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Every classifiable simple C*-algebra has a Cartan subalgebra

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Abstract We construct Cartan subalgebras in all classifiable stably finite C*-algebras. Together with known constructions of Cartan subalgebras in all UCT Kirchberg algebras, this shows that every classifiable simple C*-algebra has a Cartan subalgebra.

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1 Introduction

Classification of C*-algebras has seen tremendous advances recently. In the unital case, the classification of unital separable simple nuclear $\mathcal{Z}$-stable C*-algebras satisfying the UCT is by now complete. This is the culmination of work by many mathematicians. The reader may consult [12,20,24,34,44] and the references therein. In the stably projectionless case, classification results are being developed (see [13–15,18,19]). It is expected that—once the stably projectionless case is settled—the final result will classify all separable simple nuclear $\mathcal{Z}$-stable C*-algebras satisfying the UCT by their Elliott invariants. This class of C*-algebras is what we refer to as "classifiable C*-algebras".

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To complete these classification results, it is important to construct concrete models realizing all possible Elliott invariants by classifiable C*-algebras. Such models have been constructed—in the greatest possible generality—in [11] (see also [43] which covers special cases). In the stably finite unital case, the reader may also find such range results in [20], where the construction follows the ideas in [11] (with slight modifications, so that the models belong to the special class considered in [20]). In the stably projectionless case, models have been constructed in a slightly different way in [19] (again to belong to the special class of algebras considered) under the additional assumption of a trivial pairing between K-theory and traces.

Recently, the notion of Cartan subalgebras in C*-algebras [25,36] has attracted attention, due to connections to topological dynamics [26–28] and the UCT question [3,4]. In particular the reformulation of the UCT question in [3,4] raises the following natural question (see [29, Question 5.9], [42, Question 16] and [5, Problems 1 and 2]):

Question 1.1 *Which classifiable C*-algebras have Cartan subalgebras?*

By [25,36], we can equally well ask for groupoid models for classifiable C*-algebras. In the purely infinite case, groupoid models and hence Cartan subalgebras have been constructed in [41] (see also [29, § 5]). For special classes of stably finite unital C*-algebras, groupoid models have been constructed in [8,35] using topological dynamical systems. Using a new approach, the goal of this paper is to answer Question 1.1 by constructing Cartan subalgebras in all the C*-algebra models constructed in [11,19,20], covering all classifiable stably finite C*-algebras, in particular in all classifiable unital C*-algebras. Generally speaking, Cartan subalgebras allow us to introduce ideas from geometry and dynamical systems to the study of C*-algebras. More concretely, in view of [3,4], we expect that our answer to Question 1.1 will lead to progress on the UCT question.

The following two theorems are the main results of this paper. The reader may consult [25,36] for the definition of twisted groupoids and their relation to Cartan subalgebras, and [38, § 2.2], [32, § 8.4], [18–20] for the precise definition of the Elliott invariant.

Theorem 1.2 (unital case) *Given*

- *a weakly unperforated, simple scaled ordered countable abelian group* (G_0, G_0^+, u),
- *a non-empty metrizable Choquet simplex* T,
- *a surjective continuous affine map* $r : T \to S(G_0)$,
- *a countable abelian group* G_1,

there exists a twisted groupoid (G, Σ) *such that*

- *G is a principal étale second countable locally compact Hausdorff groupoid,*
- *$C_r^*(G, \Sigma)$ is a simple unital C*-algebra which can be described as the inductive limit of subhomogeneous C*-algebras whose spectra have dimension at most* 3*,*
- *the Elliott invariant of $C_r^*(G, \Sigma)$ is given by*

$$(K_0(C_r^*(G, \Sigma)), K_0(C_r^*(G, \Sigma))^+, [1_{C_r^*(G,\Sigma)}], T(C_r^*(G, \Sigma)), r_{C_r^*(G,\Sigma)},$$
$$K_1(C_r^*(G, \Sigma))) \cong (G_0, G_0^+, u, T, r, G_1).$$

Theorem 1.3 (stably projectionless case) *Given*

- *countable abelian groups G_0 and G_1,*
- *a non-empty metrizable Choquet simplex T,*
- *a homomorphism $\rho : G_0 \to \mathrm{Aff}(T)$ which is weakly unperforated in the sense that for all $g \in G_0$, there is $\tau \in T$ with $\rho(g)(\tau) = 0$*

there exists a twisted groupoid (G, Σ) such that

- *G is a principal étale second countable locally compact Hausdorff groupoid,*
- *$C_r^*(G, \Sigma)$ is a simple stably projectionless C*-algebra with continuous scale in the sense of* [18,19,30,31] *which can be described as the inductive limit of subhomogeneous C*-algebras whose spectra have dimension at most* 3*,*
- *the Elliott invariant of $C_r^*(G, \Sigma)$ is given by*

$$(K_0(C_r^*(G, \Sigma)), K_0(C_r^*(G, \Sigma))^+, T(C_r^*(G, \Sigma)), \rho_{C_r^*(G,\Sigma)},$$
$$K_1(C_r^*(G, \Sigma))) \cong (G_0, \{0\}, T, \rho, G_1).$$

The condition on ρ means that the pairing between K-theory and traces is weakly unperforated, in the sense of [11]. It has been shown in [14, § A.1] that this condition of weak unperforation is necessary in the classifiable setting (i.e., it follows from finite nuclear dimension, or $\mathcal{Z}$-stability).

It is worth pointing out that in the main theorems, the twisted groupoids are constructed explicitly in such a way that the inductive limit structure with subhomogeneous building blocks will become visible at the groupoid level.

Remark 1.4 The original building blocks in [11] have spectra with dimension at most two. The reason three-dimensional spectra are needed in this paper is because it is not clear how to realize all possible connecting maps at the level of K_1 by Cartan-preserving homomomorphisms using the building blocks in [11]. Therefore, the building blocks have to be modified (see Sect. 3). Roughly speaking, the idea is to realize all possible connecting maps in K_1 at the level of topological spaces. This however requires three-dimensional spectra because "nice" topological spaces (say CW-complexes) of dimension two or lower have torsion-free K^1 (because cohomology is torsion-free in all odd degrees for these spaces). The dimension can be reduced to two if K_1 is torsion-free (see Corollary 1.8 and Remark 3.9).

In particular, together with the classification results in [12,20,24,34,44], the groupoid models in [41], and [3, Theorem 3.1], we obtain the following

Corollary 1.5 *A unital separable simple C*-algebra with finite nuclear dimension has a Cartan subalgebra if and only if it satisfies the UCT.*

The only reason we restrict to the unital case here is that classification in the stably projectionless case has not been completed yet.

The constructions of the twisted groupoids in Theorems 1.2 and 1.3 yield the following direct consequences:

Corollary 1.6 *In the situation of Theorem 1.2, suppose that in addition to (G_0, G_0^+, u), T, r and G_1, we are given a topological cone $\tilde{T}$ with base T and a lower semicontinuous affine map $\tilde{\gamma} : \tilde{T} \to [0, \infty]$. Then there exists a twisted groupoid $(\tilde{G}, \tilde{\Sigma})$ such that*

- *$\tilde{G}$ is a principal étale second countable locally compact Hausdorff groupoid,*
- *$C_r^*(\tilde{G}, \tilde{\Sigma})$ is a non-unital hereditary sub-C*-algebra of $C_r^*(G, \Sigma) \otimes \mathcal{K}$,*
- *the Elliott invariant of $C_r^*(\tilde{G}, \tilde{\Sigma})$ is given by*

$$(K_0(C_r^*(\tilde{G}, \tilde{\Sigma})), K_0(C_r^*(\tilde{G}, \tilde{\Sigma}))^+, \tilde{T}(C_r^*(G, \Sigma)), \Sigma_{C_r^*(\tilde{G}, \tilde{\Sigma})}, r_{C_r^*(\tilde{G}, \tilde{\Sigma})},$$
$$K_1(C_r^*(\tilde{G}, \tilde{\Sigma}))) \cong (G_0, G_0^+, \tilde{T}, \tilde{\gamma}, r, G_1).$$

Corollary 1.7 *In the situation of Theorem 1.3, suppose that in addition to G_0, G_1, T and ρ, we are given a topological cone $\tilde{T}$ with base T and a lower semicontinuous affine map $\tilde{\gamma} : \tilde{T} \to [0, \infty]$. Then there exists a twisted groupoid $(\tilde{G}, \tilde{\Sigma})$ such that*

- *$\tilde{G}$ is a principal étale second countable locally compact Hausdorff groupoid,*
- *$C_r^*(\tilde{G}, \tilde{\Sigma})$ is a hereditary sub-C*-algebra of $C_r^*(G, \Sigma) \otimes \mathcal{K}$,*
- *the Elliott invariant of $C_r^*(\tilde{G}, \tilde{\Sigma})$ is given by*

$$(K_0(C_r^*(\tilde{G}, \tilde{\Sigma})), K_0(C_r^*(\tilde{G}, \tilde{\Sigma}))^+, \tilde{T}(C_r^*(\tilde{G}, \tilde{\Sigma})), \Sigma_{C_r^*(\tilde{G}, \tilde{\Sigma})}, \rho_{C_r^*(\tilde{G}, \tilde{\Sigma})},$$
$$K_1(C_r^*(\tilde{G}, \tilde{\Sigma})) \cong (G_0, \{0\}, \tilde{T}, \tilde{\gamma}, \rho, G_1).$$

Note that all the groupoids in Theorems 1.2, 1.3 and Corollaries 1.6, 1.7 are necessarily minimal and amenable. Theorem 1.2 and Corollary 1.6, together with

[41], imply that every classifiable C*-algebra which is not stably projectionless has a Cartan subalgebra. Once the classification of stably projectionless C*-algebras is completed, Theorem 1.3 and Corollary 1.7 will imply that every classifiable stably projectionless C*-algebra has a Cartan subalgebra. Actually, using $\mathcal{Z}$-stability, we see that all of the above-mentioned classifiable C*-algebras have infinitely many non-isomorphic Cartan subalgebras (compare [29, Proposition 5.1]). Moreover, the constructions in this paper show that in every classifiable stably finite C*-algebra, we can even find C*-diagonals (and even infinitely many non-isomorphic ones).

Moreover, more can be said about the twist, and also about the dimension of the spectra of our Cartan subalgebras.

Corollary 1.8 *The twisted groupoids* (G, Σ) *constructed in the proofs of Theorems 1.2 and 1.3 have the following additional properties:*

(i) *If* G_0 *is torsion-free, then the twist* Σ *is trivial, i.e.,* $\Sigma = \mathbb{T} \times G$.
(ii) *If* G_1 *has torsion, then* $C_r^*(G, \Sigma)$ *is an inductive limit of subhomogeneous C*-algebras whose spectra are three-dimensional, and* $\dim(G^{(0)}) = 3$.
(iii) *If* G_1 *is torsion-free and* G_0 *has torsion,* $C_r^*(G, \Sigma)$ *is an inductive limit of subhomogeneous C*-algebras whose spectra are two-dimensional, and* $\dim(G^{(0)}) = 2$.
(iv) *If both* G_0 *and* G_1 *are torsion-free with* $G_1 \ncong \{0\}$*, then* $C_r^*(G, \Sigma)$ *is an inductive limit of subhomogeneous C*-algebras whose spectra are one-dimensional, and* $\dim(G^{(0)}) = 1$.
(v) *If* G_0 *is torsion-free and* $G_1 \cong \{0\}$*, then* $C_r^*(G, \Sigma)$ *is an inductive limit of one-dimensional non-commutative finite CW-complexes, with* $\dim(G^{(0)}) \leq 1$ *in Theorem 1.2 and* $\dim(G^{(0)}) = 1$ *in Theorem 1.3.*

In particular, Corollary 1.8 implies the following:

Corollary 1.9 *The Jiang–Su algebra* $\mathcal{Z}$*, the Razak–Jacelon algebra* $\mathcal{W}$ *and the stably projectionless version* $\mathcal{Z}_0$ *of the Jiang–Su algebra of* [19, Definition 7.1] *have C*-diagonals with one-dimensional spectra. The corresponding twisted groupoids* (G, Σ) *can be chosen so that* Σ *is trivial, i.e.,* $\Sigma = \mathbb{T} \times G$.

Concrete groupoid models for $\mathcal{Z}$, $\mathcal{W}$ and $\mathcal{Z}_0$ are described in Sect. 8. It is worth pointing out that a groupoid model has been constructed for $\mathcal{Z}$ in [8] using a different construction (but the precise dimension of the unit space has not been determined in [8]). Moreover, G. Szabó and S. Vaes independently found groupoid models for $\mathcal{W}$, again using constructions different from ours. Furthermore, independently from [4] and the present paper, similar tools to the ones in [4, § 3] were developed in [2], which give rise to groupoid models for $\mathcal{Z}$ and $\mathcal{W}$ as well as other examples.

The key tool for all the results in this paper is an improved version of [4, Theorem 3.6], which allows us to construct Cartan subalgebras in inductive limit C*-algebras. The C*-algebraic formulation reads as follows.

Theorem 1.10 *Let* (A_n, B_n) *be Cartan pairs with normalizers* $N_n := N_{A_n}(B_n)$ *and faithful conditional expectations* $P_n : A_n \twoheadrightarrow B_n$. *Let* $\varphi_n : A_n \to A_{n+1}$ *be injective *-homomorphisms with* $\varphi_n(B_n) \subseteq B_{n+1}$, $\varphi_n(N_n) \subseteq N_{n+1}$ *and* $P_{n+1} \circ \varphi_n = \varphi_n \circ P_n$ *for all* n. *Then* $\varinjlim \{B_n; \varphi_n\}$ *is a Cartan subalgebra of* $\varinjlim \{A_n; \varphi_n\}$.

If all B_n *are C*-diagonals, then* $\varinjlim \{B_n; \varphi_n\}$ *is a C*-diagonal of* $\varinjlim \{A_n; \varphi_n\}$.

A special case of this theorem is proved in [6].

Actually, in addition to Theorem 1.10, much more is accomplished: Groupoid models are developed for *-homomorphisms such as φ_n, and the twisted groupoid corresponding to $\left(\varinjlim \{A_n; \varphi_n\}, \varinjlim \{B_n; \varphi_n\}\right)$ as in Theorem 1.10 is described explicitly. These results (in Sect. 5) might be of independent interest.

Applications of these explicit descriptions of groupoid models (for homomorphisms and Cartan pairs) and Theorem 1.10 include a unified approach to Theorems 1.2, 1.3, and explicit constructions of the desired twisted groupoids. The strategy is as follows: C*-algebras with prescribed Elliott invariant have been constructed in [11] (see also [20, § 13] for the unital case). These C*-algebras have all the desired properties as in Theorems 1.2 and 1.3 and are constructed as inductive limits of subhomogeneous C*-algebras. However, the connecting maps in [11] and [20, § 13] do not preserve the canonical Cartan subalgebras in these building blocks in general. Therefore, a careful choice or modification of the building blocks and connecting maps in the constructions in [11,20] is necessary in order to allow for an application of Theorem 1.10. The modification explained in Remark 4.1 is particularly important. Actually, a more general result is established in Sect. 4.2, where a class of inductive limits of subhomogeneous C*-algebras is identified, which encompasses all the C*-algebras in Theorems 1.2, 1.3 and Corollaries 1.6, 1.7, where we can apply Theorem 1.10.

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2 The constructions of Elliott and Gong–Lin–Niu

Let us briefly recall the constructions in [11] (see also [16] for simplifications and further explanations) and [20, § 13].

2.1 The unital case

Let us describe the construction in [20, § 13], which is based on [11] (with slight modifications). Given $(G_0, G_0^+, u, T, r, G_1)$ as in Theorem 1.2, write $G = G_0$, $K = G_1$, and let $\rho : \ G \to \mathrm{Aff}(T)$ be the dual map of r. Choose a dense subgroup $G' \subseteq \mathrm{Aff}(T)$. Set $H := G \oplus G'$,

$$H^+ := \{(0,0)\} \cup \left\{(g, f) \in G \oplus G' \colon \rho(g)(\tau) + f(\tau) > 0 \; \forall \, \tau \in T\right\},$$

and view u in G as an element of H. Then (H, H^+, u) becomes a simple ordered group, inducing the structure of a dimension group on $H/\mathrm{Tor}(H)$. Now construct a commutative diagram

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & G_n \arrow[r] \arrow[d] & G_{n+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & G \arrow[d] \\
H_1 \arrow[r, "\gamma_1"] \arrow[d] & \cdots \arrow[r, "\gamma_{n-1}"] & H_n \arrow[r, "\gamma_n"] \arrow[d] & H_{n+1} \arrow[r, "\gamma_{n+1}"] \arrow[d] & \cdots \arrow[r] & H \arrow[d] \\
H_1/G_1 \arrow[r] & \cdots \arrow[r] & H_n/G_n \arrow[r] & H_{n+1}/G_{n+1} \arrow[r] & \cdots \arrow[r] & H/G
\end{tikzcd}$$

where:

- H_n is a finitely generated abelian group with $H_n = \bigoplus_i H_n^i$, where for one distinguished index $\boldsymbol{i}$, $H_n^{\boldsymbol{i}} = \mathbb{Z} \oplus \mathrm{Tor}(H_n)$, and for all other indices, $H_n^i = \mathbb{Z}$;
- with $(H_n^{\boldsymbol{i}})^+ := \{(0,0)\} \cup (\mathbb{Z}_{>0} \oplus \mathrm{Tor}(H_n))$, $(H_n^i)^+ := \mathbb{Z}_{\geq 0}$ for all $i \neq \boldsymbol{i}$, $H_n^+ := \bigoplus_i (H_n^i)^+ \subseteq H_n^{\boldsymbol{i}} \oplus \bigoplus_{i \neq \boldsymbol{i}} H_n^i = H_n$ and $u_n = (([n, \boldsymbol{i}], \tau_n), ([n, i])_{i \neq \boldsymbol{i}}) \in H_n^+$, we have

$$\varinjlim \left\{(H_n, H_n^+, u_n); \gamma_n\right\} \cong (H, H^+, u); \tag{1}$$

- with $G_n := (\gamma_n^{\infty})^{-1}(G)$, where $\gamma_n^{\infty} : \ H_n \to H$ is the map provided by (1), and $G_n^+ := G_n \cap H_n^+$, we have $u_n \in G_n \subseteq H_n$, and (1) induces $\varinjlim \left\{(G_n, G_n^+, u_n); \gamma_n\right\} \cong (G, G^+, u)$;
- the vertical maps are the canonical ones.

Let $\hat{\gamma}_n \ : \ \ H_n/\mathrm{Tor}(H_n) =: \hat{H}_n = \bigoplus_i \hat{H}_n^i \to \bigoplus_j \hat{H}_{n+1}^j = \hat{H}_{n+1} := H_{n+1}/\mathrm{Tor}(H_{n+1})$ be the homomorphism induced by γ_n, where $\hat{H}_n^i = \mathbb{Z} =$

$\hat{H}_{n+1}^{j}$ for all i and j. For fixed n, the map $\hat{\gamma} = \hat{\gamma}_n$ is given by a matrix $(\hat{\gamma}_{ji})$, where we can always assume that $\hat{\gamma}_{ji} \in \mathbb{Z}_{>0}$ (considered as a map $\hat{H}_n^i = \mathbb{Z} \to \mathbb{Z} = \hat{H}_{n+1}^j$). Then $\gamma_n = \hat{\gamma} + \tau + t$ for homomorphisms $\tau : \mathrm{Tor}(H_n) \to \mathrm{Tor}(H_{n+1})$ and $t : \hat{H}_n \to \mathrm{Tor}(H_{n+1})$. Here we think of $\hat{H}_n$ as a subgroup (actually a direct summand) of H_n. As explained in [20, § 6], given a positive constant Γ_n depending on n, we can always arrange that

$$(\hat{\gamma}_n)_{ji} \geq \Gamma_n \text{ for all } i \text{ and } j. \tag{2}$$

Also, let K_n be finitely generated abelian groups and $\chi_n : K_n \to K_{n+1}$ homomorphisms such that $K \cong \varinjlim \{K_n; \chi_n\}$.

