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SMOOTH, NONSYMPLECTIC EMBEDDINGS OF RATIONAL BALLS IN THE COMPLEX PROJECTIVE PLANE

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ABSTRACT. We exhibit an infinite family of rational homology balls which embed smoothly but not symplectically in the complex projective plane. We also obtain a new lattice embedding obstruction from Donaldson's diagonalisation theorem, and use this to show that no two of our examples may be embedded disjointly.

1. INTRODUCTION

A Markov triple is a positive integer solution (p_1, p_2, p_3) to the Markov equation

$$(1) \quad p_1^2 + p_2^2 + p_3^2 = 3p_1p_2p_3.$$

Each Markov triple gives rise to an embedding

$$(2) \quad \bigsqcup_{i=1}^3 B_{p_i, q_i} \hookrightarrow \mathbb{CP}^2$$

of a disjoint union of three rational homology balls in the complex projective plane. Here $B_{p,q}$ is the rational homology ball smoothing of the quotient singularity $\frac{1}{p^2}(1, pq-1)$. The embedding in (2) arises by smoothing the three singular points in the weighted projective space $\mathbb{P}(p_1^2, p_2^2, p_3^2)$, and the numbers q_i are given by

$$q_i = \pm 3p_j/p_k \pmod{p_i},$$

where i, j, k is a permutation of $1, 2, 3$. The apparent sign ambiguity here is due to the fact that $B_{p,q} \cong B_{p,p-q}$.

Hacking and Prokhorov proved in [5] that any projective surface with quotient singularities which admits a smoothing to \mathbb{CP}^2 is \mathbb{Q} -Gorenstein deformation equivalent to some $\mathbb{P}(p_1^2, p_2^2, p_3^2)$ as above. Evans and Smith proved in [4] that any disjoint union $\bigsqcup_{i \in \mathcal{I}} B_{p_i, q_i}$ which admits a symplectic embedding in \mathbb{CP}^2 arises in this way, with $|\mathcal{I}| \leq 3$.

Let $F(2n-1)$ denote the n th odd Fibonacci number, defined by the recursion

$$(3) \quad F(2n+3) = 3F(2n+1) - F(2n-1), \quad F(1) = 1, \quad F(3) = 2.$$

Then $(1, F(2n-1), F(2n+1))$ is a Markov triple for each $n \in \mathbb{N}$, showing in particular that $B_{F(2n+1), F(2n-3)}$ admits a symplectic embedding in \mathbb{CP}^2 for each $n > 1$.

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In [8] we mentioned but overlooked the significance of the following result. Here $\Delta_{p,q}$ is a properly embedded surface in the 4-ball whose double branched cover is $B_{p,q}$, and P_+ is the unknotted Möbius band in the 4-ball with normal Euler number 2; see [8] for further details.

Theorem 1. *For each $n \in \mathbb{N}$, the slice surface $\Delta_{F(2n+1), F(2n-1)}$ admits a simple embedding as a sublevel surface of the unknotted Möbius band P_+ . Taking double branched covers yields a simple smooth embedding*

$$B_{F(2n+1), F(2n-1)} \hookrightarrow \mathbb{CP}^2.$$

If $n > 1$, then $B_{F(2n+1), F(2n-1)}$ does not embed symplectically in \mathbb{CP}^2 .

Theorem 1 gives the first-known smooth embeddings of rational balls $B_{p,q}$ in the complex projective plane that do not arise from symplectic embeddings. This shows that the smooth embedding problem has an as-yet-unknown solution which differs from that to the symplectic problem solved by Evans-Smith. Bulent Tosun has informed the author that work of Nemirovski-Segal [7] implies the existence of a rational ball, bounded by a Seifert fibred space with 3 exceptional fibres, which embeds smoothly but not symplectically in \mathbb{CP}^2 . Most of the embeddings obtained in [8], but not those given in Theorem 1, have since been reproved and generalised by different methods in [9].

A conjecture of Kollár [6] would imply that at most three rational balls B_{p_i, q_i} may embed smoothly and disjointly in \mathbb{CP}^2 . The following result gives some mild support to this conjecture.

Theorem 2. *It is not possible to smoothly embed a disjoint union $\bigsqcup_{i \in \mathcal{I}} B_{p_i, q_i}$ of two or more of the balls from Theorem 1 in \mathbb{CP}^2 , where each (p_i, q_i) is a consecutive pair of odd Fibonacci numbers.*

This result uses a new obstruction derived from Donaldson's diagonalisation theorem [3]. This is stated in Proposition 3.2.

Corrigendum to [8]. In [8, sentence after Theorem 5, and Remark 4.1] we incorrectly stated that $B_{F(2n+1), F(2n-1)}$ embeds symplectically in \mathbb{CP}^2 . I am very grateful to Giancarlo Urzúa who reminded me that the Markov triple $(1, F(2n-1), F(2n+1))$ gives rise to a symplectic embedding in \mathbb{CP}^2 of $B_{F(2n+1), F(2n-3)}$, and not of $B_{F(2n+1), F(2n-1)}$.

