_\theta}{\gamma}[\varrho_\theta^2 + \eta_\theta^2]^{-\frac{1}{2}}. \tag{E.3}$$

The derivative of Ω^* with respect to λ_1 can be decomposed as follows

$$\frac{\partial\Omega^*}{\partial\lambda_1} = C_1\frac{\partial I_1^*}{\partial\lambda_1} + C_2\frac{\partial I_2^*}{\partial\lambda_1} + \frac{\partial\left(\beta[\boldsymbol{C}\,\mathbb{D}]\cdot\mathbb{D}\right)}{\partial\lambda_1}, \tag{E.4}$$

where

$$\frac{\partial I_1^*}{\partial\lambda_1} = 2\lambda_1 + \frac{\partial(\lambda_2^{*2})}{\partial\lambda_1} - \frac{2}{\lambda_1^3\lambda_2^{*2}} - \frac{1}{\lambda_1^2\lambda_2^{*4}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}, \tag{E.5}$$

$$\frac{\partial I_2^*}{\partial\lambda_1} = -\frac{2}{\lambda_1^3} - \frac{\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}}{\lambda_2^{*4}} + \lambda_1^2\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1} + 2\lambda_1\lambda_2^{*2} \tag{E.6}$$

and

$$\frac{\partial\left(\beta[\boldsymbol{C}\,\mathbb{D}]\cdot\mathbb{D}\right)}{\partial\lambda_1} = \frac{C_1\mathcal{E}}{4}\left[2\lambda_1\lambda_2^{*2} + \lambda_1^2\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right], \tag{E.7}$$

where

$$
\begin{aligned}
\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1} =& -\frac{PH[8\alpha\lambda_1 - 2\mathcal{E}\lambda_1]}{R_b[4\lambda_1 + 4\alpha\lambda_1^2 - \mathcal{E}\lambda_1^2]^2} \\
&+ \frac{\frac{P^2H^2}{R_b^2}\lambda_1 + 32\alpha^2\lambda_1^3 - 8\mathcal{E}\alpha\lambda_1^3 + 32\alpha\lambda_1 - 4\mathcal{E}\lambda_1}{\left[\sqrt{\frac{P^2H^2}{R_b^2}\lambda_1^2 - 4\alpha\mathcal{E}\lambda_1^4 + 16\alpha^2\lambda_1^4 + 32\alpha\lambda_1^2 - 4\mathcal{E}\lambda_1^2 + 16}\right][4\lambda_1 + 4\alpha\lambda_1^3 - \mathcal{E}\lambda_1^3]} \\
&- \frac{\left[\frac{PH}{R_b}\lambda_1 + \sqrt{\frac{P^2H^2}{R_b^2}\lambda_1^2 - 4\alpha\mathcal{E}\lambda_1^4 + 16\alpha^2\lambda_1^4 + 32\alpha\lambda_1^2 - 4\mathcal{E}\lambda_1^2 + 16}\right][12\alpha\lambda_1^2 - 3\mathcal{E}\lambda_1^2 + 4]}{[4\lambda_1 + 4\alpha\lambda_1^3 - \mathcal{E}\lambda_1^3]^2}. \quad \text{(E.8)}
\end{aligned}
$$

Then, the total derivatives of the two terms with respect to θ are calculated as

$$
\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial\Omega^*}{\partial\,\varrho_\theta}\right) = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\varrho_\theta}{\gamma}[\varrho_\theta^2 + \eta_\theta^2]^{-\frac{1}{2}}\right)\frac{\partial\Omega^*}{\partial\lambda_1} + \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial\Omega^*}{\partial\lambda_1}\right)\underbrace{\left[\frac{\varrho_\theta}{\gamma}[\varrho_\theta^2 + \eta_\theta^2]^{-\frac{1}{2}}\right]}_{\mathcal{W}_1}, \quad \text{(E.9)}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial\Omega^*}{\partial\eta_\theta}\right) = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\eta_\theta}{\gamma}[\varrho_\theta^2 + \eta_\theta^2]^{-\frac{1}{2}}\right)\frac{\partial\Omega^*}{\partial\lambda_1} + \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial\Omega^*}{\partial\lambda_1}\right)\underbrace{\left[\frac{\eta_\theta}{\gamma}[\varrho_\theta^2 + \eta_\theta^2]^{-\frac{1}{2}}\right]}_{\mathcal{W}_2}. \quad \text{(E.10)}
$$