Let $F_n = \bigoplus_i F_n^i$ be C*-algebras, where $F_n^{\boldsymbol{i}}$ is a homogeneous C*-algebra of the form $F_n^{\boldsymbol{i}} = P_n^{\boldsymbol{i}} M_\infty(C(Z_n^{\boldsymbol{i}})) P_n^{\boldsymbol{i}}$ for a connected compact space $Z_n^{\boldsymbol{i}}$ with base point $\theta_n^{\boldsymbol{i}}$ and a projection $P_n^{\boldsymbol{i}} \in M_\infty(C(Z_n^{\boldsymbol{i}}))$, while for all other indices $i \neq \boldsymbol{i}$, F_n^i is a matrix algebra, $F_n^i = M_{[n,i]}$. We require that $(K_0(F_n^{\boldsymbol{i}}), K_0(F_n^{\boldsymbol{i}})^+, [1_{F_n^{\boldsymbol{i}}}]) \cong (H_n^{\boldsymbol{i}}, (H_n^{\boldsymbol{i}})^+, ([n, \boldsymbol{i}], \tau_n))$ and $K_1(F_n^{\boldsymbol{i}}) \cong K_n$, so that $(K_0(F_n), K_0(F_n)^+, [1_{F_n}], K_1(F_n)) \cong (H_n, H_n^+, u_n, K_n)$.

Let ψ_n be a unital homomorphism $F_n \to F_{n+1}$ which induces γ_n in K_0 and χ_n in K_1. We write $F_n = P_n C(Z_n) P_n$ where $Z_n = Z_n^{\boldsymbol{i}} \amalg \coprod_{i \neq \boldsymbol{i}} \{\theta_n^i\}$, and $P_n = (P_n^{\boldsymbol{i}}, (1_{[n,i]})_{i \neq \boldsymbol{i}}) \in M_\infty(C(Z_n^{\boldsymbol{i}})) \oplus \bigoplus_{i \neq \boldsymbol{i}} M_{[n,i]}(C(\{\theta_n^i\}))$. Thus evaluation in θ_n^i induces a quotient map $\pi_n : F_n \to \hat{F}_n := \bigoplus_i \hat{F}_n^i$, where $\hat{F}_n^i = M_{[n,i]}$. We require that ψ_n induce homomorphisms $\hat{\psi}_n : \hat{F}_n \to \hat{F}_{n+1}$ so that we obtain a commutative diagram

$$\begin{tikzcd}
F_n \arrow[r, "\psi_n"'] \arrow[d, "\pi_n"'] & F_{n+1} \arrow[d, "\pi_{n+1}"] \\
\hat{F}_n \arrow[r, "\hat{\psi}_n"] & \hat{F}_{n+1}
\end{tikzcd}$$

which induces in K_0

$$\begin{tikzcd}
H_n \arrow[r, "\gamma_n"'] \arrow[d] & H_{n+1} \arrow[d] \\
H_n/\mathrm{Tor}(H_n) \arrow[r, "\hat{\gamma}_n"] & H_{n+1}/\mathrm{Tor}(H_{n+1})
\end{tikzcd}$$

where the vertical arrows are the canonical projections. As $\mathrm{Tor}(H_n) \subseteq G_n$, H_n/G_n is torsion-free, and there is a canonical projection $H_n/\mathrm{Tor}(H_n) \to H_n/G_n$. Now let $E_n := \bigoplus_p E_n^p$, $E_n^p = M_{\{n,p\}}$, so that $K_0(E_n) \cong H_n/G_n$, and for fixed n, let $\beta_0, \beta_1 : \hat{F}_n \to E_n$ be unital homomorphisms which yield the commutative diagram

$$\begin{tikzcd} K_0(\hat{F}_n) \arrow[r, "(\beta_0)_* - (\beta_1)_*"'] \arrow[d, "\cong"'] & K_0(E_n) \arrow[d, "\cong"] \\ H_n/\mathrm{Tor}(H_n) \arrow[r, "\hat{\gamma}_n"] & H_n/G_n. \end{tikzcd}$$

We can assume $\beta_0 \oplus \beta_1 : \hat{F}_n \to E_n \oplus E_n$ to be injective, because only the difference $(\beta_0)_* - (\beta_1)_*$ matters.

Define

$$\begin{aligned} A_n &:= \{(f, a) \in C([0,1], E_n) \oplus F_n\colon f(t) = \beta_t(\pi_n(a)) \text{ for } t = 0, 1\}, \\ \hat{A}_n &:= \left\{(f, \hat{a}) \in C([0,1], E_n) \oplus \hat{F}_n\colon f(t) = \beta_t(\hat{a}) \text{ for } t = 0, 1\right\}. \end{aligned}$$

As $\beta_0 \oplus \beta_1$ is injective, we view $\hat{A}_n$ as a subalgebra of $C([0,1], E_n)$ via $(f, \hat{a}) \mapsto f$.

Choose for each n a unital homomorphism $\hat{\varphi}_n : \hat{A}_n \to \hat{A}_{n+1}$ such that the composition with the map $C([0,1], E_{n+1}) \twoheadrightarrow C([0,1], E^q_{n+1})$ induced by the canonical projection $E_{n+1} \twoheadrightarrow E^q_{n+1}$,

$$\hat{A}_n \xrightarrow{\hat{\varphi}_n} \hat{A}_{n+1} \hookrightarrow C([0,1], E_{n+1}) \twoheadrightarrow C([0,1], E^q_{n+1}),$$

is of the form

$$C([0,1], E_n) \supseteq \hat{A}_n \ni f \mapsto u^* \begin{pmatrix} V(f) & \\ & D(f) \end{pmatrix} u, \tag{3}$$

where u is a continuous path of unitaries $[0,1] \to U(E^q_{n+1})$,

$$V(f) = \begin{pmatrix} \pi_1(f) & & \\ & \pi_2(f) & \\ & & \ddots \end{pmatrix}$$

for some $\pi_\bullet$ of the form $\pi_\bullet : \hat{A}_n \to \hat{F}_n \twoheadrightarrow \hat{F}^i_n$, where the first map is given by $(f, \hat{a}) \mapsto \hat{a}$ and the second map is the canonical projection, and

$$D(f) = \begin{pmatrix} f \circ \lambda_1 & & \\ & f \circ \lambda_2 & \\ & & \ddots \end{pmatrix}$$

for some continuous maps $\lambda_\bullet : [0,1] \to [0,1]$ with $\lambda_\bullet^{-1}(\{0,1\}) \subseteq \{0,1\}$. We require that the diagram

$$\begin{tikzcd}
\hat{A}_n \arrow[r, "\hat{\varphi}_n"'] \arrow[d] & \hat{A}_{n+1} \arrow[d] \\
\hat{F}_n \arrow[r, "\hat{\psi}_n"] & \hat{F}_{n+1}
\end{tikzcd}$$

commute, where the vertical maps are given by $(f, \hat{a}) \mapsto \hat{a}$.

Then there exists a unique homomorphism $\varphi_n : A_n \to A_{n+1}$ which fits into the commutative diagram

$$\begin{tikzcd}
A_n \arrow[rr, "\varphi_n"'] \arrow[dr] \arrow[dd] & & A_{n+1} \arrow[dr] \arrow[dd] & \\
& \hat{A}_n \arrow[rr, "\hat{\varphi}_n"] \arrow[dd] & & \hat{A}_{n+1} \arrow[dd] \\
F_n \arrow[rr, "\psi_n"'] \arrow[dr] & & F_{n+1} \arrow[dr] & \\
& \hat{F}_n \arrow[rr, "\hat{\psi}_n"] & & \hat{F}_{n+1}
\end{tikzcd}$$

where all the unlabelled arrows are given by the canonical maps.

By construction, $\varinjlim \{A_n; \varphi_n\}$ has the desired Elliott invariant (in particular, the canonical map $\varinjlim \{A_n; \varphi_n\} \to \hat{F} := \varinjlim \left\{\hat{F}_n; \hat{\psi}_n\right\}$ induces $T(\varinjlim \{A_n; \varphi_n\}) \cong T(\hat{F})$). However, this is not a simple C*-algebra. Thus a further modification is needed to enforce simplicity. To this end, choose $\boldsymbol{I}_n \subseteq (0,1)$ and $\boldsymbol{Z}_n^{\boldsymbol{i}} \subseteq Z_n^{\boldsymbol{i}}$ $\frac{1}{n}$-dense and replace $\varphi_n : A_n \to A_{n+1}$ by the unital homomorphism $\xi_n : A_n \to A_{n+1}$ such that:

- the compositions

$$A_n \xrightarrow{\xi_n} A_{n+1} \to F_{n+1} \twoheadrightarrow F_{n+1}^j \quad \text{and}$$
$$A_n \xrightarrow{\varphi_n} A_{n+1} \to F_{n+1} \twoheadrightarrow F_{n+1}^j \text{ coincide except for one index } j_\xi \neq \boldsymbol{j};$$

- the composition

$$A_n \xrightarrow{\xi_n} A_{n+1} \to F_{n+1} \twoheadrightarrow F_{n+1}^{j_\xi}$$

is of the form

$$A_n \ni (f, a) \mapsto u^* \begin{pmatrix} I(f) & & \\ & Z(a) & \\ & & P(a) \end{pmatrix} u,$$

where u is a permutation matrix in $M_{[n+1, j_\xi]}$,

$$I(f) = \begin{pmatrix} f^{p_1}(t_1) & & \\ & f^{p_2}(t_2) & \\ & & \ddots \end{pmatrix}$$

for indices $p_\bullet$ and $t_\bullet \in \boldsymbol{I}_n$ such that all possible pairs $p_\bullet, t_\bullet$ appear (f^p is the component of f in $C([0, 1], E_n^p)$),

$$Z(a) = \begin{pmatrix} \tau_1(a(z_1)) & & \\ & \tau_2(a(z_2)) & \\ & & \ddots \end{pmatrix} \tag{4}$$

for $z_\bullet \in \boldsymbol{Z}_n$ and isomorphisms $\tau_\bullet : \; P_n^{\boldsymbol{i}}(z_\bullet) M_\infty P_n^{\boldsymbol{i}}(z_\bullet) \cong \hat{F}_n^{\boldsymbol{i}} = M_{[n, \boldsymbol{i}]}$, and

$$P(a) = \begin{pmatrix} \pi_n^{i_1}(a) & & \\ & \pi_n^{i_2}(a) & \\ & & \ddots \end{pmatrix},$$

where π_n^i is the canonical projection $F_n \twoheadrightarrow \hat{F}_n \twoheadrightarrow \hat{F}_n^i$;

- for every q, the composition

$$A_n \xrightarrow{\xi_n} A_{n+1} \to C([0, 1], E_{n+1}) \twoheadrightarrow C([0, 1], E_{n+1}^q)$$

is of the form

$$A_n \ni (f, a) \mapsto u^* \begin{pmatrix} \Phi(f) & \\ & \Xi(a) \end{pmatrix} u,$$

where u is a continuous path of unitaries $[0, 1] \to U(E_{n+1}^q)$, $\Phi(f)$ is of the same form

$$\begin{pmatrix} V(f) & \\ & D(f) \end{pmatrix}$$

as in (3),

$$\Xi(a)(t) = \begin{pmatrix} \tau_1(t)(a(z_1(t))) & & \\ & \tau_2(t)(a(z_2(t))) & \\ & & \ddots \end{pmatrix}$$

for continuous maps $z_\bullet : [0,1] \to Z_n^{\boldsymbol{i}}$, each of which is either a constant map with value in $\boldsymbol{Z}_n$ or connects $\theta_n^{\boldsymbol{i}}$ with $z_\bullet \in \boldsymbol{Z}_n$, and isomorphisms $\tau_\bullet(t) : P_n^{\boldsymbol{i}}(z_\bullet(t)) M_\infty P_n^{\boldsymbol{i}}(z_\bullet(t)) \cong \hat{F}_n^{\boldsymbol{i}}$ depending continuously on $t \in [0,1]$ such that for $t \in \{0,1\}$, $\tau_\bullet(t) = \mathrm{id}$ if $z_\bullet(t) = \theta_n^{\boldsymbol{i}}$ and $\tau_\bullet(t) = \tau_\bullet$ if $z_\bullet(t) = z_\bullet$, where $\tau_\bullet$ is as in (4).

Then $\varinjlim \{A_n; \xi_n\}$ is a simple unital C*-algebra with prescribed Elliott invariant.

2.2 The stably projectionless case

We follow [11] (see also [16]), with slight modifications as in the unital case. Let (G_0, T, ρ, G_1) be as in Theorem 1.3.

Write $G = G_0$ and $K = G_1$. Choose a dense subgroup $G' \subseteq \mathrm{Aff}(T)$. Set $H := G \oplus G'$,

$$H^+ := \{(0,0)\} \cup \left\{(g,f) \in G \oplus G' : \rho(g)(\tau) + f(\tau) > 0 \; \forall \, \tau \in T\right\}.$$

Then (H, H^+) becomes a simple ordered group, inducing the structure of a dimension group on $H/\mathrm{Tor}(H)$. Now construct a commutative diagram

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d] & \dots \arrow[r] & G_n \arrow[r] \arrow[d] & G_{n+1} \arrow[r] \arrow[d] & \dots \arrow[r] & G \arrow[d] \\
H_1 \arrow[r, "\gamma_1"] \arrow[d] & \dots \arrow[r, "\gamma_{n-1}"] & H_n \arrow[r, "\gamma_n"] \arrow[d] & H_{n+1} \arrow[r, "\gamma_{n+1}"] \arrow[d] & \dots \arrow[r] & H \arrow[d] \\
H_1/G_1 \arrow[r] & \dots \arrow[r] & H_n/G_n \arrow[r] & H_{n+1}/G_{n+1} \arrow[r] & \dots \arrow[r] & H/G
\end{tikzcd}$$

where

- H_n is a finitely generated abelian group with $H_n = \bigoplus_i H_n^i$, where for one distinguished index $\boldsymbol{i}$, $H_n^{\boldsymbol{i}} = \mathbb{Z} \oplus \mathrm{Tor}(H_n)$, and for all other indices, $H_n^i = \mathbb{Z}$;
- with $(H_n^{\boldsymbol{i}})^+ := \{(0,0)\} \cup (\mathbb{Z}_{>0} \oplus \mathrm{Tor}(H_n))$, $(H_n^i)^+ := \mathbb{Z}_{\geq 0}$ for all $i \neq \boldsymbol{i}$, and $H_n^+ := \bigoplus_i (H_n^i)^+ \subseteq H_n^{\boldsymbol{i}} \oplus \bigoplus_{i \neq \boldsymbol{i}} H_n^i = H_n$ we have

$$\varinjlim \left\{(H_n, H_n^+); \gamma_n\right\} \cong (H, H^+); \tag{5}$$

- with $G_n := (\gamma_n^\infty)^{-1}(G)$, where $\gamma_n^\infty : H_n \to H$ is the map provided by (5), we have $G_n \cap H_n^+ = \{0\}$, and (1) induces $\varinjlim \{G_n; \gamma_n\} \cong G$;
- the vertical maps are the canonical ones.

Let $\hat{\gamma}_n : H_n/\mathrm{Tor}(H_n) =: \hat{H}_n = \bigoplus_i \hat{H}_n^i \to \bigoplus_j \hat{H}_{n+1}^j = \hat{H}_{n+1} := H_{n+1}/\mathrm{Tor}(H_{n+1})$ be the homomorphism induced by γ_n, where $\hat{H}_n^i = \mathbb{Z} = \hat{H}_{n+1}^j$ for all i and j. For fixed n, the map $\hat{\gamma} = \hat{\gamma}_n$ is given by a matrix $(\hat{\gamma}_{ji})$, where we can always assume that $\hat{\gamma}_{ji} \in \mathbb{Z}_{>0}$ (considered as a map $\hat{H}_n^i = \mathbb{Z} \to \mathbb{Z} = \hat{H}_{n+1}^j$). Then $\gamma_n = \hat{\gamma} + \tau + t$ for homomorphisms $\tau : \mathrm{Tor}(H_n) \to \mathrm{Tor}(H_{n+1})$ and $t : \hat{H}_n \to \mathrm{Tor}(H_{n+1})$. Here we think of $\hat{H}_n$ as a subgroup of H_n. As in the unital case (see [20, § 6]), given a positive constant Γ_n depending on n, we can always arrange that

$$(\hat{\gamma}_n)_{ji} \geq \Gamma_n \text{ for all } i \text{ and } j. \tag{6}$$

Also, let K_n be finitely generated abelian groups and $\chi_n : K_n \to K_{n+1}$ homomorphisms such that $K \cong \varinjlim \{K_n; \chi_n\}$.

Let $F_n = \bigoplus_i F_n^i$ be C*-algebras, where $F_n^{\boldsymbol{i}}$ is a homogeneous C*-algebra of the form $F_n^{\boldsymbol{i}} = P_n^{\boldsymbol{i}} M_\infty(C(Z_n^{\boldsymbol{i}})) P_n^{\boldsymbol{i}}$ for a connected compact space $Z_n^{\boldsymbol{i}}$ with base point $\theta_n^{\boldsymbol{i}}$ and a projection $P_n^{\boldsymbol{i}} \in M_\infty(C(Z_n^{\boldsymbol{i}}))$, while for all other indices $i \neq \boldsymbol{i}$, F_n^i is a matrix algebra, $F_n^i = M_{[n,i]}$. We require that $(K_0(F_n^{\boldsymbol{i}}), K_0(F_n^{\boldsymbol{i}})^+) \cong (H_n^{\boldsymbol{i}}, (H_n^{\boldsymbol{i}})^+)$ and $K_1(F_n^{\boldsymbol{i}}) \cong K_n$, so that $(K_0(F_n), K_0(F_n)^+, K_1(F_n)) \cong (H_n, H_n^+, K_n)$.

Let ψ_n be a unital homomorphism $F_n \to F_{n+1}$ which induces γ_n in K_0 and χ_n in K_1. We write $F_n = P_n C(Z_n) P_n$ where $Z_n = Z_n^{\boldsymbol{i}} \amalg \coprod_{i \neq \boldsymbol{i}} \{\theta_n^i\}$, and $P_n = (P_n^{\boldsymbol{i}}, (1_{[n,i]})_{i \neq \boldsymbol{i}}) \in M_\infty(C(Z_n^{\boldsymbol{i}})) \oplus \bigoplus_{i \neq \boldsymbol{i}} M_{[n,i]}(C(\{\theta_n^i\}))$. Thus, evaluation in θ_n^i induces a quotient map $\pi_n : F_n \to \hat{F}_n := \bigoplus_i \hat{F}_n^i$, where $\hat{F}_n^i = M_{[n,i]}$. We require that ψ_n induce homomorphisms $\hat{\psi}_n : \hat{F}_n \to \hat{F}_{n+1}$ so that we obtain a commutative diagram

$$\begin{tikzcd} F_n \arrow[r, "\psi_n"'] \arrow[d, "\pi_n"'] & F_{n+1} \arrow[d, "\pi_{n+1}"] \\ \hat{F}_n \arrow[r, "\hat{\psi}_n"] & \hat{F}_{n+1} \end{tikzcd}$$

which induces in K_0

$$\begin{tikzcd} H_n \arrow[r, "\gamma_n"'] \arrow[d] & H_{n+1} \arrow[d] \\ H_n/\mathrm{Tor}(H_n) \arrow[r, "\hat{\gamma}_n"] & H_{n+1}/\mathrm{Tor}(H_{n+1}) \end{tikzcd}$$

where the vertical arrows are the canonical projections. As $\mathrm{Tor}(H_n) \subseteq G_n$, H_n/G_n is torsion-free, and there is a canonical projection $H_n/\mathrm{Tor}(H_n) \to H_n/G_n$. Now let $E_n := \bigoplus_p E_n^p$, $E_n^p = M_{\{n,p\}}$, such that $K_0(E_n) \cong H_n/G_n$, and for fixed n, let $\beta_0, \beta_1 : \hat{F}_n \to E_n$ be (necessarily non-unital) homomorphisms which yield the commutative diagram

$$\begin{array}{ccc} K_0(\hat{F}_n) & \xrightarrow[(\beta_0)_* - (\beta_1)_*]{} & K_0(E_n) \\ \cong \downarrow & & \downarrow \cong \\ H_n/\mathrm{Tor}(H_n) & \xrightarrow{\hat{\gamma}_n} & H_n/G_n. \end{array}$$

As in the unital case, we may assume $\beta_0 \oplus \beta_1 : \hat{F}_n \to E_n \oplus E_n$ to be injective.

Define

$$A_n := \{(f, a) \in C([0,1], E_n) \oplus F_n \colon f(t) = \beta_t(\pi_n(a)) \text{ for } t = 0, 1\},$$
$$\hat{A}_n := \left\{(f, \hat{a}) \in C([0,1], E_n) \oplus \hat{F}_n \colon f(t) = \beta_t(\hat{a}) \text{ for } t = 0, 1\right\}.$$

As $\beta_0 \oplus \beta_1$ is injective, we view $\hat{A}_n$ as a subalgebra of $C([0,1], E_n)$ via $(f, \hat{a}) \mapsto f$.

