Deep Recessions[*]

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Abstract

This paper demonstrates how a 'modest' financial shock can trigger a deep recession. We suggest that two factors can help generate it. The first is that the economy has accumulated a moderately high level of private debt by the time the adverse shock occurs. The second factor is when monetary policy, set under discretion, is restricted by the zero lower bound. These factors can result in a sharp contraction in output. Perhaps surprisingly, we use a standard DSGE model with financial frictions along the lines of Jermann and Quadrini (2012) to demonstrate this result and so do not need to rely on dysfunctional interbank markets.

Key Words: financial frictions, credit boom, stagnation, ZLB

JEL Reference Numbers: E23, E32, E44, G01, G32

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1 Introduction

A decade has passed since the onset of the Great Recession in 2008, but for a number of economies recovery was slow and fragile. In fact, some worry it ushered in an era of permanently lower trend growth. The origins of some of those concerns are reflected in Figure 1, which plots GDP growth in all post-war recessions for the UK, US and Japan.[1] The Great Recession stands out for at least two reasons – it was both deep and protracted: First, in each country the average recession was not as deep as was experienced in the Great Recession. Second, the average recession delivered its lowest output growth rate at the onset of the recession (period zero in the figure); thereafter, economies tended return to their previous growth trajectory within one to two quarters. In contrast, following the onset of the Great Recession growth reached its minimum some two to four quarters later and growth persisted at the lower end of the typical post-war experience; recovery was slow by recent historical standards.

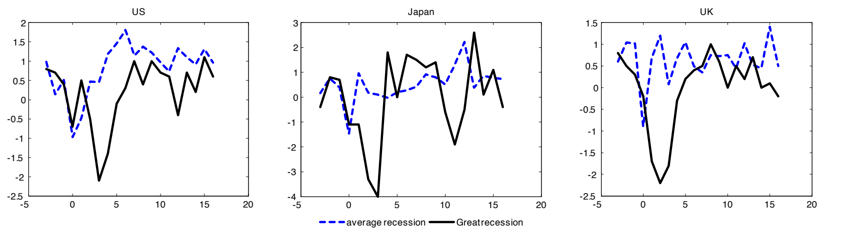
A great deal of research has followed trying to document further and explain these developments, in particular the unexpectedly sharp and prolonged reduction in output. For example For example Boissay et al. (2016) consider about 200 events and identify this broad pattern, see Table 1. Duncan and Nolan (2018) is a detailed review of that literature. They show that, whatever controversies remain amongst scholars of financial crises, there is a widespread consensus that recessions following banking and financial crises are typically worse than 'normal' recessions. For example, Reinhart and Rogoff (2009, 2013) observe that financial crises result in deeper recessions, with contractions in economic activity reaching 9% in some severe episodes. This is substantially more than output reductions more commonly encountered during ordinary recessions. It has also been observed that financial crises often follow periods of strong credit growth. Working with long-run data, Schularick and Taylor (2012) argue that credit growth is a powerful predictor of financial crises. However, such credit booms may not on their own necessarily cause financial crises; some periods of high credit growth do not appear to end badly. Gorton and Ordoñez (2020) argue that credit booms typically start with positive productivity growth, which subsequently falls much faster during credit booms that end badly.

The role of 'financial frictions' in explaining these observed patterns has been identified as central by many researchers and policymakers and there is now a large and growing body of research which seeks to provide quantitative macroeconomic models to explain how seemingly

[1]Episods in Figure 1 can be identified in the available data series in the FRED FRB St. Louis database. The data series are NAEXKP01JPQ657S (Japan 1960Q1-2014Q4), NAEXKP01GBQ652S_PCH (UK 1955Q1-2014Q4) and GDPC1_PCH (US 1947Q1-2015Q1). All series are seasonally adjusted.

well-functioning economies might 'unexpectedly' end up with financial crises. The role of financial shocks in generating deep recessions is discussed in Nolan and Thoenissen (2009) , Christiano et al. (2014), Mumtaz and Zanetti (2016) and Merola (2015) to mention only a few. Many develop non-trivial models of financial intermediaries that help to explain how even relatively 'small' shocks can have large effects. For example, Boissay et al. (2016) develop a model of interbank market which can come to a complete still and demonstrate how an 'innocuous' sequences of positive productivity shocks can expose economies to rare but deep financial crises. Jermann and Quadrini (2012) propose a simpler, linear, macroeconomic model with financial frictions which is quite closely aligned with the US data since the early 1990s, although it fails to replicate the large reduction in output and hours worked observed during the Great Recession.

Figure 1: Post-war recessions in the US, Japan and UK



Note: The period zero corresponds to the start of each of post-war recessions.

Table 1: Statistics on recessions

		Financial	Other	All
No. of Events		41	155	196
Frequency		2.36	8.93	11.29
Duration (years)		2.32	1.65	1.79
Magnitude	$\Delta_{p,t}y$, %	-6.84	-3.75	-4.40
Credit boom	$\Delta_{p-2,p}k$, %	4.55	0.18	1.13

Note: $\Delta_{p,t}x$ ($\Delta_{p-2,p}x$) is the percentage change of variable x from peak to trough (respectively from peak-2 years to peak) where y is output and k is a measure of credit per capita. The table is based on data in Boissay et al. (2016).

In this paper we argue that simple linear models with financial frictions are nevertheless capable of generating quantitatively large adjustments in output following small financial shocks once a couple of important mechanisms are activated. Specifically, we demonstrate how a simple log-linear DSGE model with financial frictions *à la* Jermann and Quadrini (2012) and with nominal rigidities *à la* Rotemberg (1983) is capable of generating the observed deep recessions that are present in the data. In our model, the recession is triggered by a small 'capital quality' shock which accelerates capital obsolescence (Gertler et al. 2012) and reveals that the existing level of debt is out of kilter with the stock of underlying capital. The level of accumulated debt may become 'too high' just like prior to the Great Recession in many developed countries, see e.g. Schularick and Taylor (2012). We demonstrate how this measure of 'overlending' can lead to recession, in line with the important and insightful analysis in Jermann and Quadrini (2012) who identify an increase in probability of default on the existing corporate debt during the financial crisis of 2008. The implied reduction in the stock of debt requires substantial deleveraging in the economy. Following the shock, the reduction in debt presages a fall in the capital stock. In turn the lower capital stock requires less finance and these two effects reinforce one another and the sluggish adjustment of both stocks results in a much greater reduction of capital, output and hours worked. As a result, output falls substantially. Moreover, if monetary policy is conducted under discretion (as argued in Chen et al. 2017) then optimal monetary policy will seek to lower interest rates to below the zero lower bound. If that constraint is binding, higher-than-desired real interest rates disrupt the process of deleveraging, and firms have to reduce labour and capital by much more, contributing further to the depth of the ensuing recession (see also Francis et al. 2020).

The ability of our model to generate quantitatively substantial reductions in output is interesting as some have argued that the onset of the crisis was essentially an adverse credit event (see e.g., Taylor, 2009), whilst others attribute a large role to adverse selection and moral hazard problems in the interbank/money markets (Boissay et al., 2016). No doubt both explanations played some role, but a contribution of this paper is simply to argue that standard models of financial frictions explain more of the initial contraction at the onset of the Great Recession than perhaps is generally realized.

The paper is organized as follows. In the next section we present the model. We discuss the empirical evidence and the corresponding calibration of the model in section 3. In section 4 we discuss the sequence of policy experiments, which show how to generate stylized, post-crisis dynamics like those with which we motivated this paper. Section 5 concludes and notes an important aspect of the data that still cannot be explained by the model in its current formulation.

2 The Model

We present a simple variant of an increasingly familiar type of a model with firms' borrowing constraints *à la* Jermann and Quadrini (2012) and with nominal rigidities *à la* Rotemberg (1983).

The economy is populated by households and firms. Firms use labor and capital to produce differentiated goods. Firms issue equity and debt and use intra-period loans to finance working capital. Firms face credit restrictions because financial intermediaries fear they may not repay those loans. The detailed model of the economy is presented in this section.

2.1 Households

There is a continuum of homogeneous households of measure one. Households are indexed by j. The typical household seeks to maximize the following utility function:

$$\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t U\left(C_t^j, n_t^j\right) \tag{1}$$

where $\mathbb{E}_t$ indicates expectations conditional on information available at time t; $0 < \beta < 1$ is the discount factor; C_t and n_t are a consumption aggregate and labor supply in period t, respectively. The period utility is:

$$U\left(C_t^j, n_t^j\right) = \frac{C_t^{j1-\sigma}}{1-\sigma} - \alpha \frac{n_t^{j1+\psi}}{1+\psi}$$

where σ is the coefficient of relative risk aversion, ψ is the elasticity of labour supply and α is a 'preference' parameter. The households are assumed to hold equities and corporate bonds. The household's budget constraint in real terms can be written as:

$$w_t n_t^j + b_t + s_t \left(d_t + \bar{p}_t\right) + \Phi_t + T_t = \frac{b_{t+1}}{1+r_t} + s_{t+1}\bar{p}_t + C_t^j \tag{2}$$

where w_t is the real wage rate, r_t is the real interest rate, b_t is the real value of one-period bonds held by the households, s_t is equity holdings, d_t is the real dividend, $\bar{p}_t$ is the real market price of shares, P_t is the price of goods, T_t is a government transfer and Φ_t is real profit from the ownership of final goods firms.

