Mitigating Bunching with Bus-following Models and Bus-to-bus Cooperation

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***Abstract*—Bus bunching is an instability problem where buses operating on high-frequency public transport lines arrive at stops in bunches. This work unveils that *bus-following models* can be used to design bus-to-bus cooperative control strategies and mitigate bunching. The use of bus-following models avoids the explicit modelling of bus-stops, which would render the resulting problem discrete, with events occurring at arbitrary time intervals. In a *follow-the-leader* two-bus system, bus-to-bus communication allows the driver of the following bus to observe (from a remote distance) the position and speed of the leading bus operating in the same transport line. The information transmitted from the leader is then used to control the speed of the follower to eliminate bunching. A platoon of buses operating in the same transit line can be then controlled as *leader-follower dyads*. In this context, we propose practical control laws to regulate speeds, which would lead to bunching cure. A combined state estimation and remote control scheme is developed to capture the effect of disturbances and randomness in passenger arrivals. To investigate the performance of the developed schemes the 9-km 1-California line in San Francisco with about 50 arbitrary spaced bus stops is used. Simulations with empirical passenger data are carried out. Results show bunching avoidance and improvements in terms of schedule reliability of bus services and delays. The proposed control is robust, scalable in terms of transit network size, and thus easy to deploy by transit agencies to improve communication and guidance to drivers, and reduce costs.**

***Index Terms*—Bus bunching; bus-following models; bus-to-bus cooperation; bus schedule reliability; linear gaussian control.**

I. INTRODUCTION

BUSES can randomly fall behind schedule due to the variability of passenger's demand at bus stops and speed heterogeneity [1]. In a high-frequency transit line with two buses, if the lead bus is delayed at bus stops (or due to traffic congestion) then the next bus stops will flock more passengers waiting for boarding that would lead to further delays due to extended on-off passenger activity. Moreover, the following bus operating in the same line will pick up fewer passengers, causing its commercial speed to be higher than normal. Eventually, this positive feedback loop would result in the two buses meeting up, with the lead bus having more passengers than the following bus. Therefore, the two buses pair up and travel as a single unit, although in reality overtaking is possible. This reduces the reliability of the transit network and forces passengers allotting more time at bus stops to buffer this instability. This instability problem of high-frequency transit lines is known as *bus bunching* [1], [2].

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Transit agencies often attempt to mitigate bus bunching by including *slack-time* into their schedules and then asking bus drivers to be punctual at specific control points along the route (hold the bus up to a certain amount of time). This cure is of limited effectiveness because the medicine (slack) is sometimes worse than the illness (irregular headways) [2], [3]. A number of other control strategies have been proposed in the literature to mitigate bunching and improve the reliability of transit operations. Basic schemes include *patching* and *stop-skipping*. Patching is a scheme where buses are dispatched from a depot to fill gaps as they arise. Patching is reactive and not optimal but can be used in case of emergency. Stop skipping allows buses to recover from delays at the cost of reliability and customer satisfaction [4].

Schedule- or headway-based *holding strategies* have several forms with varying levels of complexity [5], [6], [7], [3], [8]. The most basic version involves having a bus waiting for an amount of time before continuing on the loop and it is therefore useful for both headway and schedule adherence. While holding can be effective at reducing bus bunching it reduces commercial speed. Commercial speed and headway variance can be thought of as inversely proportional when a basic holding plan is in place and it is up to the bus agencies to find the right compromise between schedule reliability and commercial speed. Static holding involves fixed time buffers whereas dynamic holding responds to system events and adjusts buffers in real-time. However, static slack does not prevent large events from disrupting headways [7]. Dynamic holding can respond better to large headway disruptions. Recently a number of works showed that dynamic holding strategies for transit lines run with or without a schedule would increase commercial speed and return to equilibrium after large disruptions [3], [8], [9].

The recent advances in bus automation and communication systems and the availability of real-time locational data enabled the development of advanced control methods for bunching cure. Modern buses are equipped with surveillance and measurement equipment such as Automated Vehicle Location (AVL), Global Positioning System (GPS) and Automated Passenger Counter (APC) devices, and telecommunications equipment to transmit information to traffic control centres in real-time. Thus it is possible to fully observe and communicate the true state of transit vehicles (speed, position, passenger load) operating in the same transit line in real-time. In this framework, cooperative two-way-looking strategies based on the spacings in the front and back of each bus can be developed. A continuum idealisation of the bus bunching problem is proposed in [3]. This model considers that bus stops are evenly spaced and bus delays due to on-off passenger activity are analogous to the number of passengers boarding the bus. In the same vein, a self-coordinating bus scheme where headways are dynamically self-equalising and the natural headway of the system tends to emerge spontaneously is proposed in [10]. A hybrid model predictive control scheme for Traffic Signal Priority (TSP) and for regulating bus headways is proposed in [11], [12]. TSP can be used to improve bus service reliability and alleviate bunching as shown in [13], [14], [15], [16].

This work proposes for the first time in literature and demonstrates that *bus-following models* (analogous to car-following) can be used to design bus-to-bus cooperative control strategies and mitigate bunching. Car-following models were traditionally employed to describe the behaviour of closely spaced two-vehicle or stream of vehicles systems. The basic assumption of a *follow-the-leader model* is that the following driver reacts to a stimulus from the lead car to maintain a specific (space or time [17]) headway between the two cars. This behaviour assumes driver's perception based on visual contact between the two cars and very short reaction times, which circumvents instabilities in case of stopped cars or stop-and-go phenomena. In a bus-following model, the visual contact is missing because the two buses are far apart from each other. Moreover, a continuous-time bus-following model includes some simplification because it does not take into account the bus stops, where buses normally stop at arbitrary time intervals. Remarkably, this work unveils that bus-to-bus cooperation and driver's response through real-time information permit the use of continuous-time bus-following models for bunching modelling and remote feedback control.

The rest of the paper is organised as follows. Section II introduces the proposed bus-following model involving leaders and followers that can be used to design advanced bus cooperation strategies. It also presents nonlinear and linear control laws to regulate space (or time) headways and speeds, which would lead to bunching cure remarkably without holding at bus stops. Section III presents a rigorous combined state estimation and control scheme based on Linear-Quadratic Gaussian (LQG) control. This scheme captures the effect of bus stops, traffic disturbances, driver/motor errors and randomness in passenger arrivals by introducing stochastic variables in a bus-following model. Section IV presents a simulation study for the 9-km 1-California line in San Francisco and a comparative study of the proposed follow-the-leader-based control laws with schedule- and headway-based holding strategies proposed in the literature. The case study benefits from empirical passenger data obtained from the San Francisco Municipal Transportation Agency. Conclusions are given in Section V.