Choose for each n a homomorphism $\hat{\varphi}_n : \hat{A}_n \to \hat{A}_{n+1}$ such that the composition with the map $C([0,1], E_{n+1}) \twoheadrightarrow C([0,1], E_{n+1}^q)$ induced by the canonical projection $E_{n+1} \twoheadrightarrow E_{n+1}^q$,

$$\hat{A}_n \xrightarrow{\hat{\varphi}_n} \hat{A}_{n+1} \hookrightarrow C([0,1], E_{n+1}) \twoheadrightarrow C([0,1], E_{n+1}^q),$$

is of the form

$$C([0,1], E_n) \supseteq \hat{A}_n \ni f \mapsto u^* \begin{pmatrix} V(f) & \\ & D(f) \end{pmatrix} u,$$

where u is a continuous path of unitaries $[0,1] \to U(E_{n+1}^q)$,

$$V(f) = \begin{pmatrix} \pi_1(f) & & \\ & \pi_2(f) & \\ & & \ddots \end{pmatrix}$$

for some $\pi_\bullet$ of the form $\pi_\bullet : \hat{A}_n \to \hat{F}_n \twoheadrightarrow \hat{F}_n^i$, where the first map is given by $(f, \hat{a}) \mapsto \hat{a}$ and the second map is the canonical projection, and

$$D(f) = \begin{pmatrix} f \circ \lambda_1 & & \\ & f \circ \lambda_2 & \\ & & \ddots \end{pmatrix}$$

for some continuous maps $\lambda_\bullet : [0, 1] \to [0, 1]$ with $\lambda_\bullet^{-1}(\{0, 1\}) \subseteq \{0, 1\}$. We require that

$$\begin{tikzcd} \hat{A}_n \arrow[r, "\hat{\varphi}_n"'] \arrow[d] & \hat{A}_{n+1} \arrow[d] \\ \hat{F}_n \arrow[r, "\hat{\psi}_n"] & \hat{F}_{n+1} \end{tikzcd}$$

commutes, where the vertical maps are given by $(f, \hat{a}) \mapsto \hat{a}$.

Then there exists a unique homomorphism $\varphi_n : A_n \to A_{n+1}$ which fits into the commutative diagram

$$\begin{tikzcd} A_n \arrow[rr, "\varphi_n"'] \arrow[dr] \arrow[dd] & & A_{n+1} \arrow[dr] \arrow[dd] & \\ & \hat{A}_n \arrow[rr, "\hat{\varphi}_n"] \arrow[dd] & & \hat{A}_{n+1} \arrow[dd] \\ F_n \arrow[rr, "\psi_n"'] \arrow[dr] & & F_{n+1} \arrow[dr] & \\ & \hat{F}_n \arrow[rr, "\hat{\psi}_n"] & & \hat{F}_{n+1} \end{tikzcd}$$

where all the unlabelled arrows are given by the canonical maps.

By construction, $\varinjlim \{A_n; \varphi_n\}$ has the desired Elliott invariant (the details are as in the unital case, see [20, § 13]). The same modification as in the unital case produces new connecting maps $\xi_n : A_n \to A_{n+1}$ such that $\varinjlim \{A_n; \xi_n\}$ is a simple (stably projectionless) C*-algebra with prescribed Elliott invariant. Moreover, choosing ξ_n with the property that strictly positive elements are sent to strictly positive elements, $\varinjlim \{A_n; \xi_n\}$ will have continuous scale by [18, Theorem 9.3] (compare also [19, § 6]). In addition, we choose ξ_n such that full elements are sent to full elements.

Remark 2.1 In an earlier version of this paper, we modified the construction in [19, § 6] instead, which covers all Elliott invariants for stably projectionless

C*-algebras with trivial pairing between K-theory and traces ($\rho = 0$). I would like to thank the referee for pointing out that [11] (see also [16]) describes a general construction exhausting all possible Elliott invariants with weakly unperforated pairing between K-theory and traces (in the stably projectionless case, this is precisely the condition that ρ is weakly unperforated as in Theorem 1.3).

3 Concrete construction of AH-algebras

We start with the following standard fact.

Lemma 3.1 *Given an integer $N > 1$, let $\mu_N : S^1 \to S^1$, $z \mapsto z^N$, and set $X_N := D^2 \cup_{\mu_N} S^1$, where we identify $z \in S^1 = \partial D^2$ with $\mu_N(z) \in S^1$. Then*

$$H^\bullet(X_N) \cong \begin{cases} \mathbb{Z} & \text{if } \bullet = 0; \\ \mathbb{Z}/N & \text{if } \bullet = 2; \\ \{0\} & \text{else.} \end{cases}$$

Moreover, $(K_0(C(X_N)), K_0(C(X_N))^+, [1_{C(X_N)}], K_1(C(X_N))) \cong (\mathbb{Z} \oplus \mathbb{Z}/N, \{(0,0)\} \cup (\mathbb{Z}_{>0} \oplus \mathbb{Z}/N), (1,0), \{0\})$.

In the following, we view S^2 as the one point compactification of $\mathring{D}^2$, $S^2 = \mathring{D}^2 \cup \{\infty\}$.

Lemma 3.2 *Let $X_N \twoheadrightarrow S^2$ be the continuous map sending $\mathring{D}^2 \subseteq D^2$ identically to $\mathring{D}^2 \subseteq S^2$, ∂D^2 to ∞ and S^1 to ∞. Let p_{X_N} be the pullback of the Bott line bundle on S^2 (see for instance* [39, § 6.2]*) to X_N via this map. We view p_{X_N} as a projection in $M_2(C(X_N))$. Then there is an isomorphism $K_0(C(X_N)) \cong \mathbb{Z} \oplus \mathbb{Z}/N$ identifying the class of $1_{C(X_N)}$ with the generator of $\mathbb{Z}$ and the class of p_{X_N} with* $(1, 1)$.

Proof Just analyse the K-theory exact sequence attached to $0 \to C_0(\mathring{D}^2) \to C(X_N) \to C(S^1) \to 0$. □

We recall another standard fact.

Lemma 3.3 *Given an integer $N > 1$, let $Y_N := \Sigma X_N \cong D^3 \cup_{\Sigma\mu_N} S^2$, where we identify $z \in S^2 = \partial D^3 \cong \Sigma S^1$ with $(\Sigma\mu_N)(z) \in \Sigma S^1 \cong S^2$. (Here Σ stands for suspension.) Then*

$$H^\bullet(Y_N) \cong \begin{cases} \mathbb{Z} & \text{if } \bullet = 0; \\ \mathbb{Z}/N & \text{if } \bullet = 3; \\ \{0\} & \text{else.} \end{cases}$$

Moreover, $K_0(C(Y_N)) = \mathbb{Z}[1_{C(Y_N)}]$ and $K_1(C(Y_N)) \cong \mathbb{Z}/N$.

In the following, we view S^3 as the one point compactification of $\mathring{D}^3$, $S^3 = \mathring{D}^3 \cup \{\infty\}$.

Lemma 3.4 *Let $Y_N \twoheadrightarrow S^3$ be the continuous map sending $\mathring{D}^3 \subseteq D^3$ identically to $\mathring{D}^3 \subseteq S^3$, ∂D^3 to ∞ and S^2 to ∞. Then the dual map $C(S^3) \to C(Y_N)$ induces in K_1 a surjection $K_1(C(S^3)) \cong \mathbb{Z} \twoheadrightarrow \mathbb{Z}/N \cong K_1(C(Y_N))$.*

Proof Just analyse the K-theory exact sequence attached to $0 \to C_0(\mathring{D}^3) \to C(Y_N) \to C(S^2) \to 0$.

Analysing K-theory exact sequences, the following is a straightforward observation.

Lemma 3.5 *Let $N, N' \in \mathbb{Z}_{>1}$ and $m \in \mathbb{Z}_{>0}$ with $N' \mid m \cdot N$, say $m \cdot N = m' \cdot N'$. Define a continuous map*

$$\Psi_m^* : X_{N'} = D^2 \cup_{\mu_{N'}} S^1 \to D^2 \cup_{\mu_N} S^1 = X_N$$

by sending $x \in D^2$ to $x^m \in D^2$ and $z \in S^1$ to $z^{m'} \in S^1$. Then the dual map $\Psi_m : C(X_N) \to C(X_{N'})$ induces in K_0 the homomorphism

$$K_0(C(X_N)) \cong \mathbb{Z} \oplus \mathbb{Z}/N \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right)} \mathbb{Z} \oplus \mathbb{Z}/N' \cong K_0(C(X_{N'})).$$

Naturality of suspension yields

Lemma 3.6 *Let $N, N' \in \mathbb{Z}_{>1}$ and $m \in \mathbb{Z}_{>0}$ with $N' \mid m \cdot N$, say $m \cdot N = m' \cdot N'$. Let $\Sigma\Psi_m : C(Y_N) \to C(Y_{N'})$ be the map dual to $\Sigma\Psi_m^* : Y_{N'} \cong \Sigma X_{N'} \to \Sigma X_N \cong Y_N$. Then $\Sigma\Psi_m$ induces in K_1 the homomorphism*

$$K_1(C(Y_N)) \cong \mathbb{Z}/N \xrightarrow{m} \mathbb{Z}/N' \cong K_1(C(Y_{N'})).$$

In the following, we view X_N and Y_N as pointed spaces, with base point $1 = (1, 0) \in S^1 = \partial D^2 \subseteq D^2$ in X_N and base point $(1, 0, 0) \in S^2 = \partial D^3 \subseteq D^3$ in Y_N. Note that Ψ_m and $\Sigma\Psi_m$ preserve base points. Moreover, if θ denotes the base point of X_N, then the projection p_{X_N} in Lemma 3.2 satisfies

$$p_{X_N}(\theta) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right). \tag{7}$$

Now let $H_n = H_n^{\boldsymbol{i}} \oplus \bigoplus_{i \neq \boldsymbol{i}} H_n^i$, $H_{n+1} = H_{n+1}^{\boldsymbol{j}} \oplus \bigoplus_{j \neq \boldsymbol{j}} H_{n+1}^j$ be abelian groups with $H_n^{\boldsymbol{i}} = \mathbb{Z} \oplus T_n$, $H_{n+1}^{\boldsymbol{j}} = \mathbb{Z} \oplus T_{n+1}$ for finitely generated torsion groups T_n, T_{n+1}, and $H_n^i = \mathbb{Z}$, $H_{n+1}^j = \mathbb{Z}$ for all $i \neq \boldsymbol{i}$, $j \neq \boldsymbol{j}$. Let $(H_n^{\boldsymbol{i}})^+ := \{(0,0)\} \cup (\mathbb{Z}_{>0} \oplus T_n)$, $(H_n^i)^+ := \mathbb{Z}_{\geq 0}$ for all $i \neq \boldsymbol{i}$, $H_n^+ := \bigoplus_i (H_n^i)^+ \subseteq H_n^{\boldsymbol{i}} \oplus \bigoplus_{i \neq \boldsymbol{i}} H_n^i = H_n$ and $u_n = (([n, \boldsymbol{i}], \tau_n), ([n, i])_{i \neq \boldsymbol{i}}) \in H_n^+$. Similarly, define

$(H_{n+1}^j)^+$, $H_{n+1}^+ := \bigoplus_i (H_{n+1}^j)^+$, and let $u_{n+1} = (([n+1, \boldsymbol{j}], \tau_{n+1}), ([n+1, j])_{j \neq \boldsymbol{j}}) \in H_{n+1}^+$. Let $T_n = \bigoplus_k T_n^k$, where $T_n^k = \mathbb{Z}/N_n^k$, and $T_{n+1} = \bigoplus_k T_{n+1}^l$, where $T_{n+1}^l = \mathbb{Z}/N_{n+1}^l$. Let $\gamma_n : H_n \to H_{n+1}$ be a homomorphism with $\gamma_n(u_n) = u_{n+1}$. (In the stably projectionless case, these order units are not part of the given data, but we can always choose such order units.) Let us fix n, and suppose that $\gamma = \gamma_n$ induces a homomorphism $\hat{\gamma} : H_n/\mathrm{Tor}(H_n) = \hat{H}_n = \bigoplus_i \hat{H}_n^i \to \bigoplus_j \hat{H}_{n+1}^j = \hat{H}_{n+1} = H_{n+1}/\mathrm{Tor}(H_{n+1})$, where $\hat{H}_n^i = \mathbb{Z} = \hat{H}_{n+1}^j$ for all i and j. Viewing $\hat{H}_n$ as a subgroup (actually a direct summand) of H_n, we obtain that $\gamma_n = \hat{\gamma} + \tau + t$ for homomorphisms $\tau : \mathrm{Tor}(H_n) \to \mathrm{Tor}(H_{n+1})$ and $t : \hat{H}_n \to \mathrm{Tor}(H_{n+1})$. $\hat{\gamma}$ is given by an integer matrix $(\hat{\gamma}_{ji})$. Similarly, τ is given by an integer matrix (τ_{lk}), where we view τ_{lk} as a homomorphism $T_n^k \to T_{n+1}^l$. Also, t is given by an integer matrix (t_{li}), where we view t_{li} as a homomorphism $H_n^i \to T_{n+1}^l$. Clearly, we can always arrange $\tau_{lk}, t_{li} > 0$ for all l, k, i, and because of (2) and (6), we can also arrange

$$\hat{\gamma}_{ji} > 0 \text{ and } \hat{\gamma}_{\boldsymbol{ji}} \geq \#_0(k) + 1. \tag{8}$$

Here $\#_0(k)$ is the number of direct summands in T_n (i.e., the number of indices k).

We have the following direct consequence of Lemma 3.1.

Lemma 3.7 *Let $X_n^{\boldsymbol{i}} := \bigvee_k X_{N_n^k}$, where we take the wedge sum with respect to the base points of the individual $X_{N_n^k}$. Denote the base point of $X_n^{\boldsymbol{i}}$ by $\theta_n^{\boldsymbol{i}}$. Set $X_n := X_n^{\boldsymbol{i}} \amalg \coprod_{i \neq \boldsymbol{i}} \{\theta_n^i\}$. Then*

$$(K_0(C(X_n)), K_0(C(X_n))^+, K_1(C(X_n))) \cong (H_n, H_n^+, \{0\}).$$

Define X_{n+1} in an analogous way, i.e., $X_{n+1}^{\boldsymbol{j}} := \bigvee_l X_{N_{n+1}^l}$, and $X_{n+1} := X_{n+1}^{\boldsymbol{j}} \amalg \coprod_{j \neq \boldsymbol{j}} \{\theta_{n+1}^j\}$. Now, for fixed n, our goal is to construct a homomorphism ψ realizing the homomorphism γ in K_0.

The map $\bigvee_l \Psi_{\tau_{lk}}^* : \bigvee_l X_{N_{n+1}^l} \to X_{N_n^k}$ induces the dual homomorphism $\psi_\tau^k : C(X_{N_n^k}) \to C(X_{n+1}^{\boldsymbol{j}})$. Here $\Psi_{\tau_{lk}}$ are the maps from Lemma 3.5. The direct sum $\bigoplus_k \psi_\tau^k : \bigoplus_k C(X_{N_n^k}) \to M_{\#_0(k)}(C(X_{n+1}^{\boldsymbol{j}}))$ restricts to a homomorphism $\psi_\tau : C(X_n^{\boldsymbol{i}}) = C(\bigvee_k X_{N_n^k}) \to M_{\#_0(k)}(C(X_{n+1}^{\boldsymbol{j}}))$.

Let $p^{(\boldsymbol{i})} \in M_2(C(X_{n+1}^{\boldsymbol{j}})) = M_2(C(\bigvee_l X_{N_{n+1}^l}))$ be given by $p^{(\boldsymbol{i})}|_{C(X_{N_{n+1}^l})} = M_2(\Psi_{t_{l\boldsymbol{i}}})(p_{X_{N_{n+1}^l}})$. Define ψ_t as the composite

$$C(X_n^{\boldsymbol{i}}) \xrightarrow{\mathrm{ev}_{\theta_n^{\boldsymbol{i}}}} \mathbb{C} \to M_2(C(X_{n+1}^{\boldsymbol{j}})), \text{ where the second map is given by } 1 \mapsto p^{(\boldsymbol{i})}.$$

Moreover, define $\psi_{\boldsymbol{ji}} : C(X_n^{\boldsymbol{i}}) \to M_{\hat{\gamma}_{\boldsymbol{ji}}+1}(C(X_{n+1}^{\boldsymbol{j}}))$ by setting

$$\psi_{\boldsymbol{ji}}(f) = \begin{pmatrix} f(\theta_n^{\boldsymbol{i}}) & & & & \\ & \ddots & & & \\ & & f(\theta_n^{\boldsymbol{i}}) & & \\ & & & \psi_\tau(f) & \\ & & & & \psi_t(f) \end{pmatrix}$$

where we put $\hat{\gamma}_{\boldsymbol{ji}} - \#_0(k) - 1$ copies of $f(\theta_n^{\boldsymbol{i}})$ on the diagonal.

For $i \neq \boldsymbol{i}$, let $p^{(i)} \in M_2(C(X_{n+1}^{\boldsymbol{j}})) = M_2(C(\bigvee_l X_{N_{n+1}^l}))$ be given by $p^{(i)}|_{C(X_{N_{n+1}^l})} = M_2(\Psi_{t_{li}})(p_{X_{N_{n+1}^l}})$. Define

$$\psi_{\boldsymbol{j}i} : C(\{\theta_n^i\}) = \mathbb{C} \to M_{\hat{\gamma}_{\boldsymbol{j}i}+1}(C(X_{n+1}^{\boldsymbol{j}})) \text{ by sending } 1 \in \mathbb{C} \text{ to } \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & p^{(i)} \end{pmatrix},$$

where we put $\hat{\gamma}_{\boldsymbol{j}i} - 1$ copies of 1 on the diagonal.

For $j \neq \boldsymbol{j}$, define

$$\psi_{j\boldsymbol{i}} : C(X_n^{\boldsymbol{i}}) \to M_{\hat{\gamma}_{j\boldsymbol{i}}}(C(\{\theta_{n+1}^j\})),\ f \mapsto \begin{pmatrix} f(\theta_n^{\boldsymbol{i}}) & & \\ & \ddots & \\ & & f(\theta_n^{\boldsymbol{i}}) \end{pmatrix},$$

where we put $\hat{\gamma}_{j\boldsymbol{i}}$ copies of $f(\theta_n^{\boldsymbol{i}})$ on the diagonal.

For $i \neq \boldsymbol{i}$ and $j \neq \boldsymbol{j}$, define

$$\psi_{ji} : C(\{\theta_n^i\}) \to M_{\hat{\gamma}_{ji}}(C(\{\theta_{n+1}^j\})),\ 1 \mapsto \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix},$$

where we put $\hat{\gamma}_{ji}$ copies of 1 on the diagonal.

To unify notation, let us set $X_n^i = \{\theta_n^i\}$, $X_{n+1}^j = \{\theta_{n+1}^j\}$.

For $u_n = (([n, \boldsymbol{i}], \tau_n), ([n, i])_{i \neq \boldsymbol{i}}) \in H_n^+$, let $s(n, \boldsymbol{i})$ be a positive integer and $P_n^{\boldsymbol{i}} \in M_{s(n,\boldsymbol{i})}(C(X_n^{\boldsymbol{i}}))$ a projection such that:

- $P_n^{\boldsymbol{i}}$ is a sum of line bundles;
- $[P_n^{\boldsymbol{i}}]$ corresponds to $([n, \boldsymbol{i}], \tau_n)$ under the identification in Lemma 3.7;

- $P_n^{\boldsymbol{i}}(\theta_n^{\boldsymbol{i}}) = 1_{[n,\boldsymbol{i}]}$ is of the form

$$u^* \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} u, \quad \text{where } u \text{ is a permutation matrix.}$$

$P_n^{\boldsymbol{i}}$ exists because of Lemma 3.2. Moreover, we can extend $P_n^{\boldsymbol{i}}$ by $1_{[n,i]}$ to a projection in $\bigoplus_i M_{s(n,i)}(C(X_n^i))$ such that $[P_n]$ corresponds to u_n under the isomorphism in Lemma 3.7. Here $s(n,i) = [n,i]$ whenever $i \neq \boldsymbol{i}$. Then

$$\left(M_{s(n,i)}(\psi_{ji})\right)_{ji} : \bigoplus_i M_{s(n,i)}(C(X_n^i)) \to \bigoplus_j M_{s(n+1,j)}(C(X_{n+1}^j)) \quad (9)$$

sends P_n to P_{n+1}, where P_{n+1} is of the same form as P_n, with $[P_{n+1}]$ corresponding to u_{n+1} under the isomorphism from Lemma 3.7. Hence the map in (9) restricts to a unital homomorphism

$$P_n M_\infty(C(X_n)) P_n \to P_{n+1} M_\infty(C(X_{n+1})) P_{n+1} \quad (10)$$

which in K_0 induces γ by Lemma 3.5.