Upon explicitly computing the first full derivative terms in both equations and separating the coefficients of $\varrho_{\theta\theta}$ and $\eta_{\theta\theta}$ yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\varrho_\theta}{\gamma}[\varrho_\theta^2 + \eta_\theta^2]^{-\frac{1}{2}}\right) &= \frac{\varrho_{\theta\theta}}{\gamma}[\varrho_\theta^2 + \eta_\theta^2]^{-\frac{1}{2}} - \frac{\varrho_\theta}{\gamma}[\varrho_\theta\,\varrho_{\theta\theta} + \eta_\theta\eta_{\theta\theta}][\varrho_\theta^2 + \eta_\theta^2]^{-\frac{3}{2}} \\
&= \underbrace{\left[\frac{1}{\gamma[\varrho_\theta^2 + \eta_\theta^2]^{\frac{1}{2}}} - \frac{\varrho_\theta^2}{\gamma[\varrho_\theta^2 + \eta_\theta^2]^{\frac{3}{2}}}\right]}_{\mathcal{U}_1}\varrho_{\theta\theta} \underbrace{- \frac{\varrho_\theta\,\eta_\theta}{\gamma[\varrho_\theta^2 + \eta_\theta^2]^{\frac{3}{2}}}}_{\mathcal{U}_2}\eta_{\theta\theta}, \quad \text{(E.11)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\eta_\theta}{\gamma}[\varrho_\theta^2 + \eta_\theta^2]^{-\frac{1}{2}}\right) &= \frac{\eta_{\theta\theta}}{\gamma}[\varrho_\theta^2 + \eta_\theta^2]^{-\frac{1}{2}} - \frac{\eta_\theta}{\gamma}[\varrho_\theta\,\varrho_{\theta\theta} + \eta_\theta\eta_\theta][\varrho_\theta^2 + \eta_\theta^2]^{-\frac{3}{2}}. \\
&= \underbrace{\left[\frac{1}{\gamma[\varrho_\theta^2 + \eta_\theta^2]^{\frac{1}{2}}} - \frac{\eta_\theta^2}{\gamma[\varrho_\theta^2 + \eta_\theta^2]^{\frac{3}{2}}}\right]}_{\mathcal{V}_2}\eta_{\theta\theta} \underbrace{- \frac{\varrho_\theta\,\eta_\theta}{\gamma[\varrho_\theta^2 + \eta_\theta^2]^{\frac{3}{2}}}}_{\mathcal{V}_1}\varrho_{\theta\theta}\,. \quad \text{(E.12)}
\end{aligned}
$$

The remaining full derivative term $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial\Omega^*}{\partial\lambda_1}\right)$ is expressed as

$$
\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial\Omega^*}{\partial\lambda_1}\right) = C_1\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial {I_1}^*}{\partial\lambda_1}\right) + C_2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial {I_2}^*}{\partial\lambda_1}\right) + \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{C_1\mathcal{E}}{4}\left[2\lambda_1\lambda_2^{*2} + \lambda_1^2\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right]\right), \quad \text{(E.13)}
$$