The standard maximization of intertemporal utility (1) subject to the sequence of budget constraints (2), yields the following system of first order conditions:

$$\begin{aligned}
0 &= U_{n,t} + U_{C,t} w_t; && (3)\\
1 &= \beta \mathbb{E}_t \frac{U_{C,t+1}}{U_{C,t}} \left(1 + r_t\right); && (4)\\
\bar{p}_t &= \beta \mathbb{E}_t \frac{U_{C,t+1}}{U_{C,t}} \left(d_{t+1} + \bar{p}_{t+1}\right); && (5)\\
b_t &= \frac{b_{t+1}}{1+r_t} + s_{t+1}\bar{p}_t + C_t^j + T_t - s_t\left(d_t + \bar{p}_t\right) - w_t n_t^j - \Phi_t. && (6)
\end{aligned}$$

Here U_C and U_n denote derivatives of $U\left(C_t^j, n_t^j\right)$ with respect to C_t^j and n_t^j respectively. The above equations reflect optimal decisions with respect to labour supply, intertemporal resource allocation, equity purchases and bond purchases.

2.2 Firms

There are two types of firms in this economy. There are flexible price intermediate goods producers and monopolistically competitive final goods retailers. We discuss each in turn.

2.2.1 Intermediate goods producers

Intermediate goods producers have access to a standard production technology

$$y_t = F(e^{z_t}, k_t, n_t) = Ae^{z_t} k_t^{\theta} n_t^{1-\theta} \tag{7}$$

where A is a constant productivity shifter, e^{z_t} is stochastic productivity common to all firms, n_t is the labor input which can be flexibly changed at time t, k_t is the capital stock determined at time $t-1$ and θ is the capital share. Capital in process for period $t+1$ is transformed into capital for production after the realization of a multiplicative shock to capital quality, ϑ_{t+1}:

$$k_{t+1} = \left(\left(1-\delta\right) k_t + I_t\right) \vartheta_{t+1} \tag{8}$$

where I_t is investment and δ is the depreciation rate. Following Gertler et al. (2012),[2] we introduce the capital quality shock, ϑ_t, which is an i.i.d. process allowing for occasional disasters in the form of sharp contractions in the quality of capital as we describe later. The random variable ϑ_t captures some form of economic obsolescence, as opposed to physical depreciation. This disaster shock serves to reveal 'overlending' once the quality of underlying capital has fallen.

Firms use equity and debt to finance their operations. They prefer debt, b_t, to equity because of debt's tax advantage, see Jermann and Quadrini (2012). The budget constraint in real terms can be written as:

$$X_t F(e^{z_t}, k_t, n_t) + \frac{b_{t+1}}{R_t} = b_t + w_t n_t + I_t + \Psi_t \tag{9}$$

where $X_t = \frac{P_{mt}}{P_t}$ is the relative price of produced intermediate goods, P_t is price of final goods, $R_t = 1 + r_t(1-\tau)$ is the after tax return on bonds and Ψ_t is the real payout to shareholders.

We assume that firms raise funds via intertemporal debt, b_t, and an intra-period loan, L_t, to finance working capital. They pay back the interest-free intra-period loan at the end of the period.

[2]See also Gertler and Karadi (2011), Brunnermeier and Sannikov (2014), Gourio (2012) and Afrin (2017).

Specifically, firms start the period with intertemporal debt b_t and they choose labour, investment in capital, the dividend, d_t, and new intertemporal debt, b_{t+1}, *before* producing. Therefore the payments to workers $w_t n_t$, suppliers of investment goods I_t, shareholders Ψ_t and bondholders b_t, are made ahead of the realization of revenues. The loan will cover these costs as follows (in real terms):

$$L_t = I_t + w_t n_t + \Psi_t + b_t - \frac{b_{t+1}}{R_t}.$$

Combining this constraint with the budget constraint yields total liquid real resources of the firm $L_t = X_t F(e^{z_t}, k_t, n_t)$ which can be used to repay the loan at the end of the period.

The ability of firms to borrow is bounded because they may (choose to) default on their debt. Default arises after the realization of revenues but before repaying the intra-period loan. The total real liabilities of the firm at that time are $L_t + \frac{b_{t+1}}{R_t}$, as it will need to pay back the loan and redeem all the bonds. The total liquid real resources of the firm, L_t, can be 'diverted' by the firm and so cannot be recovered by the lender after a default. Then, the only asset available to the lender is capital, k_{t+1}. Following Jermann and Quadrini (2012), we assume that the liquidation value of capital is unknown at the moment of contracting the loan. With probability Ξe^{ξ_t} the full value of k_{t+1} will be recovered, but with probability $1 - \Xi e^{\xi_t}$ the liquidation value is zero. Therefore the enforcement constraint will be as follows:

$$\Xi e^{\xi_t} \left(k_{t+1} - \frac{b_{t+1}}{1 + r_t} \right) \geq X_t F(e^{z_t}, k_t, n_t). \tag{10}$$

Whilst Ξe^{ξ_t} is stochastic and depends on (uncertain) markets conditions, it is the same for all firms. This variable is what is identified as a 'financial shock'.

The transmission of financial shocks is affected by the ease with which firms can switch between debt and equity. There are any number of reasons why firms in reality are unable easily to substitute between such sources of external finance (see Frank and Goyal 2008). In any case, one way to regulate that flexibility in the model is to introduce an adjustment cost to dividend payments. That cost reflects a rigidity that affects the substitution between debt and equity. We assume that the firm's nominal payout to shareholders is assumed to be subject to a quadratic adjustment cost set on the nominal payoff:

$$\Psi_t = d_t + \kappa \left(\frac{d_t}{d_{t-1}} (1 + \pi_t) - 1 \right)^2 d_t$$

where $\kappa \geq 0$ is a dividend adjustment parameter and $\pi_t = \frac{P_t}{P_{t-1}} - 1$ is the inflation rate. The exact functional form of the adjustment cost matters less, as the flexibility can be regulated by the appropriate choice of parameter κ. Our preferred functional form introduces history dependence in dividend payments; another possible way to model it would be to penalize deviations of dividends from the steady state (and both have some support, see Frank and Goyal 2008).

Each firm maximizes profit subject to budget constraint (9) and enforcement constraint (10), so the Lagrangian is:

$$\begin{aligned} L &= \mathbb{E}_t \sum_{t=0}^{\infty} m_{0,t} \left(d_t + \mu_t \left(\Xi e^{\xi_t} \left(k_{t+1} - \frac{b_{t+1}}{1+r_t} \right) - X_t F(e^{z_t}, k_t, n_t) \right) \right. \\ & \quad + \lambda_t \left((1-\delta) k_t + X_t F(e^{z_t}, k_t, n_t) + \frac{b_{t+1}}{R_t} - b_t - w_t n_t \right. \\ & \quad \left. \left. - \frac{k_{t+1}}{\vartheta_{t+1}} - \left(d_t + \kappa \left(\frac{d_t}{d_{t-1}} (1+\pi_t) - 1 \right)^2 d_t \right) \right) \right) \end{aligned}$$

where $m_{0,t} = m_{0,t-1} m_{t-1,t} = \beta^t \frac{U_c(c_t,n_t)}{U_c(c_0,n_0)}$ is the stochastic discount factor, and μ_t and λ_t are Lagrange multipliers.

The first order conditions include (9), (10) and derivatives with respect to n_t, k_{t+1}, b_{t+1}, and d_t:

$$0 = (\lambda_t - \mu_t) X_t F_n(e^{z_t}, k_t, n_t) - \lambda_t w_t; \tag{11}$$

$$0 = \mu_t \Xi e^{\xi_t} - \frac{\lambda_t}{\vartheta_{t+1}} + \mathbb{E}_t m_{t,t+1} \left((\lambda_{t+1} - \mu_{t+1}) X_{t+1} F_k(e^{z_{t+1}}, k_{t+1}, n_{t+1}) + \lambda_{t+1} (1-\delta) \right); \tag{12}$$

$$0 = \frac{\lambda_t}{R_t} - \mu_t \Xi e^{\xi_t} \frac{1}{1+r_t} - \mathbb{E}_t m_{t,t+1} \lambda_{t+1}; \tag{13}$$

$$\begin{aligned} 0 &= 1 + \mathbb{E}_t m_{t,t+1} \lambda_{t+1} 2\kappa \left(\frac{d_{t+1}}{d_t} (1+\pi_{t+1}) - 1 \right) \left(\frac{d_{t+1}}{d_t} \right)^2 (1+\pi_{t+1}); \\ & \quad - \lambda_t \left(1 + 2\kappa \left(\frac{d_t}{d_{t-1}} (1+\pi_t) - 1 \right) \frac{d_t}{d_{t-1}} (1+\pi_t) + \kappa \left(\frac{d_t}{d_{t-1}} (1+\pi_t) - 1 \right)^2 \right), \end{aligned} \tag{14}$$

where F_n and F_k are derivatives of $F(e^{z_t}, k_t, n_t)$ with respect to n and k. Absent distortions due to the tax code and on dividend payouts, and the role of capital as collateral, the debt-equity mix would be a matter of irrelevance in the maximization of firm value; the quantity of investment could be decided with no mind to the debt-equity mix. Whilst the tax code would induce firms only to issue debt, the risk of that debt not being honored provides a role for equity. Moreover, the quadratic adjustment cost on dividends provides yet another consideration on the debt-equity mix. A simple way to summarize what is happening, then, is that bigger firms (those with a larger capital stock) can take on more debt other things constant, and they will try to smooth dividend payments. Both features have been observed in the data.