Now we turn to K_1. Assume $K_n = \bigoplus_i K_n^i$ is an abelian group, where for a distinguished index $\boldsymbol{i}$, $K_n^{\boldsymbol{i}} = T_n$ is a finitely generated torsion group $T_n = \bigoplus_k T_n^k$, $T_n^k = \mathbb{Z}/N_n^k$, and $K_n^i = \mathbb{Z}$ for all $i \neq \boldsymbol{i}$. Similarly, let $K_{n+1} = \bigoplus_j K_{n+1}^j$ be an abelian group, where for a distinguished index $\boldsymbol{j}$, $K_{n+1}^{\boldsymbol{j}} = T_{n+1}$ is a finitely generated torsion group $T_{n+1} = \bigoplus_l T_{n+1}^l$, $T_{n+1}^l = \mathbb{Z}/N_{n+1}^l$, and $K_{n+1}^j = \mathbb{Z}$ for all $j \neq \boldsymbol{j}$. For fixed n, let $\chi : K_n \to K_{n+1}$ be a homomorphism which is a sum $\chi = \hat{\chi} + \tau + t$, where $\hat{\chi} : \bigoplus_{i \neq \boldsymbol{i}} K_n^i \to \bigoplus_{j \neq \boldsymbol{j}} K_{n+1}^j$ is given by an integer matrix $(\hat{\chi}_{ji})$ (viewing $\hat{\chi}_{ji}$ as a homomorphism $K_n^i \to K_{n+1}^j$), $\tau : T_n \to T_{n+1}$ is given by an integer matrix (τ_{lk}) (viewing τ_{lk} as a homomorphism $T_n^k \to T_{n+1}^l$), and $t : \bigoplus_{i \neq \boldsymbol{i}} K_n^i \to T_{n+1}$ is given by an integer matrix (t_{li}) (viewing t_{li} as a homomorphism $K_n^i \to T_{n+1}^l$). We can always arrange that all the entries of these matrices are positive integers.

The following is a direct consequence of Lemma 3.3.

Lemma 3.8 *Let $Y_n^{\boldsymbol{i}} = \bigvee_k Y_{N_n^k}$ and $Y_n = Y_n^{\boldsymbol{i}} \vee \bigvee_{i \neq \boldsymbol{i}} S^3$. Then $K_0(C(Y_n)) \cong \mathbb{Z}$ and $K_1(C(Y_n)) \cong K_n$.*

We view Y_n as a pointed space, and let θ_n be the base point of Y_n. Now let $\psi_\tau^k : C(Y_{N_n^k}) \to C(\bigvee_l Y_{N_{n+1}^l}) = C(Y_{n+1}^{\boldsymbol{j}})$ be the dual homomorphism of the map $\bigvee_l \Sigma(\Psi_{\tau_{lk}}^*) : Y_{n+1}^{\boldsymbol{j}} = \bigvee_l Y_{N_{n+1}^l} \to Y_{N_n^k}$. Here $\Sigma(\Psi_{\tau_{lk}}^*)$ are the maps from Lemma 3.6. The direct sum $\bigoplus_k \psi_\tau^k : \bigoplus_k C(Y_{N_n^k}) \to M_{\#_1(k)}(C(Y_{n+1}^{\boldsymbol{j}}))$ restricts to a homomorphism $\psi_{\boldsymbol{i}} : C(Y_n^{\boldsymbol{i}}) = C(\bigvee_k Y_{N_n^k}) \to M_{\#_1(k)}(C(Y_{n+1}^{\boldsymbol{j}})) \hookrightarrow M_{\#_1(k)}(C(Y_{n+1}))$.

For $i \neq \boldsymbol{i}$, define $\psi_{\boldsymbol{j}i} : C(Y_n^i) = C(S^3) \to C(Y_{n+1}^{\boldsymbol{j}})$ as the dual map of the composite

$$Y_{n+1}^{\boldsymbol{j}} = \bigvee_l Y_{N_{n+1}^l} \xrightarrow{\bigvee_l \Sigma(\psi_{\tau_{li}}^*)} \bigvee_l Y_{N_{n+1}^l} \xrightarrow{\bigvee_l \Omega_l^*} S^3,$$

where Ω_l^* is the map $Y_{N_{n+1}^l} \to S^3$ constructed in Lemma 3.4.

For $i \neq \boldsymbol{i}$ and $j \neq \boldsymbol{j}$, define $\psi_{ji} : C(Y_n^i) = C(S^3) \to C(S^3) = C(Y_{n+1}^j)$ as the dual map of $\Sigma\Sigma\mu_{\hat{\chi}_{ji}} : S^3 \cong \Sigma\Sigma S^1 \to \Sigma\Sigma S^1 \cong S^3$, where $\mu_{\hat{\chi}_{ji}}$ is the map from Lemma 3.1.

For every $i \neq \boldsymbol{i}$, we thus obtain the direct sum $\bigoplus_j \psi_{ji} : C(Y_n^i) \to \bigoplus_j C(Y_{n+1}^j)$ with image in $C(Y_{n+1}) = C(\bigvee_j Y_{n+1}^j) \subseteq \bigoplus_j C(Y_{n+1}^j)$. Hence we obtain a homomorphism $\psi_i : C(Y_n^i) \to C(Y_{n+1})$.

Now let $\#_1(i)$ be the number of summands of K_n. Then let $\psi : C(Y_n) \to M_{\#_1(k)+\#_1(i)-1}(C(Y_{n+1}))$ be the restriction of $\bigoplus_i \psi_i$ to $C(Y_n) = C(\bigvee_i Y_n^i) \subseteq \bigoplus_i C(Y_n^i)$. By construction, and using Lemmas 3.4 and 3.6 , ψ induces χ in K_1.

We now combine our two constructions. Define $Z_n = X_n \vee Y_n$, where we identify the base point $\theta_n^{\boldsymbol{i}} \in X_n^{\boldsymbol{i}} \subseteq X_n$ with $\theta_n \in Y_n$. We extend P_n from X_n constantly to Y_n (with value $P_n(\theta_n^{\boldsymbol{i}})$). Note that $\mathrm{rk}\,(P_{n+1}(\theta_{n+1}^{\boldsymbol{j}})) = \hat{\gamma}_{\boldsymbol{ji}} \cdot \mathrm{rk}\,(P_n(\theta_n^{\boldsymbol{i}}))$. Because of (2) and (6), we can arrange $\hat{\gamma}_{\boldsymbol{ji}} \geq \#_1(k)+\#_1(i)-1$. By adding ev_{θ_n} on the diagonal if necessary, we can modify ψ to a homomorphism $\psi : C(Y_n) \to M_{\hat{\gamma}_{\boldsymbol{ji}}}(C(Y_{n+1}))$ which induces γ in K_1. We can thus think of $M_{[n,\boldsymbol{i}]}(\psi)$ as a unital homomorphism $P_n(\theta_n^{\boldsymbol{i}})M_{s(n,\boldsymbol{i})}(C(Y_n))P_n(\theta_n^{\boldsymbol{i}}) \to P_{n+1}(\theta_{n+1}^{\boldsymbol{j}})M_{s(n+1,\boldsymbol{j})}(C(Y_{n+1}))P_{n+1}(\theta_{n+1}^{\boldsymbol{j}})$, i.e., as a unital homomorphism $P_n M_{s(n,\boldsymbol{i})}(C(Y_n))P_n \to P_{n+1}M_{s(n+1,\boldsymbol{j})}(C(Y_{n+1}))P_{n+1}$. In combination with the homomorphism (10), we obtain a unital homomorphism

$$P_n M_\infty(C(Z_n))P_n \to P_{n+1}M_\infty(C(Z_{n+1}))P_{n+1}$$

which induces γ in K_0, sending u_n to u_{n+1}, and χ in K_1.

Evaluation at $\theta_n^{\boldsymbol{i}} = \theta_n$ and θ_n^i (for $i \neq \boldsymbol{i}$) induces a quotient homomorphism which fits into a commutative diagram

$$\begin{tikzcd} P_n M_\infty(C(Z_n)) P_n \arrow[r] \arrow[d] & P_{n+1} M_\infty(C(Z_{n+1})) P_{n+1} \arrow[d] \\ \bigoplus_i M_{[n,i]} \arrow[r] & \bigoplus_j M_{[n+1,j]} \end{tikzcd} \tag{11}$$

which induces in K_0

$$\begin{tikzcd} H_n \arrow[r, "\gamma"'] \arrow[d] & H_{n+1} \arrow[d] \\ \hat{H}_n \arrow[r, "\hat{\gamma}"] & \hat{H}_{n+1}. \end{tikzcd}$$

Remark 3.9 If all K_n are torsion-free, then we can replace S^3 by S^1 in our construction of Y_n.

4 The complete construction

4.1 The general construction with concrete models

Applying our construction in Sect. 3, we obtain concrete models for F_n, $\hat{F}_n$, γ_n and $\hat{\gamma}_n$ which we now plug into the general construction in Sects. 2.1 and 2.2. Note that it is crucial that we work with these concrete models from Sect. 3. The reason is that only for these models can we provide groupoid descriptions of the C*-algebras and their homomorphisms which arise in the general construction (see Sect. 6).

Note that with these concrete models, the composition

$$M_{[n,i]} \hookrightarrow \hat{F}_n \xrightarrow{\beta_\bullet} E_n \twoheadrightarrow M_{\{n,p\}},$$

where the first and third maps are the canonical ones, is of the form

$$x \mapsto u^* \begin{pmatrix} x & & & & \\ & \ddots & & & \\ & & x & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} u$$

for a permutation matrix u.

Apart from inserting these concrete models, we keep the same construction as in Sects. 2.1 and 2.2.

4.2 Summary of the construction

In both the unital and stably projectionless cases, the C*-algebra with the prescribed Elliott invariant which we constructed is an inductive limit whose building blocks are of the form

$$A_n = \{(f, a) \in C([0, 1], E_n) \oplus F_n \colon f(t) = \beta_t(a) \text{ for } t = 0, 1\}, \tag{12}$$

where:

- E_n is finite dimensional;
- F_n is homogeneous of the form $P_n M_\infty(Z_n) P_n$, where P_n is a sum of line bundles, and there are points $\theta_n^i \in Z_n$, one for each connected component, and all connected components just consist of θ_n^i with the only possible exception being the component of a distinguished point $\theta_n^{\boldsymbol{i}}$;
- both β_0 and β_1 are compositions of the form $F_n \to \bigoplus_i M_{[n,i]} \to E_n$, where the first homomorphism is given by evaluation in $\theta_n^i \in Z_n$ and the second homomorphism is determined by the composites $M_{[n,i]} \hookrightarrow \bigoplus_i M_{[n,i]} \to E_n \twoheadrightarrow E_n^p$ (where E_n^p is a matrix block of E_n), which are of the form

$$x \mapsto v^* \begin{pmatrix} x & & & & & \\ & \ddots & & & & \\ & & x & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} v$$

  for a permutation matrix v.

The connecting maps φ_n of our inductive limit can be described as two parts:

$$A_n \to A_{n+1} \twoheadrightarrow F_{n+1}; \tag{13}$$
$$A_n \to A_{n+1} \twoheadrightarrow C([0, 1], E_{n+1}). \tag{14}$$

Both parts send $(f, a) \in A_n$ to an element which is in diagonal form up to permutation, i.e.,

$$u^* \begin{pmatrix} * & & \\ & * & \\ & & \ddots \end{pmatrix} u, \tag{15}$$

where for the entries on the diagonal, there are the following possibilities:

- a map of the form

$$[0, 1] \ni t \mapsto f^p(\lambda(t)), \text{ for a continuous map } \lambda : [0, 1] \to [0, 1]$$
$$\text{with } \lambda^{-1}(\{0, 1\}) \subseteq \{0, 1\}, \tag{16}$$

  where f^p is the image of f under the canonical projection $C([0, 1], E_n) \twoheadrightarrow C([0, 1], E_n^p)$;
- a map of the form

$$[0, 1] \ni t \mapsto \tau(t)a(x(t)), \tag{17}$$

  where $x : [0, 1] \to Z_n$ is continuous and $\tau(t) : P_n(x(t))M_\infty P_n(x(t)) \cong P_n(\theta_n^i)M_\infty P_n(\theta_n^i)$ is an isomorphism depending continuously on t, with θ_n^i in the same connected component as $x(t)$, and $\tau(t) = \mathrm{id}$ if $x(t) = \theta_n^i$;
- an element of $P_{n+1}M_\infty(C(Z_{n+1}))P_{n+1}$ with support in an isolated point θ_{n+1}^j, which is of the form

$$f^p(\boldsymbol{t}), \text{ for some } \boldsymbol{t} \in (0, 1), \tag{18}$$

  where f^p is the image of f under the canonical projection $C([0, 1], E_n) \twoheadrightarrow C([0, 1], E_n^p)$;
- an element of $P_{n+1}M_\infty(C(Z_{n+1}))P_{n+1}$ with support in an isolated point θ_{n+1}^j, which is of the form

$$\tau(a(x)) \quad \text{for some } x \in Z_n \text{ with } x \notin \{\theta_n^i\}_i$$
$$\text{and an isomorphism } \tau : P_n(x)M_\infty P_n(x) \cong P_n(\theta_n^i)M_\infty P_n(\theta_n^i), \tag{19}$$

  where θ_n^i is in the same connected component as x;
- an element of $P_{n+1}M_\infty(C(Z_{n+1}))P_{n+1}$ with support in an isolated point θ_{n+1}^j, which is of the form

$$a(\theta_n^i), \text{ where } \theta_n^i \text{ is an isolated point in } Z_n; \tag{20}$$

- an element of $P_{n+1}M_\infty(C(Z_{n+1}))P_{n+1}$, which is of the form

$$(a_{ij} \cdot q)_{ij}, \text{ where } q \text{ is a line bundle over } Z_{n+1}, \text{ and } (a_{ij}) = a(\theta_n^i); \tag{21}$$

- an element of $P_{n+1}M_\infty(C(Z_{n+1}))P_{n+1}$ of the form

$$a \circ \lambda, \tag{22}$$

where $\lambda : Z_{n+1} \to Z_n$ is a continuous map whose image is only contained in one wedge summand of Z_n (see our constructions in Sect. 3).

Note that in (17) and (19), we identify $P_n(\theta_n^{\boldsymbol{i}})M_\infty P_n(\theta_n^{\boldsymbol{i}})$ with $M_{[n,\boldsymbol{i}]}$ via a fixed isomorphism.

Let $P^{\boldsymbol{a}} \in M(A_{n+1})$ be projections, with $\sum_{\boldsymbol{a}} P^{\boldsymbol{a}} = 1$, giving rise to the diagonal form in (15), and let $\varphi^{\boldsymbol{a}}$ be the homomorphism $A_n \to P_{\boldsymbol{a}}A_{n+1}P_{\boldsymbol{a}}$, $x \mapsto P^{\boldsymbol{a}}u\varphi(a)u^*P^{\boldsymbol{a}}$. Since each of the $P^{\boldsymbol{a}}$ either lies in $C([0,1], E_{n+1}^q)$ or F_{n+1}, we have $\mathrm{im}\,(\varphi^{\boldsymbol{a}}) \subseteq P^{\boldsymbol{a}}C([0,1], E_{n+1}^q)P^{\boldsymbol{a}}$ or $\mathrm{im}\,(\varphi^{\boldsymbol{a}}) \subseteq P^{\boldsymbol{a}}F_{n+1}P^{\boldsymbol{a}}$. Then both maps in (13), (14) are of the form $u^*(\bigoplus_{\boldsymbol{a}} \varphi^{\boldsymbol{a}})u$. The unitary u is a permutation matrix for the map in (13) and is a unitary in $C([0,1], E_{n+1})$ such that $u(0)$ and $u(1)$ are permutation matrices for the map in (14).

Remark 4.1 Let us write $C_n := C([0,1], E_n)$ and $u_{n+1} \in C_{n+1}$ for the unitary for the map in (14). The only reason we need u_{n+1} is to ensure that we send (f, a) to an element satisfying the right boundary conditions at $t = 0$ and $t = 1$. For this, only the values $u_{n+1,t} := u_{n+1}(t)$ at $t \in \{0,1\}$ matter. Therefore, by an iterative process, we can change β_t in order to arrange $u_{n+1} = 1$ for the map in (14): First of all, it is easy to see that φ_n extends uniquely to a homomorphism $\Phi_n : C_n \oplus F_n \to C_{n+1} \oplus F_{n+1}$. Let us write Φ_n^C and Φ_n^F for the composites

$$C_n \oplus F_n \xrightarrow{\Phi_n} C_{n+1} \oplus F_{n+1} \twoheadrightarrow C_{n+1} \text{ and}$$
$$C_n \oplus F_n \xrightarrow{\Phi_n} C_{n+1} \oplus F_{n+1} \twoheadrightarrow F_{n+1}.$$

As φ_n sends strictly positive elements to strictly positive elements, Φ_n is unital. Now, for all n, let $\Lambda_n(t) \subseteq [0,1]$ be a finite set such that for all $(f_n, a_n) \in A_n$ with $\varphi(f_n, a_n) = (f_{n+1}, a_{n+1}) \in A_{n+1}$, $f_n|_{\Lambda_n(t)} \equiv 0$ implies $f_{n+1}(t) = 0$. In other words, $\Lambda_n(t)$ are the evaluation points for $f_{n+1}(t)$. Similarly, let $T_n \subseteq (0,1)$ be such that for all $(f_n, a_n) \in A_n$ with $\varphi(f_n, a_n) = (f_{n+1}, a_{n+1}) \in A_{n+1}$, $f_n|_{T_n} \equiv 0$ and $a_n = 0$ imply $a_{n+1} = 0$. Now we choose inductively on n unitaries $v_n \in U(C_n)$ and $u_{n+1} \in U(C_{n+1})$ such that, for all n, $v_n(s) = 1$ for all $s \in (\Lambda_n(0) \cup \Lambda_n(1) \cup T_n) \setminus \{0,1\}$, $u_{n+1}(t) = u_{n+1,t}$ for $t \in \{0,1\}$, and $v_{n+1} = \Phi_n^C(v_n, 1)u_{n+1}^*$: Simply start with $v_1 := 1$, and if v_n and u_n have been chosen, choose $u_{n+1} \in U(C_{n+1})$ such that $u_{n+1}(t) = u_{n+1,t}$ for all $t \in \{0,1\}$ and $u_{n+1}(s) = \Phi_n^C(v_n, 1)(s)$ for all $s \in (\Lambda_n(0) \cup \Lambda_n(1) \cup T_n) \setminus \{0,1\}$, and set $v_{n+1} := \Phi_n^C(v_n, 1)u_{n+1}^*$. If we now take this u_{n+1} for the map in (14) giving rise to φ_n and Φ_n, then we obtain a commutative diagram

$$\begin{tikzcd}
C_n \oplus F_n \arrow[r, "\Phi_n"] \arrow[d, "(v_n^* \sqcup v_n)\oplus \mathrm{id}"', "\cong"] & C_{n+1} \oplus F_{n+1} \arrow[d, "\cong"', "(v_{n+1}^* \sqcup v_{n+1})\oplus \mathrm{id}"] \\
C_n \oplus F_n \arrow[r, "(u_{n+1},1)\Phi_n(u_{n+1},1)^*"] & C_{n+1} \oplus F_{n+1}
\end{tikzcd}$$

which restricts to

$$\begin{tikzcd}
A_n \arrow[r, "\Phi_n"] \arrow[d, "(v_n^* \sqcup v_n)\oplus \mathrm{id}"', "\cong"] & A_{n+1} \arrow[d, "\cong"', "(v_{n+1}^* \sqcup v_{n+1})\oplus \mathrm{id}"] \\
\bar{A}_n \arrow[r, "\bar{\varphi}_n"] & \bar{A}_{n+1}
\end{tikzcd}$$

where the unitary $\bar{u}_{n+1}$ for the map in (14) for $\bar{\varphi}_n$ is now trivial, $\bar{u}_{n+1} = 1$, and $\bar{A}_n$ is of the same form (12) as A_n, with $\bar{\beta}_t = v_n(t)^* \beta_t v_n(t)$ of the same form as β_t for $t = 0, 1$ (the point being that $v_n(t)$ is a permutation matrix). Obviously, we have $\varinjlim \left\{\bar{A}_n; \bar{\varphi}_n\right\} \cong \varinjlim \{A_n; \varphi_n\}$.

