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The Fubini Product and Its Applications

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Abstract
The Fubini product of operator spaces provides a powerful tool for analysing properties of tensor products. In this paper, we apply the theory of Fubini products to the problem of computing invariant parts of dynamical systems. In particular, we study the invariant translation approximation property of discrete groups.

Keywords Fubini product · Slice map property · Invariant translation approximation property

Mathematics Subject Classification Primary 46B28; Secondary 46L07 · 47L25 · 20F65

1 Introduction

The Fubini product of C^*-algebras was first defined and studied by Tomiyama [20–23] and Wassermann [24]. It has been used in the study of operator algebras and operator spaces for a long time. See, for instance, [1,3,10,11,13–15].

In this paper, we use old and new results about Fubini products for the study of the invariant translation approximation property. In Sect. 2, we begin with a brief survey of the theory of Fubini products and its applications. Most of these results appear scattered

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in numerous articles, most notably [6,9,10,12,15,21,24]. After briefly recalling some definitions concerning operator spaces in Sect. 2.1, we define the Fubini product and prove its fundamental properties regarding functoriality, intersections, kernels, relative commutants, invariant elements, combinations in Sects. 2.2–2.8. In Sect. 2.9, we review the relation between the operator approximation property and the slice map property.

In Sect. 3, we apply the results in Sect. 2 to the study of groups with the invariant translation approximation property (ITAP) of Roe [17, Section 11.5.3]. In Sect. 3.1, we recall the definition of the uniform Roe algebra. In Sect. 3.2, we analyse the uniform Roe algebra of a product space. Finally, in Sect. 3.3, we study the ITAP of product groups. We show that for countable discrete groups G and H, if G has the approximation property (AP) of Haagerup–Kraus [6], the product $G \times H$ has the ITAP if and only if H has the ITAP.

Finally, in Sect. 4, we study the crossed product version of the Fubini product.

2 The Fubini Product

In this section, we recall the *Fubini product* and prove its fundamental properties. We study intersections, kernels, relative commutants, invariant elements, combinations in terms of the *slice map property* in Sects. 2.2–2.8. In Sect. 2.9, we review the relation between the operator approximation property and the slice map property.

2.1 Operator Spaces

For the sake of completeness, we start with some notations, definitions and results concerning operator spaces and their tensor products. We omit the standard proofs (see [4,16] for a complete treatment).

Let $\mathscr{H}$ be a Hilbert space and let $B(\mathscr{H})$ denote the Banach space of bounded linear operators on $\mathscr{H}$. The ideal of compact operators on $\mathscr{H}$ is denoted $K(\mathscr{H}) \subseteq B(\mathscr{H})$.

Recall that an *operator space* on $\mathscr{H}$ is a closed subspace of $B(\mathscr{H})$. For an operator space A and a subspace $S \subseteq A$, we write $\overline{S}$ for the norm closure of S in A.

For $a \in B(\mathscr{H})$ and $b \in B(\mathscr{K})$, we write $a \otimes b$ for the corresponding element in $B(\mathscr{H} \otimes \mathscr{K})$. For subsets $A \subseteq B(\mathscr{H})$ and $B \subseteq B(\mathscr{K})$, we write $A \times B$ for the subset $\{a \otimes b \mid a \in A, b \in B\} \subseteq B(\mathscr{H} \otimes \mathscr{K})$ and $A \odot B$ for the linear span of $A \times B$.

Definition 2.1 Let $A \subseteq B(\mathscr{H})$ and $B \subseteq B(\mathscr{K})$ be operator spaces. We define the *tensor product* $A \otimes B := \overline{A \odot B}$ as the norm closure of $A \odot B$ or, equivalently, the closed linear span of $A \times B$ in $B(\mathscr{H} \otimes \mathscr{K})$.

This is called the spatial tensor product and the norm is well defined. The following property is often used implicitly.

Lemma 2.2 ([20, Lemma 3]) *Suppose S, $A \subseteq B(\mathscr{H})$ and T, $B \subseteq B(\mathscr{K})$. If $S \otimes T \subseteq A \otimes B$, then $S \subseteq A$ and $T \subseteq B$.*

We say that a linear map $\phi \colon A \to B$ of operator spaces is *completely bounded* if the algebraic tensor product map $\phi \odot \mathrm{id}_D \colon A \odot D \to B \odot D$ is bounded for all operator spaces D. We write $\phi\otimes\mathrm{id}_D$ for the extension $A\otimes D\to B\otimes D$. Completely contractive and completely isometric maps are defined similarly. The *completely bounded norm* of a completely bounded map $\phi\colon A\to B$ is defined as

$$\|\phi\|_{\mathrm{cb}} := \left\|\phi\otimes\mathrm{id}_{K(\mathscr{H})}\colon A\otimes K(\mathscr{H})\to B\otimes K(\mathscr{H})\right\|,$$

where $\mathscr{H}$ is a separable infinite-dimensional Hilbert space. This definition is justified as follows. For any operator space D, we have $\|\phi\otimes\mathrm{id}_D\|\leq\|\phi\|_{\mathrm{cb}}$. For an operator space A, we write A^* for the space of bounded linear functionals on A. We note that bounded linear functionals are automatically completely bounded.

2.2 The Fubini Product

We recall the Fubini product and the slice map property. See [10,21,24] for more details.

Definition 2.3 Let $S\subseteq A$ and $T\subseteq B$ be operator spaces. The *Fubini product* $F(S,T,A\otimes B)$ of S and T in $A\otimes B$ is defined as the set of all $x\in A\otimes B$ such that $(\phi\otimes\mathrm{id}_B)(x)\in T$ for all $\phi\in A^*$ and $(\mathrm{id}_A\otimes\psi)(x)\in S$ for all $\psi\in B^*$.

Remark 2.4 We have

$$S\otimes T\subseteq F(S,T,A\otimes B)\subseteq A\otimes B.$$

Lemma 2.5 *Let K,L be subspaces of A^*,B^*, respectively, such that the closed unit ball of K and L is weak-$*$-dense in the unit balls of A^* and B^*, respectively. Then for any $S\subseteq A$, $T\subseteq B$ operator spaces the Fubini product $F(S,T,A\otimes B)$ equals the set*

$$\{x\in A\otimes B \mid (\phi\otimes\mathrm{id}_B)(x)\in T \text{ and } (\mathrm{id}_A\otimes\psi)(x)\in S \text{ for all } \phi\in K,\psi\in L\}.$$

The assumptions are, for instance, satisfied if K and L are the set of normal linear functionals in faithful representations of A and B.

Proof Suppose that $(\phi\otimes\mathrm{id}_B)(x)\in T$ for all $\phi\in L$. We need to show that $(\phi\otimes\mathrm{id}_B)(x)\in T$ for all $\phi\in A^*$. Let $\phi\in A^*$ and let $(\phi_n)\subseteq L$ be a bounded sequence (or net) converging pointwise (i.e. in the weak-$*$-sense) to ϕ. Then $(\phi_n\otimes\mathrm{id}_B)(z)\to(\phi\otimes\mathrm{id}_B)(z)$ for all $z\in A\odot B$, and using norm boundedness of $\phi_n\otimes\mathrm{id}_B$, an $\varepsilon/3$-argument shows that the same holds for $z\in A\otimes B$. Since $(\phi_n\otimes\mathrm{id}_B)(x)\in T$ by assumption, we conclude that $(\phi\otimes\mathrm{id}_B)(x)\in T$. □

Definition 2.6 We say that $(S,T,A\otimes B)$ has the *slice map property* if

$$F(S,T,A\otimes B)=S\otimes T.$$

2.3 Functoriality

The Fubini product enjoys functoriality with respect to completely bounded maps.