2.2.2 Retailers

Firms' pricing decisions are subject to nominal rigidities *à la* Rotemberg (1983). We take the standard approach, see e.g. Gertler and Karadi (2011). Each final good producer i buys intermediate goods at price P_{mt} and transforms them into final goods. That good may also be costlessly transformed into capital. It sets its optimal price p_t^i and produces quantity y_t^i. The firm faces monopolistic demand for its good,

$$y_t^i = \left(\frac{p_t^i}{P_t}\right)^{-\varepsilon} y_t.$$

Here the elasticity of substitution between any pair of goods is given by $\varepsilon > 1$. The firm chooses price p_t^i which solves the following optimization problem:

$$\begin{aligned} & \mathbb{E}_t \sum_{\tau=t}^{\infty} m_{t,\tau} \left(\frac{p_t^i}{P_t} y_t^i \left(1-\tau_s\right) - X_t y_t^i - \frac{\omega}{2} \left(\frac{p_t^i}{p_{t-1}^i} - 1 \right)^2 y_t \right) \\ = \; & \mathbb{E}_t \sum_{\tau=t}^{\infty} m_{t,\tau} \left(\left(\frac{p_t^i}{P_t} \right)^{1-\varepsilon} y_t \left(1-\tau_s\right) - X_t \left(\frac{p_t^i}{P_t} \right)^{-\varepsilon} y_t - \frac{\omega}{2} \left(\frac{p_t^i}{p_{t-1}^i} - 1 \right)^2 y_t \right) \end{aligned}$$

where $\omega > 0$ is a price adjustment cost parameter and τ_s denotes the rate at which sales-revenue is taxed.

As is well known, the aggregate implications of that pricing decision give rise to a Phillips-type relation:

$$\left(\pi_t + 1\right) \pi_t = \frac{\left(1-\varepsilon\right)\left(1-\tau_s\right) + \varepsilon X_t}{\omega} + \mathbb{E}_t m_{t,t+1} \left(\pi_{t+1} + 1\right) \pi_{t+1} \frac{y_{t+1}}{y_t}. \tag{15}$$

where the sales revenue tax can be chosen to offset the monopolistic markup.

Finally, the profit Φ_t in the household budget constraint can be found from the aggregation of firms' budget constraints:

$$\Phi_t = y_t \left(1-\tau_s\right) - X_t y_t - \frac{\omega}{2} \pi_t^2 y_t. \tag{16}$$

2.2.3 Government

The role of the government in this economy is limited. The government collects interest rate taxes, pays a labor subsidy designed to eliminate the monopolistic distortion, and rebates its revenue as a lump-sum transfer to households:

$$r_t \tau b_t + y_t \tau_s = T_t.$$

The government budget is therefore balanced at all times.

2.3 Private Sector Equilibrium and Market Clearing

After some substitutions, the system (9), (10), (11)-(14), (15), (3)-(6) collapses to the following system:

$$
\begin{aligned}
0 &= (\lambda_t - \mu_t) X_t F_n(e^{z_t}, k_t, n_t) - \lambda_t w_t; && (17)\\
0 &= -\frac{\lambda_t}{\vartheta_{t+1}} + \mu_t \Xi e^{\xi_t} + \beta \mathbb{E}_t \frac{U_{C,t+1}}{U_{C,t}} \left((\lambda_{t+1} - \mu_{t+1}) X_{t+1} \theta \frac{y_{t+1}}{k_{t+1}} + \lambda_{t+1} (1-\delta) \right); && (18)\\
0 &= \frac{\lambda_t}{R_t} - \frac{\mu_t \Xi e^{\xi_t}}{1+r_t} - \beta \mathbb{E}_t \frac{U_{C,t+1}}{U_{C,t}} \lambda_{t+1}; && (19)
\end{aligned}
$$

$$
\begin{aligned}
0 &= 1 + 2\kappa \beta \mathbb{E}_t \frac{U_{C,t+1}}{U_{C,t}} \lambda_{t+1} \left(\frac{d_{t+1}}{d_t} (1+\pi_{t+1}) - 1 \right) \left(\frac{d_{t+1}}{d_t} \right)^2 (1+\pi_{t+1}) && (20)\\
&\quad - \lambda_t \left(1 + 2\kappa \left(\frac{d_t}{d_{t-1}} (1+\pi_t) - 1 \right) \frac{d_t}{d_{t-1}} (1+\pi_t) + \kappa \left(\frac{d_t}{d_{t-1}} (1+\pi_t) - 1 \right)^2 \right);\\
X_t y_t &= \Xi e^{\xi_t} \left(k_{t+1} - \frac{b_{t+1}}{1+r_t} \right); && (21)
\end{aligned}
$$

$$
\begin{aligned}
\frac{b_{t+1}}{R_t} &= b_t + w_t n_t + \frac{k_{t+1}}{\vartheta_{t+1}} - (1-\delta) k_t + d_t \left(1 + \kappa \left(\frac{d_t}{d_{t-1}} (1+\pi_t) - 1 \right)^2 \right) - X_t y_t; && (22)\\
(1+\pi_t)\pi_t &= \frac{(1-\varepsilon)(1-\tau_s) + \varepsilon X_t}{\omega} + \beta \mathbb{E}_t \frac{U_{C,t+1}}{U_{C,t}} (\pi_{t+1}+1) \pi_{t+1} \frac{y_{t+1}}{y_t}; && (23)\\
\frac{1}{1+i_t} &= \beta \mathbb{E}_t \frac{U_{C,t+1}}{U_{C,t}} \frac{1}{1+\pi_{t+1}}; && (24)\\
\alpha n_t^{\psi} &= w_t C_t^{-\sigma}, && (25)
\end{aligned}
$$

where $1 + r_t = \frac{1+i_t}{1+\pi_{t+1}}$.

Finally, the resource constraint yields

$$
y_t = C_t + \frac{k_{t+1}}{\vartheta_{t+1}} - (1-\delta) k_t + \kappa \left(\frac{d_t}{d_{t-1}} (1+\pi_t) - 1 \right)^2 d_t + \frac{\omega}{2} \pi_t^2 y_t \qquad (26)
$$

and system (17)-(26) determines the private sector equilibrium $\{\lambda_t, \mu_t, X_t, C_t, k_t, \pi_t, n_t, d_t, w_t, b_t\}_{t=0}^{\infty}$ given policy instruments i_t and τ_t and three exogenous processes ϑ_t, z_t and ξ_t.

2.4 Linearization

We linearize system (17)-(26) around the efficient steady state, which is the flexile price equilibrium without financial and capital quality shocks. Specifically, for every variable V_t we define $\hat{v}_t = \log \frac{V_t}{V}$ where V is the steady state level. We then define $\tilde{v}_t = \hat{v}_t - \hat{v}_t^n$ where $\hat{v}_t^n$ is the log-deviation from the efficient steady state. The full linearized system is reported in he appendix.

2.5 Policy

Monetary policy is assumed to minimize an *ad hoc* welfare loss, given by the objective

$$L = \mathbb{E}_0 \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\tilde{\pi}_t^2 + \varkappa \tilde{y}_t^2 \right).$$

Thus we assume that the policymaker has objectives defined simply over inflation and output. Such an objective has long been of interest to researchers in applied and theoretical analyses of (monetary) policy problems. Of course, that assumption is open to some obvious criticisms, which we readily acknowledge; but we also note that this policy objective has significant empirical support, a recent discussion can be found in Chen et al. (2017).[3]

Importantly, note that the interest rate may be constrained by the ZLB. We compute numerically the implications of such a restriction using the approach developed in Laseen and Svensson (2011) and extended to the case of discretion. We detail in an Appendix B to this paper how to do that extension as this may be of additional interest.

3 Calibration

The model is calibrated to a quarterly frequency. We fix $\beta = 0.9825$. The capital depreciation rate is set to $\delta = 0.025$. The capital ratio in the production function is set to $\theta = 0.36$, and the mean value of A is normalized to 1. The tax wedge which introduces, ceteris paribus, an advantage of debt over equity is determined to be $\tau = 0.35$, and the dividend adjustment cost parameter set to $\kappa = 0.146$ as in Jermann and Quadrini (2012).

We calibrate the steady state debt to output ratio to match the data. The quarterly ratio of debt to output for the non-financial business sector is 3.25 over the sample period 1984:I-2010:II, see the top panel in Figure 2. In order to match that, we set the steady state value of the financial variable, Ξ, to 0.1779.[4]