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2. SMOOTH EMBEDDINGS

In this section we prove Theorem 1, using the method from [8].

We refer the reader to [1] for an excellent and readable source on Markov numbers. Suppose that (p, a, b) is a solution to the Markov equation (1) with $p > a, b$. By [1, Corollary 3.4], the integers in a Markov triple are pairwise relatively prime, so that there are unique solutions $x = u, u'$ to

$$b \equiv \pm xa \pmod{p}.$$

These satisfy $u + u' \equiv 0 \pmod{p}$, so that one of them (say u) is between 0 and $p/2$; we call this number u the characteristic number of the Markov triple (p, a, b) . The Markov equation gives $a^2 + b^2 \equiv 0 \pmod{p}$, from which it follows that

$$u^2 \equiv -1 \pmod{p}.$$

I am grateful to Jonny Evans for helping me to see the following result.

Lemma 2.1. *Let $n \in \mathbb{N}$. The rational ball $B_{F(2n+1), F(2n-1)}$ embeds symplectically in \mathbb{CP}^2 if and only if $n = 1$.*

Proof. From [4, Theorem 4.15] we have that $B_{p,q}$ embeds symplectically in \mathbb{CP}^2 if and only if p is the maximum of a Markov triple (a, b, p) , and $q = \pm 3b/a \pmod{p}$. Then in fact $q = \pm 3u$, where u is the characteristic number of the Markov triple.

For $n > 1$, the odd Fibonacci number $F(2n + 1)$ is the maximum of the Markov triple $(1, F(2n - 1), F(2n + 1))$, from which it follows that $B_{F(2n+1), F(2n-1)}$ embeds symplectically. Also note that the characteristic number of this Markov triple is $F(2n - 1)$, and $F(2n - 1)^2 \equiv -1 \pmod{F(2n + 1)}$.

Then $B_{F(2n+1), F(2n-1)}$ embeds symplectically if and only if $F(2n + 1)$ is the maximum of another Markov triple $(a, b, F(2n + 1))$, and $F(2n - 1) = \pm 3u$, where u is the characteristic number of the triple $(a, b, F(2n + 1))$. This would imply that

$$-1 \equiv F(2n - 1)^2 \equiv 9u^2 \equiv -9 \pmod{F(2n + 1)}.$$

The only odd Fibonacci numbers which divide 8 are $F(1) = 1$ and $F(3) = 2$, so we conclude that $n = 1$.

Finally, $F(3) = 2$ is the maximum of the Markov triple $(1, 1, 2)$ and $B_{F(3), F(1)} = B_{2,1}$ does embed symplectically. \square

Proof of Theorem 1. As noted in the proof of Lemma 2.1, the Markov triple $(1, 1, 2)$ gives rise to an embedding of $B_{F(3), F(1)} = B_{2,1}$ in \mathbb{CP}^2 . Suppose now that $n > 1$. Induction using (3) yields the Hirzebruch-Jung continued fraction expansion

$$\frac{F(2n + 1)}{F(2n - 1)} = [3^{n-1}, 2].$$

Now using [8, Lemma 3.1] we have

$$\frac{F(2n + 1)^2}{F(2n + 1)F(2n - 1) - 1} = [3^{n-1}, 5, 3^{n-2}, 2].$$

These continued fractions may be used to describe the surface $\Delta_{F(2n+1), F(2n-1)}$, as described in [8].

The proof that $\Delta_{F(2n+1), F(2n-1)}$ is a sublevel surface of P_+ is a minor modification of the proof of [8, Theorem 5]. We refer the reader to that source for details.

Consider the first diagram shown in Figure 1. This represents a surface Σ bounded by the unknot, which we claim is P_+ . Note first that the band move corresponding to the blue band labelled 0 converts the diagram to one of $\Delta_{F(2n+1), F(2n-1)}$, which is the slice disk described by Casson and Harer [2] for the two-bridge knot $S(F(2n+1)^2, F(2n+1)F(2n-1)-1)$. This shows that $\Delta_{F(2n+1), F(2n-1)}$ is a sublevel surface of the surface Σ . It remains to see that Σ is the unknotted Möbius band P_+ whose double branched cover is \mathbb{CP}^2 minus a 4-ball.