II. MODELLING AND CONTROL BUNCHING WITH BUS-FOLLOWING MODELS

A. Problem formulation

Consider two buses n (leader) and $n+1$ (follower) operating in the same public transport line. Assume that both buses are equipped with GPS devices reporting position $x(t)$ and speed $\dot{x}(t) = v(t)$ at any time t. Also that bus-to-bus communication allows the driver of the following bus to observe the position and speed of his own bus and (from a remote distance) that of a leading bus operating in the same line. We call this class of *follow-the-leader* two-bus systems with a remote sensing capability as *bus-following models*. A platoon of buses operating in the same transit line can be then modelled as *leader-follower dyads*.

Fig. 1 illustrates the typical block diagram of a two-bus system with bus-to-bus cooperation. The kernel of the block diagram is the control strategy (or controller), whose task is to specify in real time the control inputs (e.g., acceleration or deceleration), based on available measurements (GPS and APC of the two-bus system), so as to achieve pre-specified goals (e.g., maintain a desired space or time headway [17]) despite the influence of the various disturbances (observation, motor, and traffic). More precisely, GPS (and APC) information from the lead bus is transmitted to the following bus (with some noise). GPS signals can be corrupted unintentionally by external interfering sources such as tinted vehicle windows and buildings in urban canyons that block satellite transmission. Also APCs usually track every person who gets on or off the bus (including the operator) with some noise, which may lead to some irregularities in the data. The idea is to use the information transmitted from the leading bus to control the speed of the following bus in order to eliminate bunching. The driver of the following bus responds with a reaction time and a muscular response to control the speed. Speed control can be effectuated by a continuous adjustment of acceleration (or deceleration) of the following bus (via the acceleration and brake pedals), which is denoted $\ddot{x}(t)$ at time t. The applied control to bus dynamics is being corrupted by noise, including motor noise and traffic disturbances. Finally, the driver bus system includes a closed-loop structure to provide feedback. The feedback loop can inform the driver of the following bus when they are ahead or behind schedule (or have too small or too large a time or space headway) at all points along the route (by comparison of bus position and speed with the lead GPS data). Feedback can provide also information that may be useful for avoiding bunching by appropriate instructions (e.g., follow a desired speed that allows fast recovery).

B. Deterministic control laws based on bus-following models

A first intuitive control law of a two-bus transit system considers that driver's response is proportional to the speed difference and the difference between their actual space headway and a desired (scheduled) headway given by,

$$\ddot{x}_{n+1}(t+T) = l_1 \Delta \dot{x}_{n,n+1}(t) + l_2 \left[\Delta x_{n,n+1}(t) - x^d \right], \quad (1)$$

where $\Delta \dot{x}_{n,n+1}(t) = \dot{x}_n(t) - \dot{x}_{n+1}$ and $\Delta x_{n,n+1}(t) = x_n(t) - x_{n+1}$; T is the reaction time of the bus driver $n+1$; l_1, l_2 are control parameters; and x^d is the desired space headway. The desired headway x^d can be specified from the scheduled time headway of the corresponding bus line and an average operational speed. Note that the desired space headway in (1) (and subsequent laws below) might be non-identical on different parts of the route due to uneven spacing of bus stops. In this case the desired space headway can be determined as a function of the time, i.e. $x^d(t)$, by taking into account

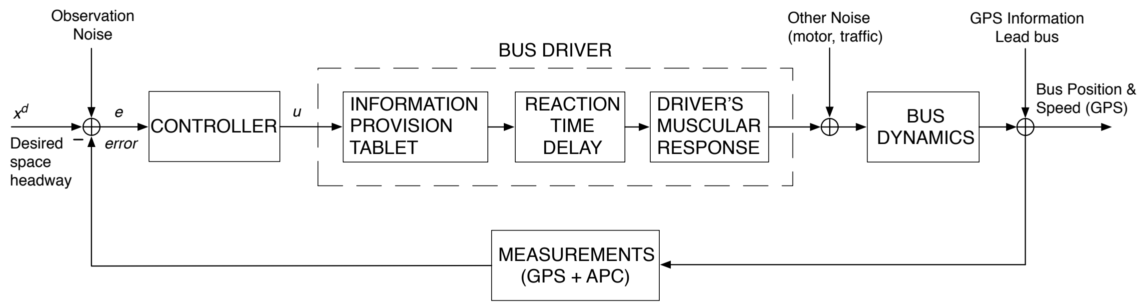


Fig. 1. Block diagram of the bus-following model with bus-to-bus cooperation.

the spacing between consecutive bus stops, day-to-day traffic, weather and other factors. The choice of the control parameters l_1 and l_2 is performed via a trial-and-error procedure so as to achieve a satisfactory control behaviour for a given two-bus system; although optimised control gains could be determined by appropriate control methodologies as in Section III. If the two buses travel with the same speed the first term in (1) is negligible and acceleration (or deceleration) is based on the actual space headway of the two buses and a desired space headway. A similar control law was first proposed by [18] for car-following models. A difference with (1) is that the reaction time T of the driver-bus system is in general higher than of a driver-car system (approximately 1.5 s, see e.g., [19]). In a bus-following model the visual contact is missing (the two buses are far from each other) and reaction time depends on how often GPS locational data from the leading bus are transmitted (e.g. via wireless communication) to the follower and the state of the following bus (traveling or stopped to drop-off or pick up passengers). Obviously the driver cannot react whenever the bus is stopped at bus stops. Bus-following models can be validated with the use of empirical GPS data of bus-to-bus cooperation to determine the reaction time T.

A family of more complex nonlinear control laws can be developed by assuming that driver's response is proportional to the velocity difference and inversely proportional to the difference between their actual space headway and a desired headway given by,

$$\ddot{x}_{n+1}(t+T) = \beta^{m,l}_{n+1}(t+T)\big[\Delta\dot{x}_{n,n+1}(t) - v^d\big], \quad (2)$$

where $\beta^{m,l}_{n+1}$ is the sensitivity of bus $n+1$ and is given by,

$$\beta^{m,l}_{n+1}(t+T) = \begin{cases} \frac{\lambda \dot{x}^m_{n+1}(t+T)}{\left[\Delta x_{n,n+1}(t) - x^d\right]^l}, & \text{if } \Delta x_{n,n+1}(t) \neq x^d, \\ 0, & \text{otherwise}, \end{cases} \quad (3)$$

where λ is a control gain, m, l are (positive integer or real) constants, and v^d is an additional parameter. The correction term v^d is introduced to circumvent instabilities whenever the lead bus is stopped for dropping off or picking up passengers and the corresponding speed is virtually zero. The product $\lambda\dot{x}_{n+1}(t+T)$ in (2) guarantees good control behaviour whenever $\dot{x}_{n+1}(t+T) \approx 0$. Similar models (without considering a desired space headway and correction term v^d) for car-following have been proposed in [20], [21].