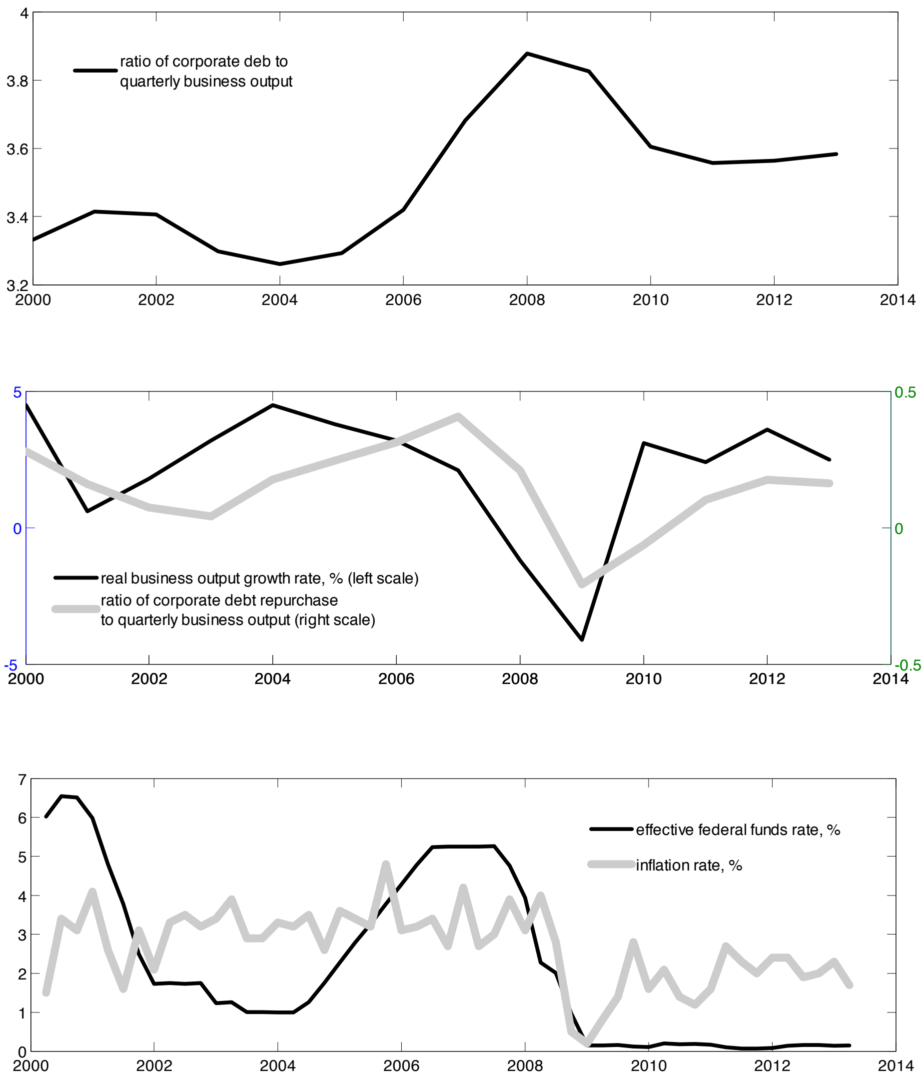
Parameters of the household utility function are determined as follows. The calibration of the Frisch elasticity of substitution in labor supply, ψ, is assumed to be equal to 1 and the risk aversion parameter is $\sigma = 1$. The relative weight on the disutility of labour, $\alpha = 1.8834$, is chosen so as to set steady state hours worked equal to 0.3.

We calibrate the measure of price stickiness, $\omega = 80$, in a way that corresponds to the probability of firms changing prices every 3 quarters (that is, in a corresponding Calvo model).

[3] See also estimation of policy objectives in e.g. Dennis (2006), Ilbas (2010) and Givens (2012).

[4] Data sources: NIPA and FoF tables. The calculations follow Jermann and Quadrini (2012).

Figure 2: Historical data in the US



The elasticity of substitution between any pair of goods ε is equal to 11 in steady state which gives a 10% mark up.

Parameters of the policy objective function are chosen to be $\varkappa = 0.5$, see Chen et al. (2017).[5]

Calibration of technology and financial shocks is taken from Jermann and Quadrini (2012). Specifically, technology and financial shocks $\hat{z}_t$ and $\tilde{\xi}_t$ are correlated AR(1) processes with transition matrix [0.9457, -0.0091; 0.0321, 0.9703] and standard errors of innovations of 0.0045 and 0.0098 respectively, see Table 2 in Jermann and Quadrini (2012). Following Gertler et al. (2012) we assume that the capital quality shock ϑ_t follows an i.i.d. process and allow for randomly arriving rare 'disasters'. Specifically, $\vartheta_t = \tilde{\vartheta}_t \tilde{\vartheta}_t^J$ with $\tilde{\vartheta}_t = \bar{\vartheta} e^{\epsilon_t}$ and $\tilde{\vartheta}_t^J = e^{\eta_t}$ where ϵ_t is normally distributed, $\epsilon_t \sim N(0,1)$ and η_t is binomially distributed

$$\eta_t = \begin{cases} -(1-q)\,\Delta \text{ with prob } q \\ q\Delta \text{ with prob } 1-q \end{cases}$$

where Δ is a positive number so that the disaster innovation $-(1-q)\,\Delta$ is negative. The mean of η_t is zero and the variance is $q\,(1-q)\,\Delta^2$.We calibrate these parameters $q = 0.01$, $\Delta = 0.001$. Parameter $\bar{\vartheta} = 1$.

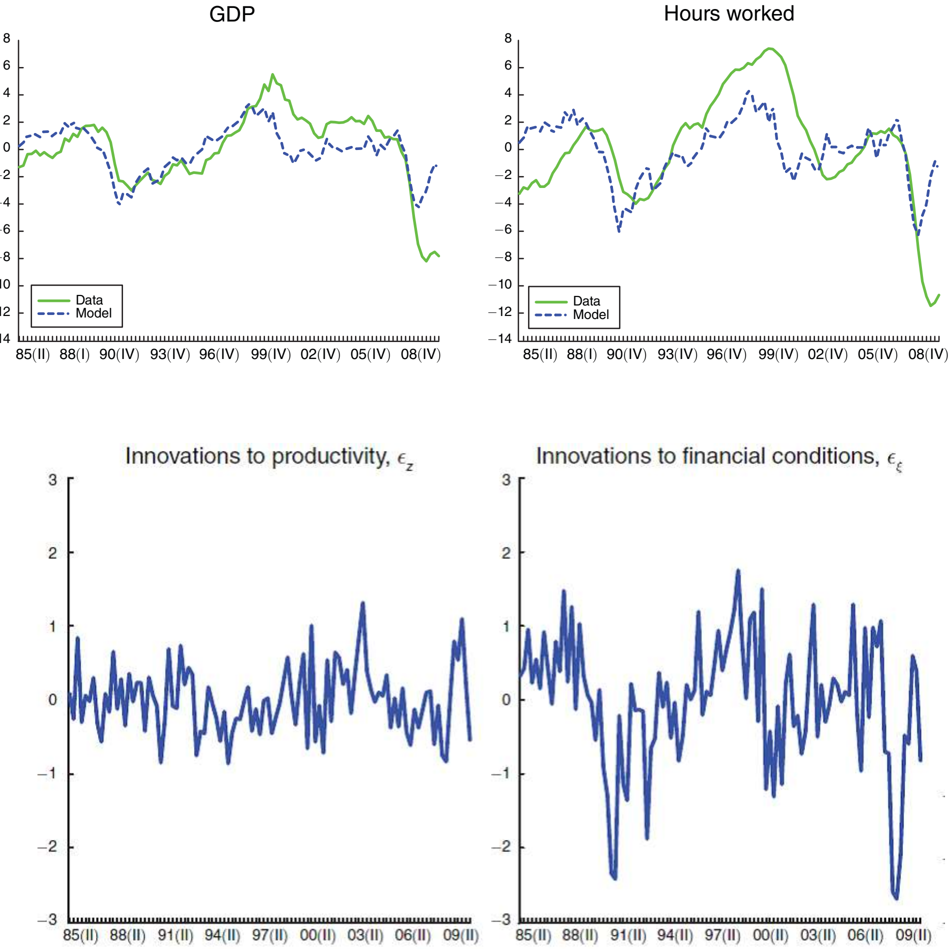
4 Discussion

Figures 3 are taken from Jermann and Quadrini (2012). Its top panel demonstrates the ability of an RBC-type model with financial frictions to explain the three episodes of recessions. As the authors note, it is evident that the model fails to account for the depth of the Great Recession. Its bottom panel identifies innovations to technology and financial shocks around the time of the largest fall in output. Specifically, innovations to productivity seem to be negative, around one standard error in size, i.e. about -0.005. The innovation to the financial shock process is about -0.025. We call them 'crisis innovations' and use them further in our discussion below.

In this paper we claim that deep recessions can still be generated by a very similar model. To support this claim we run three numerical experiments. We start with an interesting experiment in Jermann and Quadrini (2012) with negative initial impact of technology and financial shocks, determined by the 'crisis innovations'. We confirm that the model, indeed, generates only a moderate reduction in output. We then demonstrate that if a capital quality shock happens at about the same time, it creates a situation of 'overlending' which requires greater deleveraging (and possibly implies an increase in probability of default, so triggering a financial shock). In this

[5] The results are robust to the wide range of parameters $\varkappa$ between zero and one.

Figure 3: Response of Jermann and Quadrini (2012) model to Financial Shocks



Source: Figure 4 and Figure 2 in Jermann and Quadrini (2012).

case a deeper recession follows. These events may trigger an aggressive relaxation in monetary policy; indeed the optimal nominal interest rate may be negative – below the ZLB. In the final experiment we demonstrate how that ZLB on interest rates reduces output even further.

4.1 Capital quality shock and 'overlending'

We start with the baseline scenario, where the economy is initially in the steady state but then it is hit by both technology and financial shocks with 'crisis innovations'. This scenario is plotted with solid lines in Figure 4. The dynamics of the model economy is mostly affected by the negative financial shock. Such a shock reduces the probability that, in the event of default, banks will be able to recover in liquidation the full value of capital. Banks lend to firms at the beginning of the period, so that firms are able to pay wages. As the financial shock reduces the probability of that debt being honored, the amount of bank lending falls. Firms which are not able to obtain funds up front have to delever or reduce production. Firms reduce their labour demand, produce less output and also pay lower wages. The equilibrium prices of intermediate and final goods both fall as a result of lower income and lower demand. Firms also reduce the amount of borrowing. Both constraints for firms are tightened, as the values of the Lagrange multipliers indicate.

This is, of course, different from the scenario studied in Jermann and Quadrini (2012) where the 'crisis innovations' further amplify already existing deviation of the economy from the steady state created by highly persistent autoregressive shocks. Nevertheless, in our case output and labour fall by about 2%, which is significant, but still short of the magnitude generated by the model reported in Figure 3.

In response to lower inflation and output the central bank reduces the nominal interest rate sufficiently to guarantee a reduction in the real rate. Initially, consumption rises just above the steady state, but then falls consistently with the low real interest rate.

Lower interest rates also make it easier for firms to meet obligations arising from the existing stock of corporate debt, and so help to reduce the debt stock relatively quickly. Output falls by less than wages and so firms' profits fall as do their dividend payout.