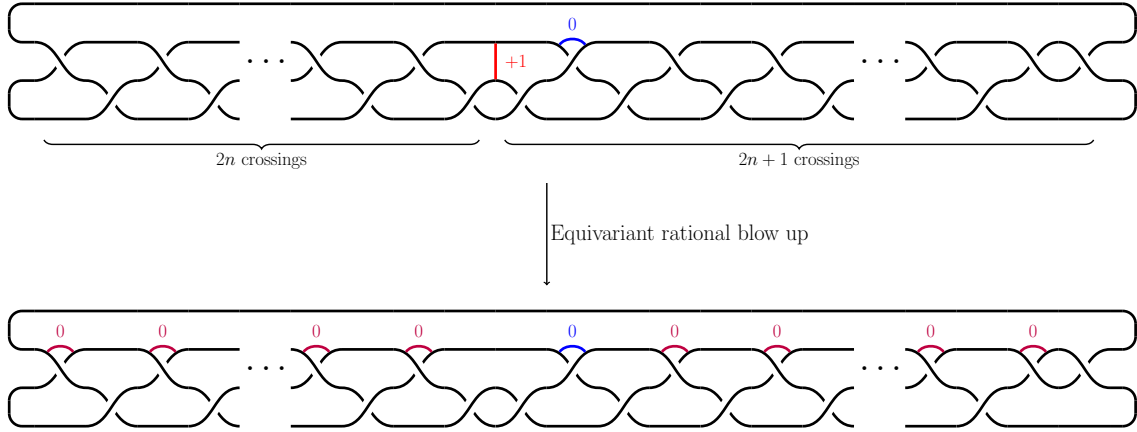


FIGURE 1. **The slice disk $\Delta_{F(2n+1), F(2n-1)}$ as a sublevel surface of P_+ , and the resulting equivariant rational blow up.** Numbers beside bands give the signed count of half-twists or crossings.

Figure 2 shows a sequence of isotopies and band slides converting Σ to P_+ in the first case of interest which is $n = 2$. Taking double branched covers we see that $B_{5,2}$ admits a smooth embedding in \mathbb{CP}^2 . The proof for $n > 2$ follows by an induction argument involving band slides similar to those in Figure 2. The inductive step is shown in Figure 3.

Recall that an embedding of $B_{p,q}$ in a 4-manifold Z is called simple if the resulting rational blow up of Z is obtainable by a sequence of ordinary blow ups. The proof that the embeddings described above are simple follows as in [8, Proposition 5.1]; we again refer the reader to [8] for more details on equivariant rational blow up, and to Section 3 for a description of rational blow up. We describe here a slightly shorter version of the proof at the level of double branched covers. The second diagram in Figure 1 represents the surface in the 4-ball pushed in from the black surface of the

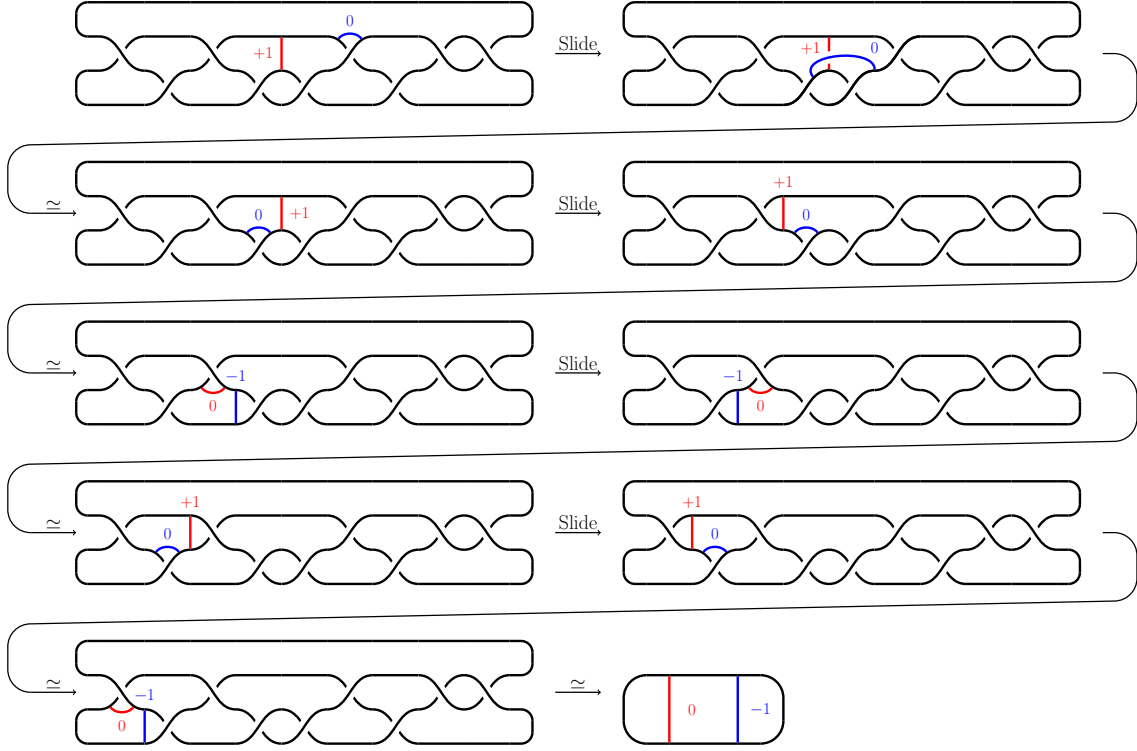


FIGURE 2. The slice disk $\Delta_{5,2}$ is a sublevel surface of P_+ .