An additional family of linear and nonlinear bus-following-based control laws can be obtained from,

$$\ddot{x}_{n+1}(t+T) = \beta^{m,l}_{n+1}(t+T)\big[\Delta\dot{x}_{n,n+1}(t) - v^d\big] + \mu\big[\Delta x_{n,n+1}(t) - x^d\big]^p, \quad (4)$$

where μ is a an additional control gain, p is (a positive integer or real) constant, and $\beta^{m,l}_{n+1}$ is given by (3). Control law (4) is similar (excluding the last term) to the so-called General Motors Nonlinear (GM) model [21]. Note that for different values of m, l and p in (2)–(4) different control laws can be produced. Typical values are $m = 1$, $l = 2$, and $p = 1$ or $p = 2$. Remarkably, the bus-following control law in (4) (with $\mu = 0$) can be used to derive (under certain conditions) a macroscopic relationship describing speed and flow of a platoon of buses operating in dedicated bus lanes. Such connection is in agreement with the recently proposed three-dimensional vehicle and passenger Macroscopic Fundamental Diagram (3D-vMFD and 3D-pMFD) for mixed traffic, bi-modal urban road networks [22], [23]. Stability criteria of the nonlinear control laws in (2)–(4) are difficult to prove, thus the control gains λ and μ should be carefully specified with respect to the physics of traffic and reaction time T. More rigorous bus-following control laws can be developed by considering motor system dynamics, stochastic variables and delays, as described in Section III.

III. STOCHASTIC BUS-FOLLOWING MODELS & LINEAR-QUADRATIC GAUSSIAN CONTROL

Human operator behaviour for *follow-the-leader* models in presence of control being corrupted by noise has been studied extensively in the 60s and later on [24], [25], [26], [27], [28]. These models assume closely spaced two-vehicle systems with basic vehicle motion dynamics, where each driver is required to track only the vehicle ahead. In the sequel, we propose a continuous-time bus-following model for bunching modelling that considers basic bus motion dynamics and uncertainties of different variables of the problem (bus stops, traffic disturbances, randomness in passenger arrivals). A combined state estimation and remote feedback control scheme based on Linear-Quadratic Gaussian (LQG) control [29], [30] is also developed. This unveils that the bus bunching problem can be viewed as a classic problem of relative space (or time) and speed regulation with noise in control theory.

Consider the bus-following model with bus-to-bus cooperation as in Fig. 1. The lead bus dynamics can be represented by vehicle motion as,

$$x_n(t+T) = a_{n,1}x_n(t) + \varepsilon_{n,1}(t), \tag{5}$$

$$\dot{x}_n(t+T) = a_{n,2}\dot{x}_n(t) + \varepsilon_{n,2}(t), \tag{6}$$

where T is the reaction time, $x_n(t)$, $\dot{x}_n(t) = v_n(t)$ are state variables, $a_{n,1}$, $a_{n,2}$ are constant modelling parameters and $\varepsilon_{n,1}$, $\varepsilon_{n,2}$ are zero-mean white gaussian noise with known variance. This linear model provides a mathematical description of the leading bus motion without considering any forces that affect the motion, i.e., a control term in (5)–(6) is absent. The basic idea of the bus-following model is to control the speed of the following bus (based on the location and speed of the leader) in order to eliminate bus bunching. Thus the following bus dynamics can be given by,

$$x_{n+1}(t+T) = a_1x_{n+1}(t) + b_1u_{n+1}(t) + \varepsilon_{n+1,1}(t), \tag{7}$$

$$\dot{x}_{n+1}(t+T) = a_2\dot{x}_{n+1}(t) + b_2u_{n+1}(t) + \varepsilon_{n+1,2}(t), \tag{8}$$

where $x_{n+1}(t)$, $\dot{x}_{n+1}(t) = v_{n+1}(t)$ are state variables, $u_{n+1}(t) = \ddot{x}_{n+1}(t)$ is the control variable, a_1, a_2, b_1, b_2 are constant modelling parameters and $\varepsilon_{n+1,1}$, $\varepsilon_{n+1,2}$ are zero-mean white gaussian noise with known variance. In the bus model (7)–(8), $u_{n+1}(t)$ represents the control applied by the driver (acceleration or deceleration) via the acceleration and brake pedals. The stochastic variables $\varepsilon_{n+i,j}$, for all $i = 0, 1$, $j = 1, 2$ are introduced to capture the effect of bus stops, traffic disturbances, driver errors and randomness in passenger arrivals (other noise in Fig. 1). From acceleration or deceleration a target velocity for the follower can be calculated that is easily understood by a driver.

Expressions (5)–(8) provide a continuous-time bus-following model that does not allow for direct consideration of the bus stops (where buses stop at arbitrary time intervals) and passenger loads. Nevertheless, uncertainty and variability of different variables of the problem can be captured by the noise terms in (5)–(8). The number of bus stops (fixed and known) and historical (or fusion with real-time) data of passenger activity for a specific line can reflect the typical time variation of passenger arrivals. In addition, traffic and driver/motor noise can be obtained with different sensors in real-time. This is the price to pay for avoiding the explicit modelling of bus-stops which would render the resulting problem discrete (Discrete Event Dynamic System) and would lead to complex models difficult to use for control purposes without relaxations.

Expressions (5)–(8) can be combined into a discrete-time[1] linear system with noise given by,

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) + \boldsymbol{\gamma}(k), \tag{9}$$

where $\mathbf{x} \in \mathbb{R}^{4\times 1}$ (with elements $x_{n+i,j}$, $\dot{x}_{n+i,j}$ for all $i = 0, 1$, $j = 1, 2$) is the state vector of both leader and follower position and velocity, and $u \equiv u_{n+1}$ is the control; $\mathbf{A} \in \mathbb{R}^{4\times 4}$ (with elements $a_{n+i,j}$, for all $i = 0, 1$, $j = 1, 2$) and $\mathbf{B} \in \mathbb{R}^{4\times 1}$ (with elements $b_{n+1,i}, i = 1, 2$) are the state and control matrices, respectively; $\boldsymbol{\gamma} \in \mathbb{R}^4$ (with elements $\varepsilon_{n+i,j}$, for all $i = 0, 1$, $j = 1, 2$) is a Gaussian noise process with known covariance matrix $\boldsymbol{\Gamma}$ given by $\mathbf{E}\{\boldsymbol{\gamma}(k)\} = \mathbf{0}$, $\mathbf{E}\big\{\boldsymbol{\gamma}(k_i)\cdot\boldsymbol{\gamma}^{\mathsf{T}}(k_j)\big\} = \boldsymbol{\Gamma}\cdot\delta_{ij}$, $\boldsymbol{\Gamma} \succeq 0$, where δ is the Kronecker delta function. We also assume that the initial state $\mathbf{x}(0)$ has a known mean $\mathbf{x}_0$ and covariance matrix $\boldsymbol{\Pi}_0$ given by, $\mathbf{E}\{\mathbf{x}(0)\} = \mathbf{x}_0$, $\mathbf{E}\Big\{[\mathbf{x}(0) - \mathbf{x}_0]\cdot[\mathbf{x}(0) - \mathbf{x}_0]^{\mathsf{T}}\Big\} = \boldsymbol{\Pi}_0$.