We now allow for a 'rare' capital quality shock to occur; this happens in the initial period of time. Our calibration implies that the level of ϑ is reduced by 0.1%. The quality of capital shock has a negative impact in two ways. First, this shock has a direct impact on the productivity of capital in the production of output. Second, it also causes lenders to reconsider the effectiveness of capital in its role as collateral. When the quality of capital falls, the existing stock of debt remains unchanged. In other words, agents suddenly realize that the level of accumulated debt

Figure 4: Productivity and financial shocks with 'crisis innovations'. Percentage deviations from the initial (steady) state under discretionary monetary policy.

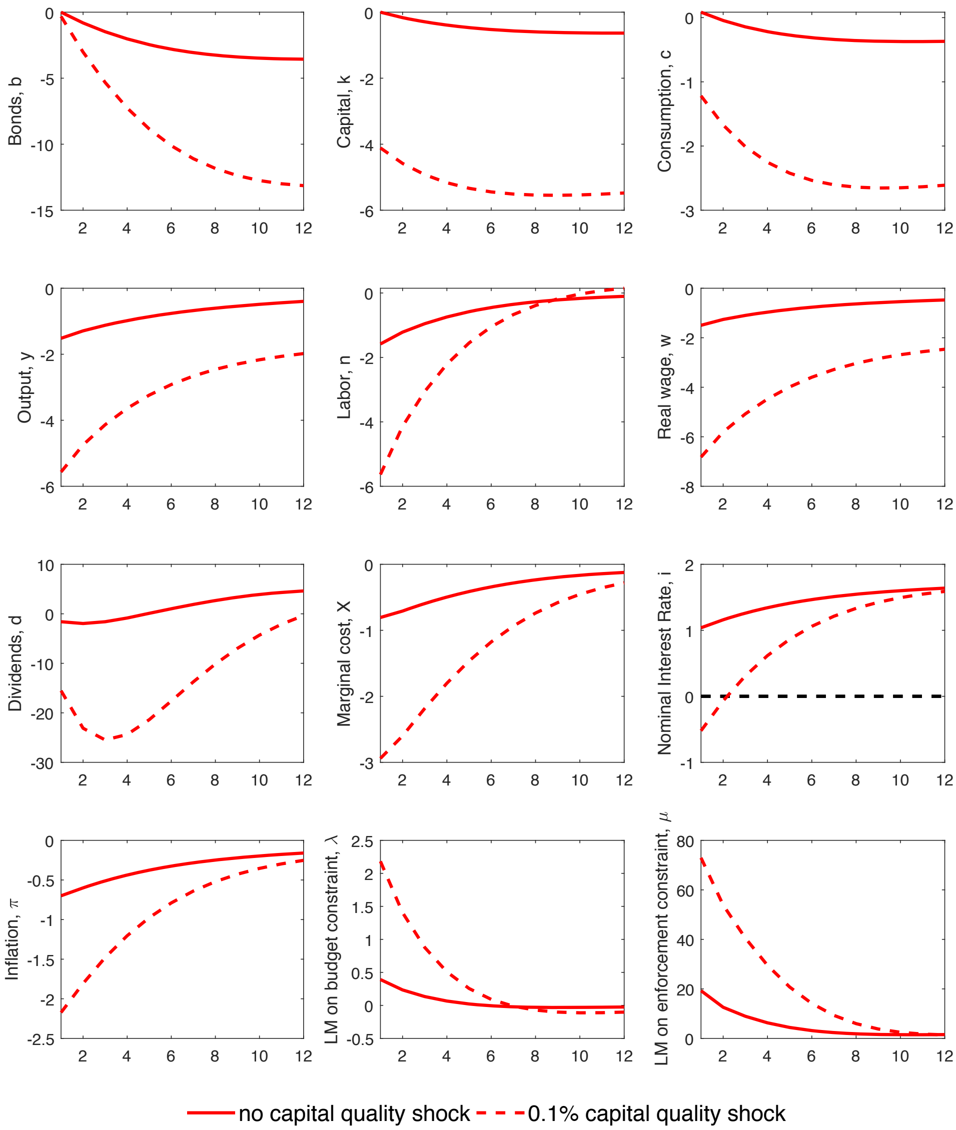
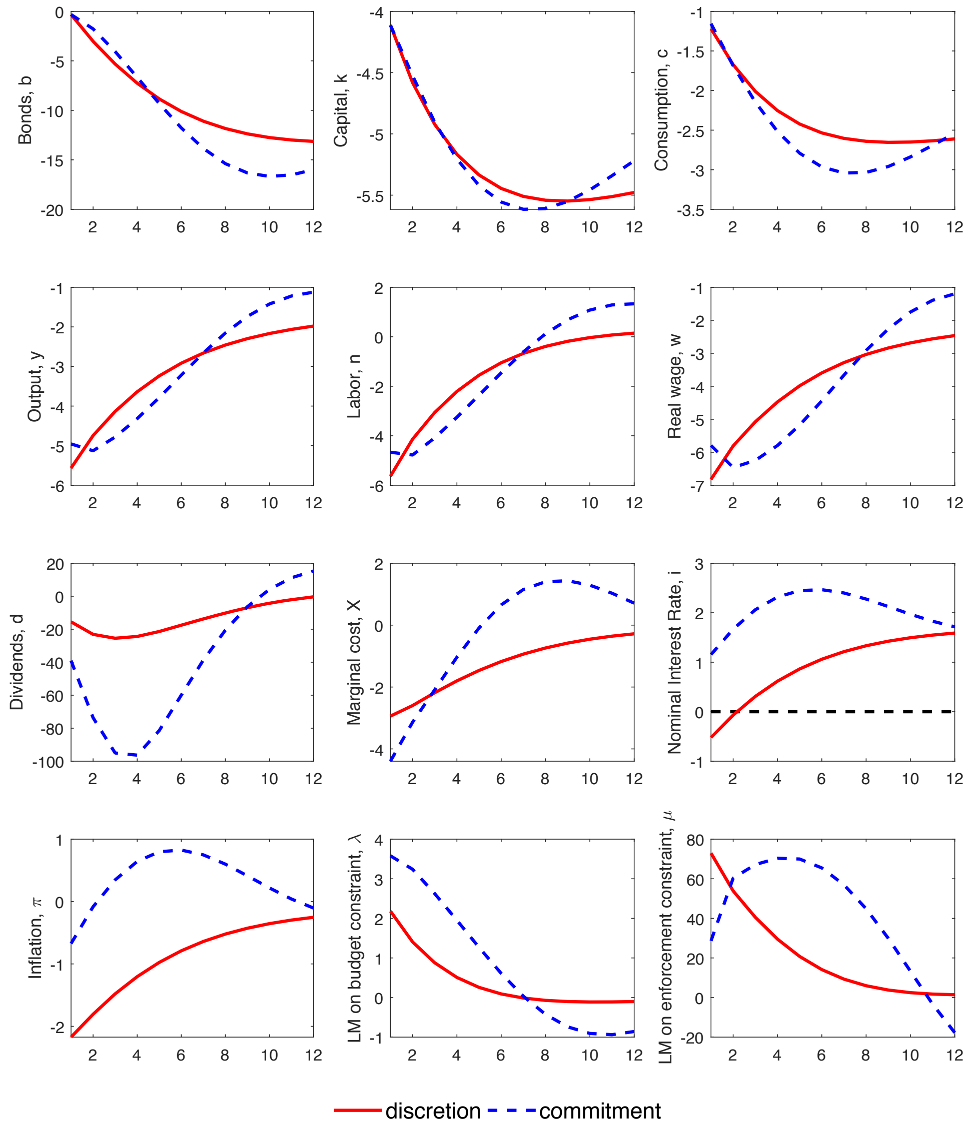


Figure 5: Commitment vs. Discretion. The scenario with all three shocks, percentage deviation from the steady state.



is excessive in comparison to the existing capital stock. Such a situation in our model may be considered to be one of 'overlending' and is close to what Gorton and Ordoñez (2020) call a 'bad credit boom' [6]. In any case, the stock of debt needs to be brought back in line with the underlying capital, so sharper deleveraging is required. As before, the scenario assumes that both technology and financial shocks apply. This allows us to quantify the impact of the capital quality shock. Also, although the model treats all shocks in a similar way – they are exogenous – a more elaborate model may provide a link from the capital quality shock to an increase in the probability of default.

This scenario is plotted in Figure 4 with the dashed lines. The required deleveraging brings the level of debt down towards the steady state level of new quality-adjusted capital. The value of capital is re-evaluated instantaneously, while the amount of debt in circulation is predetermined and is about 10% above the new steady state, consistent with data presented in the first panel in Figure 2. This, now excessively high, initial level of debt requires greater deleveraging than in the default scenario. Lower debt reduces the capital stock and the lower capital stock requires less financing. These two effects reinforce each other, and as a result there is greater reduction in labour demand and wages, and much lower inflation. The interest rate falls by more, not only because it tries to stabilize inflation, but because it also helps to stabilize debt–and thus all policy-relevant variables–faster. Adding a rare capital quality shock, therefore, results in reduction in output of about 6%, which is much more consistent with the evidence presented in Figure 3.

The type of optimal monetary policy does not affect the resulting reduction in output, see Figure 5, which compares policies under discretion and commitment. Although adjustments of real variables are very similar, commitment policy does not allow the interest rate to move below the ZLB, unlike discretionary policy. Most empirical analysis to date suggest the US Fed operates under discretion rather than commitment, see e.g. Dennis (2006), Givens (2012), Chen et al. (2017). Therefore, next we discuss the implications of the ZLB on the dynamics of the economy under discretionary monetary policy.