two-bridge diagram shown, using a chessboard colouring in which the unbounded region is white. The rational blow up of \mathbb{CP}^2 , minus a 4-ball, is the double cover X of the 4-ball branched along this black surface, which in turn is the plumbing of disk bundles over S^2 corresponding to the linear graph with weights

$$(-3)^{n-1}, -2, -1, (-3)^{n-2}, -2,$$

where $(-3)^m$ denotes -3 repeated m times. A sequence of -1 blow downs reduces this to the linear plumbing with weights -3 and 0 , which is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, again minus a ball. It follows that

$$X \cong \mathbb{CP}^2 \# (2n-1) \overline{\mathbb{CP}^2}.$$

Together with Lemma 2.1, this completes the proof of Theorem 1. \square

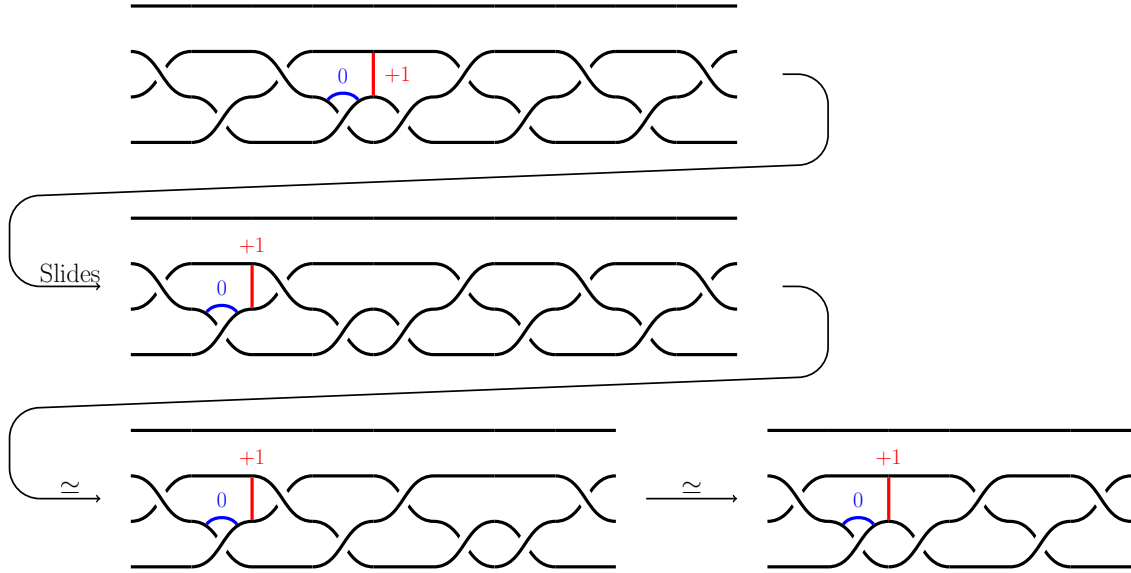


FIGURE 3. **The inductive step.** The band slides are similar to those shown in Figure 2. This shows how to transform the first diagram in Figure 1 with $n = k$, to the same diagram with $n = k - 1$.

3. AN OBSTRUCTION FROM DONALDSON'S DIAGONALISATION THEOREM

In this section, we derive a lattice embedding obstruction to smoothly embedding a rational homology ball bounded by a lens space, or a disjoint union of such, in \mathbb{CP}^2 . We begin by setting some conventions and terminology.

All homology and cohomology groups in this section have integer coefficients. Recall that if X is a smooth 4-manifold, possibly with boundary, then its intersection lattice Λ_X consists of the free abelian group $H_2(X)/Tors$ together with the symmetric bilinear intersection pairing. The term lens space will be used here to refer to $L(p, q)$ with $p > q \geq 1$; in particular not S^3 or $S^2 \times S^1$. Given integers a_1, \dots, a_k , the linear lattice $\Lambda(a_1, \dots, a_k)$ is defined to be the free abelian group with generators v_1, \dots, v_n , and with symmetric bilinear pairing given by

$$(4) \quad v_i \cdot v_j = \begin{cases} a_i & \text{if } i = j; \\ -1 & \text{if } |i - j| = 1; \\ 0 & \text{if } |i - j| > 1. \end{cases}$$

As this is the lattice associated to a weighted linear graph, we often refer to the generators v_1, \dots, v_k as vertices. Recall that a lens space $L(p, q)$ is the boundary of a plumbing C of disk bundles over spheres determined by the weighted linear graph

with weights $a_1, \dots, a_k \geq 2$ where

$$\frac{p}{p-q} = [a_1, a_2, \dots, a_k] := a_1 - \frac{1}{a_2 - \dots - \frac{1}{a_k}}.$$

The intersection lattice of C is then $\Lambda(a_1, \dots, a_k)$.