where

$$C_1\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial {I_1}^*}{\partial\lambda_1}\right)=C_1\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\left[2\lambda_1+\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}-\frac{2}{\lambda_1^3\lambda_2^{*2}}-\frac{1}{\lambda_1^2\lambda_2^{*4}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right]\right)$$
$$=2C_1\lambda_{1\theta}+C_1\left(\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right)_\theta+C_1\left[\frac{6\lambda_{1\theta}}{\lambda_1^4\lambda_2^{*2}}+\frac{2(\lambda_2^{*2})_\theta}{\lambda_1^3\lambda_2^{*4}}\right]$$
$$-C_1\left[\frac{1}{\lambda_1^2\lambda_2^{*4}}\left(\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right)_\theta-\frac{2\lambda_{1\theta}}{\lambda_1^3\lambda_2^{*4}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}-\frac{2(\lambda_2^{*2})_\theta}{\lambda_1^2\lambda_2^{*6}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right]$$
$$=C_1\Bigg[\left[2+\frac{6}{\lambda_1^4\lambda_1^{*2}}+\frac{2}{\lambda_1^3\lambda_1^{*4}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right]+\left[\frac{2}{\lambda_1^3\lambda_1^{*4}}+\frac{2}{\lambda_1^2\lambda_1^{*6}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right]\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}$$
$$+\left[1-\frac{1}{\lambda_1^2\lambda_2^{*4}}\right]\frac{\partial^2(\lambda_2^{*2})}{\partial\lambda_1^2}\Bigg]\lambda_{1\theta}, \tag{E.14}$$

$$C_2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial {I_2}^*}{\partial\lambda_1}\right)=\frac{\mathrm{d}}{\mathrm{d}\theta}\left(C_2\left[-\frac{2}{\lambda_1^3}-\frac{1}{\lambda_2^{*4}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}+\lambda_1^2\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}+2\lambda_1\lambda_2^{*2}\right]\right)$$
$$=C_2\frac{6}{\lambda_1^4}\lambda_{1\theta}-C_2\left[\frac{1}{\lambda_2^{*4}}\left(\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right)_\theta-\frac{2(\lambda_2^{*2})_\theta}{\lambda_2^{*6}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right]$$
$$+C_2\left[\lambda_1^2\left(\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right)_\theta+2\lambda_1\lambda_{1\theta}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right]+C_2\left[2\lambda_1(\lambda_2^{*2})_\theta+2\lambda_2^{*2}\lambda_{1\theta}\right]$$
$$=C_2\Bigg[\left[\frac{6}{\lambda_1^4}+2\lambda_1\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}+2\lambda_2^{*2}\right]+\left[\frac{2}{\lambda_2^{*6}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}+2\lambda_1\right]\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}$$
$$+\left[\lambda_1^2-\frac{1}{\lambda_2^{*4}}\right]\frac{\partial^2(\lambda_2^{*2})}{\partial\lambda_1^2}\Bigg]\lambda_{1\theta}, \tag{E.15}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{C_1\mathcal{E}}{4}\left[2\lambda_1\lambda_2^{*2}+\lambda_1^2\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right]\right)$$
$$=\frac{C_1\mathcal{E}}{4}\left[2\lambda_2^{*2}\lambda_{1\theta}+2\lambda_1(\lambda_2^{*2})_\theta+2\lambda_1\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\lambda_{1\theta}+\lambda_1^2\left(\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right)_\theta\right]$$
$$=\frac{C_1\mathcal{E}}{4}\left[2\lambda_2^{*2}+4\lambda_1\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}+\lambda_1^2\frac{\partial^2(\lambda_2^{*2})}{\partial\lambda_1^2}\right]\lambda_{1\theta}. \tag{E.16}$$

The term

$$\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}=-\frac{PH[8\alpha\lambda_1-2\mathcal{E}\lambda_1]}{R_b[4\lambda_1+4\alpha\lambda_1^3-\mathcal{E}\lambda_1^3]^2}$$
$$+\underbrace{\frac{\frac{P^2H^2}{R_b^2}\lambda_1+32\alpha^2\lambda_1^3-8\mathcal{E}\alpha\lambda_1^3+32\alpha\lambda_1-4\mathcal{E}\lambda_1}{\left[\sqrt{\frac{P^2H^2}{R_b^2}\lambda_1^2-4\alpha\mathcal{E}\lambda_1^4+16\alpha^2\lambda_1^4+32\alpha\lambda_1^2-4\mathcal{E}\lambda_1^2+16}\right][4\lambda_1+4\alpha\lambda_1^3-\mathcal{E}\lambda_1^3]}}_{\mathcal{C}}$$
$$-\underbrace{\frac{\left[\frac{PH}{R_b}\lambda_1+\sqrt{\frac{P^2H^2}{R_b^2}\lambda_1^2-4\alpha\mathcal{E}\lambda_1^4+16\alpha^2\lambda_1^4+32\alpha\lambda_1^2-4\mathcal{E}\lambda_1^2+16}\right][12\alpha\lambda_1^2-3\mathcal{E}\lambda_1^2+4]}{[4\lambda_1+4\alpha\lambda_1^3-\mathcal{E}\lambda_1^3]^2}}_{\mathcal{D}}, \tag{E.17}$$