Lemma 2.7 *For $i = 1, 2$, let $S_i \subseteq A_i$ and $T_i \subseteq B_i$ be operator spaces. Suppose that completely bounded maps $\sigma\colon A_1 \to A_2$ and $\tau\colon B_1 \to B_2$ satisfy $\sigma(S_1) \subseteq S_2$ and $\tau(T_1) \subseteq T_2$. Then*

$$(\sigma \otimes \tau)(F(S_1, T_1, A_1 \otimes B_1)) \subseteq F(S_2, T_2, A_2 \otimes B_2).$$

Proof Let $x \in F(S_1, T_1, A_1 \otimes B_1)$. Let $\phi_2 \in A_2^*$ and let $\phi_1 := \phi_2 \circ \sigma \in A_1^*$. Then $(\phi_1 \otimes \mathrm{id}_{B_1})(x) \in T_1$, thus

$$(\phi_2 \otimes \mathrm{id}_{B_2})[(\sigma \otimes \tau)(x)] = \tau[(\phi_1 \otimes \mathrm{id}_{B_1})(x)]$$

belongs to T_2. Similarly for $\psi_2 \in B_2^*$, we have $(\mathrm{id}_{A_2} \otimes \psi_2)[(\sigma \otimes \tau)(x)] \in S_2$. Thus $(\sigma \otimes \tau)(x) \in F(S_2, T_2, A_2 \otimes B_2)$. □

Lemma 2.8 *Let $S \subseteq A$ and $T \subseteq B$ be operator spaces. Let $\sigma\colon A \to A$ and $\tau\colon B \to B$ be completely bounded maps. If σ restricts to the identity on S and τ on T, then $\sigma \otimes \tau$ restricts to the identity on $F(S, T, A \otimes B)$.*

Proof For $x \in F(S, T, A \otimes B)$ and $\phi \in A^*$ and $\psi \in B^*$, we have

$$\begin{aligned}\langle \phi \otimes \psi, (\sigma \otimes \tau)(x)\rangle &= \langle \phi, \sigma[(\mathrm{id}_A \otimes (\psi \circ \tau))(x)]\rangle \\ &= \langle \phi, (\mathrm{id}_A \otimes (\psi \circ \tau))(x)\rangle \\ &= \langle \psi, \tau[(\phi \otimes \mathrm{id}_B)(x)]\rangle \\ &= \langle \phi \otimes \psi, x\rangle.\end{aligned}$$

Thus $(\sigma \otimes \tau)(x) = x$. □

This implies the following corollaries.

Corollary 2.9 (cf. [10, Lemma 2]) *For $i = 1, 2$, let $S \subseteq A_i$ and $T \subseteq B_i$ be operator spaces. Suppose that completely bounded maps $\sigma_1\colon A_1 \to A_2$, $\sigma_2\colon A_2 \to A_1$ and $\tau_1\colon B_1 \to B_2$, $\tau_2\colon B_2 \to B_1$ satisfy, for $i = 1, 2$, $\sigma_i(s) = s$ for $s \in S$ and $\tau_i(t) = t$ for $t \in T$, then $\sigma_1 \otimes \tau_1$ restricts to an isomorphism*

$$F(S, T, A_1 \otimes B_1) \cong F(S, T, A_2 \otimes B_2).$$

Corollary 2.10 (cf. [21, Proposition 3.7]) *Let $S \subseteq A$ and $T \subseteq B$ be operator spaces. If there exist completely bounded projections $A \to S$ and $B \to T$, then $(S, T, A \otimes B)$ has the slice map property:*

$$F(S, T, A \otimes B) = S \otimes T.$$

2.4 Intersections

The Fubini product is compatible with intersections of operator spaces.

Lemma 2.11 *For operator spaces $S \subseteq A$ and $T \subseteq B$, we have*

$$F(S, T, A \otimes B) = F(A, T, A \otimes B) \cap F(S, B, A \otimes B).$$

More generally, for families of operator spaces $\{S_\alpha \subseteq A\}$ and $\{T_\beta \subseteq B\}$, we have

$$F(\cap_\alpha S_\alpha, \cap_\beta T_\beta, A \otimes B) = \cap_{\alpha,\beta} F(S_\alpha, T_\beta, A \otimes B).$$

Proof Clear from the definitions. □

As a corollary, we express the compatibility between intersections and tensor products in terms of a slice map property.

Corollary 2.12 (cf. [24, Corollary 5]) *Let $S_1, S_2 \subseteq A$ and $T_1, T_2 \subseteq B$ be operator spaces. If $(S_1 \cap S_2, T_1 \cap T_2, A \otimes B)$ has the slice map property, then*

$$(S_1 \cap S_2) \otimes (T_1 \cap T_2) = (S_1 \otimes T_1) \cap (S_2 \otimes T_2).$$

Proof We have

$$\begin{aligned}(S_1 \cap S_2) \otimes (T_1 \cap T_2) &\subseteq (S_1 \otimes T_1) \cap (S_2 \otimes T_2) \\ &\subseteq F(S_1, T_1, A \otimes B) \cap F(S_2, T_2, A \otimes B) \\ &= F(S_1 \cap S_2, T_1 \cap T_2, A \otimes B).\end{aligned}$$

□

2.5 Kernels

The Fubini product is compatible with kernels of completely bounded maps.

Proposition 2.13 *Let A, B and D be operator spaces and let $\tau : B \to D$ be a completely bounded map. Then*

$$F(A, \ker(\tau), A \otimes B) = \ker(\mathrm{id}_A \otimes \tau).$$

Proof Let $x \in A \otimes B$. For $\phi \in A^*$, we have

$$\tau[(\phi \otimes \mathrm{id}_B)(x)] = (\phi \otimes \mathrm{id}_D)[(\mathrm{id}_A \otimes \tau)(x)].$$

Thus $x \in F(A, \ker(\tau), A \otimes B)$ iff $x \in \ker(\mathrm{id}_A \otimes \tau)$. □

Corollary 2.14 (cf. [21, Corollary 1]) *Let A, B and D be operator spaces and let $\tau\colon B \to D$ be a completely bounded map. Then the equality*

$$A \otimes \ker(\tau) = \ker(\mathrm{id}_A \otimes \tau)$$

holds if and only if the triple $(A, \ker(\tau), A \otimes B)$ has the slice map property.