Let B be a rational homology ball with lens space boundary. Given an embedding $B \hookrightarrow \mathbb{CP}^2$, we let M be the complement $\mathbb{CP}^2 \setminus B$ and “rationally blow up” to obtain the closed positive-definite manifold $M \cup C$, where C is the positive-definite plumbed manifold bounded by ∂B . Donaldson’s diagonalisation theorem then implies the existence of a lattice embedding

$$(5) \quad \Lambda_M \oplus \Lambda_C \hookrightarrow \mathbb{Z}^m,$$

where Λ_M and Λ_C are the intersection lattices of M and C respectively, and m is the sum of their ranks.

The reader familiar with the use of such lattice obstructions will note that since M is a submanifold of \mathbb{CP}^2 , and since $Y = \partial B$ bounds a rational ball, each of Λ_M and Λ_C admit finite-index embeddings in diagonal unimodular lattices, so that an embedding as in (5) must in fact exist, with the first factor embedding in \mathbb{Z} and the second in the orthogonal \mathbb{Z}^{m-1} . We will show that simple topological considerations place further restrictions on the lattice embedding in (5), giving rise to a useful obstruction, which also extends to the case of an embedding of a disjoint union of rational balls.

Lemma 3.1. *Let B_i be rational homology balls bounded by lens spaces for $i = 1, \dots, n$, and suppose that the disjoint union $\bigsqcup_i B_i$ embeds smoothly in \mathbb{CP}^2 . Then the complement $M = \mathbb{CP}^2 \setminus \bigsqcup_i B_i$ has $H_1(M; \mathbb{Z}) = 0$ and $H_2(M; \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. We use the Mayer-Vietoris sequence and induction. The base case is $n = 0$ and $M = \mathbb{CP}^2$.

Now suppose $M' = \mathbb{CP}^2 \setminus \bigsqcup_{i=1}^{n-1} B_i$ has $H_1(M'; \mathbb{Z}) = 0$ and $H_2(M'; \mathbb{Z}) \cong \mathbb{Z}$. Then

$$M' = M \cup_Y B_n,$$

where $Y = L(p_n^2, q_n)$ has $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}/p_n^2\mathbb{Z}$. We have $H_2(B_n; \mathbb{Z}) = 0$, since it is a torsion subgroup of $H_2(M'; \mathbb{Z}) \cong \mathbb{Z}$; then from the long exact sequence of the pair (B_n, Y) , we have $H_1(B_n; \mathbb{Z}) \cong \mathbb{Z}/p_n\mathbb{Z}$. The Mayer-Vietoris sequence, with integer coefficients, shows that $H_2(M)$ is a finite-index subgroup of \mathbb{Z} , hence $H_2(M) \cong \mathbb{Z}$. The same sequence shows that there is a surjection from $\mathbb{Z}/p_n^2\mathbb{Z}$ to $H_1(M) \oplus \mathbb{Z}/p_n\mathbb{Z}$, from which it follows that the latter direct sum is finite cyclic and also that cyclic summands of $H_1(M)$ have orders dividing p_n . We conclude that $H_1(M)$ must be trivial. \square

We recall the notion of rational blow up, and modify and generalise it for our convenience. If a disjoint union $\bigsqcup_i B_i$ embeds smoothly in some 4-manifold Z , where each B_i is a rational ball bounded by a lens space $L(p_i, q_i)$, then we may excise each B_i

and replace it by the positive-definite plumbed manifold C_i bounded by $L(p_i, p_i - q_i)$ to obtain a new manifold

$$X = M \cup C,$$

called the positive rational blow up of Z . Here M is the complement of $\bigsqcup_{i=1}^n B_i$ in Z , and C is the disjoint union $\bigsqcup_{i=1}^n C_i$ of plumbed manifolds. We assume that all weights in each plumbing C_i are at least 2.

Proposition 3.2. *Let B_i be rational homology balls bounded by lens spaces for $i = 1, \dots, n$, and suppose that the disjoint union $\bigsqcup_i B_i$ embeds smoothly in \mathbb{CP}^2 . Let $X = M \cup C$ be the resulting positive rational blow up of \mathbb{CP}^2 . Then there exists a finite-index lattice embedding*

$$(6) \quad \Lambda_M \oplus \Lambda_C \hookrightarrow \mathbb{Z}^m,$$

such that each unit vector $e \in \mathbb{Z}^m$ has nonzero pairing with each of Λ_M and Λ_C . Moreover the image of the generator of Λ_M is a primitive vector in \mathbb{Z}^m .