Remark 4.2 Note that the construction described in Sect. 4.2 also encompasses (a slight modification of) the C*-algebra construction in [19, § 6]. (In particular, one obtains model algebras of rational generalized tracial rank one, in the sense of [19].)

5 Inductive limits and Cartan pairs revisited

In this section, we improve the main result in [4, § 3] and give a C*-algebraic interpretation. Let us first recall [4, Theorem 3.6]. We use the same notations and definitions as in [4,36]. We start with the following

Remark 5.1 We can drop the assumptions of second countability for groupoids and separability for C*-algebras in [36] if we replace "topologically principal" by "effective" throughout. In other words, given a twisted étale effective groupoid (G, Σ), i.e., a twisted étale groupoid (G, Σ) where G is effective (not necessarily second countable), $(C_r^*(G, \Sigma), C_0(G^{(0)}))$ is a Cartan pair; and conversely, given a Cartan pair (A, B) (where A is not necessarily separable), the Weyl twist $(G(A, B), \Sigma(A, B))$ from [36] is a twisted étale effective groupoid. These constructions are inverse to each other, i.e., there are canonical isomorphisms $(G, \Sigma) \cong (G(C_r^*(G, \Sigma), C_0(G^{(0)})), \Sigma(C_r^*(G, \Sigma), C_0(G^{(0)})))$ (provided by [36, 4.13, 4.15, 4.16]) and $(A, B) \cong (C_r^*(G(A, B), \Sigma(A, B)), C_0(G(A, B)^{(0)}))$ (provided by [36, 5.3, 5.8, 5.9]). Similarly, everything in [4, § 3] works without the assumption of second countability. In particular, [4, Theorem 3.6] holds for general twisted étale groupoids if we replace "topologically principal" by "effective". This is why in this section, we formulate everything for twisted étale effective groupoids and general Cartan pairs. In our applications later on, however, we will only consider second countable groupoids and separable C*-algebras.

Now suppose that (A_n, B_n) are Cartan pairs, let (G_n, Σ_n) be their Weyl twists, and set $X_n := G_n^{(0)}$. Let $\varphi_n : A_n \to A_{n+1}$ be injective *-homomorphisms. Assume that there are twisted groupoids (H_n, T_n), with $Y_n := H_n^{(0)}$, together with twisted groupoid homomorphisms $(i_n, \imath_n) : (H_n, T_n) \to (G_{n+1}, T_{n+1})$ and $(\dot{p}_n, p_n) : (H_n, T_n) \to (G_n, T_n)$ such that $i_n : H_n \to G_{n+1}$ is an embedding with open image, and $\dot{p}_n : H_n \to G_n$ is surjective, proper, and fibrewise bijective (i.e., for every $y \in Y_n$, $\dot{p}_n|_{(H_n)_y}$ is a bijection onto $(G_n)_{\dot{p}_n(y)}$). Suppose that $\varphi_n = (\imath_n)_* \circ p_n^*$ for all n. Further assume that condition (LT) is satisfied, i.e., for every continuous section $\rho : U \to \rho(U)$ for the canonical projection $\Sigma_n \twoheadrightarrow G_n$, where $U \subseteq G_n$ is open, there is a continuous section $\tilde{\rho} : \dot{p}_n^{-1}(U) \to \tilde{\rho}(\dot{p}_n^{-1}(U))$ for the canonical projection $T_n \twoheadrightarrow H_n$ such that $\tilde{\rho}(\dot{p}_n^{-1}(U)) \subseteq \dot{p}_n^{-1}(\rho(U))$ and $p_n \circ \tilde{\rho} = \rho \circ \dot{p}_n$. Also assume that condition (E) is satisfied, i.e., for every continuous section $t : U \to t(U)$ for the source map of G_n, where $U \subseteq X_n$ and $t(U) \subseteq G_n$ are open, there is a continuous section $\tilde{t} : \dot{p}_n^{-1}(U) \to \tilde{t}(\dot{p}_n^{-1}(U))$ for the source map of H_n such that $\tilde{t}(\dot{p}_n^{-1}(U)) \subseteq \dot{p}_n^{-1}(t(U))$ and $\dot{p}_n \circ \tilde{t} = t \circ \dot{p}_n$.

In this situation, define

$$
\begin{aligned}
&\Sigma_{n,0} := \Sigma_n \text{ and } \Sigma_{n,m+1} := p_{n+m}^{-1}(\Sigma_{n,m}) \subseteq T_{n+m} \text{ for all } n \text{ and } m = 0, 1, \dots, \\
&G_{n,0} := G_n \text{ and } G_{n,m+1} := \dot{p}_{n+m}^{-1}(G_{n,m}) \subseteq H_{n+m} \text{ for all } n \text{ and } m = 0, 1, \dots, \\
&\bar{\Sigma}_n := \varprojlim_m \left\{\Sigma_{n,m}; p_{n+m}\right\} \text{ and } \bar{G}_n := \varprojlim_m \left\{G_{n,m}; \dot{p}_{n+m}\right\} \text{ for all } n. \qquad (23)
\end{aligned}
$$

Then [4, Theorem 3.6] tells us that

(a) $(\bar{G}_n, \bar{\Sigma}_n)$ are twisted groupoids, and $(i_n, \imath_n)$ induce twisted groupoid homomorphisms $(\bar{i}_n, \bar{\imath}_n) : (\bar{G}_n, \bar{\Sigma}_n) \to (\bar{G}_{n+1}, \bar{\Sigma}_{n+1})$ such that $\bar{i}_n$ is an embedding with open image for all n, and

$$
\bar{\Sigma} := \varinjlim \left\{\bar{\Sigma}_n; \bar{\imath}_n\right\} \text{ and } \bar{G} := \varinjlim \left\{\bar{G}_n; \bar{i}_n\right\} \qquad (24)
$$

defines a twisted étale groupoid $(\bar{G}, \bar{\Sigma})$,

(b) & (c) $(\varinjlim \{A_n; \varphi_n\}, \varinjlim \{B_n; \varphi_n\})$ is a Cartan pair whose Weyl twist is given by $(\bar{G}, \bar{\Sigma})$.

Remark 5.2 It is clear that the proof of [4, Theorem 3.6] shows that if all B_n are C*-diagonals, i.e., all G_n are principal, then $\bar{G}$ is principal, i.e., $\varinjlim \{B_n; \varphi_n\}$ is a C*-diagonal.

It turns out that conditions (LT) and (E) are redundant.

Lemma 5.3 *In the situation above, conditions (LT) and (E) are automatically satisfied.*

Proof To prove condition (LT), let $\rho : U \to \rho(U)$ be a continuous section for the canonical projection $\pi_n : \Sigma_n \twoheadrightarrow G_n$, where $U \subseteq G_n$ is open. Let $\pi_{n+1} : \Sigma_{n+1} \twoheadrightarrow G_{n+1}$ be the canonical projection. Then $\pi_{n+1}|_{p_n^{-1}(\rho(U))} : p_n^{-1}(\rho(U)) \to \dot{p}_n^{-1}(U)$ is bijective. Indeed, given $\tau_1, \tau_2 \in p_n^{-1}(\rho(U))$ with $\pi_{n+1}(\tau_1) = \pi_{n+1}(\tau_2) =: \eta \in H_n$, we must have $\tau_2 = z \cdot \tau_1$ for some $z \in \mathbb{T}$. Also, $\pi_n(p_n(\tau_1)) = \dot{p}_n(\eta) = \pi_n(p_n(\tau_1))$. As $\pi_n|_{\rho(U)} : \rho(U) \to U$ is bijective (with inverse ρ), we deduce $p_n(\tau_1) = p_n(\tau_2)$. Hence $p_n(\tau_1) = p_n(\tau_2) = z \cdot p_n(\tau_1)$, which implies $z = 1$, i.e., $\tau_2 = \tau_1$. This proves injectivity, and surjectivity is easy to see. As π_{n+1} is open, $\tilde{\rho} := (\pi_{n+1}|_{p_n^{-1}(\rho(U))})^{-1} : \dot{p}_n^{-1}(U) \to p_n^{-1}(\rho(U))$ is the continuous section we are looking for.

To verify (E), let $t : U \to t(U)$ be a continuous section for the source map s_n of G_n, where $U \subseteq X_n$ and $t(U) \subseteq G_n$ are open. Let s_{n+1} be the source map of H_n. Then $s_{n+1}|_{\dot{p}_n^{-1}(t(U))} : \dot{p}_n^{-1}(t(U)) \to \dot{p}_n^{-1}(U)$ is bijective. Indeed, given $\eta_1, \eta_2 \in \dot{p}_n^{-1}(t(U))$ with $s_{n+1}(\eta_1) = s_{n+1}(\eta_2) =: y \in Y_n$, we must have $s_n(\dot{p}_n(\eta_1)) = \dot{p}_n(y) = s_n(\dot{p}_n(\eta_2))$. As $s_n|_{t(U)} : t(U) \to U$ is bijective (with inverse t), we deduce $\dot{p}_n(\eta_1) = \dot{p}_n(\eta_2)$. Since $\dot{p}_n$ is fibrewise bijective, this implies $\eta_1 = \eta_2$. This proves injectivity, and surjectivity is easy to see. As $\dot{p}_n^{-1}(t(U))$ is open and s_{n+1} is open, $\tilde{t} := (s_{n+1}|_{\dot{p}_n^{-1}(t(U))})^{-1} : \dot{p}_n^{-1}(U) \to \dot{p}_n^{-1}(t(U))$ is the continuous section we are looking for.

Let us now determine which *-homomorphisms are of the form $\imath_* \circ p^*$. Let (A, B) and $(\hat{A}, \hat{B})$ be Cartan pairs with normalizers $N := N_A(B)$, $\hat{N} := N_{\hat{A}}(\hat{B})$ and faithful conditional expectations $P : A \twoheadrightarrow B$, $\hat{P} : \hat{A} \twoheadrightarrow \hat{B}$. Let (G, Σ) and $(\hat{G}, \hat{\Sigma})$ be the Weyl twists of (A, B) and $(\hat{A}, \hat{B})$. Suppose that $\varphi : A \to \hat{A}$ is an injective *-homomorphism.

Proposition 5.4 *The following are equivalent:*

(i) $\varphi(B) \subseteq \hat{B}$, $\varphi(N) \subseteq \hat{N}$, $\hat{P} \circ \varphi = \varphi \circ P$;

(ii) *There exists a twisted étale effective groupoid (H, T) and twisted groupoid homomorphisms $(i, \imath) : (H, T) \to (\hat{G}, \hat{\Sigma})$, $(\dot{p}, p) : (H, T) \to (G, \Sigma)$, where i is an embedding with open image and $\dot{p}$ is surjective, proper and fibrewise bijective, such that $\varphi = \imath_* \circ p^*$.*

Proof (ii) $\Rightarrow$ (i): It is easy to see that $(\imath_* \circ p^*)(B) \subseteq \hat{B}$. Given an open bisection S of G, $\dot{p}^{-1}(S)$ is an open bisection of H, and then $i(\dot{p}^{-1}(S))$ is an open bisection of $\hat{G}$. Therefore, $(\imath_* \circ p^*)(N) \subseteq \hat{N}$. Finally, we have $\hat{P} \circ (\imath_* \circ p^*) = (\imath_* \circ p^*) \circ P$ because $\dot{p}^{-1}(G^{(0)}) = H^{(0)}$.

(i) $\Rightarrow$ (ii): Let $\breve{B}$ be the ideal of $\hat{B}$ generated by $\varphi(B)$, and $\breve{A} := C^*(\varphi(A), \breve{B})$. Then $(\breve{A}, \breve{B})$ is a Cartan pair: It is clear that $\breve{B}$ contains an approximate unit for $\breve{A}$. To see that $\breve{B}$ is maximal abelian, take $a \in \breve{A} \cap (\breve{B})'$. Let $b \in \hat{B}$, and take an approximate unit $(h_\lambda) \subseteq \breve{B}$ for $\breve{A}$. Then $ba = \lim_\lambda bh_\lambda a = \lim_\lambda abh_\lambda = \lim_\lambda ah_\lambda b = ab$. Hence $a \in \breve{A} \cap (\hat{B})' = \breve{A} \cap \hat{B} = \breve{B}$ (the last equality holds because $\breve{B}$ contains an approximate unit for $\breve{A}$, and $\breve{B} \cdot \hat{B} \subseteq \breve{B}$). This shows $\breve{A} \cap (\breve{B})' = \breve{B}$. Moreover, we have $\varphi(N) \subseteq \breve{N} := N_{\breve{A}}(\breve{B})$: Let $n \in \varphi(N)$, $b \in \breve{B}$, and $(h_\lambda) \subseteq B$ be an approximate unit for A. Then $nbn^* \in \hat{B}$ as $n \in \varphi(N) \subseteq \hat{N}$, and thus $nbn^* = \lim_\lambda \varphi(h_\lambda)nbn^* \subseteq \overline{\varphi(B) \cdot \hat{B}} \subseteq \breve{B}$. Finally, it is clear that $\breve{P} := \hat{P}|_{\breve{A}}$ is a faithful conditional expectation onto $\breve{B}$.

Let (H, T) be the Weyl twist attached to $(\breve{A}, \breve{B})$, and write $X := G^{(0)}$, $Y := H^{(0)}$ and $\hat{X} := \hat{G}^{(0)}$. It is easy to see that $\breve{N} \subseteq \hat{N}$. Hence we may define maps

$$
\begin{aligned}
& i: \ H \to \hat{G},\ [x, \alpha_n, y] \mapsto [x, \alpha_n, y] \\
& \text{and} \imath: \ T \to \hat{\Sigma},\ [x, n, y] \mapsto [x, n, y],\ \text{ for } n \in \breve{N}.
\end{aligned}
$$

Clearly, i and $\imath$ are continuous groupoid homomorphisms. i is injective since $[x, \alpha_n, y] = [x', \alpha_{n'}, y']$ in $\hat{G}$ implies $x = x'$, $y = y'$ and $\alpha_n = \alpha_{n'}$ on a neighbourhood $U \subseteq \hat{X}$ of y, so that $\alpha_n = \alpha_{n'}$ on $U \cap Y$, which is a neighbourhood of y in Y, and hence $[x, \alpha_n, y] = [x', \alpha_{n'}, y']$ in H. The image of i is given by $\bigcup_{n \in \breve{N}} \{[\alpha_n(y), \alpha_n, y]\colon y \in \text{dom}\,(n)\}$ which is clearly open in $\hat{G}$. Finally, it is easy to see that we have a commutative diagram

$$
\begin{tikzcd}
\breve{A} \arrow[r, "\subseteq"'] \arrow[d, "\cong"'] & \hat{A} \arrow[d, "\cong"] \\
C_r^*(H, T) \arrow[r, "\imath_*"] & C_r^*(\hat{G}, \hat{\Sigma}),
\end{tikzcd}
$$

where the upper horizontal map is given by inclusion, and the vertical isomorphisms are as in [36, Definition 5.4].

We now proceed to construct $(\dot{p}, p)$. Since $A = C^*(N)$ and $\varphi(N) \subseteq \breve{N}$, it is easy to see that $\breve{A} = \overline{\text{span}}(\varphi(N) \cdot \breve{B})$. It follows that for every $\breve{n} \in \breve{N}$ and $y \in \text{dom}\,(\breve{n})$, there is $n \in \varphi(N)$ such that $y \in \text{dom}\,(n)$ and $[x, \breve{n}, y] = [x, n, y]$ in T. Indeed, for $a \in \text{span}(\varphi(N) \cdot \breve{B})$ it is clear that $a \equiv 0$ on $T \setminus \left(\bigcup_{n \in \varphi(N)} \{[\alpha_n(y), n, y]\colon y \in \text{dom}\,(n)\}\right)$. As the latter set is closed in T, we must have $a \equiv 0$ on $T \setminus \left(\bigcup_{n \in \varphi(N)} \{[\alpha_n(y), n, y]\colon y \in \text{dom}\,(n)\}\right)$ for all $a \in \breve{A}$. Hence $T = \bigcup_{n \in \varphi(N)} \{[\alpha_n(y), n, y]\colon y \in \text{dom}\,(n)\}$. This observation allows us to define the maps

$$
\begin{aligned}
& \dot{p}: \ H \to G,\ [x, \alpha_{\varphi(n)}, y] \mapsto [\varphi^*(x), \alpha_n, \varphi^*(y)] \text{and} \\
& p: \ T \to \Sigma,\ [x, \varphi(n), y) \mapsto [\varphi^*(x), n, \varphi^*(y)],\ \text{ for } n \in N,
\end{aligned}
$$

where $\varphi^* : Y \to X$ is the map dual to $B \to \breve{B}$, $b \mapsto \varphi(b)$ determined by $\varphi(b) = b \circ \varphi^*$ for all $b \in B$. Note that φ^* exists since $\varphi(B)$ is full in $\breve{B}$. p is well-defined because $[x, \varphi(m), y] = [x, \varphi(n), y]$ implies $\hat{P}(\varphi(n)^*\varphi(m))(y) > 0$, so that $P(n^*m)(\varphi^*(y)) = \varphi(P(n^*m))(y) = \hat{P}(\varphi(n)^*\varphi(m))(y) > 0$, which in turn yields $[\varphi^*(x), \alpha_m, \varphi^*(y)] = [\varphi^*(x), \alpha_n, \varphi^*(y)]$. Similarly, $\dot{p}$ is well-defined. Clearly, $(\dot{p}, p)$ is a twisted groupoid homomorphism. As φ is injective, φ^* is surjective, so that $\dot{p}$ is surjective.

To see that $\dot{p}$ is proper, let $K \subseteq G$ be compact. Given $n \in N$, write $U(n) := \{[\alpha_n(y), \alpha_n, y] \colon y \in \mathrm{dom}\,(n)\}$ and $K(n) := (s|_{U(n)})^{-1}(s(K))$. As K is compact, there exists a finite set $\{n_i\} \subseteq N$ such that $K \subseteq \bigcup_i U(n_i)$, so that $K = \bigcup_i U(n_i) \cap K \subseteq \bigcup_i K(n_i)$. Now given $m \in N$, $\dot{p}([x, \alpha_{\varphi(m)}, y]) \in K(n)$ implies $\varphi^*(y) \in s(K)$, i.e., $y \in (\varphi^*)^{-1}(s(K))$, $\dot{p}([x, \alpha_{\varphi(m)}, y]) = [\varphi^*(x), \alpha_m, \varphi^*(y)] = [\varphi^*(x), \alpha_n, \varphi^*(y)]$, so that $P(n^*m)(\varphi^*(y)) \neq 0$, which yields $\hat{P}(\varphi(n)^*\varphi(m))(y) = \varphi(P(n^*m))(y) \neq 0$, thus $[x, \alpha_{\varphi(m)}, y] = [x, \alpha_{\varphi(n)}, y]$. Hence $\dot{p}^{-1}(K(n)) \subseteq \left\{[\alpha_{\varphi(n)}(y), \varphi(n), y] \colon y \in (\varphi^*)^{-1}(s(K))\right\} = (s|_{U(\varphi(n))})^{-1}((\varphi^*)^{-1}(s(K)) =: \breve{K}(n)$. As φ^* is proper, $\breve{K}(n)$ is compact for all $n \in N$. Hence $\dot{p}^{-1}(K) \subseteq \bigcup_i \dot{p}^{-1}(K(n_i)) \subseteq \bigcup_i \breve{K}(n_i)$ is a closed subset of a compact set, thus compact itself.

Moreover, given $y \in Y$, $\dot{p}([w, \alpha_{\varphi(m)}, y]) = \dot{p}([x, \alpha_{\varphi(n)}, y])$ implies $[\varphi^*(w), \alpha_m, \varphi^*(y)] = [\varphi^*(x), \alpha_n, \varphi^*(y)]$, so that $\hat{P}(\varphi(n)^*\varphi(m))(y) = P(n^*m)(\varphi^*(y)) \neq 0$, so that $[w, \alpha_{\varphi(m)}, y] = [x, \alpha_{\varphi(n)}, y]$. This shows injectivity of $\dot{p}|_{H_y}$, and it is clear that $\dot{p}(H_y) = G_{\dot{p}(y)}$. Thus $\dot{p}$ is fibrewise bijective.

Finally, it is easy to see that we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"'] \arrow[d, "\cong"'] & \breve{A} \arrow[d, "\cong"] \\ C_r^*(G, \Sigma) \arrow[r, "p^*"] & C_r^*(H, T), \end{tikzcd}$$

where the vertical isomorphisms are as in [36, Definition 5.4].

Remark 5.5 In Proposition 5.4, φ sends full elements to full elements if and only if we have $i(H^{(0)}) = \hat{G}^{(0)}$.

Theorem 1.10 now follows from [4, Theorem 3.6], Lemma 5.3, Proposition 5.4 and Remark 5.2.