The system output vector $\mathbf{y} \in \mathbb{R}^2$ for the augmented system (9) is given by,

$$\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k) + \boldsymbol{\zeta}(k), \tag{10}$$

where $\boldsymbol{\zeta}$ is the observation noise (see feedback loop in Fig. 1) and $\mathbf{H}$ is the observation matrix consisting of 0's and 1's such that a number of elements or linear combinations of elements of $\mathbf{x}$ are correlated in (10). More precisely, $\mathbf{H}$ matrix allows the lead bus state variables to be compared with those of the following bus. An appropriate choice is,

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & -1 & -\alpha h^d \\ 0 & 1 & 0 & -1 \end{bmatrix}, \tag{11}$$

where α is a constant parameter and h^d is a desired (scheduled) time headway between two buses, which is known from the timetable. Parameter α is introduced to capture the correlation between actual space headway $x_n - x_{n+1}$ and a desired headway x^d, where x^d is assumed to be a linear function of the following bus velocity $\dot{x}_{n+1}$ and the desired time headway h^d, i.e. $x^d \propto \alpha\dot{x}_{n+1}(t)h^d$. Note that in general, the desired spacing is a nonlinear function of the velocity of the following bus, time headway, passenger load and traffic conditions in the transit line.

Noise vector $\boldsymbol{\zeta} \in \mathbb{R}^2$ is a Gaussian process with known covariance matrix $\mathbf{Z}$ given by $\mathbf{E}\{\boldsymbol{\zeta}(k)\} = \mathbf{0}$, $\mathbf{E}\big\{\boldsymbol{\zeta}(k_i)\cdot\boldsymbol{\zeta}^{\mathsf{T}}(k_j)\big\} = \mathbf{Z}\cdot\delta_{ij}$, $\mathbf{Z} \succ 0$. We assume that the system (9)–(10) is reachable and observable; and the initial state condition $\mathbf{x}_0$ is uncorrelated with the input noise $\boldsymbol{\gamma}(k)$ and observation noise $\boldsymbol{\zeta}(k)$, i.e., $\mathbf{E}\big\{\boldsymbol{\gamma}(k)\cdot\mathbf{x}_0^{\mathsf{T}}\big\} = \mathbf{E}\big\{\boldsymbol{\zeta}(k)\cdot\mathbf{x}_0^{\mathsf{T}}\big\} = \mathbf{0}$. Finally both noise inputs are assumed uncorrelated (although can be correlated), i.e., $\mathbf{E}\big\{\boldsymbol{\gamma}(k)\cdot\boldsymbol{\zeta}^{\mathsf{T}}(k)\big\} = \mathbf{0}$.

The control objective is to regulate the speed of the following bus with minimum effort so as to maintain the actual space (or time) headway of a two-bus system to a desired pre-specified headway, and as a consequence to avoid bunching. A quadratic criterion that considers this objective has the form,

$$\mathcal{L} = \mathbf{E}\Bigg\{\sum_{k=0}^{\infty}\Big\{q_1\big[x_n(k) - x_{n+1}(k) - x^d\big]^2 + q_2\big[\dot{x}_n(k) - \dot{x}_{n+1}(k)\big]^2 + ru(k)^2\Big\}\Bigg\}, \tag{12}$$

where $q_1 \geq 0$, $q_2 \geq 0$ and $r > 0$ are weighting constants. The first term in (12) is responsible for the minimisation of deviations of the space headway from a desired spacing. The second term is responsible for normalising the speeds between the two buses. Note that an additional correction term v^d (cf. (2)) may be introduced in the second term to circumvent speed instabilities whenever the lead bus is stopped for collecting passengers and the corresponding speed is

[1]For control implementation purposes, we define a discrete time index k such that $t = kT$ with T the sampling period (reaction time) and denote $x(k) = x(kT)$ analogously for all variables.

virtually zero. Clearly the third term is responsible for avoiding bunching with minimum control effort. Weights q_1, q_2 must be chosen such that the corresponding subgoals are satisfied. More precisely, $q_1 = 1/x^d$ and $q_2 = 1/v^c$ are appropriate values, where v^c is the commercial speed. Commercial speed is defined as the average operational speed of the buses, including cruising, passenger alighting and boarding. Weight r influences the magnitude of the control reactions and is selected through a trial-and-error procedure so as to achieve a satisfactory control behaviour for a given application transport line. The infinite time horizon in (12) is taken in order to obtain a time-invariant feedback law according to the LQG theory.

To design the combined state estimation and control scheme, we assume at time $k > k_0$ availability of portable information $\mathcal{I}(k) = \{\mathbf{y}(k_0), u(k_0), \mathbf{y}(k_0+1), u(k_0+1), \ldots, \mathbf{y}(k)\}$. Based on this information and the system (9)–(10) we aim to derive a control law,

$$u(k) = \mathcal{R}\big[\mathcal{I}(k-1), k\big], \tag{13}$$

such that the cost criterion (12) is minimised. According to the LQG theory [29], [30] the control law (13) has the form,

$$u(k) = -\mathbf{L}\hat{\mathbf{x}}(k), \tag{14}$$

where $\mathbf{L} \in \mathbb{R}^{1\times 4}$ (depends on $\mathbf{A}, \mathbf{B}$ and q_1, q_2, r) is a time-invariant control gain, which is calculated by the solution of the corresponding deterministic Linear-Quadratic (LQR) control problem (see [31]); and $\hat{\mathbf{x}}(k)$ is the output of the optimal state estimator, i.e. Kalman Filter. The estimate $\hat{\mathbf{x}}(k)$ is generated in real-time by the recursive estimator:

$$\hat{\mathbf{x}}(k) = \mathbf{A}\hat{\mathbf{x}}(k-1) + \mathbf{B}u(k-1) + \mathbf{K}[\mathbf{y}(k-1) - \mathbf{H}\hat{\mathbf{x}}(k-1)], \tag{15}$$

where $\hat{\mathbf{x}}(0) = \mathbf{x}_0$ is known and $\mathbf{K} \in \mathbb{R}^{4\times 2}$ (depends on $\mathbf{A}, \mathbf{H}, \mathbf{\Gamma}, \mathbf{Z}$) is a time-invariant (might be time-variant) estimator gain, which is calculated by a recursive Riccati equation according to the Kalman Filter theory [31]. Due to the celebrated Separation Theorem the two problems namely control and estimation can be solved separately by forgetting completely the stochastic aspects and control problem, respectively. The Separation Theorem guarantees that the overall LQG control design is optimal in the sense that the control law (14) minimises the cost criterion (12). The stability of control law (14) may be mathematically guaranteed under certain assumptions (presence of white noise disturbances). In case of strong plant uncertainty and lack of robustness of the LQG, advanced LQG/LTR (Loop Transfer Recovery) control [32], [33], [34] or robust control [35] schemes can be employed.