Remark 3.3. *Let A be the matrix of the embedding in (6) in terms of a basis v_1, \dots, v_m for Λ_X , where $v_1 \in \Lambda_M$ and $v_2, \dots, v_m \in \Lambda_C$, and an orthonormal basis for \mathbb{Z}^m . Then the proposition states that each row of A has at least two nonzero entries including one in the first column, and also that the entries of the first column of A , which are all nonzero, have no common divisor.*

The known embeddings mentioned earlier in this section each give rise to a block diagonal matrix A which does not satisfy the condition in the proposition.

Proof of Proposition 3.2. Let Y denote the union of lens spaces which is the common boundary of M and C . Let e be a unit vector in Λ_X . We may write

$$e = e_M + e_C,$$

where $e_M \in H_2(M, Y)$ and $e_C \in H_2(C, Y)$. There are no unit vectors in Λ_M , which is a rank one lattice whose generator squared is the order of the first homology of Y . There are also no unit vectors in Λ_C since we assumed all weights in each plumbing are at least 2. It follows that e_M and e_C are both nonzero.

Since $H_1(M) = 0$ by Lemma 3.1, all homology groups of M are in fact torsion-free by standard arguments using universal coefficients, Poincaré-Lefschetz duality, and the long exact sequence of the pair. It follows that the second homology group $H_2(M)$ is the underlying group of the lattice Λ_M , and the relative homology group $H_2(M, Y)$ is the underlying group of the dual lattice Λ_M^* via the universal coefficient theorem. Then since Λ_M is positive definite, we see that an element of $H_2(M, Y)$ is nonzero if and only if it has nonzero intersection with some element in $H_2(M)$. Thus in particular e_M and also e has nonzero intersection with some element of $H_2(M)$. The same argument applies to Λ_C , so that e_C and also e has nonzero pairing with some element of $H_2(C)$ which is the underlying group of Λ_C .

Finally let v denote the image in $H_2(X)$ of the generator of Λ_M , and suppose that $v = kw$ for some $k \in \mathbb{N}$ and $w \in H_2(X)$. As above we write $w = w_M + w_C$ and we conclude that $w_C = 0$ since it has zero pairing with all of Λ_C . This implies $w = w_M \in \Lambda_M$, but then $k = 1$ since v is the generator. \square

In what follows we study lattice embeddings $\Lambda \hookrightarrow \mathbb{Z}^m$ up to lattice automorphisms of \mathbb{Z}^m , or in other words, up to reordering of the orthonormal basis e_1, \dots, e_m , and/or changing signs of some orthonormal basis elements. Embeddings of linear lattices all of whose weights are 2 or 3 are very restricted, since up to $\text{Aut}(\mathbb{Z}^m)$, vectors $v \in \mathbb{Z}^m$ with $v \cdot v = 2$ or $v \cdot v = 3$ take the form $v = e_1 + e_2$ or $v = e_1 + e_2 + e_3$.

Example 3.4. *The rational ball $B_{3,1}$ does not embed smoothly in \mathbb{CP}^2 .*

Proof. The boundary of $B_{3,1}$ is the lens space $L(9, 2)$, which also bounds the positive-definite plumbing C with weights $[2, 2, 2, 3]$. Let v_2, \dots, v_5 be the generators of the linear lattice $\Lambda_C = \Lambda(2, 2, 2, 3)$ as in (4), and let v_1 be the generator of the rank one lattice $\Lambda_M = \Lambda(9)$. Let e_1, \dots, e_5 be an orthonormal basis for \mathbb{Z}^5 . There is, up to lattice automorphisms of \mathbb{Z}^5 , a unique embedding

$$\Lambda_M \oplus \Lambda_C \hookrightarrow \mathbb{Z}^5;$$

this takes v_1 to $3e_1$, v_i to $-e_i + e_{i+1}$ for $2 \leq i \leq 4$, and v_5 to $e_2 + e_3 + e_4$. This does not satisfy the conditions of Proposition 3.2, since e_i has zero pairing with Λ_M for $i > 1$ and e_1 has zero pairing with Λ_C . \square

Lemma 3.5. *Up to $\text{Aut}(\mathbb{Z}^m)$, there are precisely two ways to embed the linear lattice $\Lambda(2, 2, 2)$ in \mathbb{Z}^m , where $m \geq 4$. The first has image in a \mathbb{Z}^3 sublattice of \mathbb{Z}^m , and its orthogonal complement in this sublattice is the zero sublattice. The second has image in a \mathbb{Z}^4 sublattice, and its orthogonal complement in \mathbb{Z}^4 is spanned by a vector w with $w \cdot w = 4$.*

Let $n > 1$ and let Λ denote the linear lattice $\Lambda(3^{n-1}, 2, 2, 3^{n-1}, 2)$, with rank $r = 2n + 1$. Up to $\text{Aut}(\mathbb{Z}^m)$, there are precisely three ways to embed Λ in \mathbb{Z}^m , where $m \in \mathbb{N}$ is sufficiently large. The first has image in a \mathbb{Z}^r sublattice, and its orthogonal complement in this sublattice is the zero sublattice. The second has image in a \mathbb{Z}^{r+1} sublattice, and its orthogonal complement in \mathbb{Z}^{r+1} is spanned by a vector w with $w \cdot w = F(2n + 1)^2$. The third has image in a \mathbb{Z}^{4n} sublattice, and its orthogonal complement in \mathbb{Z}^{4n} contains no unit vectors.