Theorem 2.15 *Let A, B, C and D be operator spaces and let $\{\sigma_\alpha\colon A \to C\}$ and $\{\tau_\beta\colon B \to D\}$ be families of completely bounded maps. Then*

$$F(\cap_\alpha \ker(\sigma_\alpha), \cap_\beta \ker(\tau_\beta), A \otimes B) = (\cap_\alpha \ker(\sigma_\alpha \otimes \mathrm{id}_B)) \cap (\cap_\beta \ker(\mathrm{id}_A \otimes \tau_\beta)).$$

Proof Follows from Lemma 2.11 and Proposition 2.13. □

2.6 Relative Commutants

For operators $a, b \in B(\mathscr{H})$, we write $[a, b] = ab - ba \in B(H)$ for the commutator. For an operator space $S \subseteq B(\mathscr{H})$, we write

$$S' := \{b \in B(\mathscr{H}) \mid [b, s] = 0 \text{ for all } s \in S\}$$

for the commutant of S in $B(\mathscr{H})$. As a corollary of Sect. 2.5, we obtain the following.

Proposition 2.16 *Let S, $A \subseteq B(\mathscr{H})$ and T, $B \subseteq B(\mathscr{K})$ be operator spaces. Then we have*

$$F(S' \cap A, T' \cap B, A \otimes B) = (S \otimes \mathbb{C}1_{B(\mathscr{K})} + \mathbb{C}1_{B(\mathscr{H})} \otimes T)' \cap (A \otimes B).$$

Proof For $s \in S$, let $\sigma_s := [-, s]\colon A \to B(\mathscr{H})$. Then σ_s is completely bounded and

$$\sigma_s \otimes \mathrm{id}_B = [-, s \otimes 1_{B(\mathscr{K})}]\colon A \otimes B \to B(\mathscr{H}) \otimes B.$$

Moreover, we have

$$\cap_{s \in S} \ker(\sigma_s) = S' \cap A \quad \text{and}$$
$$\cap_{s \in S} \ker(\sigma_s \otimes \mathrm{id}_B) = (S \otimes \mathbb{C}1_{B(\mathscr{K})})' \cap (A \otimes B).$$

Similarly, $\tau_t := [-, t]\colon B \to B(\mathscr{K})$, $t \in T$, are completely bounded and

$$\cap_{t \in T} \ker(\tau_t) = T' \cap B \quad \text{and}$$
$$\cap_{t \in T} \ker(\mathrm{id}_A \otimes \tau_t) = (\mathbb{C}1_{B(\mathscr{H})} \otimes T)' \cap (A \otimes B).$$

Thus Theorem 2.15 completes the proof. □

This proves the following corollaries.

Corollary 2.17 (cf. [9, Theorem 1] and [21, Corollary 2]) *Let* S, $A \subseteq B(\mathscr{H})$ *and* T, $B \subseteq B(\mathscr{K})$ *and suppose* $1_{B(\mathscr{H})} \in S$ *and* $1_{B(\mathscr{K})} \in T$. *Then the equality*

$$(S' \cap A) \otimes (T' \cap B) = (S \otimes T)' \cap (A \otimes B)$$

holds if and only if the triple $(S' \cap A, T' \cap B, A \otimes B)$ *has the slice map property.*

Proof Follows from Proposition 2.16, since under the unitality conditions, we have

$$(S \otimes T)' = (S \otimes \mathbb{C}1_{B(\mathscr{K})} + \mathbb{C}1_{B(\mathscr{H})} \otimes T)'.$$

□

Corollary 2.18 (cf. [9, Corollary 1]) *Let* A *and* B *be* C^**-algebras. Then*

$$Z(A) \otimes Z(B) \cong Z(A \otimes B),$$

where Z *denotes the centre.*

Proof Follows from Corollary 2.17 and the slice map property for abelian algebras discussed in Example 2.30 in Sect. 2.9. □

2.7 Invariant Elements

Now we turn our attention to dynamical systems and invariant elements. This subsection will play an important role in the study of the invariant translation approximation property in Sect. 3.3.

Let A be an operator space equipped with an action of a discrete group G by completely bounded maps. We write $A^G := \{a \in A \mid ga = g \text{ for all } g \in G\}$ for the invariant part.

Proposition 2.19 *Let* A *and* B *be operator spaces. Suppose that group* G *acts on* A *and group* H *acts on* B *by completely bounded maps. Then* $G \times H$ *acts on* $A \otimes B$ *by completely bounded maps and we have*

$$F(A^G, B^H, A \otimes B) = (A \otimes B)^{G \times H}.$$

Proof For $g \in G$, let $\sigma_g := (\mathrm{id}_A - g)\colon A \to A$. Then σ_g is completely bounded and

$$\sigma_g \otimes \mathrm{id}_B = (\mathrm{id}_{A\otimes B} - 1_H \times g)\colon A \otimes B \to A \otimes B.$$

Moreover, we have

$$\cap_{g\in G} \ker(\sigma_g) = A^G$$
$$\cap_{g\in G} \ker(\sigma_g \otimes \mathrm{id}_B) = (A \otimes B)^{G\times\{1_H\}}.$$

Similarly, $\tau_h := (\mathrm{id}_B - h)\colon B \to B$, $h \in H$, are completely bounded and

$$\cap_{h\in H} \ker(\tau_h) = B^H$$
$$\cap_{h\in H} \ker(\mathrm{id}_A \otimes \tau_h) = (A \otimes B)^{\{1_G\}\times H}.$$

Now Theorem 2.15 completes the proof, since

$$(A \otimes B)^{G\times H} = (A \otimes B)^{G\times\{1_H\}} \cap (A \otimes B)^{\{1_G\}\times H}.$$

□

Corollary 2.20 (cf. [24, Corollary 7]) *Let A and B be operator spaces. Suppose that group G acts on A and group H acts on B by completely bounded maps. Then the equality*

$$A^G \otimes B^H = (A \otimes B)^{G\times H}$$

holds if and only if the triple $(A^G, B^H, A \otimes B)$ has the slice map property.

2.8 Combinations

Sometimes it is useful to be able to combine slice map properties.

Lemma 2.21 *Let $S \subseteq A$ and $T \subseteq B$ be operator spaces. Then*

$$F(S, T, A \otimes T) = (A \otimes T) \cap F(S, B, A \otimes B).$$

Proof Clear since any element of T^* extends to an element of B^*. □

Lemma 2.22 ([12, Lemma 2.6]) *Let $S \subseteq A$ and $T \subseteq B$ be operator spaces. If $(A, T, A \otimes B)$ has the slice map property, then*

$$F(S, T, A \otimes T) = F(S, T, A \otimes B)$$

Proof We have

$$\begin{aligned} F(S, T, A \otimes T) &= (A \otimes T) \cap F(S, B, A \otimes B) \\ &= F(A, T, A \otimes B) \cap F(S, B, A \otimes B) \\ &= F(S, T, A \otimes B), \end{aligned}$$

by Lemmas 2.21 and 2.11. □

Proposition 2.23 ([12, Corollary 2.7]) *Let $S \subseteq A$ and $T \subseteq B$ be operator spaces. If the triples $(A, T, A \otimes B)$ and $(S, T, A \otimes T)$ have the slice map property, then $(S, T, S \otimes B)$ and $(S, T, A \otimes B)$ also have the slice map property. Conversely, if $(S, T, A \otimes B)$ has the slice map property, then $(S, T, A \otimes T)$ and $(S, T, S \otimes B)$ also have the slice map property.*

Proof Follows from Lemma 2.22 and the following commutative diagram of inclusions:

$$\begin{array}{ccc} F(S,T,A\otimes T) & \hookrightarrow & F(S,T,A\otimes B) \\ \uparrow & & \uparrow \\ S\otimes T & \hookrightarrow & F(S,T,S\otimes B) \end{array}$$

□

2.9 The Operator Approximation Property

In this subsection, we briefly recall the connection between the operator approximation property and the slice map property. This is included partly for completeness and partly because we use it in Sect. 3. The results are mostly due to Kraus [15] and Haagerup–Kraus [6].