The final control law reads,

$$u(k) = l_1 \Delta\hat{\dot{x}}_{n,n+1}(k) + l_2[\Delta\hat{x}_{n,n+1}(k) - l_3\hat{\dot{x}}_{n+1}(k)], \tag{16}$$

where l_1, l_2 and l_3 are elements of matrix $\mathbf{L}$. Note that (16) is similar to (1) with $l_3 = 0$ and deterministic state vector, which corresponds to the standard linear-quadratic control solution without the Kalman Filter. The real-time application of the LQG scheme calls for estimates of $\hat{\mathbf{x}}(k)$ via (15) (starting from known $\hat{\mathbf{x}}_0$) based on which the scheme executes the control law (16) and returns the control (acceleration or deceleration) for application to the driver. Thus the required calculations in real-time are limited to the parallel execution of (15)–(16), while the control and estimation gains $\mathbf{L}$ and $\mathbf{K}$ are calculated off-line (although may be time-variant and updated on-line). It should be noted that the bus-following model given by (9) is generic and could be applied to transit lines with arbitrary geometry and characteristics. Although the first bus in service for a particular transit line remains uncontrolled (see (5)–(6)), all the other buses in service can be controlled as *leader-follower dyads*. Thus a platoon of buses operating in the same transit line can be modelled and controlled as leader-follower dyads using (9), (15), (16). If essential, the first bus in service can be under control too, e.g. by including a control term in the state space model (5)–(6) of the leading bus.

IV. EMPIRICAL PASSENGER DATA & APPLICATION

A. Simulation environment and setup

A simulation environment was designed in Matlab to reflect real world bus lines including arbitrary placed stops and on-off passenger activity. It emulates a closed-loop bus line and includes a depot where buses leave the depot at pre-specified time intervals (schedule) and return to the depot when they complete one loop. In case of bunching, overtaking is allowed as in reality. Variability in the headways is introduced via randomness in passenger arrivals at bus stops. The probability of passenger arrival is variable with time and between different bus stops to emulate passenger behaviour in on/off-peak times. In utilising empirical data, the stop locations, speed limits, schedule, bus capacity, and passenger behaviour are accounted for and, when applicable, are time-variant. The simulator is stochastic, i.e., the same seed is initialised with different random number generators so that different replications to produce different results for the same demand scenario.

The simulator assumes that delays at bus stops due to on-off passenger activity are analogous to the number of passengers alighting and boarding the bus. Alighting and boarding times of the passengers are set to 1-3 s and 2-5 s, respectively. In case that real passenger activity data at stops are not available, the number of people getting off at a bus stop is a random number between 0 and the number of passengers at the bus. Bus capacity is 30 pax. The maximum speed of a bus is 45 km/h. The controlled acceleration is bounded in the interval $[-2.0, 2.5]$ m/s^2. In addition, a random but bounded time delay is added to simulate buses slowing to a stop (in the interval $[0, 10]$ s) and accelerating to operating speed after stopping (in the interval $[0, 20]$ s). The simulator includes an Application Programming Interface (API) that allows the simulator to interact in real-time with an external control strategy or algorithm, exchange data, pass inputs (i.e., controls) and receive outputs (i.e., position, velocity, on-off passenger activity and load). At regular intervals headway data, state and control trajectories are gathered for post-analysis and visualisation. The simulator allows the user to adjust bus schedules and speed, start, pause and exit the simulation. A Graphical User Interface (GUI) displays the location of bus stops and passenger queues, bus positions and passenger load.

B. Empirical passenger data and control strategies

To investigate the efficiency of the developed approaches to the problem of bus bunching, two different public transport lines are considered. The first case is a 2-km closed-loop transport line with 10 arbitrary spaced bus stops. The second case is the 9-km 1-California line in San Francisco (service area of California St, Clay St, and Sacramento St) with about 50 arbitrary spaced bus stops, depicted in Fig. 2. For each line, we compare the behaviour of the *follow-the-leader control laws* (2) (denoted "Lambda"), (4) (denoted "GM"), and (16) (denoted "LQR") with four different control strategies proposed previously in the literature, namely *conventional schedule-based control*, *forward headway-based control* [7], *backward headway-based control* [10], and *two-way looking headway-based control* [3]. A *no-control* case where buses traveling with a nominal speed is used as baseline for comparison. The schedule-based control is applied with holding at bus stops to meet scheduled departure times (if applicable).

To fine tune the proposed controllers different passenger demand scenarios were used in the simulator. A trial-and-error procedure involves the suitable choice of the control gains l_1 and l_2 (l_3 can be obtained via l_2) for (16), λ and μ for (2) or (4), so as to achieve a satisfactory control behaviour and performance with respect to schedule adherence and reliability of bus service. Initially the 2-km line was used to investigate the behaviour of the deterministic control laws for the bus-following model with only two buses operating in the same line, for different demand scenarios with time horizon of 2 h. The most satisfactory results with respect to schedule adherence, bunching avoidance, and reliability of bus service were obtained with control gains $(l_1, l_2) = (0.075, 2.5)$ for the control law (16), $\lambda = 0.005$ for the control law (2), and $(\lambda, \mu) = (0.00045, 0.5)$ for the control law (4) with $m = 1$, $l = 2$, and $p = 1$. The optimal solution for control law (16) with $(l_1, l_2) = (0.075, 2.5)$ is obtained by the LQG theory with $q_1 = 1/x^d$, $q_2 = 1/v^c$ and $r = 10^{-4}$, where commercial speed $v^c = 10$ km/h, desired time headway $h^d = 7$ min and $x^d \propto \alpha \dot{x}_{n+1}(t) h^d = \alpha v^c h^d$, where α is a unit correction parameter. Then tests for the 1-California line were conducted with real passenger (on-off activity) data obtained from the San Francisco Municipal Transportation Agency (SFMTA) [36]. See the caption in Fig. 2 for details on the data format. Aggregated data were also be available for specific time periods (AM peak, PM peak, Midday, School) and for a whole day. Therefore tests were conducted for different peak periods (with a time horizon of 3-4 hours) and a whole day. For all reported results the reaction time T of the bus driver is set to 30 s (10 s and 60 s were also tested), this includes GPS latency and delays due to stops. Note that this time is significantly higher than of a driver-car system (closely spaced cars).

C. Results and sensitivity analysis

Fig. 3 displays the control trajectories (acceleration or deceleration) for two different sets of control gains (l_1, l_2) (control law (16)) and λ (control law (2)). It can be seen that the control trajectories in LQR are smoother than in the Lambda. Note that control is activated every 30 s but the trajectories include also the acceleration or deceleration of the bus due to stops. Both controllers apply mainly deceleration to avoid bunching (the following bus is delayed), although acceleration is observed when is possible. Note that bunching is observed under no-control in this scenario. The spike (oscillatory) behaviour of the Lambda controller is attributed to its nonlinear nature. In general, the Lambda controller is very sensitive to the choice of gain λ and its stability cannot be studied easily.