Proof. For the first case, we can either map the vertices of $\Lambda(2, 2, 2)$ to $-e_1 + e_2, -e_2 + e_3, e_1 + e_2$ or to $-e_1 + e_2, -e_2 + e_3, -e_3 + e_4$. It is straightforward to see there are no other possibilities.

In the second case we begin by embedding the two adjacent vertices of weight two. Up to automorphism of \mathbb{Z}^m , these are mapped to $-e_1 + e_2$ and $-e_2 + e_3$. By inspection, the linear lattice $\Lambda(3, 2, 2, 3)$, which is a sublattice of Λ , admits three

possible embeddings up to symmetry as follows:

$$\begin{aligned}
 & -e_2 - e_3 - e_4, -e_1 + e_2, -e_2 + e_3, e_1 + e_2 - e_4; \\
 (7) \quad & -e_2 - e_3 - e_4, -e_1 + e_2, -e_2 + e_3, -e_3 + e_4 + e_5; \\
 \text{or} \quad & e_1 + e_4 + e_5, -e_1 + e_2, -e_2 + e_3, -e_3 + e_6 + e_7.
 \end{aligned}$$

The first of these does not extend to an embedding of $\Lambda(3, 2, 2, 3, 2)$ or $\Lambda(3, 2, 2, 3, 3)$ so we discard it. By a simple induction argument, the second of these extends uniquely to an embedding of $\Lambda(3^{n-1}, 2, 2, 3^{n-1})$ as follows:

$$\begin{aligned}
 & -e_{2n-2} - e_{2n-1} - e_{2n}, \dots, -e_4 - e_5 - e_6, -e_2 - e_3 - e_4, -e_1 + e_2, \\
 & -e_2 + e_3, -e_3 + e_4 + e_5, -e_5 + e_6 + e_7, \dots, -e_{2n-1} + e_{2n} + e_{2n+1}.
 \end{aligned}$$

This can be extended to an embedding of Λ in precisely two ways: we may map the additional weight two vertex to $e_{2n} - e_{2n+1}$ or to $-e_{2n+1} + e_{2n+2}$. The first choice results in an embedding in \mathbb{Z}^r . The second choice results in an embedding in \mathbb{Z}^{r+1} . The orthogonal complement in \mathbb{Z}^{r+1} has rank one and so is generated by a vertex w . We may compute w and hence its square directly or use the fact that Λ is a primitive sublattice of \mathbb{Z}^{r+1} with determinant $F(2n+1)^2$, which is therefore also the determinant of its rank one orthogonal complement.

Finally another simple induction argument shows that the third embedding in (7) extends uniquely to $\Lambda(3^{n-1}, 2, 2, 3^{n-1})$, and also extends uniquely up to symmetry to give the following embedding of Λ :

$$\begin{aligned}
 (8) \quad & e_{4n-7} + e_{4n-4} - e_{4n-3}, \dots, e_5 + e_8 - e_9, e_1 + e_4 - e_5, -e_1 + e_2, -e_2 + e_3, \\
 & -e_3 + e_6 + e_7, -e_7 + e_{10} + e_{11}, \dots, -e_{4n-5} + e_{4n-2} + e_{4n-1}, -e_{4n-1} + e_{4n}.
 \end{aligned}$$

We see that each of e_1, \dots, e_{4n} appears in (8), and therefore has nonzero pairing with the image of this embedding. \square

Proof of Theorem 2. For the duration of this proof, we denote by B_n the rational ball $B_{F(2n+1), F(2n-1)}$, and by C_n the positive-definite plumbed manifold with the same boundary as B_n , for each $n \in \mathbb{N}$. For $n = 1$, the boundary of the rational ball $B_1 = B_{2,1}$ is $L(4, 1)$, and the plumbing C_1 has weights $[2, 2, 2]$. For $n > 1$, C_n is the plumbing with weights $[3^{n-1}, 2, 2, 3^{n-1}, 2]$, as may be seen using [8, Lemma 3.1].