Definition 2.24 We say that A has the *slice map property* for B if $(A, T, A \otimes B)$ has the slice map property for all operator spaces $T \subseteq B$.

We write $F(A, B)$ for the space of finite-rank maps $A \to B$. Recall that finite-rank maps of operator spaces are completely bounded.

Let A and B be operator spaces and let $x \in A \otimes B$. Define

$$F_B(x) := \overline{\{(\Phi \otimes \mathrm{id}_B)(x) \mid \Phi \in F(A, A)\}} \subseteq A \otimes B,$$
$$T_B(x) := \overline{\{(\phi \otimes \mathrm{id}_B)(x) \mid \phi \in A^*\}} \subseteq B,$$

where $\overline{T} \subseteq B$ denotes the norm closure of $T \subseteq B$.

Then we have

$$F_B(x) = A \otimes T_B(x).$$

Moreover, we have

$$\begin{aligned} F(A, T, A \otimes B) &= \{x \in A \otimes B \mid T_B(x) \subseteq T\} \\ &= \{x \in A \otimes B \mid F_B(x) \subseteq A \otimes T\}. \end{aligned}$$

Lemma 2.25 ([15, Theorem 5.4]) *Let A and B be operator spaces. Then A has the slice map property for B if and only if $x \in F_B(x)$ for all $x \in A \otimes B$.*

Proof $(\Rightarrow)$: Let $x \in A \otimes B$. Clearly, $x \in F(A, T_B(x), A \otimes B)$. Since A has the slice map property for B, we have $F(A, T_B(x), A \otimes B) = A \otimes T_B(x) = F_B(x)$. Thus $x \in F_B(x)$.

$(\Leftarrow)$: Let $T \subseteq B$ be operator subspace and let $x \in F(A, T, A \otimes B)$. Then $F_B(x) \subseteq A \otimes T$. Thus $x \in A \otimes T$. □

Definition 2.26 An operator space B is *matrix stable* if for each $n \in \mathbb{N}$, there is a completely bounded surjection $B \to B \otimes M_n$.

Definition 2.27 We say that A has the *operator approximation property* (OAP) for B if there is a net $\Phi_\alpha \in F(A, A)$ of finite-rank maps such that $\Phi_\alpha \otimes \mathrm{id}_B$ converges to $\mathrm{id}_A \otimes \mathrm{id}_B$ in point-norm topology.

Theorem 2.28 (cf. [15, Theorem 5.4]) *Let A and B be operator spaces. If A has the operator approximation property for B, then A has the slice map property for B. If B is matrix stable, then the converse also holds.*

Proof The first statement is clear from Lemma 2.25.

Now we prove the second statement. Suppose B is matrix stable. Let $x_1, \ldots, x_n \in A \otimes B$ and let $\epsilon > 0$. Let $C := B \otimes M_n$ and choose a completely bounded surjection $\pi : B \to C$.

Let $x := x_1 \oplus \cdots \oplus x_n \in A \otimes B \otimes M_n = A \otimes C$. Then there exists $y \in A \otimes B$ such that $x = (\mathrm{id}_A \otimes \pi)(y)$. By Lemma 2.25, there is $\Phi \in F(A, A)$ such that $\|(\Phi \otimes \mathrm{id}_B)(y) - y\| < \epsilon/(\|\pi\|_{\mathrm{cb}} + 1)$. Then

$$\begin{aligned}\|(\Phi \otimes \mathrm{id}_C)(x) - x\| &= \|(\Phi \otimes \mathrm{id}_C)((\mathrm{id}_A \otimes \pi)(y)) - (\mathrm{id}_A \otimes \pi)(y)\| \\ &= \|(\mathrm{id}_A \otimes \pi)((\Phi \otimes \mathrm{id}_B)(y) - y)\| \\ &< \epsilon.\end{aligned}$$

It follows that for any $1 \leq k \leq n$, we have $\|(\Phi \otimes \mathrm{id}_B)(x_k) - x_k\| < \epsilon$. □

Definition 2.29 Let $\mathscr{H}$ is a separable infinite-dimensional Hilbert space. We say that an operator space A has the *operator approximation property* (OAP) if A has the OAP for $K(\mathscr{H})$ and the *strong operator approximation property* (SOAP) if A has the OAP for $B(\mathscr{H})$.

If an operator space has the strong OAP, then it has the OAP for any B.

Example 2.30 An operator space A is said to have the completely bounded approximation property (CBAP) if the there is a constant $C > 0$ and a net $\Phi_\alpha \in F(A, A)$ of finite-rank maps with $\|\Phi_\alpha\|_{\mathrm{cb}} \leq C$, converging to id_A in the point-norm topology.

We have the implications

$$\text{CBAP} \implies \text{SOAP} \implies \text{OAP}.$$

It follows that, nuclear, in particular, abelian C^*-algebras have the slice map property for any operator space. See [3] for more details.

Definition 2.31 ([6, Definition 1.1]) A countable discrete group G has the *approximation property* (AP) if the constant function 1 is in the $\sigma(M_0A(G), Q(G))$-closure of $A(G)$ in $M_0A(G)$, where $A(G)$ is the Fourier algebra of G and $M_0A(G)$ is the space of completely bounded Fourier multipliers of G and $Q(G)$ is the standard predual of $M_0A(G)$ and $\sigma(M_0A(G), Q(G))$ is the weak topology on $M_0A(G)$ determined by $Q(G)$.

Theorem 2.32 ([6, Theorem 2.1]) *A countable discrete group G has the AP if and only if its reduced group C^*-algebra $C^*_\lambda(G)$ has the (strong) OAP.*

Proof See the original article [6] or [3, Section 12.4]. □

3 The Invariant Translation Approximation Property

In this section, we apply the results in Sect. 2 to the problem of studying the invariant part of the uniform Roe algebra.

3.1 Uniform Roe Algebras

Definition 3.1 We say that a (countable discrete) metric space is of *bounded geometry* if for any $R > 0$, there is $N_R < \infty$ such that all balls of radius at most R have at most N_R elements.

Definition 3.2 (cf. [17, Section 4.1] and [25, Section 2]) Let (X, d) be a metric space of bounded geometry and let $S \subseteq B(\mathscr{H})$ be a subset. For $R > 0$ and $M > 0$, let $A_{R,M}(X, S)$ denote the set of $X \times X$ matrices a with values in S satisfying

(i) for any $x_1, x_2 \in X$ with $d(x_1, x_2) > R$, we have $a(x_1, x_2) = 0$ and
(ii) for any $x_1, x_2 \in X$, we have $\|a(x_1, x_2)\|_{B(\mathscr{H})} \leq M$.