Remark 5.6 The Weyl twist of $(\varinjlim \{A_n; \varphi_n\}, \varinjlim \{B_n; \varphi_n\})$ in the situation of Theorem 1.10 is given by $(\bar{G}, \bar{\Sigma})$ as given in (23) and (24).

If, in Theorem 1.10, all φ_n send full elements to full elements, then $G_{n,m+1}^{(0)} = H_{n+m}^{(0)} = G_{n+m+i}^{(0)}$ (where we identify $H_{n+m}^{(0)}$ with $i_{n+m}(H_{n+m}^{(0)})$), so that $\bar{G}_n^{(0)} = \varprojlim_m \{G_{n,m}^{(0)}; \dot{p}_{n+m}\} \cong \varprojlim \{G_l^{(0)}; \dot{p}_l\}$ for all n, and thus $\bar{i}_n(\bar{G}_n^{(0)}) = \bar{G}_{n+1}^{(0)}$ for all n, which implies $\bar{G}^{(0)} \cong \varprojlim \{G_n^{(0)}; \dot{p}_n\}$.

If all B_n in Theorem 1.10 are C*-diagonals, i.e., all G_n are principal, then $\bar{G}$ is principal.

6 Groupoid models

6.1 The building blocks

We first present groupoid models for the building blocks that give rise to our AH-algebras (see Sect. 3). Let Z be a second countable compact Hausdorff space and let $p_i \in M_\infty(C(Z))$ be a finite collection of line bundles over Z. Let $P = \sum_i^{\oplus} p_i \in M_\infty(C(Z))$. The following is easy to check:

Lemma 6.1 *$\bigoplus_i p_i M_\infty(C(Z)) p_i$ is a Cartan subalgebra of $PM_\infty(C(Z))P$.*

Thus, by [36, Theorem 5.9], there exists a twisted groupoid $(\dot{\mathcal{F}}, \mathcal{F})$ (the Weyl twist) such that

$$(C_r^*(\dot{\mathcal{F}}, \mathcal{F}),\ C_0(\dot{\mathcal{F}}^{(0)})) \cong (PM_\infty(C(Z))P,\ \bigoplus_i p_i M_\infty(C(Z)) p_i).$$

Let us now describe $(\dot{\mathcal{F}}, \mathcal{F})$ explicitly. Let R be the full equivalence relation on the finite set $\{p_i\}$ (just a set with the same number of elements as the number of line bundles). Let $\dot{\mathcal{F}} = Z \times R$, which is a groupoid in the canonical way. For every p_i, let T_i be a circle bundle over Z such that $p_i = \mathbb{C} \times_{\mathbb{T}} T_i$. We form the circle bundles $T_j \cdot T_i^*$, which are given as follows: For each index i, let $\{V_{i,a}\}_a$ be an open cover of Z, and let $v_{i,a}$ be a trivialization of $T_i|_{V_{i,a}}$. We view $v_{i,a}$ as a continuous map $v_{i,a} : V_{i,a} \to M_\infty$ with values in partial isometries such that $v_{i,a}(z)$ has source projection e_{11} and range projection $p_i(z)$, so that $v_{i,a}(z) = p_i(z) v_{i,a}(z) e_{11}$. Here e_{11} is the rank one projection in M_∞ which has zero entry everywhere except in the upper left $(1,1)$-entry, where the value is 1. Then

$$T_j \cdot T_i^* = \Big(\coprod_{c,a} \mathbb{T} \times (V_{j,c} \cap V_{i,a}) \Big) \Big/ {\sim}$$

where we define $(z,x) \sim (z',x')$ if $x = x'$, and if $x \in V_{j,c} \cap V_{i,a}$, $x' \in V_{j,d} \cap V_{i,b}$, then $z' = v_{i,b} v_{j,d}^* v_{j,c} v_{i,a}^* z$.

We set

$$\mathcal{F} := \coprod_{j,i} T_j \cdot T_i^*.$$

Note that $T_i \cdot T_i^*$ is just the trivial circle bundle $\mathbb{T} \times Z$. We define a multiplication on $\mathcal{F}$: For $([z, x], (j, i))$ and $([z', x'], (j', i'))$ in $\mathcal{F}$, we can only multiply these elements if $x = x'$ and $i = j'$. In that case, write $h := i'$ and assume that $x \in V_{j,c} \cap V_{i,b}$ and $x' = x \in V_{i,b} \cap V_{h,a}$. Then we define the product as

$$([z, x], (j, i)) \cdot ([z', x'], (j', i')) = ([zz', x], (j, h)).$$

Moreover, $\mathcal{F}$ becomes a twist of $\dot{\mathcal{F}}$ via the map

$$\mathcal{F} \to \dot{\mathcal{F}},\ T_j \cdot T_i^* \ni \sigma \mapsto (\pi(\sigma), (j, i)),$$
$$\text{where } \pi : T_j \cdot T_i^* \to Z \text{ is the canonical projection.}$$

It is now straightforward to check (compare [36]) that the twisted groupoid $(\dot{\mathcal{F}}, \mathcal{F})$ is precisely the Weyl twist of $(PM_\infty(C(Z))P, \bigoplus_i p_i M_\infty(C(Z))p_i)$. More precisely, we have the following

Lemma 6.2 *We have a $C(Z)$-linear isomorphism $C_r^*(\dot{\mathcal{F}}, \mathcal{F}) \cong PM_\infty(C(Z))P$ sending $\tilde{f} \in C_c(\dot{\mathcal{F}}, \mathcal{F})$ with $\mathrm{supp}(\tilde{f}) \subseteq \left(V_{j,c} \cap V_{i,a}\right) \times \{(j, i)\} \subseteq \dot{\mathcal{F}}$ to $f v_{j,c} v_{i,a}^*$, where $f \in C(Z)$ is determined by $\tilde{f}(([z, x], (j, i)) = \bar{z} f(x)$. Moreover, this $C(Z)$-linear isomorphism identifies $C(\dot{\mathcal{F}}^{(0)})$ with $\bigoplus_i p_i M_\infty(C(Z))p_i$.*

Let us now fix n, and apply the result above to the homogeneous C*-algebra $F := F_n$ from Sect. 4.2 to obtain a twisted groupoid $(\dot{\mathcal{F}}, \mathcal{F})$ such that $C_r^*(\dot{\mathcal{F}}, \mathcal{F}) \cong F$. More precisely, we apply our construction above to the summand of F corresponding to the component of $\theta_n^{\boldsymbol{i}}$. Note that in the construction above, all our line bundles satisfy

$$p_i(\theta_n^{\boldsymbol{i}}) = e_{11} \tag{25}$$

because of (7). For the other summands, it is easy to construct a groupoid model, as these are just matrix algebras, so that we can just take the full equivalence relation on finite sets.

Now our goal is to present a groupoid model for the building block $A := A_n$ in Sect. 4.2. Let $\mathcal{R}$ be an equivalence relation (on a finite set) such that $C^*(\mathcal{R}) \cong E := E_n$. Write $\mathcal{R} = \coprod_p \mathcal{R}^p$ for subgroupoids $\mathcal{R}^p$ such that the isomorphism $C^*(\mathcal{R}) \cong E$ restricts to isomorphisms $C^*(\mathcal{R}^p) \cong E^p := E_n^p$. Set $\dot{\mathcal{C}} := [0, 1] \times \mathcal{R}$. Then $C_r^*(\dot{\mathcal{C}})$ is canonically isomorphic to $C := C([0, 1], E)$. Consider the trivial twist $\mathcal{C} := \mathbb{T} \times \dot{\mathcal{C}}$ of $\dot{\mathcal{C}}$. Clearly, we have $C_r^*(\dot{\mathcal{C}} \amalg \dot{\mathcal{F}}, \mathcal{C} \amalg \mathcal{F}) \cong C \oplus F$.

For $t = 0, 1$ and β_t as in Sect. 4.2, write

$$F \xrightarrow{\beta_t} E \twoheadrightarrow E^p$$

as the composition

$$F \to \bigoplus_{l} M_{n_l} \otimes \mathbb{C}^{I_l^p} \hookrightarrow E^p, \tag{26}$$

where each of the components $F \to M_{n_l} \otimes \mathbb{C}^{I_l^p}$ of the first map is given by

$$a \mapsto \begin{pmatrix} a(\theta^l) & & \\ & a(\theta^l) & \\ & & \ddots \end{pmatrix},$$

with $\# I_l^p$ copies of $a(\theta^l)$ on the diagonal, and the components $M_{n_l} \otimes \mathbb{C}^{I_l^p} \hookrightarrow E^p$ of the second map are given by

$$\boldsymbol{x} \mapsto u^* \begin{pmatrix} \boldsymbol{x} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} u, \tag{27}$$

where u is a permutation matrix.

Let E_t^p be the image of $\bigoplus_l M_{n_l} \otimes \mathbb{C}^{I_l^p}$ in E, and set $E_t := \bigoplus_p E_t^p \subseteq E$, for $t = 0, 1$. Let $\mathcal{R}_t^p \subseteq \mathcal{R}^p$ be subgroupoids such that the identification $C^*(\mathcal{R}^p) \cong E^p$ restricts to $C^*(\mathcal{R}_t^p) \cong E_t^p$. Write $\mathcal{R}_t := \coprod_p \mathcal{R}_t^p$, so that $C^*(\mathcal{R}) \cong E$ restricts to $C^*(\mathcal{R}_t) \cong E_t$. Let σ_t^p be the groupoid isomorphism $\coprod_l \mathcal{R}_l \times I_l^p \cong \mathcal{R}_t^p$, given by a bijection of the finite unit space, corresponding to conjugation by the unitary u in (27). Let $V_{i,a}$ and $v_{i,a}$ be as above (introduced after Lemma 6.1). We now define a map $\boldsymbol{b}_t : \mathbb{T} \times (\{t\} \times \mathcal{R}_t) \to \Sigma$ as follows: Given an index l and $(j, i) \in \mathcal{R}_l$, choose indices a and c such that $\theta^l \in V_{j,c} \cap V_{i,a}$. Then define

$$z_{j,i} := v_{j,c}(\theta^l) v_{i,a}(\theta^l)^* \in \mathbb{T}. \tag{28}$$

Here, we are using (25). If θ^l is not the distinguished point $\theta_n^{\boldsymbol{i}}$, then we set $z_{j,i} = 1$. For $z \in \mathbb{T}$ and $h \in I_l^p$, set

$$\boldsymbol{b}_t(z, t, \sigma_t^p((j, i), h)) := [z_{j,i}, \theta^l] \in T_j \cdot T_i^* \subseteq \Sigma,$$
where we view $(z_{j,i}, \theta^l)$ as an element in $\mathbb{T} \times (V_{j,c} \cap V_{i,a})$.

Define

$$\check{\Sigma} := \{x \in \mathcal{C} \amalg \mathcal{F} \colon x = (z, t, \gamma) \in \mathbb{T} \times [0, 1] \times \mathcal{R} \Rightarrow \gamma \in \mathcal{R}_t \text{ for } t = 0, 1\}$$
and $\Sigma := \check{\Sigma}/\sim$

where $\sim$ is the equivalence relation on $\check{\Sigma}$ generated by $(z, t, \gamma) \sim \boldsymbol{b}_t(z, t, \gamma)$ for all $z \in \mathbb{T}$, $t = 0, 1$ and $\gamma \in \mathcal{R}_t$. $\check{\Sigma}$ and Σ are principal $\mathbb{T}$-bundles belonging to twisted groupoids, and we denote the underlying groupoids by $\check{G}$ and G.

By construction, the canonical projection and inclusion $\Sigma \twoheadleftarrow \check{\Sigma} \hookrightarrow \mathcal{C} \amalg \mathcal{F}$ induce on the level of C*-algebras

$$\begin{tikzcd}
C_r^*(\dot{\mathcal{C}} \amalg \dot{\mathcal{F}}, \mathcal{C} \amalg \mathcal{F}) \arrow[rr, "\cong"'] & & C \oplus F \\
C_r^*(\check{G}, \check{\Sigma}) \arrow[u, "\subseteq"'] \arrow[r, "\cong"'] & \check{A} := \{(f, a) \in C \oplus F \colon f(t) \in E_t \text{ for } t = 0, 1\} \arrow[ur, "\subseteq"] & \\
C_r^*(G, \Sigma) \arrow[u, hook] \arrow[r, "\cong"] & A = \{(f, a) \in C \oplus F \colon f(t) = \beta_t(a) \text{ for } t = 0, 1\} \arrow[u, "\subseteq"] &
\end{tikzcd}$$

In particular, (G, Σ) is the desired groupoid model for our building block.

In what follows, it will be necessary to keep track of the index n, so that we will consider, for all n, twisted groupoids $(\dot{\mathcal{C}}_n \amalg \dot{\mathcal{F}}_n, \mathcal{C}_n \amalg \mathcal{F}_n)$, $(\check{G}_n, \check{\Sigma}_n)$, (G_n, Σ_n) describing the C*-algebras $C_n \oplus F_n$, $\check{A}_n$ and A_n as explained above. Moreover, for all n, let $B_n \subseteq A_n$ be the subalgebra corresponding to $C_0(G_n^{(0)})$ under the isomorphism $C_r^*(G_n, \Sigma_n) \cong A_n$.

6.2 The connecting maps

Let us now describe the connecting maps $\varphi_n : A_n \to A_{n+1}$ in the groupoid picture above. Let $P_{n+1}^{\boldsymbol{a}}$, $\varphi_n^{\boldsymbol{a}}$ be as in Sect. 4.2, so that $\varphi_n = \bigoplus_{\boldsymbol{a}} \varphi_n^{\boldsymbol{a}}$ and $\mathrm{im}\,(\varphi_n^{\boldsymbol{a}}) \subseteq P_{n+1}^{\boldsymbol{a}} A_{n+1} P_{n+1}^{\boldsymbol{a}}$. Also, let $\Phi_n : C_n \oplus F_n \to C_{n+1} \oplus F_{n+1}$ be the extension of φ_n as in Remark 4.1. Set $\Phi_n^{\boldsymbol{a}} : C_n \oplus F_n \to P_{n+1}^{\boldsymbol{a}}(C_{n+1} \oplus F_{n+1}) P_{n+1}^{\boldsymbol{a}}$, $x \mapsto P_{n+1}^{\boldsymbol{a}} \varphi_n^{\boldsymbol{a}}(x) P_{n+1}^{\boldsymbol{a}}$. We obtain $\check{\varphi}_n : \check{A}_n \to \check{A}_{n+1}$ and $\check{\varphi}_n^{\boldsymbol{a}} : \check{A}_n \to P_{n+1}^{\boldsymbol{a}} \check{A}_{n+1} P_{n+1}^{\boldsymbol{a}}$ by restricting Φ_n and $\Phi_n^{\boldsymbol{a}}$. Set

$$(C \oplus F)[\Phi_n] := \Big\{ x \in C_{n+1} \oplus F_{n+1} : x = \sum_{\boldsymbol{a}} P_{n+1}^{\boldsymbol{a}} x P_{n+1}^{\boldsymbol{a}} \Big\},$$

$$\check{A}[\check{\varphi}_n^{\boldsymbol{a}}] := \mathrm{im}\,(\check{\varphi}_n^{\boldsymbol{a}}),$$

$$\check{A}[\varphi_n] := \Big\{ x \in \check{A}_{n+1} : x = \sum_{\boldsymbol{a}} P_{n+1}^{\boldsymbol{a}} x P_{n+1}^{\boldsymbol{a}},\ P_{n+1}^{\boldsymbol{a}} x P_{n+1}^{\boldsymbol{a}} \in \check{A}[\check{\varphi}_n^{\boldsymbol{a}}] \Big\},$$

$$A[\varphi_n] := A_{n+1} \cap \check{A}[\check{\varphi}_n].$$

Note that $\check{A}[\check{\varphi}_n^{\boldsymbol{a}}] = P_{n+1}^{\boldsymbol{a}} F_{n+1} P_{n+1}^{\boldsymbol{a}}$ if $P_{n+1}^{\boldsymbol{a}} \in F_{n+1}$ and $\check{A}[\check{\varphi}_n^{\boldsymbol{a}}] = \Big\{ x \in P_{n+1}^{\boldsymbol{a}} \check{A}_{n+1} P_{n+1}^{\boldsymbol{a}} \colon x(t) \in \mathrm{im}\,(\mathrm{ev}_t \circ \check{\varphi}_n^{\boldsymbol{a}}) \text{ for } t = 0, 1 \Big\}$ if $P_{n+1}^{\boldsymbol{a}} \in C_{n+1}$.

Let $\mathcal{T}_n$ be the open subgroupoid of $\mathcal{C}_{n+1} \amalg \mathcal{F}_{n+1}$, with $\dot{\mathcal{T}}_n \subseteq \dot{\mathcal{C}}_{n+1} \amalg \dot{\mathcal{F}}_{n+1}$ correspondingly, such that $C_r^*(\dot{\mathcal{C}}_{n+1} \amalg \dot{\mathcal{F}}_{n+1}, \mathcal{C}_{n+1} \amalg \mathcal{F}_{n+1}) \cong C_{n+1} \oplus F_{n+1}$ restricts to $C_r^*(\dot{\mathcal{T}}_n, \mathcal{T}_n) \cong (C \oplus F)[\Phi_n]$. Similarly, let $\check{T}_n$ be the open subgroupoid of $\check{\Sigma}_{n+1}$, with $\check{H}_n \subseteq \check{G}_{n+1}$ correspondingly, such that $C_r^*(\check{G}_{n+1}, \check{\Sigma}_{n+1}) \cong \check{A}_{n+1}$ restricts to $C_r^*(\check{H}_n, \check{T}_n) \cong \check{A}[\check{\varphi}_n]$. For $\eta \in \check{T}_n$ and $\eta' \in \check{\Sigma}_{n+1}$, $\eta \sim \eta'$ implies that η' lies in $\check{T}_n$. It follows that $T_n = \check{T}_n/_\sim$ is an open subgroupoid of Σ_{n+1}. Define $H_n = \check{H}_n/_\sim$ in a similar way. By construction, the commutative diagram at the groupoid level

$$\begin{array}{ccccc}
\mathcal{C}_n \amalg \mathcal{F}_n & & \mathcal{T}_n & \xrightarrow[\subseteq]{} & \mathcal{C}_{n+1} \amalg \mathcal{F}_{n+1} \\
\uparrow \subseteq & & \uparrow \subseteq & & \subseteq \uparrow \\
\check{\Sigma}_n & & \check{T}_n & \xrightarrow{\subseteq} & \check{\Sigma}_{n+1} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma_n & & T_n & \xrightarrow{\subseteq} & \Sigma_{n+1}
\end{array}$$

induces at the C*-level

$$\begin{array}{ccccc}
C_n \oplus F_n & & (C \oplus F)[\Phi_n] & \xrightarrow[\subseteq]{} & C_{n+1} \oplus F_{n+1} \\
\uparrow \subseteq & & \uparrow \subseteq & & \subseteq \uparrow \\
\check{A}_n & & \check{A}[\check{\varphi}_n] & \xrightarrow[\subseteq]{} & \check{A}_{n+1} \\
\uparrow \subseteq & & \uparrow \subseteq & & \subseteq \uparrow \\
A_n & & A[\varphi_n] & \xrightarrow{\subseteq} & A_{n+1}
\end{array}$$

Let $\check{T}_n = \coprod_{\boldsymbol{a}} \check{T}_n^{\boldsymbol{a}}$ and $\check{H}_n = \coprod_{\boldsymbol{a}} \check{H}_n^{\boldsymbol{a}}$ be the decompositions into subgroupoids such that the identification $C_r^*(\check{H}_n, \check{T}_n) \cong \check{A}[\check{\varphi}_n] \subseteq \check{A}_{n+1}$ restricts to $C_r^*(\check{H}_n^{\boldsymbol{a}}, \check{T}_n^{\boldsymbol{a}}) \cong \check{A}[\check{\varphi}_n^{\boldsymbol{a}}]$. For fixed n and every $\varphi^{\boldsymbol{a}} = \varphi_n^{\boldsymbol{a}}$ from our list in Sect. 4.2, we now construct a map $p^{\boldsymbol{a}} : \check{T}_n^{\boldsymbol{a}} \to \Sigma_n$ such that

$$\begin{array}{ccc}
C_r^*(G_n, \Sigma_n) & \xrightarrow[C^*(p^{\boldsymbol{a}})]{} & C_r^*(\check{H}_n^{\boldsymbol{a}}, \check{T}_n^{\boldsymbol{a}}) \\
\downarrow \cong & & \cong \downarrow \\
A_n & \xrightarrow{\varphi^{\boldsymbol{a}}} & \check{A}[\check{\varphi}_n^{\boldsymbol{a}}]
\end{array}$$

commutes.