and hence

$$\frac{\partial^2(\lambda_2^{*2})}{\partial\lambda_1^2} = \frac{\partial}{\partial\lambda_1}\left(-\frac{PH[8\alpha\lambda_1 - 2\mathcal{E}\lambda_1]}{R_b[4\lambda_1 + 4\alpha\lambda_1^3 - \mathcal{E}\lambda_1^3]^2}\right) + \frac{\partial\mathcal{A}}{\partial\lambda_1} + \frac{\partial\mathcal{B}}{\partial\lambda_1}, \tag{E.18}$$

where

$$\frac{\partial}{\partial\lambda_1}\left(-\frac{PH[8\alpha\lambda_1 - 2\mathcal{E}\lambda_1]}{R_b[4\lambda_1 + 4\alpha\lambda_1^2 - \mathcal{E}\lambda_1^2]^2}\right) = -\frac{PH}{R_b}\left[\frac{8\alpha - 2\mathcal{E}}{[4\lambda_1 + 4\alpha\lambda_1^2 - \mathcal{E}\lambda_1^2]^2} - \frac{2[8\alpha\lambda_1 - 2\mathcal{E}\lambda_1]^2}{[4\lambda_1 + 4\alpha\lambda_1^2 - \mathcal{E}\lambda_1^2]^3}\right], \tag{E.19}$$

$$\begin{aligned}\frac{\partial\mathcal{C}}{\partial\lambda_1} =& \frac{\frac{P^2H^2}{R_b^2} + 96\alpha^2\lambda_1^2 - 24\mathcal{E}\alpha\lambda_1^2 + 32\alpha - 4\mathcal{E}}{\left[\sqrt{\frac{P^2H^2}{R_b^2}\lambda_1^2 - 4\alpha\mathcal{E}\lambda_1^4 + 16\alpha^2\lambda_1^4 + 32\alpha\lambda_1^2 - 4\mathcal{E}\lambda_1^2 + 16}\right][4\lambda_1 + 4\alpha\lambda_1^3 - \mathcal{E}\lambda_1^3]} \\ &- \frac{\left[\frac{P^2H^2}{R_b^2}\lambda_1 + 32\alpha^2\lambda_1^3 - 8\mathcal{E}\alpha\lambda_1^3 + 32\alpha\lambda_1 - 4\mathcal{E}\lambda_1\right]^2}{\left[\sqrt{\frac{P^2H^2}{R_b^2}\lambda_1^2 - 4\alpha\mathcal{E}\lambda_1^4 + 16\alpha^2\lambda_1^4 + 32\alpha\lambda_1^2 - 4\mathcal{E}\lambda_1^2 + 16}\right]^3[4\lambda_1 + 4\alpha\lambda_1^3 - \mathcal{E}\lambda_1^3]} \\ &- \frac{\left[\frac{P^2H^2}{R_b^2}\lambda_1 + 32\alpha^2\lambda_1^3 - 8\mathcal{E}\alpha\lambda_1^3 + 32\alpha\lambda_1 - 4\mathcal{E}\lambda_1\right][12\alpha\lambda_1^2 - 3\mathcal{E}\lambda_1^2 + 4]}{\left[\sqrt{\frac{P^2H^2}{R_b^2}\lambda_1^2 - 4\alpha\mathcal{E}\lambda_1^4 + 16\alpha^2\lambda_1^4 + 32\alpha\lambda_1^2 - 4\mathcal{E}\lambda_1^2 + 16}\right][4\lambda_1 + 4\alpha\lambda_1^3 - \mathcal{E}\lambda_1^3]^2},\end{aligned} \tag{E.20}$$