Let $A(X, S) := \cup_{R,M} A_{R,M}(X, S)$. For $S = \mathbb{C}$, we write $A_{R,M}(X)$ and $A(X)$.

Lemma 3.3 *Under the natural action, elements of $A_{R,M}(X, S)$ act on $l^2(X, \mathscr{H}) \cong l^2X \otimes \mathscr{H}$ as bounded operators of norm at most $M \cdot N_R$.*

Proof The action is given by

$$(a\xi)(x_1) := \sum_{x_2} a(x_1, x_2)\xi(x_2), \quad \xi \in l^2X \otimes \mathscr{H}.$$

Thus

$$\begin{aligned}\|(a\xi)(x_1)\| &\leq \sum_{d(x_1,x_2)\leq R} \|a(x_1, x_2)\xi(x_2)\| \\ &\leq M \sum_{d(x_1,x_2)\leq R} \|\xi(x_2)\| \\ &\leq M \left(N_R \sum_{d(x_1,x_2)\leq R} \|\xi(x_2)\|^2 \right)^{1/2}.\end{aligned}$$

It follows that

$$\begin{aligned}\|a\xi\|^2 &= \sum_{x_1} \|(a\xi)(x_1)\|^2 \\ &\leq M^2 N_R \sum_{x_1} \sum_{d(x_1,x_2)\leq R} \|\xi(x_2)\|^2 \\ &\leq M^2 N_R^2 \sum_{x_2} \|\xi(x_2)\|^2 \\ &= M^2 N_R^2 \|\xi\|^2.\end{aligned}$$

□

Thus we have a canonical inclusion $A(X, S) \subseteq B(l^2 X \otimes \mathscr{H})$.

Lemma 3.4 *For any $S \subseteq B(\mathscr{H})$, we have*

$$A(X) \times S \subseteq A(X, S) \subseteq A(X, \overline{S}) \subseteq \overline{A(X, S)}$$

in $B(l^2 X \otimes \mathscr{H})$.

Proof Only the last inclusion needs checking. Take $a \in A_{R,M}(X, \overline{S})$. For each positive integer $n \geq 1$, we define $a_n \in A(X, S)$ as follows. For $x_1, x_2 \in X$, if $d(x_1, x_2) > R$, then let $a_n(x_1, x_2) = 0$, and if $d(x_1, x_2) \leq R$, then choose $a_n(x_1, x_2) \in S$ to satisfy $\|a_n(x_1, x_2) - a(x_1, x_2)\| \leq 1/n$. Then $a_n \in A_{R,M+1}(X, S)$ and $\|a_n - a\| \leq N_R/n$ by Lemma 3.3. Hence, the sequence a_n converges to a in $B(l^2 X \otimes \mathscr{H})$ as $n \to \infty$ and thus $a \in \overline{A(X, S)}$. □

Definition 3.5 (cf. [17, Section 4.4] [25]) Let X be a metric space of bounded geometry and let $S \subseteq B(H)$ be an operator space. The *uniform Roe operator space* $C_u^*(X, S)$ is the closure of $A(X, S)$ in $B(l^2 X \otimes \mathscr{H})$. For $S = \mathbb{C}$, we write $C_u^*(X)$.

Lemma 3.6 *For any operator space $S \subseteq B(\mathscr{H})$, we have*

$$C_u^*(X) \otimes S \subseteq C_u^*(X, S).$$

Proof Follows from Lemma 3.4. □

3.2 Products

For metric spaces (X, d_X) and (Y, d_Y), we equip the product $X \times Y$ with the metric

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

If X and Y are of bounded geometry, then so is $X \times Y$.

Lemma 3.7 *Let X and Y be metric spaces of bounded geometry and let $S \subseteq B(\mathscr{H})$ be a subset. Then for $R, R' > 0$ and $M, M' > 0$ we have a natural inclusion*

$$A_{R,M}(X) \times A_{R',M'}(Y, S) \subseteq A_{R'',M''}(X \times Y, S),$$

where $R'' := \max\{R, R'\}$ and $M'' := M \cdot M'$. In particular, we have

$$A(X) \times A(Y, S) \subseteq A(X \times Y, S)$$

in $B(l^2 X \otimes l^2 Y \otimes \mathscr{H})$.

Proof Take $a \in A_{R,M}(X)$ and $b \in A_{R',M'}(Y, S)$ and let $R'' = \max\{R, R'\}$ and $M'' = M \cdot M'$. For $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we define

$$(a \otimes b)((x_1, y_1), (x_2, y_2)) := a(x_1, x_2) \otimes b(y_1, y_2).$$

(i) For $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ with $d((x_1, y_1), (x_2, y_2)) > R'' = \max\{R, R'\}$, we have either $d(x_1, x_2) > R'' \geq R$ thus $a(x_1, x_2) = 0$, or $d(y_1, y_2) > R'' \geq R'$ thus $b(y_1, y_2) = 0$, hence $(a \otimes b)((x_1, y_1), (x_2, y_2)) = a(x_1, x_2) \otimes b(y_1, y_2) = 0$ and

(ii) For $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we have

$$\|(a \otimes b)((x_1, y_1), (x_2, y_2))\| = \|a(x_1, x_2)\| \cdot \|b(y_1, y_2)\| \leq M \cdot M' = M''.$$

Hence, $a \otimes b$ belongs to $A_{R'',M''}(X \times Y, S)$. The last statement is clear. □

Lemma 3.8 *Let X and Y be metric spaces of bounded geometry and let $S \subseteq B(\mathscr{H})$ be a subset. Then for $R > 0$ and $M > 0$, we have a natural inclusion*

$$A_{R,M}(X \times Y, S) \subseteq A_{R,M'}(X, A_{R,M}(Y, S)),$$

where $M' := M \cdot N_R^X$. In particular, we have

$$A(X \times Y, S) \subseteq A(X, A(Y, S))$$

in $B(l^2 X \otimes l^2 Y \otimes \mathscr{H})$.

Proof Take $a \in A_{R,M}(X \times Y, S)$. Fix $x_1, x_2 \in X$ and consider

$$b_{x_1,x_2}(y_1, y_2) := a((x_1, y_1), (x_2, y_2)).$$

First we show that b_{x_1,x_2} is an element of $A_{R,M}(Y, S)$.

(i) For $y_1, y_2 \in Y$ with $d(y_1, y_2) > R$, we have $d((x_1, y_1), (x_2, y_2)) \geq d(y_1, y_2) > R$, thus $b_{x_1,x_2}(y_1, y_2) = a((x_1, y_1), (x_2, y_2)) = 0$.

(ii) For $y_1, y_2 \in Y$, we have $\|b_{x_1,x_2}(y_1, y_2)\| = \|a((x_1, y_1), (x_2, y_2))\| \leq M$.

Now we show that b is an element of $A_{R,M'}(X, A_{R,M}(Y, S))$.