Recall that $\check{\Sigma}_n \subseteq \mathcal{C}_n \amalg \mathcal{F}_n = (\mathbb{T} \times [0,1] \times \mathcal{R}_n) \amalg \mathcal{F}_n$. Also, we denote the canonical projection $\mathcal{F}_n \twoheadrightarrow \dot{\mathcal{F}}_n$ by $\sigma \mapsto \dot{\sigma}$.

- For $\varphi^{\boldsymbol{a}}$ as in (16), let $p^{\boldsymbol{a}}$ be the composite

$$\check{T}_n^{\boldsymbol{a}} \cong \mathbb{T} \times [0,1] \times \mathcal{R}_n \to \check{\Sigma}_n \xrightarrow{q} \Sigma_n,$$
$$(z,t,\gamma) \mapsto (z,\lambda(t),\gamma)$$

  where we note that the first map has image in $\check{\Sigma}_n$, so that we an apply the quotient map $q: \check{\Sigma}_n \twoheadrightarrow \Sigma_n$.
- For $\varphi^{\boldsymbol{a}}$ as in (17), let $p^{\boldsymbol{a}}$ be the composite

$$\check{T}_n^{\boldsymbol{a}} \cong \mathbb{T} \times [0,1] \times \mathcal{R}_n \to \mathcal{F}_n \xrightarrow{q} \Sigma_n,$$
$$(z,t,\gamma) \mapsto z \cdot \sigma(t,\gamma) \tag{29}$$

  where σ is a continuous groupoid homomorphism such that $\dot{\sigma}(t,\gamma) = (x(t),\gamma)$. For $x(t) = \theta^l \in \left\{\theta_n^i\right\}$ and $\gamma = (j,i)$, write

$$\sigma(t,\gamma) = [z_{j,i}, \theta^l], \tag{30}$$

  which has to match up with (28).
- For $\varphi^{\boldsymbol{a}}$ as in (18), let $p^{\boldsymbol{a}}$ be the composite

$$\check{T}_n^{\boldsymbol{a}} \cong \mathbb{T} \times \{\theta_{n+1}^j\} \times \mathcal{R}_n \to \check{\Sigma}_n \xrightarrow{q} \Sigma_n,$$
$$(z,\theta_{n+1}^j,\gamma) \mapsto (z,\boldsymbol{t},\gamma).$$

- For $\varphi^{\boldsymbol{a}}$ as in (19), let $p^{\boldsymbol{a}}$ be the composite

$$\check{T}_n^{\boldsymbol{a}} \cong \mathbb{T} \times \{\theta_{n+1}^j\} \times (\dot{\mathcal{F}}_n)_x^x \to \mathcal{F}_n \xrightarrow{q} \Sigma_n,$$
$$(z,\theta_{n+1}^j,\gamma) \mapsto z \cdot \sigma(\gamma),$$

  where $\sigma: (\dot{\mathcal{F}}_n)_x^x \to \mathcal{F}_n$ is a groupoid homomorphism with $\dot{\sigma}(\gamma) = (x,\gamma)$ matching up with σ in (29).
- For $\varphi^{\boldsymbol{a}}$ as in (20), let $p^{\boldsymbol{a}}$ be the composite

$$\check{T}_n^{\boldsymbol{a}} \cong \mathbb{T} \times \{\theta_{n+1}^j\} \times (\dot{\mathcal{F}}_n)_{\theta_n^i}^{\theta_n^i} \to \mathcal{F}_n \xrightarrow{q} \Sigma_n,$$
$$(z,\theta_{n+1}^j,\gamma) \mapsto (z,\theta_n^i,\gamma).$$

- For $\varphi^{\boldsymbol{a}}$ as in (21), let $p^{\boldsymbol{a}}$ be the composite

$$\check{T}_n^{\boldsymbol{a}} \cong \mathbb{T} \times Z_{n+1} \times (\dot{\mathcal{F}}_n)_{\theta_n^i}^{\theta_n^i} \twoheadrightarrow \mathbb{T} \times (\dot{\mathcal{F}}_n)_{\theta_n^i}^{\theta_n^i} \to \mathcal{F}_n \xrightarrow{q} \Sigma_n,$$
$$(z,\gamma) \mapsto z \cdot \sigma(\gamma),$$

where $\sigma : (\dot{\mathcal{F}}_n)_{\theta_n^i}^{\theta_n^i} \to \mathcal{F}_n$ is a groupoid homomorphism with $\dot{\sigma}(\gamma) = (\theta_n^i, \gamma)$ matching up with (28), just as (30).

- For $\varphi^{\boldsymbol{a}}$ as in (22), we have $C_r^*(\check{H}_n^{\boldsymbol{a}}, \check{T}_n^{\boldsymbol{a}}) \cong \big(\sum_i \lambda^*(p_i)\big) \cdot F_{n+1} \cdot \big(\sum_i \lambda^*(p_i)\big)$, where p_i are the line bundles such that $P_n = \sum_i p_i$ (see Sect. 3 and Sect. 6.1), and $p^{\boldsymbol{a}}$ is the composite

$$\begin{aligned} \check{T}_n^{\boldsymbol{a}} \to \mathcal{F}_n &\xrightarrow{q} \Sigma_n, \\ [z, x] &\mapsto [z, \lambda(x)], \end{aligned}$$

with $(z, x) \in \mathbb{T} \times \lambda^{-1}(V_{i,a})$ and $(z, \lambda(x)) \in \mathbb{T} \times V_{i,a}$, where for a given open cover $V_{i,a}$ and trivialization $v_{i,a}$ for $\mathcal{F}_n$, we choose the open cover $\lambda^{-1}(V_{i,a})$ and trivialization $v_{i,a} \circ \lambda$ for $\check{T}_n^{\boldsymbol{a}}$ (see Sect. 6.1).

The homomorphism

$$\coprod_{\boldsymbol{a}} p^{\boldsymbol{a}} : \check{T}_n = \coprod_{\boldsymbol{a}} \check{T}_n^{\boldsymbol{a}} \to \Sigma_n$$

must descend to $p_n : T_n \to \Sigma_n$ because $C_r^*(\coprod_{\boldsymbol{a}} p^{\boldsymbol{a}}) : C_r^*(G_n, \Sigma_n) \to C_r^*(\check{H}_n, \check{T}_n)$, $f \mapsto f \circ (\coprod_{\boldsymbol{a}} p^{\boldsymbol{a}})$ lands in $C_r^*(H_n, T_n)$. Moreover, the homomorphisms Φ_n and $\check{\varphi}_n$ admit similar groupoid models (say $\mathcal{P}_n$ and $\check{p}_n$) as φ_n, so that we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{C}_n \amalg \mathcal{F}_n & \mathcal{T}_n \arrow[l, "\mathcal{P}_n"'] \arrow[r, "\subseteq"'] & \mathcal{C}_{n+1} \amalg \mathcal{F}_{n+1} \\ \check{\Sigma}_n \arrow[u, "\subseteq"'] \arrow[d, two heads] & \check{T}_n \arrow[l, "\check{p}_n"'] \arrow[r, "\subseteq"] \arrow[u, "\subseteq"'] \arrow[d, two heads] & \check{\Sigma}_{n+1} \arrow[u, "\subseteq"] \arrow[d, two heads] \\ \Sigma_n & T_n \arrow[l, "p_n"'] \arrow[r, "\subseteq"] & \Sigma_{n+1}. \end{tikzcd}$$

7 Conclusions

Proofs of Theorems 1.2 and 1.3 All we have to do is to check the conditions in Theorem 1.10, using Proposition 5.4 and the groupoid models in Sect. 6. We treat the unital and stably projectionless cases simultaneously. Given a prescribed Elliott invariant, let A_n and φ_n be as in Sect. 4.2. Consider the groupoid models for A_n and φ_n in Sect. 6. First of all, by construction, (H_n, T_n) is a subgroupoid of (G_{n+1}, Σ_{n+1}) and $H_n \subseteq G_{n+1}$ is open. Let $(i_n, \imath_n)$ be the canonical inclusion. Secondly, p_n is proper because all the $p^{\boldsymbol{a}}$ in Sect. 6.2 are proper (they are closed, and pre-images of points are compact). Thirdly, p_n

is fibrewise bijective because this is true for $\check{p}_n$ and the canonical projections $\check{\Sigma}_n \twoheadrightarrow \Sigma_n$, $\check{T}_n \twoheadrightarrow T_n$. By construction, all the connecting maps φ_n in Sect. 4.2 are of the form $\varphi_n = (\iota_n)_* \circ (p_n)^*$. Thus, by Proposition 5.4, the conditions in Theorem 1.10 are satisfied. Hence $\varinjlim \{B_n; \varphi_n\}$, with B_n as in Sect. 6.1, is a Cartan subalgebra of $\varinjlim \{A_n; \varphi_n\}$, and actually even a C*-diagonal by Remark 5.6 because all G_n are principal.

Remark 7.1 By Remark 5.6, the twisted groupoids (G, Σ) we obtain in the proofs of Theorems 1.2 and 1.3 are given by the Weyl twists described by (23) and (24). Moreover, it is easy to see that for the groupoids in Sect. 6, we have $(\mathcal{C}_n \amalg \mathcal{F}_n)^{(0)} = \check{G}_n^{(0)}$, and since Φ_n, $\check{\varphi}_n$ and φ_n send full elements to full elements, $\dot{\mathcal{T}}_n^{(0)} = (\mathcal{C}_n \amalg \mathcal{F}_n)^{(0)}$, $\check{H}_n^{(0)} = \check{G}_{n+1}^{(0)}$ and $H_n^{(0)} = G_{n+1}^{(0)}$, for all n (by Remark 5.5). So Remark 5.6 tells us that $G^{(0)} \cong \varprojlim \{G_n^{(0)}; \dot{p}_n\}$.

We now turn to the additional statements in Sect. 1. In order to prove Corollaries 1.6 and 1.7 , we need to show the following statement. In both the unital and stably projectionless cases, let $A = C_r^*(G, \Sigma)$, $D = C_0(G^{(0)})$, and $\gamma = \tilde{\gamma}|_T$ be as in Corollaries 1.6 and 1.7 . Let $\mathcal{C}$ be the canonical diagonal subalgebra of the algebra of compact operators $\mathcal{K}$.

Proposition 7.2 *There exists a positive element $a \in D \otimes \mathcal{C} \subseteq A \otimes \mathcal{K}$ such that $d_\bullet(a) = \gamma$.*

Here $d_\bullet(a)$ denotes the function $T \ni \tau \mapsto d_\tau(a)$. For the proof, we need the following

Lemma 7.3 *Given a continuous affine map $g : T \to (0, \infty)$ and $\varepsilon > 0$, there exists $z \in D \otimes D_k \subseteq A \otimes M_k \subseteq A \otimes \mathcal{K}$ with $\|z\| = 1$, $z \geq 0$, $z \in \mathrm{Ped}(A \otimes \mathcal{K})$ such that $g - \varepsilon < d_\bullet(z) < g + \varepsilon$.*

Here D_k is the canonical diagonal subalgebra of M_k.

Proof We treat the unital and stably projectionless cases simultaneously. Let $\hat{F}_n$ be as in Sect. 2.1 and $\hat{F} := \varinjlim \hat{F}_n$. Choose $a \in \hat{F} \otimes \mathcal{K}$ with $a \geq 0$ and $d_\bullet(a) = g$. Then we can choose $b \in \hat{F}_n \otimes M_k$ (for n big enough) with $b \geq 0$, $d_\bullet(b)$ continuous and

$$g - \varepsilon < d_\bullet(b) < g + \varepsilon.$$

Using [1, Theorem 3.1] just as in [37, Proof of (6.2) and (6.3)], choose $c \in D(\hat{F}_n) \otimes D_k$ with $c \geq 0$ such that c and b are Cuntz equivalent, where $D(\hat{F}_n)$ is the canonical diagonal subalgebra of $\hat{F}_n$. Choose $d \in \mathrm{Ped}(A_n \otimes M_k)$ with $d \in D(A_n) \otimes D_k$ such that $(\pi \otimes \mathrm{id})(d) = c$, where $\pi : A_n \twoheadrightarrow F_n \twoheadrightarrow \hat{F}_n$ is the canonical projection. Let z denote the image of d under the canonical map $A_n \otimes M_k \to A \otimes M_k$. Then $z \in D \otimes D_k$. It is now straightforward to check, using the isomorphism $T(A) \cong T(\hat{F})$ from [20, § 13], that z has the desired properties.

Proof of Proposition 7.2 There is a sequence (γ_i) of continuous affine maps $T \to [0, \infty)$ with $\gamma_i \nearrow \gamma - \min(\gamma)$. Choose $\varepsilon_i > 0$ such that $\sum_i \varepsilon_i = \min(\gamma)$. Define $f_i := \gamma_i + \sum_{h=1}^{i-1} \varepsilon_h$. Then $f_i \nearrow \gamma$ and $f_i > 0$. Moreover,

$$f_{i+1} = \gamma_{i+1} + \sum_{h=1}^{i} \varepsilon_h \geq \gamma_i + \left(\sum_{h=1}^{i-1} \varepsilon_h \right) + \varepsilon_i = f_i + \varepsilon_i.$$

Using Lemma 7.3, proceed inductively on i to find $z_i \in D \otimes D_{k(i)}$ such that

$$\left(f_{i+1} - \sum_{h=1}^{i} d_\bullet(z_h) \right) - \varepsilon_{i+1} < d_\bullet(z_{i+1}) < \left(f_{i+1} - \sum_{h=1}^{i} d_\bullet(z_h) \right) + \varepsilon_{i+1}.$$

Note that $f_{i+1} - \sum_{h=1}^{i} d_\bullet(z_h) > 0$ since $\sum_{h=1}^{i} d_\bullet(z_h) < f_i + \varepsilon_i \leq f_{i+1}$.

By construction, we have

$$f_i - \varepsilon_i < \sum_{h=1}^{i} d_\bullet(z_h) < f_i + \varepsilon_i, \text{ so that } \sum_{h=1}^{i} d_\bullet(z_h) \nearrow \gamma.$$

Now set

$$a := \sum\nolimits_{h=1}^{\infty} {}^{\oplus}\, 2^{-h} z_h, \text{ where we put the elements } 2^{-h} z_h \text{ on the diagonal in } D \otimes \mathcal{C}.$$

In this way, we obtain an element $a \in D \otimes \mathcal{C} \subseteq A \otimes \mathcal{K}$ with $d_\bullet(a) = \gamma$.

Proof of Corollaries 1.6 and 1.7 Given $\tilde{\gamma}$ as in Corollaries 1.6 and 1.7, let $\gamma = \tilde{\gamma}|_T$. Using Proposition 7.2, choose a positive element $a \in D \otimes \mathcal{C}$ with $d_\bullet(a) = \gamma$. In the unital case, it is straightforward to check that we can always arrange a to be purely positive. Then it is straightforward to check that $(\overline{a(A \otimes \mathcal{K})a}, \overline{a(D \otimes \mathcal{C})a})$ is a Cartan pair. Hence, by [36, Theorem 5.9], there is a twisted groupoid $(\tilde{G}, \tilde{\Sigma})$ such that $(C_r^*(\tilde{G}, \tilde{\Sigma}), C_0(\tilde{G}^{(0)})) \cong (\overline{a(A \otimes \mathcal{K})a}, \overline{a(D \otimes \mathcal{C})a})$. It is now easy to see (compare also [19, Corollary 6.12]) that $(\tilde{G}, \tilde{\Sigma})$ has all the desired properties.

Proofs of Corollaries 1.8 and 1.9 (i) follows from the observation that we only need the twist if G_0 has torsion. The claims in (ii)–(iv) about subhomogeneous building blocks and their spectra follow immediately from our constructions (see also Remark 3.9). Moreover, the inverse limit description of the unit space in Remark 7.1 and the dimension formula for inverse limits (see for instance [17, Chapter 3, § 5.3, Theorem 22]) imply that $\dim(G^{(0)}) \leq 3$ in (ii), $\dim(G^{(0)}) \leq 2$ in (iii) and $\dim(G^{(0)}) \leq 1$ in (iv) and (v). Since $C_0(G^{(0)})$ is projectionless in Theorem 1.3, we obtain $\dim(G^{(0)}) \neq 0$, which forces $\dim(G^{(0)}) = 1$ in (v), in the situation of Theorem 1.3. In particular, this shows that $\mathcal{W}$ and $\mathcal{Z}_0$ have C*-diagonals with one-dimensional spectra. Similarly, given a groupoid G with $\mathcal{Z} \cong C_r^*(G)$, the only projections in $C(G^{(0)})$ are 0 and 1, so that $\dim(G^{(0)}) \neq 0$ and hence $\dim(G^{(0)}) = 1$. It remains to prove that $\dim(G^{(0)}) \geq 3$ in (ii), $\dim(G^{(0)}) \geq 2$ in (iii) and $\dim(G^{(0)}) \geq 1$ in (iv).

To do so, let us use the same notation as in Sect. 6, and write $X_n := G_n^{(0)}$, $Q_n := \dot{\mathcal{C}}_n^{(0)}$, and $W_n := \dot{\mathcal{F}}_n^{(0)}$. Clearly, Q_n is homotopy equivalent to a finite set of points, so that for any cohomology theory $H^\bullet$ (satisfying the Eilenberg–Steenrod axioms, see [40, Chapter 17]), we have

$$H^\bullet(Q_n) \cong \{0\} \text{ whenever } \bullet \geq 1. \tag{31}$$

Let $P_n := \left\{(t, x) \in Q_n \colon t \in \{0, 1\},\ x \in \mathcal{R}_t^{(0)}\right\}$. Then we have a pushout diagram

$$\begin{tikzcd} P_n \arrow[r] \arrow[d] & W_n \arrow[d] \\ Q_n \arrow[r] & X_n, \end{tikzcd}$$

where $P_n \to W_n$ is induced by $\boldsymbol{b}_t$ and the left vertical arrow is the canonical inclusion. The long exact (Mayer-Vietoris type) sequence attached to the pushout reads

$$\ldots \to H^{\bullet-1}(P_n) \to H^\bullet(X_n) \to H^\bullet(Q_n) \times H^\bullet(W_n) \to H^\bullet(P_n) \to H^{\bullet+1}(X_n) \to \ldots.$$

Since $H^\bullet(P_n) \cong \{0\}$ and $H^\bullet(Q_n) \cong \{0\}$ (see (31)), we deduce that the canonical map $W_n \to X_n$ induces a surjection $H^\bullet(X_n) \to H^\bullet(W_n)$ for $\bullet \geq 1$. Moreover, the map

$$Q_{n+1} \amalg W_{n+1} = (\dot{\mathcal{C}}_{n+1} \amalg \dot{\mathcal{F}}_{n+1})^{(0)} = \check{G}_{n+1}^{(0)} = \check{H}_n^{(0)} \xrightarrow{\hat{p}_n} \hat{G}_n^{(0)} = Q_n \amalg W_n$$

induces for $\bullet \geq 1$ a homomorphism $H^\bullet(\check{p}_n) : H^\bullet(W_n) \to H^\bullet(W_{n+1})$ which fits into the commutative diagram

$$\begin{tikzcd} H^\bullet(X_n) \arrow[r, two heads] \arrow[d, "H^\bullet(p_n)"'] & H^\bullet(W_n) \arrow[d, "H^\bullet(\check{p}_n)"] \\ H^\bullet(X_{n+1}) \arrow[r, two heads] & H^\bullet(W_{n+1}). \end{tikzcd}$$

Thus the canonical maps $W_n \to X_n$ induce for all $\bullet \geq 1$ surjections

$$\check{H}^\bullet(G^{(0)}) \cong \varinjlim \left\{H^\bullet(X_n);\, H^\bullet(p_n)\right\} \twoheadrightarrow \varinjlim \left\{H^\bullet(W_n);\, H^\bullet(\check{p}_n)\right\}.$$

Here $\check{H}^\bullet$ is Čech cohomology, and the first identification follows from the inverse limit description of $G^{(0)}$ in Remark 7.1 and continuity of Čech cohomology. By construction, $W_n = Z_n \times I_n$ for some finite set I_n and Z_n is as in Sect. 4.2. Now it is an immediate consequence of our construction that $\varinjlim \left\{H^\bullet(W_n);\, H^\bullet(\check{p}_n)\right\}$ surjects onto $\mathrm{Tor}(G_1)$ in case (ii) for $\bullet = 3$, $\mathrm{Tor}(G_0)$ in case (iii) for $\bullet = 2$, and G_1 in case (iv) for $\bullet = 1$. Hence it follows that $\check{H}^3(G^{(0)}) \ncong \{0\}$ in case (ii), $\check{H}^2(G^{(0)}) \ncong \{0\}$ in case (iii), and $\check{H}^1(G^{(0)}) \ncong \{0\}$ in case (iv). As cohomological dimension is always a lower bound for covering dimension, this implies $\dim(G^{(0)}) \geq 3$ in case (ii), $\dim(G^{(0)}) \geq 2$ in case (iii), and $\dim(G^{(0)}) \geq 1$ in case (iv), as desired.