TABLE I
PERFORMANCE OF DIFFERENT STRATEGIES FOR 1-CALIFORNIA LINE.

Control Strategy	Commercial Speed (km/h)	Headway Adherence	Bunching Avoidance
1. No control	14.6	0.4282	N (6)
2. Forward-headway [7]	13.2	0.2573	Y (0)
3. Backward-headway [10]	13.3	0.2665	Y (0)
4. Two-way looking [3]	13.9	0.3103	N (1)
5. Schedule-based	13.0	0.2241	Y (0)
6. Lambda Eq. (2)	13.8	0.3079	Y (0)
7. GM Eq. (4)	12.9	0.2167	Y (0)
8. LQR Eq. (16)	12.6	0.1794	Y (0)

Table I displays the performance of the six control methods for the 1-California line in terms of commercial speed (in km/h), headway adherence, and bunching avoidance. These results obtained from the average of twelve replications (different seeds used for the same demand scenario in the simulator) for each control strategy. The last column in Table I reports in parentheses the average (rounded up) number of bunches over twelve replications and if bunching is avoided (Y or N). Commercial speed is the average operational speed of the buses, including cruising, passenger alighting and boarding. Headway adherence is defined as the ratio of standard deviation of headway deviation (difference between actual and scheduled headway) over scheduled headway, i.e. $\mathrm{std}(h - h^d)/h^d$ where h and h^d are the actual and scheduled headways, respectively. The headway adherence provides insights for the reliability of the bus lines. A value close to 0 indicates a small deviation from the scheduled headway and thus high reliability of bus services. On the other hand, high values of headway adherence results to high waiting times for the passengers at bus stops. As can be seen in Table I the no control case exhibits the highest commercial speed among all control methods because no holding is applied at bus stops, as expected. However, the lower commercial speed in case of control, due to holding at bus stops in schedule-based control or acceleration and deceleration in follow-the-leader-based control laws) proves beneficial for the reliability of bus services. Thus, the ranking of the control methods with respect to headway adherence is the opposite. As can be seen, Lambda, GM, and LQR lead to a reduction of headway adherence and waiting times for the passengers (shown later in Fig. 5) compared to no control and headway-based strategies. Remarkably the schedule-based control exhibits better headway adherence compared to headway-based strategies, though it requires holding. Bunching is observed under no-control (6 times) and two-way looking headway-based control (1 time) for this particular demand scenario. The LQR control exhibits the best reliability among all strategies, while it is the slowest, slower is faster

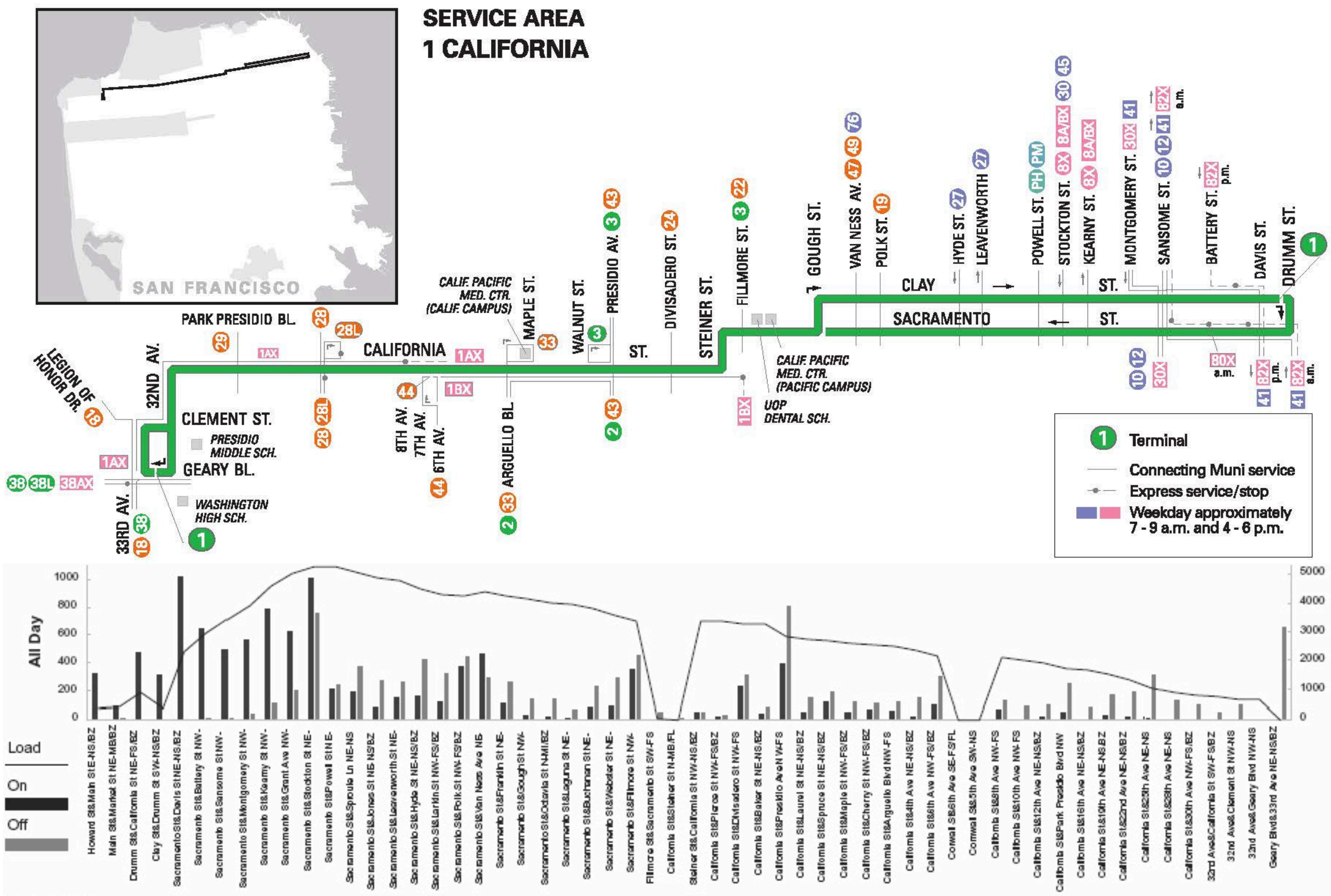


Fig. 2. The service area of 1-California line and corresponding passenger activity graph, source [36]. Left y-axis is the scale for the on-off activity at each bus stop; Right y-axis is the scale for the solid line showing total load; Solid line indicates the total number of passengers that ride through stops; The bars are the number of passengers that get on (shown in black) or off (shown in gray) at each bus stop.