4.2 'Overlending' with interest rate above the ZLB

The effect of the ZLB is illustrated in Figure 6. We compare two scenarios. The first one, represented by the solid lines, is the case of 'unconstrained discretion' with the three shocks discussed in the section above. That scenario assumes that the interest rate can move below

[6] They argue that what distinguished good booms from bad booms is essentially that under the former but not the later productivity in the economy grows more persistently. As a result, good booms end without wider economic hardship.

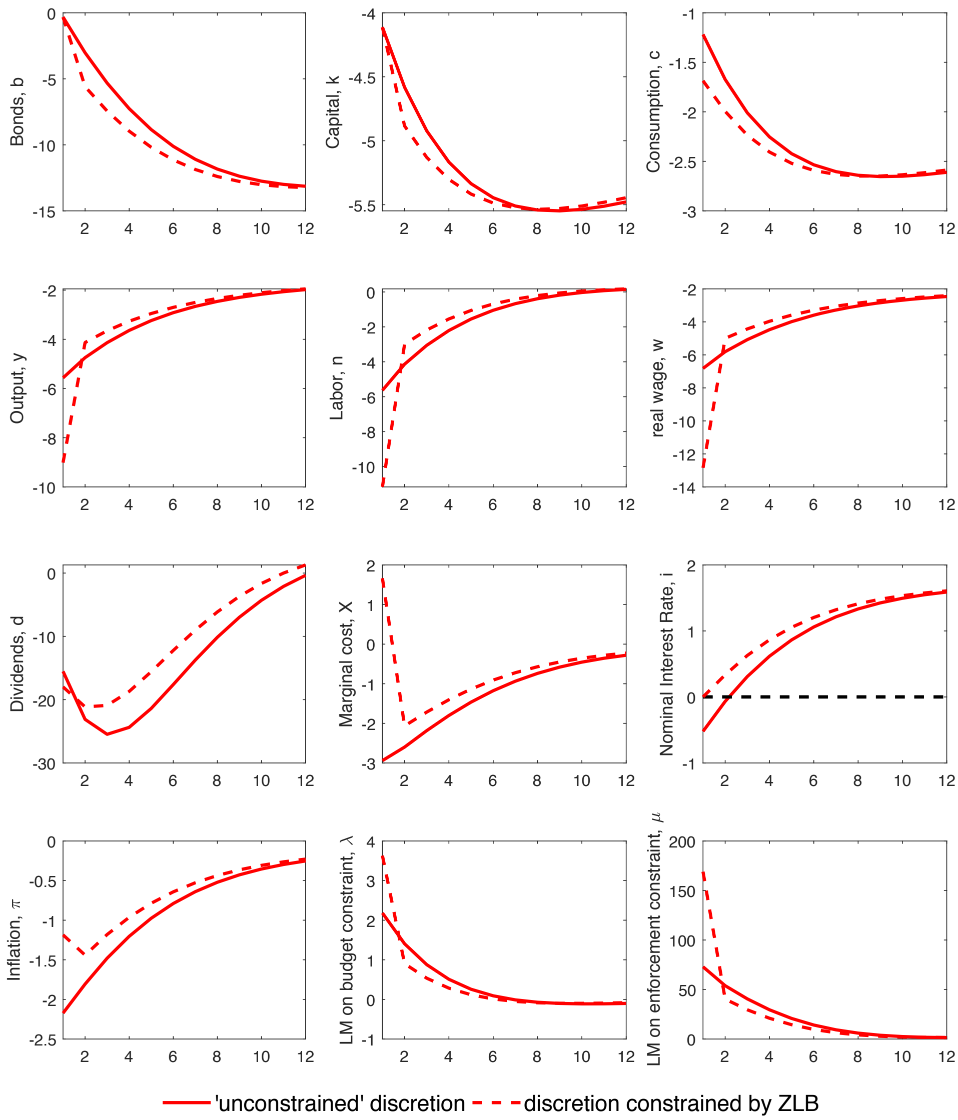
the ZLB. The second scenario, represented by the dashed lines, is where such movements are prohibited. Following Erceg and Linde (2014) and Bodenstein et al. (2013) we use the algorithm in Laseen and Svensson (2011) which was developed to solve linear-quadratic models with a ZLB on the policy instrument. The algorithm assumes that the private sector anticipates policy shocks to a policy rule so that the actual and expected paths of the interest rate do not go below the ZLB. Laseen and Svensson (2011) develop the algorithm for a commitment policy. We extend the algorithm to the case of discretionary policy which we set out in the Appendix. We also perform iteration steps described in Bodenstein et al. (2013) to endogenize the duration of the ZLB episode contingent on the realization of particular shocks.

Figure 6 suggests that when the optimal interest rate is constrained by the ZLB, the recession is deeper. Since the interest rate remains 'too high', debt deleveraging is too slow. The high interest rate also results in lower consumption and lower demand. Firms attempt to accommodate the reduction in demand and the need for deleveraging with reductions in both factors of production. Both bond and capital stocks fall quickly and by large amounts. Wages and labor fall instantaneously.

As capital and labour fall, the production of intermediate goods falls too. In the initial period, the price of intermediate goods (marginal cost) rises to compensate for the sharp reduction in supply of intermediate goods and relatively buoyant demand from final goods producers. As a result, inflation does not fall as far as it otherwise would had the ZLB not been present. However, expected inflation remains negative. Together with a relatively high nominal interest rate, that means there is only a relatively small reduction in the real interest rate and that consumption falls only slowly through time. As a result, we observe a reduction in consumption and investment reflecting the reduction in output in the first few periods following the shock.

When the financial shock occurs both constraints tighten and monetary policy is loosened. However, the interest rate cannot fall by as much as the policymaker would like as the ZLB is encountered. The enforcement constraint is tightened much more than in the 'unconstrained' policy scenario, see Figure 6. However, once the initial-periods capital and debt de-accumulation is done, the constraint is weakened substantially. There is no further need to reduce bonds and capital quickly, and no need to restrain investment as much. Investment remains negative, but somewhat higher than in the first several periods. Intermediate goods firms increase output, and the price of intermediate goods falls to equalize demand and supply. Therefore, inflation falls as costs fall. Inflation is negative and it is optimal to keep the interest rate below the steady state level in order to stabilize the economy, but the interest rate does not need to be below the ZLB.

Figure 6: The effect of ZLB.



We show that it is optimal to raise the interest rate slightly above the ZLB. The higher interest rate increases the real interest rate, but it still remains below the steady state level. As such, consumption continues falling to match the desired path for capital and supply. At some point the optimal deleveraging is achieved, the real rate becomes positive, consumption and demand start rising, prices and inflation rise and the economy converges back on the steady state. To summarize, constrained monetary intervention results in a deep recession. In our experiment the reduction in output is nearly double the fall in the previous section and replicates the data in Figure 3.

5 Conclusion

In this paper we demonstrate how a simple model with borrowing constrained firms is able to generate quantitatively large reductions in output like those observed following the 2008 financial crisis. We demonstrate that, in response to a moderate financial shock triggered by a very small reduction in capital quality, the economy may undergo a very deep recession. If monetary policy is conducted under discretion, the zero lower bound on interest rate is likely to bind.

The analysis therefore makes two key contribution. First, it shows how in a simple, increasingly familiar framework one may model excessive debt and allow for the potential of 'small' shocks to have large effects. And second, it shows how that same framework goes some way to help understanding the depth of the recession in the face of certain key factors, namely financial frictions and constraints on monetary policy.

Our research, however, shows that the frictions/model features that help us understand the depth of the Great Recession do not necessarily help us understand its persistence. Understanding that feature of the Great Recession, in combination with its 'initial' severity remains an urgent and outstanding research question.

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A Linearized model equations

The complete linearized model of the system of equations derived in the text (17)-(26) is given as follows:

$$
\begin{aligned}
\tilde{w}_t &= \frac{\mu}{1-\mu}\left(\tilde{\lambda}_t - \tilde{\mu}_t\right) + \tilde{x}_t + \hat{z}_t + \theta\tilde{k}_t - \theta\tilde{n}_t; && (27)\\
\tilde{\lambda}_t &= \beta\left(X\theta\frac{y}{k}\left(\left(\tilde{\lambda}_{t+1} - \mu\tilde{\mu}_{t+1}\right) + (1-\mu)\left(\hat{z}_{t+1} + \theta\tilde{k}_{t+1} + (1-\theta)\,\tilde{n}_{t+1} - \tilde{k}_{t+1} + \tilde{x}_{t+1}\right)\right) && (28)\\
&\quad - (1-\mu\Xi)\,\sigma\tilde{c}_{t+1} + (1-\delta)\,\tilde{\lambda}_{t+1}\right) + \hat{\vartheta}_{t+1} + \mu\Xi\left(\tilde{\mu}_t + \tilde{\xi}_t\right) + \beta\,(1-\mu\Xi)\,\sigma\tilde{c}_t; \\
0 &= \beta\left(\sigma\tilde{c}_t - \sigma\tilde{c}_{t+1} + \tilde{\lambda}_{t+1}\right) - \frac{1}{R}\left(\hat{\lambda}_t - \frac{(1-\tau)}{\beta R}\left(\tilde{\imath}_t - \tilde{\pi}_{t+1}\right) + \frac{r\tau}{R}\tilde{\tau}_t\right) && (29)\\
&\quad + \frac{\Xi\mu}{1+r}\left(\hat{\mu}_t + \tilde{\xi}_t - \tilde{\imath}_t + \tilde{\pi}_{t+1}\right); \\
\tilde{\lambda}_t &= 2\kappa\beta\left(\tilde{\pi}_{t+1} + \tilde{d}_{t+1} - \tilde{d}_t\right) - 2\kappa\left(\tilde{\pi}_t + \tilde{d}_t - \tilde{d}_{t-1}\right); && (30)
\end{aligned}
$$