8 Examples

Let us describe concrete groupoid models for the Jiang–Su algebra $\mathcal{Z}$, the Razak–Jacelon algebra $\mathcal{W}$ and the stably projectionless version $\mathcal{Z}_0$ of the Jiang–Su algebra as in [19, Definition 7.1]. These C*-algebras can be constructed in a way which fits into the framework of Sect. 4.2, so that our general machinery in Sect. 5 produces groupoid models as in Sect. 6. In the following, we focus on $\mathcal{Z}$.

First we recall the original construction of $\mathcal{Z}$ in [23]. For every $n \in \mathbb{N}$, choose natural numbers p_n and q_n such that they are relatively prime, with $p_n \mid p_{n+1}$ and $q_n \mid q_{n+1}$, such that $\frac{p_{n+1}}{p_n} > 2q_n$ and $\frac{q_{n+1}}{q_n} > 2p_n$. Then $\mathcal{Z} = \varinjlim \{A_n; \varphi_n\}$, where $A_n = \{(f, a) \in C([0,1], E_n) \oplus F_n\colon f(t) = \beta_t(a) \text{ for } t = 0, 1\}$, $E_n = M_{p_n} \otimes M_{q_n}$, $F_n = M_{p_n} \oplus M_{q_n}$, $\beta_0 : M_{p_n} \oplus M_{q_n} \to M_{p_n} \otimes M_{q_n}$, $(x, y) \mapsto x \otimes 1_{q_n}$, $\beta_1 : M_{p_n} \oplus M_{q_n} \to M_{p_n} \otimes M_{q_n}$, $(x, y) \mapsto 1_{p_n} \otimes y$.

To describe φ_n for fixed n, let $d_0 := \frac{p_{n+1}}{p_n}$, $d_1 := \frac{q_{n+1}}{q_n}$, $d := d_0 \cdot d_1$, and write $d = l_0 q_{n+1} + r_0$ with $0 \leq r_0 < q_{n+1}$, $d = l_1 p_{n+1} + r_1$ with $0 \leq r_1 < p_{n+1}$. Note that we must have $d_1 \mid r_0$ and $d_0 \mid r_1$. Then

$$\varphi_n(f) = u_{n+1}^* \cdot (f \circ \lambda_y)_{y \in \mathcal{Y}(n)} \cdot u_{n+1},$$

$$\text{where } \mathcal{Y}(n) = \{1, \ldots, d\} \text{ and } \lambda_y(t) = \begin{cases} \frac{t}{2} & \text{if } 1 \leq y \leq r_0, \\ \frac{1}{2} & \text{if } r_0 < y \leq d - r_1, \\ \frac{t+1}{2} & \text{if } d - r_1 < y \leq d. \end{cases}$$

Here we think of A_n as a subalgebra of $C([0, 1], E_n)$ via the embedding $A_n \hookrightarrow C([0, 1], E_n)$, $(f, a) \mapsto f$.

To construct groupoid models for building blocks and connecting maps, start with a set $\mathcal{X}(1)$ with $p_1 \cdot q_1$ elements, and define recursively $\mathcal{X}(n+1) := \mathcal{X}(n) \times \mathcal{Y}(n)$. Let $\mathcal{R}(n)$ be the full equivalence relation on $\mathcal{X}(n)$. Let $\mathcal{R}(n, p)$ and $\mathcal{R}(n, q)$ be the full equivalence relations on finite sets $\mathcal{X}(n, p)$ and $\mathcal{X}(n, q)$ with p_n and q_n elements. For $t = 0, 1$, let $\rho_{n+1,t}$ be the bijections corresponding to conjugation by $u_{n+1}(t)$, which induce $\sigma_{n,t} : \mathcal{R}(n, p) \times \mathcal{R}(n, q) \cong \mathcal{R}(n)$ corresponding to conjugation by $v_n(t)$ introduced in Remark 4.1. Now set

$$\begin{aligned} \check{G}_n &:= \{(t, \gamma) \in [0, 1] \times \mathcal{R}(n) \colon \gamma\} \in \sigma_{n,0}(\mathcal{R}(n, p) \\ &\qquad \times \mathcal{X}(n, q)) \text{ if } t = 0,\ \gamma \in \sigma_{n,1}(\mathcal{X}(n, p) \times \mathcal{R}(n, q)) \text{ if } t = 1, \\ G_n &:= \check{G}_n/_{\sim} \quad \text{where } \sim \text{ is given by } (0, \sigma_{n,0}(\gamma, y)) \\ &\qquad \sim (0, \sigma_{n,0}(\gamma, y')) \text{ and } (1, \sigma_{n,1}(x, \eta)) \sim (1, \sigma_{n,1}(x', \eta)). \end{aligned}$$

Define $\check{p}_n : \check{H}_n \to \check{G}_n$ as the restriction of $\mathcal{P}_n : \dot{\mathcal{T}}_n := [0, 1] \times \mathcal{R}(n) \times \mathcal{Y}(n) \to [0, 1] \times \mathcal{R}(n)$, $(t, \gamma, y) \mapsto (\lambda_y(t), \gamma)$ to $\check{H}_n := \mathcal{P}_n^{-1}(\check{G}_n)$. Set $H_n := \check{H}_n/_{\sim}$ where $\sim$ is the equivalence relation defining $G_{n+1} = \check{G}_{n+1}/_{\sim}$. The map $\check{p}_n$ descends to $p_n : H_n \to G_n$. The groupoid G with $\mathcal{Z} \cong C^*_r(G)$ is now given by (23) and (24). As explained in Remark 7.1, its unit space $X := G^{(0)}$ is given by $X \cong \varinjlim \{X_n; p_n\}$, where $X_n = G_n^{(0)}$.

To further describe X, let $\boldsymbol{p}_n$ be the set-valued function on [0, 1] defined by $\boldsymbol{p}_n(s) := \{\lambda_y(s) \colon y \in \mathcal{Y}(n)\}$. We can form the inverse limit

$$\boldsymbol{X} := \varprojlim \{[0, 1]; \boldsymbol{p}_n\} := \left\{(s_n) \in \prod_{n=1}^{\infty} [0, 1] : \ s_n \in \boldsymbol{p}_n(s_{n+1})\right\}.$$

as in [21, § 2.2]. It is easy to see that $X_n \mapsto [0, 1]$, $[(t, x)] \mapsto t$ gives rise to a continuous surjection $X \twoheadrightarrow \boldsymbol{X}$ whose fibres are all homeomorphic to the Cantor space. Moreover, $\boldsymbol{X}$ is connected and locally path connected. The space X itself is also connected. This follows easily from the construction itself (basically from $\gcd(p_n, q_n) = 1$) and also from abstract reasons because $\mathcal{Z}$ is unital projectionless. In addition, it is straightforward to check that for particular choices for $\rho_{n,t}$ and hence $\sigma_{n,t}$, our space X becomes locally path connected as well. In that case, it is a one-dimensional Peano continuum.

Every X_n is homotopy equivalent to a finite bouquet of circles. It is then easy to compute K-theory and Čech (co)homology:

$$K_0(C(X)) = \mathbb{Z}[1], \quad K_1(C(X)) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}; \tag{32}$$

$$\check{H}^{\bullet}(X) \cong \begin{cases} \mathbb{Z} & \text{for } \bullet = 0, \\ \bigoplus_{i=1}^{\infty} \mathbb{Z} & \text{for } \bullet = 1, \\ \{0\} & \text{for } \bullet \geq 2, \end{cases} \quad \text{and} \quad \check{H}_{\bullet}(X) \cong \begin{cases} \mathbb{Z} & \text{for } \bullet = 0, \\ \prod_{i=1}^{\infty} \mathbb{Z} & \text{for } \bullet = 1, \\ \{0\} & \text{for } \bullet \geq 2. \end{cases} \tag{33}$$

It follows that for choices of $\rho_{n,t}$ and $\sigma_{n,t}$ such that X is locally path connected, X must be shape equivalent to the Hawaiian earring by [7]. In particular, its first Čech homotopy group is isomorphic to the one of the Hawaiian earring, which is the canonical projective limit of non-abelian free groups of finite rank. Moreover, by [9], the singular homology $H_1(X)$ coincides with the singular homology of the Hawaiian earring, which is described in [10]. We refer the reader to [33] for more information about shape theory, which is the natural framework to study our space since it is constructed as an inverse limit.

Now we turn to $\mathcal{W}$. Recall the construction in [22]. For every $n \in \mathbb{N}$, choose integers $a_n, b_n \geq 1$ with $a_{n+1} = 2a_n + 1$, $b_{n+1} = a_{n+1} \cdot b_n$. Then $\mathcal{W} = \varinjlim \{A_n; \varphi_n\}$, where $A_n = \{(f, a) \in C([0,1], E_n) \oplus F_n \colon f\}(t) = \beta_t(a)$ for $t = 0, 1$, $E_n = M_{(a_n+1)\cdot b_n}$, $F_n = M_{b_n}$, with

$$\beta_0 : M_{b_n} \to M_{(a_n+1)\cdot b_n}, \; x \mapsto \begin{pmatrix} x & & & \\ & \ddots & & \\ & & x & \\ & & & 0 \end{pmatrix}$$

$$\text{and} \;\; \beta_1 : M_{b_n} \to M_{(a_n+1)\cdot b_n}, \; x \mapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix},$$

where we put a_n copies of x on the diagonal for β_0, and $a_n + 1$ copies of x on the diagonal for β_1. To describe φ_n for fixed n, let $d := 2a_{n+1}$. Then

$$\varphi_n(f) = u_{n+1}^* \cdot (f \circ \lambda_y)_{y \in \mathcal{Y}(n)} \cdot u_{n+1},$$

$$\text{where } \mathcal{Y}(n) = \{1, \ldots, d\} \text{ and } \lambda_y(t) = \begin{cases} \frac{t}{2} & \text{if } 1 \leq y \leq a_{n+1}, \\ \frac{1}{2} & \text{if } y = a_{n+1} + 1, \\ \frac{t+1}{2} & \text{if } a_{n+1} + 1 < y \leq d. \end{cases}$$

Here we think of A_n as a subalgebra of $C([0,1], E_n)$ via the embedding $A_n \hookrightarrow C([0,1], E_n)$, $(f, a) \mapsto f$.

To construct groupoid models, start with a set $\mathcal{X}(1)$ with $(a_1+1)\cdot b_1$ elements, and define recursively $\mathcal{X}(n+1) := \mathcal{X}(n)\times\mathcal{Y}(n)$. Let $\mathcal{R}(n)$ be the full equivalence relation on $\mathcal{X}(n)$. Let $\mathcal{R}(n,a)$ and $\mathcal{R}(n,b)$ be the full equivalence relations on finite sets $\mathcal{X}(n,a)$ and $\mathcal{X}(n,b)$ with a_n+1 and b_n elements, and let $\mathcal{X}'(n,a) \subseteq \mathcal{X}(n,a)$ be a subset with a_n elements (corresponding to the multiplicity of β_0). For $t = 0, 1$, let $\rho_{n+1,t}$ be the bijections corresponding to conjugation by $u_{n+1}(t)$, which induce $\sigma_{n,t} : \mathcal{R}(n,a)\times\mathcal{R}(n,b) \cong \mathcal{R}(n)$ corresponding to conjugation by $v_n(t)$ introduced in Remark 4.1. Set

$$\begin{aligned}&\check{G}_n := \{(t,\gamma)\in[0,1]\times\mathcal{R}(n)\colon \gamma\} \in \sigma_{n,0}(\mathcal{X}'(n,a)\times\mathcal{R}(n,b)) \text{ if } t=0,\\ &\quad \gamma\in\sigma_{n,1}(\mathcal{X}(n,a)\times\mathcal{R}(n,b)) \text{ if } t=1,\\ &G_n := \check{G}_n/_\sim \quad \text{where } \sim \text{ is given by } (t,\sigma_{n,t}(x,\gamma))\sim(t',\sigma_{n,t'}(x',\gamma)).\end{aligned}$$

Now define $\check{p}_n : \check{H}_n \to \check{G}_n$ as the restriction of $\mathcal{P}_n : \dot{\mathcal{T}}_n := [0,1]\times\mathcal{R}(n)\times\mathcal{Y}(n) \to [0,1]\times\mathcal{R}(n)$, $(t,\gamma,y)\mapsto(\lambda_y(t),\gamma)$ to $\check{H}_n := \mathcal{P}_n^{-1}(\check{G}_n)$. Set $H_n := \check{H}_n/_\sim$ where $\sim$ is the equivalence relation defining $G_{n+1} = \check{G}_{n+1}/_\sim$. The map $\check{p}_n$ descends to $p_n : H_n \to G_n$. The groupoid G with $\mathcal{W}\cong C_r^*(G)$ is now given by (23) and (24). As explained in Remark 7.1, its unit space $X := G^{(0)}$ is given by $X \cong \varprojlim\{X_n; p_n\}$, where $X_n = G_n^{(0)}$. As in the case of $\mathcal{Z}$, X surjects continuously onto $\varprojlim\{\mathbb{T}; \boldsymbol{p}_n\}$ with Cantor space fibres, where $\mathbb{T} = [0,1]/_{0\sim1}$ and $\boldsymbol{p}_n([s]) = \{[\lambda_y(s)]\colon y\in\mathcal{Y}(n)\}$. However, it is easy to see that (at least for some choices of $\rho_{n,t}$ and $\sigma_{n,t}$), X will not be connected, though its connected components all have to be non-compact.

Now let us treat $\mathcal{Z}_0$. For each $m\in\mathbb{N}$, choose integers $a_n, b_n, h_n \geq 1$ with $a_{n+1} = ((2a_n+2)h_n+1)\cdot a_n$, $b_{n+1} = ((2a_n+2)h_n+1)\cdot b_n$. Let $A_n = \{(f,a)\in C([0,1],E_n)\oplus F_n\colon f(t) = \beta_t(a) \text{ for } t=0,1\}$, with $E_n = M_{(2a_n+2)\cdot b_n}$, $F_n = M_{b_n}\oplus M_{b_n}$,

$$\beta_0 : F_n \to E_n,\ (x,y) \mapsto \begin{pmatrix} x &&&&&& \\ & \ddots &&&&& \\ && x &&&& \\ &&& 0 &&& \\ &&&& y && \\ &&&&& \ddots & \\ &&&&&& y \\ &&&&&&& 0 \end{pmatrix},$$

$$\text{and} \quad \beta_1 : F_n \to E_n,\ (x,y) \mapsto \begin{pmatrix} x &&&&& \\ & \ddots &&&& \\ && x &&& \\ &&& y && \\ &&&& \ddots & \\ &&&&& y \end{pmatrix},$$

where we put a_n copies of x and y on the diagonal for β_0, and a_n+1 copies of x and y on the diagonal for β_1. To describe the connecting maps $\varphi_n : A_n \to$

A_{n+1}, fix n and let $d := (2a_{n+1}+2)h_n+(2a_nh_n+1)$. Then $(2a_{n+1}+2)\cdot b_{n+1} = d\cdot(2a_n+2)\cdot b_n$. It is now easy to see that for suitable choices of unitaries u_{n+1}, whose values at 0 and 1 are permutation matrices, we obtain a homomorphism $\varphi_n : A_n \to A_{n+1}$ by setting

$$\varphi_n(f) := u_{n+1}^* \cdot (f \circ \lambda_y)_{y\in\mathcal{Y}(n)} \cdot u_{n+1}, \text{ for } \mathcal{Y}(n) = \{1, \ldots, d\}, \lambda_y(t) = \begin{cases} \frac{t}{2} & \text{if } 1 \leq y \leq 2a_kh_k + 2h_k + 1, \\ \frac{1}{2} & \text{if } 2a_kh_k + 2h_k + 1 < y \leq (2a_{k+1} + 2)h_k, \\ \frac{t+1}{2} & \text{if } (2a_{k+1} + 2)h_k < y \leq d. \end{cases}$$

As above, we think of A_n as a subalgebra of $C([0, 1], E_n)$ via $A_n \hookrightarrow C([0, 1], E_n)$, $(f, a) \mapsto f$. Now arguments similar to those in [22,23] show that $\varinjlim \{A_n; \varphi_n\}$ has the same Elliott invariant as $\mathcal{Z}_0$, so that $\mathcal{Z}_0 \cong \varinjlim \{A_n; \varphi_n\}$ by [37, Corollary 6.2.4] (see also [19, Theorem 12.2]).

To construct groupoid models, start with a set $\mathcal{X}(1)$ with $(2a_1 + 2) \cdot b_1$ elements, and define recursively $\mathcal{X}(n + 1) := \mathcal{X}(n) \times \mathcal{Y}(n)$. Let $\mathcal{R}(n)$ be the full equivalence relation on $\mathcal{X}(n)$. Let $\mathcal{R}(n, a, 1)$, $\mathcal{R}(n, a, 2)$, $\mathcal{R}(n, b, 1)$ and $\mathcal{R}(n, b, 2)$ be full equivalence relations on finite sets $\mathcal{X}(n, a, 1)$, $\mathcal{X}(n, a, 2)$, $\mathcal{X}(n, b, 1)$ and $\mathcal{X}(n, b, 2)$ with a_n+1, a_n+1, b_n and b_n elements, respectively. Let $\mathcal{X}_0(n, a, 1) \subseteq \mathcal{X}(n, a, 1)$ and $\mathcal{X}_0(n, a, 2) \subseteq \mathcal{X}(n, a, 2)$ be subsets with a_n elements (corresponding to the multiplicities of β_0), and set $\mathcal{X}_1(n, a, \bullet) := \mathcal{X}(n, a, \bullet)$. For $t = 0, 1$, let $\rho_{n+1,t}$ be the bijections corresponding to conjugation by $u_{n+1}(t)$, which induce $\sigma_{n,t} : \mathcal{R}(n, a, 1) \times \mathcal{R}(n, b, 1) \amalg \mathcal{R}(n, a, 2) \times \mathcal{R}(n, b, 2) \cong \mathcal{R}(n)$ corresponding to conjugation by $v_n(t)$ introduced in Remark 4.1. Set

$$\begin{aligned} \check{G}_n &:= \{(t, \gamma) \in [0, 1] \times \mathcal{R}(n) \colon \gamma\} \in \sigma_{n,t}(\mathcal{X}_t(n, a, 1) \\ &\quad \times\mathcal{R}(n, b, 1) \amalg \mathcal{X}_t(n, a, 2)\mathcal{R}(n, b, 2)) \text{ if } t \in \{0, 1\}, \\ G_n &:= \check{G}_n/_\sim \quad \text{where } \sim \text{ is given by } (t, \sigma_{n,t}(x, \gamma)) \sim (t', \sigma_{n,t'}(x', \gamma)). \end{aligned}$$

Now define $\check{p}_n : \check{H}_n \to \check{G}_n$ as the restriction of $\mathcal{P}_n : \dot{\mathcal{T}}_n := [0, 1] \times \mathcal{R}(n) \times \mathcal{Y}(n) \to [0, 1] \times \mathcal{R}(n)$, $(t, \gamma, y) \mapsto (\lambda_y(t), \gamma)$ to $\check{H}_n := \mathcal{P}_n^{-1}(\check{G}_n)$. Set $H_n := \check{H}_n/_\sim$ where $\sim$ is the equivalence relation defining $G_{n+1} = \check{G}_{n+1}/_\sim$. The map $\check{p}_n$ descends to $p_n : H_n \to G_n$. The groupoid G with $\mathcal{Z}_0 \cong C_r^*(G)$ is now given by (23) and (24). As explained in Remark 7.1, its unit space $X := G^{(0)}$ is given by $X \cong \varprojlim \{X_n; p_n\}$, where $X_n = G_n^{(0)}$. As for $\mathcal{W}$, X surjects continuously onto $\varprojlim \left\{\mathbb{T}; \boldsymbol{p}_n\right\}$ with Cantor space fibres, where $\mathbb{T} = [0, 1]/_{0\sim 1}$ and $\boldsymbol{p}_n([s]) = \left\{[\lambda_y(s)] \colon y \in \mathcal{Y}(n)\right\}$. However, it is easy to see that (at least for some choices of $\rho_{n,t}$ and $\sigma_{n,t}$), X will not be connected, though its connected components all have to be non-compact.

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