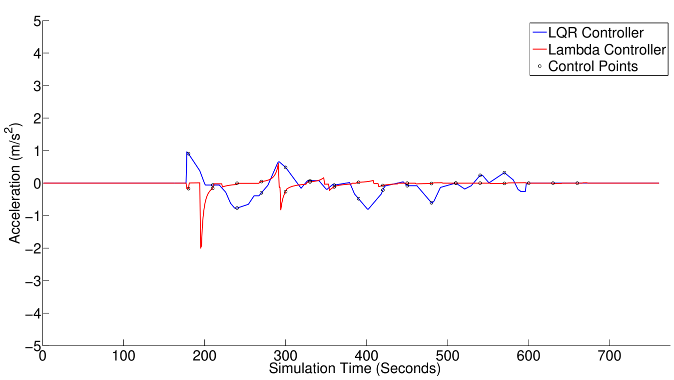


Fig. 3. Control trajectories for LQR and Lambda. Circles indicate points of control activation.

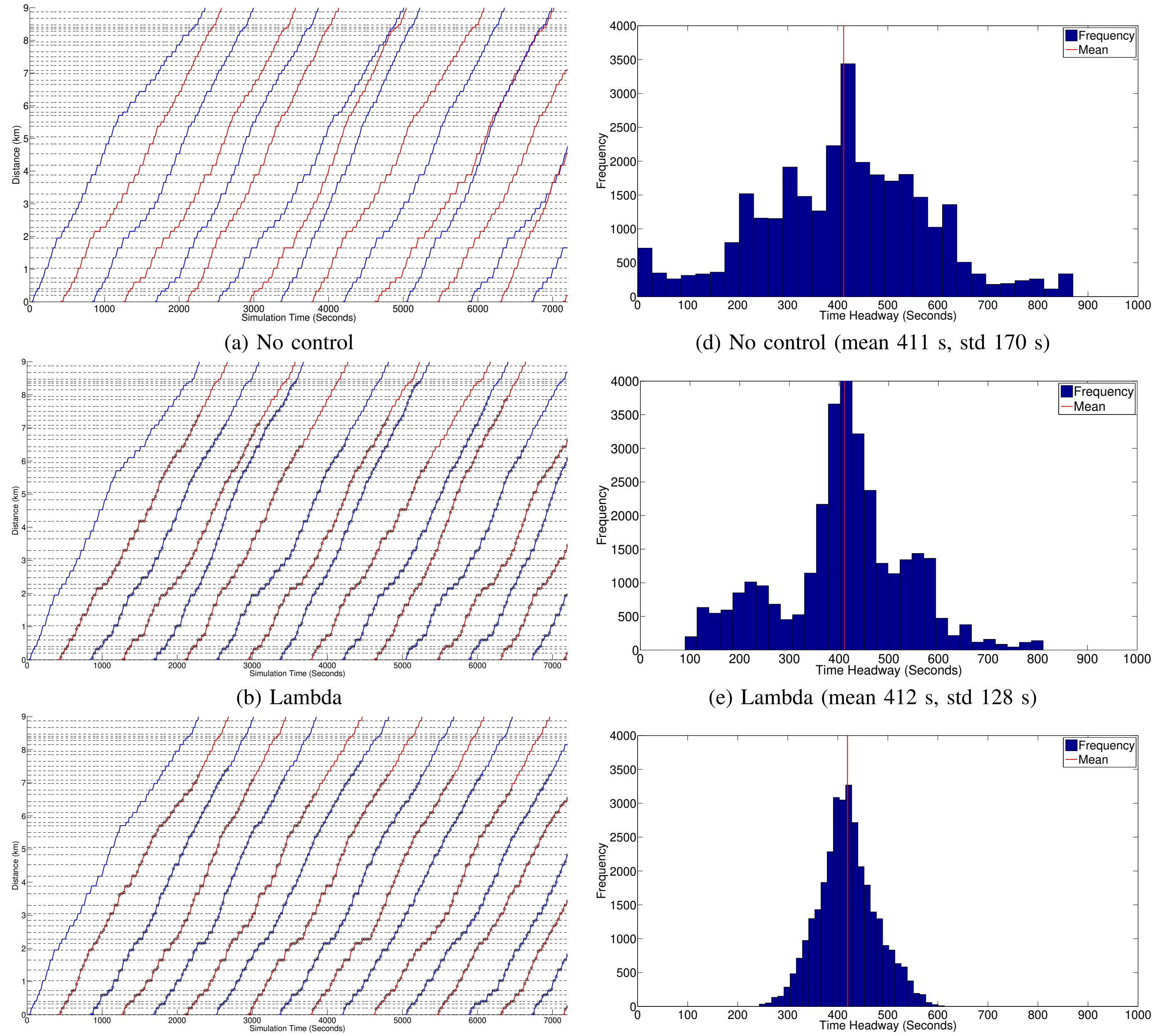
effect, i.e., better reliability and customer satisfaction.

Fig. 4 presents the results obtained from the application of the control laws (2) and (16), and no control to the 1-California line with empirical AM peak data with scheduled headway 7 min (420 s). Figs Figs 4(a)–4(c) depict the obtained trajectories of a stream of buses operating in line 1. In Figs 4(b) and 4(c) buses were controlled in pairs with a follow-the-leader like model and bus-to-bus cooperation, as in Fig. 1. Circles on the trajectories indicate the control points, thus circles disappear the time after a lead bus reach its destination (obviously no circles present on the trajectory of the first bus in the line). As can be seen, bus bunching is observed under no control (Fig. 4(a)) while it is avoided with control. LQR is seen to perform better from Lambda control (cf. Fig. 4(b) with Fig. 4(c)).

Figs 4(d)–4(f) display the headway distributions under the three control methods (headways collected every 1 s). A larger concentration of values around the scheduled headway (420 s) indicates a smaller deviation from the schedule and higher reliability. As can be seen from the distributions, LQR (mean 419 s, std 58 s) exhibits the best performance followed by Lambda (mean 412 s, std 128 s) and no control (mean 411 s, std 170 s). The ranking of the strategies with respect to headways is in agreement with the findings in Table I (see schedule adherence and reliability of bus service).

The spread of travel times between stops is a good indicator of reliability. To provide an aggregate quantification of travel times and assess the sensitivity to bus frequency and passengers demand, we plot for each bus stop the standard deviation (indicated by black bars) and calculate the mean of the standard deviations (MSD) of travel times between stops for the entire public transport line (1-California line).