$$\begin{aligned}
\theta\tilde{k}_t &= \frac{\Xi}{Xy}\left(k\left(\tilde{k}_{t+1}+\tilde{\xi}_t\right)-\frac{b}{1+r}\left(\tilde{b}_{t+1}-\tilde{\imath}_t+\tilde{\pi}_{t+1}+\tilde{\xi}_t\right)\right)-(1-\theta)\,\tilde{n}_t-\hat{z}_t-\tilde{x}_t; && (31)\\
k\tilde{k}_{t+1} &= Xy\left(\tilde{x}_t+\hat{z}_t+\theta\tilde{k}_t+(1-\theta)\,\tilde{n}_t\right)+\frac{b}{R}\left(\tilde{b}_{t+1}-\frac{(1-\tau)}{\beta R}\left(\tilde{\imath}_t-\tilde{\pi}_{t+1}\right)+\frac{r\tau}{R}\tilde{\tau}_t\right) && (32)\\
&\quad -wn\left(\tilde{w}_t+\tilde{n}_t\right)-b\tilde{b}_t-d\tilde{d}_t+(1-\delta)\,k\tilde{k}_t+k\hat{\vartheta}_{t+1}; &&\\
\tilde{\pi}_t &= \frac{\varepsilon X}{\omega}\tilde{x}_t+\beta\mathbb{E}_t\tilde{\pi}_{t+1}; && (33)\\
\tilde{c}_t &= \tilde{c}_{t+1}-\frac{1}{\sigma}\left(\tilde{\imath}_t-\tilde{\pi}_{t+1}\right); && (34)\\
\tilde{w}_t &= \psi\tilde{n}_t+\sigma\tilde{c}_t; && (35)\\
y\theta\tilde{k}_t &= C\tilde{c}_t+k\tilde{k}_{t+1}-k(1-\delta)\,\tilde{k}_t-k\hat{\vartheta}_{t+1}-y(1-\theta)\,\tilde{n}_t-y\hat{z}_t. && (36)
\end{aligned}$$

B Anticipated ZLB under discretion in LQ models

Laseen and Svensson (2011) propose a convenient algorithm to construct policy projections conditional on anticipated policy rate paths in linearized DSGE models. The algorithm expands the set of predetermined variables by adding a vector of future policy shocks to a given policy rule, that satisfies the anticipated policy rate path. Laseen and Svensson (2011) illustrate how to add an anticipated sequence of shocks to the solution under both commitment and a simple rule. We extend this algorithm to the case of discretion.[7]

Our model in equations (27)-(36) can be written in the following state-space form (the problem is certainty equivalent, so all expectation signs are omitted)

$$\begin{aligned}
\boldsymbol{X}_{t+1} &= \boldsymbol{A}_{11}\boldsymbol{X}_t+\boldsymbol{A}_{12}\,\boldsymbol{x}_t+\boldsymbol{B}_1\,\boldsymbol{u}_t+\boldsymbol{C}\boldsymbol{\varepsilon}_{t+1} && (37)\\
\boldsymbol{H}\boldsymbol{x}_{t+1} &= \boldsymbol{A}_{21}\boldsymbol{X}_t+\boldsymbol{A}_{22}\,\boldsymbol{x}_t+\boldsymbol{B}_2\,\boldsymbol{u}_t, && (38)
\end{aligned}$$

where $\boldsymbol{X}_t$ is a n_1 vector of predetermined variables; $\boldsymbol{x}_t$ is a n_2 vector of forward-looking variables; $\boldsymbol{u}_t$ is the control variable, and $\boldsymbol{\varepsilon}_t$ is a vector of zero mean k *i.i.d.* shocks. Without loss of generality, the shocks are normalized so that the covariance matrix of $\boldsymbol{\varepsilon}_t$ is the identity matrix, $\boldsymbol{I}$. Therefore, the covariance matrix of the shocks to $\boldsymbol{X}_{t+1}$ is $\mathbf{CC}'$.

The central bank has an intertemporal loss function in period t :

$$\mathbb{E}_t\sum_{s=t}^{\infty}\frac{1}{2}\beta^{s-t}\boldsymbol{L}_s,$$

where the period loss, L_s, satisfies

$$\boldsymbol{L}_s=\boldsymbol{Y}_s'\,\boldsymbol{\Lambda}\boldsymbol{Y}_s,$$

[7] The algorithm assumes anticipated shocks and in that it is different from algorithms employed to solve fully nonlinear models with occasionaally binding constraints, see e.g. Adam and Billi (2007) and Kim et al. (2010).

$\boldsymbol{\Lambda}$ is a symmetric and positive semi-definite weight matrix and $\boldsymbol{Y}_s$ is an n_Y vector of target variables

$$\boldsymbol{Y}_s = \boldsymbol{D} \begin{bmatrix} \boldsymbol{X}_s \\ \boldsymbol{x}_s \\ \boldsymbol{u}_s \end{bmatrix}.$$

It follows that the period loss function can be rewritten as

$$L_s = \begin{bmatrix} \boldsymbol{X}_s \\ \boldsymbol{x}_s \\ \boldsymbol{u}_s \end{bmatrix}' \boldsymbol{W} \begin{bmatrix} \boldsymbol{X}_s \\ \boldsymbol{x}_s \\ \boldsymbol{u}_s \end{bmatrix},$$

where $\boldsymbol{W} = \boldsymbol{D}'\boldsymbol{\Lambda}\boldsymbol{D}$ is symmetric and positive semi-definite, and

$$\boldsymbol{W} = \begin{bmatrix} Q_{XX} & Q_{Xx} & P_{Xu} \\ Q'_{Xx} & Q_{xx} & P_{xu} \\ P'_{Xu} & P'_{xu} & R \end{bmatrix}$$

is partitioned with $\boldsymbol{X}_s$, $\boldsymbol{x}_s$ and $\boldsymbol{u}_s$.

Following Laseen and Svensson (2011), we augment the predetermined variables, $\boldsymbol{X}_t$, in equations (27)-(36) by incorporating a $(T+1)$ vector of stochastic shocks, $\boldsymbol{z}^t \equiv (z_{t,t}, z_{t+1,t} \ldots z_{t+T,t})'$,which denote a projection in period t of future realizations of shocks, $z_{t+\tau,t}, \tau = 0, 1, ..., T$. Furthermore, we assume that $z_{t,t}$ follows a moving average process

$$z_{t,t} = \eta_{t,t} + \sum_{s=1}^{T} \eta_{t,t-s},$$

where $\eta_{t,t-s}, s = 0, 1, ..T$, are zero-mean *i.i.d.* shocks. For $T = 0$, $z_{t,t} = \eta_{t,t}$. For $T > 0$, the stochastic shocks following a moving average process:

$$\begin{aligned} z_{t+\tau,t+1} &= z_{t+\tau,t} + \eta_{t+\tau,t+1}, \;\; \tau = 1, ..., T \\ z_{t+T+1,t+1} &= \eta_{t+T+1,t+1}. \end{aligned}$$

The above stochastic shocks process can be rewritten in the following matrix form

$$\boldsymbol{z}^{t+1} = \boldsymbol{A}_z \boldsymbol{z}^t + \boldsymbol{\eta}^{t+1},$$

where $\boldsymbol{\eta}^{t+1} \equiv (\eta_{t+1,t+1}, \eta_{t+2,t+1} \ldots \eta_{t+T+1,t+1})'$ is a $(T+1)$ vector of *i.i.d.*shocks and A_z is $(n_1 + 1) \times (n_1 + 1)$ matrix

$$A_z = \begin{bmatrix} \mathbf{0}_{T\times 1} & \boldsymbol{I}_T \\ 0 & \mathbf{0}_{1\times T} \end{bmatrix}$$

Our model (37) incorporating the vector of stochastic shocks, $\boldsymbol{z}^t$, is subsequently augmented into the following state-space form

$$\begin{aligned} \widetilde{\boldsymbol{X}}_{t+1} &= \widetilde{A}_{11}\widetilde{\boldsymbol{X}}_t + \widetilde{A}_{12}\boldsymbol{x}_t + \widetilde{B}_1\boldsymbol{u}_t + \widetilde{C}\boldsymbol{\varepsilon}_{t+1} && (39) \\ \boldsymbol{H}\boldsymbol{x}_{t+1} &= \widetilde{A}_{21}\widetilde{\boldsymbol{X}}_t + \widetilde{A}_{22}\boldsymbol{x}_t + \widetilde{B}_2\boldsymbol{u}_t && (40) \end{aligned}$$

where $\widetilde{\boldsymbol{X}}_t = \left[\boldsymbol{z}^t, \boldsymbol{X}_t'\right]'$ is a vector of predetermined variables. The matrices of the state-space are augmented accordingly

$$\begin{aligned} \widetilde{A}_{11} &= \begin{bmatrix} A_z & 0_{T+1,n1} \\ 0_{n_1\times T+1} & A_{11} \end{bmatrix}, \widetilde{A}_{12} = \begin{bmatrix} 0_{T+1\times n_2} \\ A_{12} \end{bmatrix}, \\ \widetilde{A}_{21} &= \begin{bmatrix} 0_{n_2\times T+1} & A_{21} \end{bmatrix}, \widetilde{A}_{22} = A_{22}, \\ \widetilde{B}_1 &= \begin{bmatrix} 0_{T+1\times 1} \\ B_1 \end{bmatrix}, \widetilde{B}_2 = B_2, \widetilde{C} = \begin{bmatrix} 0_{T+1\times k} \\ C \end{bmatrix} \end{aligned}$$

The selection matrix $\boldsymbol{D}$ becomes $\widetilde{\boldsymbol{D}} = \begin{bmatrix} 0_{n_Y\times T+1} & \boldsymbol{D} \end{bmatrix}$. Subsequently, the symmetric and positive semi-definite weight matrix is now defined as $\widetilde{\boldsymbol{W}} = \widetilde{\boldsymbol{D}}'\boldsymbol{\Lambda}\widetilde{\boldsymbol{D}}$, where

$$\widetilde{\boldsymbol{W}} = \begin{bmatrix} \widetilde{Q}_{XX} & \widetilde{Q}_{Xx} & \widetilde{P}_{Xu} \\ \widetilde{Q}_{Xx}' & \widetilde{Q}_{xx} & \widetilde{P}_{xu} \\ \widetilde{P}_{Xu}' & \widetilde{P}_{xu}' & \widetilde{R} \end{bmatrix}$$

is partitioned with $\widetilde{\boldsymbol{X}}_s$, $\boldsymbol{x}_s$ and $\boldsymbol{u}_s$.