and

$$\begin{aligned}\frac{\partial\mathcal{D}}{\partial\lambda_1} =& \frac{\left[\frac{PH}{R_b}\lambda_1 + \sqrt{\frac{P^2H^2}{R_b^2}\lambda_1^2 - 4\alpha\mathcal{E}\lambda_1^4 + 16\alpha^2\lambda_1^4 + 32\alpha\lambda_1^2 - 4\mathcal{E}\lambda_1^2 + 16}\right][(24\alpha - 6\mathcal{E})\lambda_1]}{[4\lambda_1 + 4\alpha\lambda_1^3 - \mathcal{E}\lambda_1^3]^2} \\ &+ \frac{\left[\frac{PH}{R_b} + \frac{\frac{P^2H^2}{R_b^2}\lambda_1 + 32\alpha^2\lambda_1^3 - 8\mathcal{E}\alpha\lambda_1^3 + 32\alpha\lambda_1 - 4\mathcal{E}\lambda_1}{\sqrt{\frac{P^2H^2}{R_b^2}\lambda_1^2 - 4\alpha\mathcal{E}\lambda_1^4 + 16\alpha^2\lambda_1^4 + 32\alpha\lambda_1^2 - 4\mathcal{E}\lambda_1^2 + 16}}\right][12\alpha\lambda_1^2 - 3\mathcal{E}\lambda_1^2 + 4]}{[4\lambda_1 + 4\alpha\lambda_1^3 - \mathcal{E}\lambda_1^3]^2} \\ &- 2\frac{\left[\frac{PH}{R_b}\lambda_1 + \sqrt{\frac{P^2H^2}{R_b^2}\lambda_1^2 - 4\alpha\mathcal{E}\lambda_1^4 + 16\alpha^2\lambda_1^4 + 32\alpha\lambda_1^2 - 4\mathcal{E}\lambda_1^2 + 16}\right][12\alpha\lambda_1^2 - 3\mathcal{E}\lambda_1^2 + 4]^2}{[4\lambda_1 + 4\alpha\lambda_1^3 - \mathcal{E}\lambda_1^3]^3}.\end{aligned} \tag{E.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial\Omega^*}{\partial\lambda_1}\right) = C_1\mathcal{Z}\mathcal{W}_1\,\varrho_{\theta\theta} + C_1\mathcal{Z}\mathcal{W}_2\eta_{\theta\theta} \tag{E.22}$$

where

$$\begin{aligned}\mathcal{Z} =& \left[\left[2 + \frac{6}{\lambda_1^4\lambda_1^{*2}} + \frac{2}{\lambda_1^3\lambda_1^{*4}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right] + \left[\frac{2}{\lambda_1^3\lambda_1^{*4}} + \frac{2}{\lambda_1^2\lambda_1^{*6}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1}\right]\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1} + \left[1 - \frac{1}{\lambda_1^2\lambda_2^{*4}}\right]\frac{\partial^2(\lambda_2^{*2})}{\partial\lambda_1^2}\right] \\ &+ \alpha\left[\left[\frac{6}{\lambda_1^4} + 2\lambda_1\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1} + 2\lambda_2^{*2}\right] + \left[\frac{2}{\lambda_2^{*6}}\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1} + 2\lambda_1\right]\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1} + \left[\lambda_1^2 - \frac{1}{\lambda_2^{*4}}\right]\frac{\partial^2(\lambda_2^{*2})}{\partial\lambda_1^2}\right] \\ &+ \frac{\mathcal{E}}{4}\left[2\lambda_2^{*2} + 4\lambda_1\frac{\partial(\lambda_2^{*2})}{\partial\lambda_1} + \lambda_1^2\frac{\partial^2(\lambda_2^{*2})}{\partial\lambda_1^2}\right].\end{aligned} \tag{E.23}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial\Omega^*}{\partial\,\varrho_\theta}\right) = \left[\mathcal{U}_1\frac{\partial\Omega^*}{\partial\lambda_1} + C_1\mathcal{Z}\mathcal{W}_1^2\right]\varrho_{\theta\theta} + \left[\mathcal{U}_2\frac{\partial\Omega^*}{\partial\lambda_1} + C_1\mathcal{Z}\mathcal{W}_2\mathcal{W}_1\right]\eta_{\theta\theta} \tag{E.24}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial\Omega^*}{\partial\eta_\theta}\right) = \left[\mathcal{V}_1\frac{\partial\Omega^*}{\partial\lambda_1} + C_1\mathcal{Z}\mathcal{W}_1\mathcal{W}_2\right]\varrho_{\theta\theta} + \left[\mathcal{V}_2\frac{\partial\Omega^*}{\partial\lambda_1} + C_1\mathcal{Z}\mathcal{W}_2^2\right]\eta_{\theta\theta} \tag{E.25}$$