(i) For $x_1, x_2 \in X$ with $d(x_1, x_2) > R$ and for any $y_1, y_2 \in Y$, we have $d((x_1, y_1), (x_2, y_2)) \geq d(x_1, x_2) > R$, and thus

$$b_{x_1,x_2}(y_1, y_2) = a((x_1, y_1), (x_2, y_2)) = 0$$

i.e. $b_{x_1,x_2} = 0$

(ii) For $x_1, x_2 \in X$, we have $\|b_{x_1,x_2}\| \leq M \cdot N_R^X = M'$ by Lemma 3.3.

□

Theorem 3.9 *Let X and Y be metric spaces of bounded geometry and let $S \subseteq B(H)$ be an operator space. Then we have a natural inclusion*

$$C_u^*(X) \otimes C_u^*(Y, S) \subseteq C_u^*(X \times Y, S) \subseteq C_u^*(X, C_u^*(Y, S)).$$

Proof Follows from Lemmas 3.7 and 3.8. □

3.3 The Invariant Translation Approximation Property

Let G be a countable discrete group. A length function on G is a function $l\colon G \to \{0, 1, 2, \dots\}$ such that $l(s) = 0$ if and only if $s = 1_G$, and $l(st) \leq l(s) + l(t)$ and $l(s) = l(s^{-1})$ for all $s, t \in G$. We say that the length function l is proper if, in addition, $\lim_{s\to\infty} l(s) = +\infty$.

Let $|G|$ denote the metric space of bounded geometry on G associated with a proper length function. Then G acts on $|G|$ isometrically by right translations.

Example 3.10 Let G be a countable discrete group. Then $A(G)$ is the $*$-algebra generated by $\mathbb{C}[G]$ and $l^\infty(G)$, and therefore $C^*_u|G| = C^*(l^\infty(G), \lambda(G))$. It is well known that $C^*_u|G| \cong l^\infty(G) \rtimes_r G$ and $(C^*_u|G|)^G = L(G) \cap C^*_u|G|$, where $L(G)$ denote the von Neumann algebra generated by $C^*_\lambda(G)$ and the fix points are taken under the right translation. See [3, Section 5.1].

Definition 3.11 Let S be an operator space. We say G has the *invariant translation approximation property* (ITAP) for S, if we have

$$C^*_\lambda(G) \otimes S = C^*_u(|G|, S)^G.$$

Definition 3.12 We say G has the *(strong) invariant translation approximation property* if it has the ITAP for (all operator spaces $S \subseteq B(\mathscr{H})$) $\mathbb{C}$.

The following theorem connects the strong ITAP to the AP.

Theorem 3.13 ([25]) *Let G be a countable discrete group. Then G has the AP if and only if G is exact and has the strong ITAP.*

Theorem 3.14 *Any subgroup of a group with ITAP has ITAP.*

Proof Let H be a subgroup of G and suppose that G has the ITAP. Using the coset decomposition of G, one checks that we have inclusions $C^*_\lambda(H) \subseteq C^*_\lambda(G)$, $L(H) \subseteq L(G)$ and $C^*_u|H| \subseteq C^*_u|G|$. Now the multiplier with respect to the indicator function of H, being positive definite, induces conditional expectations $E_L\colon L(G) \to L(H)$, $E_u\colon C^*_u|G| \to C^*_u|H|$ and $E_\lambda\colon C^*_\lambda(G) \to C^*_\lambda(H)$. The restrictions of E_L and E_u to $C^*_\lambda(G)$ are equal to E_λ. Now let $x \in L(H) \cap C^*_u|H|$ then $x \in L(G) \cap C^*_u|G| = C^*_\lambda(G)$ and $E_L(x) = x$ as well as $E_u(x) = x$. It follows that $E_\lambda(x) = x$ i.e. $x \in C^*_\lambda(H)$ and thus H also has the ITAP. □

Now we consider products.

Lemma 3.15 *Let G and H be countable discrete groups equipped with proper length functions* $l_G\colon G \to \mathbb{R}_{\geq 0}$ *and* $l_H\colon G \to \mathbb{R}_{\geq 0}$. *Then* $l_{G\times H}\colon G \times H \to \mathbb{R}_{\geq 0}$ *given by* $l_{G\times H}(g, h) = \max\{l_G(g), l_H(h)\}$ *is a proper length function and* $|G \times H| = |G| \times |H|$.

Proof First we check that l is a length function. Indeed, we have

$$l_{G\times H}(e, e) = \max\{l_G(e), l_H(e)\} = 0$$

and

$$\begin{aligned} l_{G\times H}(g^{-1}, h^{-1}) &= \max\{l_G(g^{-1}), l_H(h^{-1})\} \\ &= \max\{l_G(g), l_H(h)\} \\ &= l_{G\times H}(g, h) \end{aligned}$$

and finally

$$\begin{aligned} l_{G\times H}(gg', hh') &= \max\{l_G(gg'), l_H(hh')\} \\ &\leq \max\{l_G(g) + l_G(g'), l_H(h) + l_H(h')\} \\ &\leq \max\{l_G(g), l_H(h)\} + \max\{l_G(g'), l_H(h')\} \\ &= l_{G\times H}(g, h) + l_{G\times H}(g', h'). \end{aligned}$$

Moreover, it is clear that l is proper. Finally, we have

$$\begin{aligned} d_{|G\times H|}((g_1, h_1), (g_2, h_2)) &= l_{G\times H}(g_1 g_2^{-1}, h_1 h_2^{-1}) \\ &= \max\{l_G(g_1 g_2^{-1}), l_H(h_1 h_2^{-1})\} \\ &= \max\{d_{|G|}(g_1, g_2), d_{|H|}(h_1, h_2)\} \\ &= d_{|G|\times|H|}((g_1, h_1), (g_2, h_2)). \end{aligned}$$

□

Proposition 3.16 *Let G and H be countable discrete groups. If $G \times H$ has the ITAP, then G and H also have the ITAP and the triple*

$$((C_u^*|G|)^G, (C_u^*|H|)^H, C_u^*|G| \otimes C_u^*|H|)$$

has the slice map property.

Proof By Theorem 3.9, we have

$$\begin{aligned} C_\lambda^*(G \times H) &= C_\lambda^*(G) \otimes C_\lambda^*(H) \\ &\subseteq (C_u^*|G|)^G \otimes (C_u^*|H|)^H \\ &\subseteq (C_u^*|G| \otimes C_u^*|H|)^{G\times H} \\ &\subseteq (C_u^*|G \times H|)^{G\times H} \\ &= C_\lambda^*(G \times H). \end{aligned}$$

Thus G and H have the ITAP by Lemma 2.2 (this is also immediate from Theorem 3.14). The last statement follows from Corollary 2.20. □

Theorem 3.17 *Let G be a countable discrete group with the AP and let B be an operator space equipped with an action of a group H by completely bounded maps. Then we have*

$$C_\lambda^*(G) \otimes B^H = C_u^*(|G|, B)^{G\times H}.$$

Proof Clearly, we have

$$C^*_\lambda(G) \otimes B^H \subseteq (C^*_u|G|)^G \otimes B^H \subseteq (C^*_u|G| \otimes B)^{G\times H} \subseteq C^*_u(|G|, B)^{G\times H}.$$