To impose the ZLB constraint on the nominal interest rate $\boldsymbol{u}_t = \tilde{\imath}_t$, we set the first element in $\widetilde{P}_{Xu}'$ to -1 to load $z_{t,t}$, which is the first element in the vector of predetermined variables $\widetilde{\boldsymbol{X}}_t = \left[z^t, \boldsymbol{X}_t'\right]'$.The projection of the future stochastic shocks is then chosen to ensure that $i_t \geqslant 0$, for $\tau = 0, 1, ..., T$.

Suppose that the reaction of the private sector is given by the following linear rule

$$\boldsymbol{x}_{t+1} = -N\widetilde{\boldsymbol{X}}_{t+1}, \tag{41}$$

We can rewrite equation (41) into an equivalent form in terms of predetermined variables and controls (as did Oudiz and Sachs, 1985) by substituting for $\widetilde{\boldsymbol{X}}_{t+1}$ using (39):

$$\boldsymbol{x}_{t+1} = -N\widetilde{\boldsymbol{X}}_{t+1} = -N\left(\widetilde{A}_{11}\widetilde{\boldsymbol{X}}_t + \widetilde{A}_{12}\boldsymbol{x}_t + \widetilde{B}_1\boldsymbol{u}_t\right)$$

Combining this with equation (40) we obtain:

$$\boldsymbol{x}_t = -J\widetilde{\boldsymbol{X}}_t - K\boldsymbol{u}_t,$$

where

$$J = \left(HN\widetilde{A}_{12} + \widetilde{A}_{22}\right)^{-1}\left(\widetilde{A}_{21} + HN\widetilde{A}_{11}\right), \tag{42}$$

$$K = \left(HN\widetilde{A}_{12} + \widetilde{A}_{22}\right)^{-1}\left(\widetilde{B}_{2} + HN\widetilde{B}_{1}\right) \tag{43}$$

The policymaker maximizes its objective function with respect to $\boldsymbol{u}_t$, taking the time-consistent reaction $\boldsymbol{x}_t$ as given, and recognizing the dependence of $\boldsymbol{x}_t$ on policy $\boldsymbol{u}_t$. We define the following Lagrangian with constraints capturing the evolution of the state variables in the economy, as well as the private sector's response to policy,

$$H_s = \mathbb{E}_t \sum_{s=t}^{\infty} \frac{1}{2}\beta^{s-t}\boldsymbol{L}_s + \lambda'_{s+1}\left(\widetilde{A}_{11}\widetilde{\boldsymbol{X}}_s + \widetilde{A}_{12}\boldsymbol{x}_s + \widetilde{B}_1\boldsymbol{u}_s - \widetilde{\boldsymbol{X}}_{s+1}\right) + \mu'_s\left(\boldsymbol{x}_s + J\widetilde{\boldsymbol{X}}_s + K\boldsymbol{u}_s\right),$$

where λ_s and μ_s are Lagrange multipliers. First order conditions are the following

$$\begin{aligned}
\frac{\partial H_s}{\partial \boldsymbol{u}_s} &= \beta^{s-t}\left(\widetilde{P}'_{Xu}\widetilde{\boldsymbol{X}}_s + \widetilde{P}'_{xu}\boldsymbol{x}_s + \widetilde{R}\boldsymbol{u}_s\right) + \widetilde{B}'_1\lambda_{s+1} + K'\mu_s = 0\\
\frac{\partial H_s}{\partial \widetilde{\boldsymbol{X}}_s} &= \beta^{s-t}\left(\widetilde{Q}_{XX}\widetilde{\boldsymbol{X}}_s + \widetilde{Q}_{Xx}\boldsymbol{x}_s + \widetilde{P}_{Xu}\boldsymbol{u}_s\right) + \widetilde{A}'_{11}\lambda_{s+1} - \lambda_s + J'\mu_s = 0\\
\frac{\partial H_s}{\partial \boldsymbol{x}_s} &= \beta^{s-t}\left(\widetilde{Q}'_{Xx}\widetilde{\boldsymbol{X}}_s + \widetilde{Q}_{xx}\boldsymbol{x}_s + \widetilde{P}_{xu}\boldsymbol{u}_s\right) + \widetilde{A}'_{12}\lambda_{s+1} + \mu_s = 0\\
\frac{\partial H_s}{\partial \lambda_{s+1}} &= \widetilde{A}_{11}\widetilde{\boldsymbol{X}}_s + \widetilde{A}_{12}\boldsymbol{x}_s + \widetilde{B}_1\boldsymbol{u}_s - \widetilde{\boldsymbol{X}}_{s+1} = 0\\
\frac{\partial H_s}{\partial \mu_{s+1}} &= \boldsymbol{x}_s + J\widetilde{\boldsymbol{X}}_s + K\boldsymbol{u}_s = 0
\end{aligned}$$

By substituting out μ_s and $\boldsymbol{x}_s$ from the above equations, we obtain the following three equations:

$$\begin{aligned}
\beta B^{*\prime}\boldsymbol{\xi}_{s+1} &= -P^{*\prime}\widetilde{\boldsymbol{X}}_s - R^{*}\boldsymbol{u}_s,\\
\beta A^{*\prime}\boldsymbol{\xi}_{s+1} &= -Q^{*}\widetilde{\boldsymbol{X}}_s - P^{*}\boldsymbol{u}_s + \boldsymbol{\xi}_s,\\
\widetilde{\boldsymbol{X}}_{s+1} &= A^{*}\widetilde{\boldsymbol{X}}_s + B^{*}\boldsymbol{u}_s,
\end{aligned}$$

where $\xi_s = \beta^{-s+t}\lambda_s$, $\xi_{s+1} = \beta^{-s-1+t}\lambda_{s+1}$, and

$$\begin{aligned}
Q^* &= \widetilde{Q}_{XX} - \widetilde{Q}_{Xx}J - J'\widetilde{Q}_{xX} + J'\widetilde{Q}_{xx}J,\\
P^* &= J'\widetilde{Q}_{xx}K - \widetilde{Q}_{Xx}K + \widetilde{P}_{Xu} - J'\widetilde{P}_{xu},\\
R^* &= K'\widetilde{Q}_{xx}K + \widetilde{R} - K'\widetilde{P}_{xu} - \widetilde{P}'_{xu}K,\\
A^* &= \widetilde{A}_{11} - \widetilde{A}_{12}J,\\
B^* &= \widetilde{B}_1 - \widetilde{A}_{12}K.
\end{aligned}$$

These then can be cast into the following matrix form

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & \beta B^{*\prime} \\ 0 & 0 & \beta A^{*\prime} \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{X}}_{t+1} \\ \boldsymbol{u}_{t+1} \\ \boldsymbol{\xi}_{t+1} \end{bmatrix} = \begin{bmatrix} A^{*} & B^{*} & 0 \\ -P^{*\prime} & -R^{*} & 0 \\ -Q^{*} & -P^{*} & I \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{X}}_{t} \\ \boldsymbol{u}_{t} \\ \boldsymbol{\xi}_{s} \end{bmatrix} \tag{44}$$

A solution to linear system (44) will necessarily have a linear form of

$$\begin{bmatrix} \boldsymbol{u}_{t} \\ \boldsymbol{\xi}_{t} \end{bmatrix} = \begin{bmatrix} -F \\ S \end{bmatrix} \widetilde{\boldsymbol{X}}_{t} \tag{45}$$

It is straightforward to show that system matrices in (45) satisfy the following Riccati equations describing the solution to the discretionary policy problem.

$$S = Q^{*} + \beta A^{*\prime} S A^{*} - \left(P^{*\prime} + \beta B^{*\prime} S A^{*}\right)\left(R^{*} + \beta B^{*\prime} S B^{*}\right)^{-1}\left(P^{*\prime} + \beta B^{*\prime} S A^{*}\right) \tag{46}$$

$$F = \left(R^{*} + \beta B^{*\prime} S B^{*}\right)^{-1}\left(P^{*\prime} + \beta B^{*\prime} S A^{*}\right) \tag{47}$$

To sum up, our ZLB algorithm under discretion constrains the nominal interest rate to remain at ZLB during the periods where particular shocks drive the nominal interest rate below ZLB, otherwise the interest rate is not bounded. To endogenize the duration of the ZLB period contingent on particular shocks, we adopt the iteration steps described in Bodenstein et al. (2013). This procedure initially chooses the periods that ZLB binds by setting a sequence of $\eta_{t,t-s}$, based on the periods during which interest rates would fall below ZLB when no such constraint was imposed. It then revises the sequence of $\eta_{t,t-s}$ to withdraw the ZLB constraint in periods where the interest rates turn out to be above ZLB in the chosen periods. We find that Bodenstein et al. (2013)'s iteration steps work well with our model and our ZLB algorithm.