One introduces a term $\mathcal{Y}$ to cancel out the material property C_1:

$$\mathcal{Y} = \frac{\partial \Omega^*}{\partial \lambda_1} / C_1. \tag{E.26}$$

The governing equations are now written as ODEs:

$$[1+\gamma\cos\theta]\left[\left[\mathcal{U}_1\mathcal{Y}+\mathcal{Z}\mathcal{W}_1^2\right]\varrho_{\theta\theta}+[\mathcal{U}_2\mathcal{Y}+\mathcal{Z}\mathcal{W}_2\mathcal{W}_1]\,\eta_{\theta\theta}\right]-\sin\theta\varrho_\theta[\varrho_\theta^2+\eta_\theta^2]^{-\frac{1}{2}}\mathcal{Y}+\frac{P\,\varrho\,\eta_\theta}{\gamma}=0 \tag{E.27}$$

$$[1+\gamma\cos\theta]\left[[\mathcal{V}_1\mathcal{Y}+\mathcal{Z}\mathcal{W}_1\mathcal{W}_2]\,\varrho_{\theta\theta}+\left[\mathcal{V}_2\mathcal{Y}+\mathcal{Z}\mathcal{W}_2^2\right]\eta_{\theta\theta}\right]-\sin\theta\eta_\theta[\varrho_\theta^2+\eta_\theta^2]^{-\frac{1}{2}}\mathcal{Y}-\frac{P\,\varrho\,\varrho_\theta}{\gamma}=0. \tag{E.28}$$

The coefficients in ODEs system (34) are modified as

$$\begin{aligned}
A_1^* &= [1+\gamma\cos\theta]\left[\mathcal{U}_1\mathcal{Y}+\mathcal{Z}\mathcal{W}_1^2\right],\\
A_2^* &= [1+\gamma\cos\theta]\left[\mathcal{U}_2\mathcal{Y}+\mathcal{Z}\mathcal{W}_1\mathcal{W}_2\right],\\
A_3^* &= -\sin\theta\varrho_\theta[\varrho_\theta^2+\eta_\theta^2]^{-\frac{1}{2}}\mathcal{Y}+\frac{P\,\varrho\,\eta_\theta}{\gamma},
\end{aligned} \tag{E.29}$$

and

$$\begin{aligned}
B_1^* &= [1+\gamma\cos\theta]\left[\mathcal{V}_1\mathcal{Y}+\mathcal{Z}\mathcal{W}_1\mathcal{W}_2\right],\\
B_2^* &= [1+\gamma\cos\theta]\left[\mathcal{V}_2\mathcal{Y}+\mathcal{Z}\mathcal{W}_2^2\right],\\
B_3^* &= -\sin\theta\eta_\theta[\varrho_\theta^2+\eta_\theta^2]^{-\frac{1}{2}}\mathcal{Y}-\frac{P\,\varrho\,\varrho_\theta}{\gamma}.
\end{aligned} \tag{E.30}$$