Since G has AP, it has the strong ITAP by Theorem 3.13, thus

$$C^*_u(|G|, B)^G = C^*_\lambda(G) \otimes B.$$

Moreover, the C^*-algebra $C^*_\lambda(G)$ has the strong OAP by Theorem 2.32; thus by Proposition 2.19, we have

$$(C^*_\lambda(G) \otimes B)^H = F(C^*_\lambda(G), B^H, C^*_\lambda(G) \otimes B) = C^*_\lambda(G) \otimes B^H.$$

It follows that

$$\begin{aligned} C^*_u(|G|, B)^{G\times H} &= \left(C^*_u(|G|, B)^G\right)^H \\ &\subseteq (C^*_\lambda(G) \otimes B)^H \\ &= C^*_\lambda(G) \otimes B^H. \end{aligned}$$

This completes the proof. □

Theorem 3.18 *Let G and H be countable discrete groups. If G has the AP and H has the ITAP, then $G \times H$ has the ITAP.*

Proof By Theorem 3.9, we have

$$C^*_u|G \times H| \subseteq C^*_u\left(|G|, C^*_u|H|\right).$$

Thus, by Theorem 3.17, we have

$$(C^*_u|G \times H|)^{G\times H} \subseteq C^*_u\left(|G|, C^*_u(|H|)\right)^{G\times H} = C^*_\lambda(G) \otimes (C^*_u|H|)^H.$$

Since H has ITAP, we have $(C^*_u|H|)^H = C^*_\lambda(H)$. Thus, we obtain

$$\begin{aligned} C^*_\lambda(G) \otimes C^*_\lambda(H) &= C^*_\lambda(G \times H) \\ &\subseteq (C^*_u|G \times H|)^{G\times H} \\ &\subseteq C^*_\lambda(G) \otimes C^*_\lambda(H), \end{aligned}$$

so that $G \times H$ has the ITAP. □

Corollary 3.19 *Let G and H be countable discrete groups. Suppose G has the AP. Then $G \times H$ has the ITAP if and only if H has the ITAP.*

4 The Fubini Crossed Product

In this section, we study the crossed product version of the Fubini product.

4.1 The Fubini Crossed Product

Let G be a countable discrete group. A G-*operator space* is an operator space equipped with a completely isometric action of G. A map $\phi\colon B \to D$ of G-operator spaces is G-equivariant if for any $g \in G$ and $b \in B$, we have $\phi(gb) = g\phi(b)$.

Let $A \subseteq B(\mathscr{H})$ be an operator space equipped with a completely isometric action α of G. Define a new action π of A on $\mathscr{H} \otimes l^2G$ by $\pi(a)(v \otimes \delta_g) := \alpha_{g^{-1}}(a)v \otimes \delta_g$ and let G act on $\mathscr{H} \otimes l^2G$ by $\lambda(g)(v \otimes \delta_h) := v \otimes \delta_{gh}$.

The *reduced crossed product* $A \rtimes_r G$ is defined as the operator space spanned by

$$\{\pi(a)\lambda(g) \in B(\mathscr{H} \otimes l^2(G)) \mid a \in A, g \in G\}.$$

Definition 4.1 ([14, Lemma 2.1]) For any $\psi \in B(l^2G)_*$, there is a natural, completely bounded slice map $\mathrm{id}_A \rtimes_r \psi\colon A \rtimes_r G \to A$. If $S \subseteq A$ is a G-invariant operator subspace, then the *Fubini crossed product* $F(S, A \rtimes_r G)$ is defined as the set of all $x \in A \rtimes_r G$ such that $(\mathrm{id}_A \rtimes_r \psi)(x) \in S$ for all $\psi \in B(l^2G)_*$.

The slice map $\mathrm{id}_A \rtimes_r \psi$ is given by the restriction of the von Neumann slice map which maps $A^{**}\overline{\otimes}B(l^2G)$ to A^{**}.

Remark 4.2 We have

$$S \rtimes_r G \subseteq F(S, A \rtimes_r G) \subseteq A \rtimes_r G.$$

In fact, many of the formal properties of the Fubini product hold for Fubini crossed products, usually with the same proof.

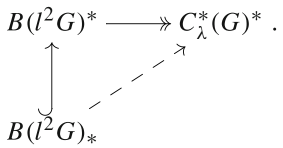
Definition 4.3 We say that $(S, A \rtimes_r G)$ has the *slice map property* if

$$F(S, A \rtimes_r G) = S \rtimes_r G.$$

Lemma 4.4 *If the action of G on A is trivial, then*

$$F(S, A \rtimes_r G) = F(S, C^*_\lambda(G), A \otimes C^*_\lambda(G)).$$

Proof Since the action of G on A is trivial, we have $A \rtimes_r G = A \otimes C^*_\lambda(G)$. The inclusion $C^*_\lambda(G) \hookrightarrow B(l^2G)$ gives the diagram



Hence, we have $F(S, C^*_\lambda(G), A \otimes C^*_\lambda(G)) \subseteq F(S, A \rtimes_r G)$. By Goldstine's theorem for any functional $\psi \in C^*_\lambda(G)^*$, there exists a bounded net $\psi_n \in B(l^2(G))_*$ with $\|\psi_n\| \leq \|\psi\|$, converging to ψ in the weak* topology. Now it is easy to see that for any $x \in A \otimes C^*_\lambda(G)$, the elements $\mathrm{id}_A \otimes \psi_n(x)$ converge to $\mathrm{id}_A \rtimes \psi(x)$ in norm. It follows that $F(S, A \rtimes_r G) \subseteq F(S, C^*_\lambda(G), A \otimes C^*_\lambda(G))$. □

Example 4.5 Let $l^\infty(G)$ act on l^2G by multiplication. Then G acts on $l^\infty(G)$ by left multiplication and $c_0(G) \subseteq l^\infty(G)$ is an invariant C^*-subalgebra. The Fubini crossed product $F(c_0(G), l^\infty(G) \rtimes_r G)$ is the ideal of all ghost operators on $|G|$. Thus G is exact if and only if $(c_0(G), l^\infty(G) \rtimes_r G)$ has the slice map property by Roe and Willett [18].

Lemma 4.6 *For families of G-invariant operator subspaces $\{S_\alpha \subseteq A\}$, we have*

$$F(\cap_\alpha S_\alpha, A \rtimes_r G) = \cap_\alpha F(S_\alpha, A \rtimes_r G).$$

Proposition 4.7 (cf. [14, Proposition 2.2]) *Let B and D be G-operator spaces and let $\sigma : B \to D$ be a completely bounded G-equivariant map. Then*

$$F(\ker(\sigma), B \rtimes_r G) = \ker(\sigma \rtimes_r G).$$

Theorem 4.8 *Let G and H be countable discrete groups and let A be a $(G \times H)$-operator space. Then*

$$F(A^H, A \rtimes_r G) = (A \rtimes_r G)^H.$$

Proof For $h \in H$, let $\alpha_h : A \to A$ denote the action by h. Then α_h is G-equivariant. Let $\sigma_h := \mathrm{id}_A - \alpha_h$. Then $\sigma_h \rtimes_r G = \mathrm{id}_{A \rtimes_r G} - \alpha_h \rtimes_r G : A \rtimes_r G \to A \rtimes_r G$. The proof follows from Proposition 4.7 and Lemma 4.6, since

$$\cap_{h \in H} \ker(\sigma_h) = A^H,$$
$$\cap_{h \in H} \ker(\sigma_h \rtimes_r G) = (A \rtimes_r G)^H.$$

□

Example 4.9 Let $l^\infty(G)$ act on l^2G by multiplication. Then G acts on $l^\infty(G)$ by left multiplication. As already pointed out $C^*_u|G| \cong l^\infty(G) \rtimes_r G$; moreover, thinking of $\mathbb{C}$ as embedded into $l^\infty(G)$ via the constant functions we have $C^*_u(|G|)^G = F(\mathbb{C}, l^\infty(G) \rtimes_r G)$. Thus G has the ITAP if and only if $(\mathbb{C}, l^\infty(G) \rtimes_r G)$ has the slice map property.