Figs 5(a)–5(c) display the obtained results for the three



(a) No control

(d) No control (mean 411 s, std 170 s)

(b) Lambda

(e) Lambda (mean 412 s, std 128 s)

(c) LQR

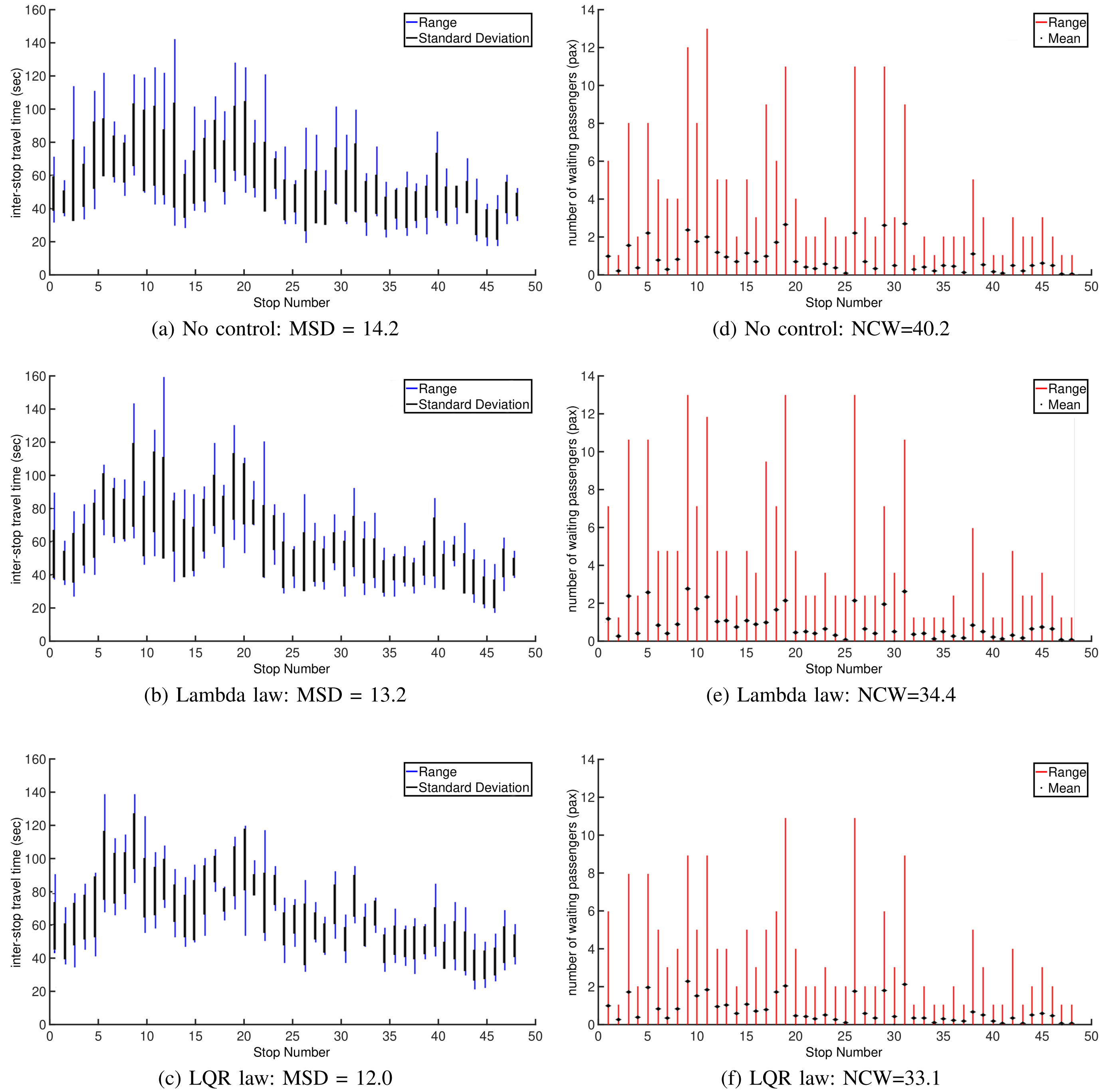
(f) LQR (mean 419 s, std 58 s)

Fig. 4. (a)–(c): Time-space diagrams depicting the trajectories of a stream of buses operating in the same line under different control methods. Horizontal dotted lines indicate the location of bus stops; circles indicate points of control activation. (d)–(f): Distribution of bus headways under different control methods.

control cases. As can be seen, the smallest deviation (i.e., most reliable control method) is the LQR method (see Fig. 5(c)), followed the Lambda law, and no control, respectively. Similar to the travel times, graphs of customers waiting per stop are only interesting in relative terms. Thus, we plot the customers waiting per stop and calculate a system-wide mean of number of customers waiting (NCW). Note that more popular stops have higher means and maximums regardless of control method and that there is large variability across space. Figs 5(d)–5(f) display the obtained results for the three control cases. Note that NCW is highly reliant on stochastic arrival events and thus the differences are quite moderate. Concluding, the ranking of the strategies in Fig. 5 is in agreement with the findings in Table I, confirming that the variability of travel times between stops is a good measure of schedule adherence.

V. CONCLUSIONS AND OUTLOOK

The paper addressed the problem of bus bunching in transit lines. The presented methodological framework combines bus-following models and bus-to-bus cooperation. In contrast to previous works, the use of bus-following models avoids the explicit modelling of bus-stops which would render the resulting problem discrete, with events occurring at arbitrary time intervals. The proposed framework unveils that the bunching problem can be viewed as a regulation problem in control theory. It allows for the devise of practical linear and nonlinear laws for remote bus control to regulate space or time headways



(a) No control: MSD = 14.2

(d) No control: NCW=40.2

(b) Lambda law: MSD = 13.2

(e) Lambda law: NCW=34.4

(c) LQR law: MSD = 12.0

(f) LQR law: NCW=33.1

Fig. 5. (a)–(c): Comparison of inter-stop travel times; (d)–(f): Comparison of waiting customers data.

and speeds, which would lead to bunching cure. Remote control can be combined with state estimation for reliable bus operations. Results from the application of the proposed framework to the 9-km 1-California line in San Francisco with about 50 arbitrary spaced bus stops showed bunching avoidance and significant improvements in terms of schedule reliability of bus services and delays.

The proposed control laws are easy to implement and can be used for real-time bunching control in real-life settings. For implementation, a tablet, smartphone, or other external device to be mounted in each bus could be utilised. The system provides speed assistance to the bus driver by displaying the reference speed obtained from the developed control laws. Of course a successful deployment of such a system assumes that drivers are complying with the instructions of the system. With the emergence of connected and automated vehicles (e.g., conditional level 3 or high automation level 4), it is envisaged that the developed framework will be able to provide a nominal reference speed to a cooperative adaptive cruise control system for buses.

The validation of bus-following models with respect to the bus drivers reaction time to instructions given by an automatic control strategy with empirical GPS data of bus-to-bus cooperation via wireless communication should be studied.

We also aim to investigate the compliance of bus drivers to instructions and its impact to control and the role of feedback. Finally, stability of bus-following models and the impact of time-delays on stability are future research directions of particular importance.

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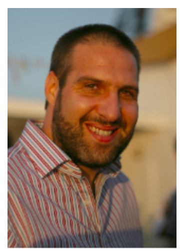

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