Suppose first that $B_1 \sqcup B_n$ embeds smoothly in \mathbb{CP}^2 . Let $r_1 = 3$ and r_2 denote the ranks of Λ_{C_1} and Λ_{C_n} respectively. By Proposition 3.2, there is a finite-index lattice embedding

$$\Lambda_M \oplus \Lambda_{C_1} \oplus \Lambda_{C_n} \hookrightarrow \mathbb{Z}^m,$$

where $m = r_1 + r_2 + 1 = r_2 + 4$. By Lemma 3.5, the restriction of this to Λ_{C_1} is either contained in a \mathbb{Z}^3 or is contained in a \mathbb{Z}^4 , spanned by e_1, \dots, e_4 say, with orthogonal complement spanned by a vector w of self-pairing 4. Since the image of the generator of Λ_M is orthogonal to the image of Λ_{C_1} and has nonzero pairing with every unit

vector in \mathbb{Z}^m by Proposition 3.2, it must be the second possibility. The image of Λ_{C_2} lies in the orthogonal complement to that of Λ_{C_1} . If it is contained in the span of e_5, \dots, e_m then this is a finite-index embedding in \mathbb{Z}^{r_2} which again contradicts the fact that the image of the generator of Λ_M has nonzero pairing with every unit vector. Thus at least one vertex of Λ_{C_2} contains a nonzero multiple of w . This vertex then has self-pairing greater than that of w , contradicting the fact that the vertices of Λ_{C_2} all have self-pairing 2 or 3.

We next suppose that $B_k \sqcup B_n$ embeds smoothly in \mathbb{CP}^2 with $n \geq k > 1$. Let $r_1 = 2k + 1$ and $r_2 = 2n + 1$ denote the ranks of Λ_{C_k} and Λ_{C_n} respectively. By Proposition 3.2, there is a finite-index lattice embedding

$$\Lambda_M \oplus \Lambda_{C_k} \oplus \Lambda_{C_n} \hookrightarrow \mathbb{Z}^m,$$

where $m = r_1 + r_2 + 1 = 2k + 2n + 3$.

Arguing as in the previous case, we see that the restriction of this embedding to Λ_{C_k} (respectively Λ_{C_n}) cannot have image in either \mathbb{Z}^{r_1} or \mathbb{Z}^{r_1+1} (respectively \mathbb{Z}^{r_2} or \mathbb{Z}^{r_2+1}). By Lemma 3.5, this leaves the possibility that the restriction to Λ_{C_k} lies in a \mathbb{Z}^{4k} sublattice, and similarly the restriction to Λ_{C_n} lies in a \mathbb{Z}^{4n} sublattice, in both cases with the orthogonal complement in said sublattice containing no unit vectors. In particular we have

$$4n \leq 2k + 2n + 3,$$

and hence n is either k or $k + 1$.

If $n = k + 1$, we have $m = 4n + 1$. Up to $\text{Aut}(\mathbb{Z}^m)$, we may suppose that the \mathbb{Z}^{4n-4} sublattice containing the image of Λ_{C_k} includes the vectors $-e_1 + e_2, -e_2 + e_3$ as the image of the two adjacent weight two vertices. The \mathbb{Z}^{4n} sublattice of \mathbb{Z}^{4n+1} containing the image of Λ_{C_n} has to intersect the \mathbb{Z}^3 sublattice spanned by e_1, e_2, e_3 nontrivially. This means that some vertex of Λ_{C_n} maps to a vector of the form $v + a(e_1 + e_2 + e_3)$, where v is a nonzero vector in the span of e_4, \dots, e_m and $a \neq 0$, noting that the image of this vertex is orthogonal to $-e_1 + e_2, -e_2 + e_3$ and has pairing -1 with a neighbouring vertex. This contradicts the fact that all vertices in Λ_{C_n} have weight 2 or 3.

Finally if $n = k$ then $m = 4n + 3$. We keep the notation Λ_{C_n} and Λ_{C_k} to distinguish the two copies of Λ_{C_n} . We may suppose that the \mathbb{Z}^{4n} sublattice containing the image of Λ_{C_k} is the span of e_1, \dots, e_{4n} , and that it includes the vectors $-e_1 + e_2, -e_2 + e_3$ as the image of the two adjacent weight two vertices. Arguing as in the case $n = k + 1$, the image of Λ_{C_n} has to be orthogonal to the span of e_1, e_2, e_3 , and so is contained in the span of e_4, \dots, e_m . We may also suppose that the two adjacent weight two vertices in Λ_{C_n} map to $-e_{4n+1} + e_{4n+2}, -e_{4n+2} + e_{4n+3}$. We consider the image of Λ_{C_n} and Λ_{C_k} under the projection to \mathbb{Z}^{4n-3} spanned by e_4, e_5, \dots, e_{4n} . From (8) we see that each of these is isomorphic to the orthogonal direct sum $\Lambda(3^{n-2}, 2) \oplus \Lambda(2, 3^{n-2}, 2)$, and has rank $2n - 1$. This leads to a contradiction, since it is not possible to orthogonally embed two lattices of rank $2n - 1$ in \mathbb{Z}^{4n-3} . \square

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