Proposition 4.10 *Let G be a discrete group with the AP. Then for any $S \subseteq A$, we have $F(S, A \rtimes_r G) = S \rtimes_r G$.*

Proof Since G has the AP, there exists a net (u_α) in $M_0A(G) \cap c_c(G)$ converging to $1 \in M_0A(G)$ in the $\sigma(M_0A(G), Q(G))$ topology. As explained in [25], this implies that the net of Schur multipliers $\hat{M}_{u_\alpha} \in CB(C^*_u(|G|, A))$ given by $\hat{M}_{u_\alpha}([a_{s,t}]) = [u_\alpha(st^{-1})a_{s,t}]$ converges to the identity map in the point-norm topology. Fixing a faithful representation $A \hookrightarrow B(K)$, we can think of $A \rtimes_r G \subseteq B(K \otimes \ell^2(G))$ as matrices indexed by G with entries in A. Since the $\langle \delta_s, \cdot\, \delta_t \rangle$ is a normal functional on $B(\ell^2(G))$ whose slice map gives the s, t entry of the matrix in $B(K \otimes \ell^2(G))$ we can characterise $F(S, A \rtimes_r G)$ as those matrices in $A \rtimes_r G$ with entries in S. Thus it is clear that $F(S, A \rtimes_r G) \subseteq C_u^*(|G|, A)$. Moreover, since each u_α has finite support, it is easy to check that $\hat{M}_{u_\alpha}(F(S, A \rtimes_r G))$ is contained in $\mathrm{span}\{\pi(s)\lambda(g) \mid s \in S,\ g \in G\} \subseteq S \rtimes_r G$. It follows that $S \rtimes_r G \ni \hat{M}_\alpha(x) \to x$ for any $x \in F(S, A \rtimes G)$, concluding the proof. □

Proposition 4.11 *If $F(S, A \rtimes_r G) = S \rtimes_r G$ for all $S \subseteq A$, then G has the AP.*

Proof Considering trivial actions, we see that $C_\lambda^*(G)$ has the slice map property for any operator space A by Lemma 4.4, thus having the OAP by Kraus' theorem (Theorem 2.28). Hence, G has the AP by Haagerup–Kraus theorem (Theorem 2.32). □

4.2 Functoriality

Lemma 4.12 *Let $S \subseteq A$ and $T \subseteq B$ be G-operator spaces. If a completely bounded map $\sigma : A \to B$ maps $\sigma(S) \subseteq T$, then*

$$(\sigma \rtimes_r G)(F(S, A \rtimes_r G)) \subseteq F(T, B \rtimes_r G).$$

Proof Let $x \in F(S, A \rtimes_r G)$. For any $\psi \in B(l^2 G)_*$, we have $(\mathrm{id}_A \rtimes_r \psi)(x) \in S$, and thus

$$(\mathrm{id}_A \rtimes_r \psi)[(\sigma \rtimes_r G)(x)] = \sigma[(\mathrm{id}_A \rtimes_r \psi)(x)]$$

belongs to T. Hence, the lemma holds. □

Lemma 4.13 *Let $S \subseteq A$ be G-operator spaces. Let $\sigma : A \to A$ be a completely bounded G-equivariant map. If σ restricts to the identity on S, then $\sigma \rtimes_r G$ restricts to the identity on $F(S, A \rtimes_r G)$.*

Proof For $x \in F(S, A \rtimes_r G)$ and $\psi \in B(l^2 G)_*$, we have

$$(\mathrm{id}_A \rtimes_r \psi)[(\sigma \rtimes_r G)(x)] = \sigma[(\mathrm{id}_A \rtimes_r \psi)(x)] = (\mathrm{id}_A \rtimes_r \psi)(x).$$

Thus $(\sigma \rtimes_r G)(x) = x$. □

Corollary 4.14 (cf. [10, Lemma 2]) *Let $S \subseteq A$ and $S \subseteq B$ be G-operator spaces. If $\phi : A \to B$ and $\psi : B \to A$ are completely bounded G-equivariant maps such that $(\psi \circ \phi)_{|S} = \mathrm{id}_S$ and $(\phi \circ \psi)_{|S} = \mathrm{id}_S$, then there is a completely bounded G-equivariant isomorphism $F(S, A \rtimes_r G) \to F(S, B \rtimes_r G)$ which is the identity on $S \rtimes_r G$.*

Let S be an operator space. We say that S is injective if for any operator spaces $T \subseteq B$ and a completely bounded map $\phi : T \to S$, there exists a completely bounded map $\psi : B \to S$ such that $\psi_{|T} = \phi$ and $\|\psi\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{cb}}$.

An extension of S is a pair (Z, κ) of an operator space Z and a completely isometric embedding $\kappa\colon S \to Z$. An extension (Z, κ) of S is called an injective envelope of S if Z is injective and id_Z is the only completely contractive map from Z to Z which extends $\mathrm{id}_{\kappa(S)}\colon \kappa(S) \to Z$ from $\kappa(S)$ to Z.

Injective envelopes of C^*-algebras, operator systems and operator spaces have been considered by various authors [2,5,7,19]. The G-injective envelope has only been defined for operator systems in the literature [8]. However, the definitions and constructions are all analogous. First one has to find an injective extension of the given object in the appropriate category and then minimise it in such a way that uniqueness is automatic.

Replacing operator spaces by G-operator spaces and completely bounded maps by completely bounded G-equivariant maps, we obtain the definition of a G-injective envelope, which exists and is unique up to isomorphism by the same argument. For a G-operator space S, we denote its G-injective envelope by $I_G(S)$. We write $I(S)$ if G is trivial.

The following lemma follows from functoriality.

Lemma 4.15 (cf. [10, Theorem 4]) *Let S be a G-operator space and let A be a G-injective operator space containing S. Then the Fubini crossed product $F(S, A \rtimes_r G)$ is independent of A.*

Definition 4.16 The Fubini crossed product $F(S, A \rtimes_r G)$ is called the *universal Fubini crossed product* of S by G and denoted $F(S, G)$. It is the largest Fubini crossed product of S.

We say that S has the *universal slice map property* for G if $S \rtimes_r G = F(S, G)$.

Example 4.17 For a discrete group G, the space $c_0(G)$ has the universal slice map property if and only if G is exact and the space $\mathbb{C}$ has the universal slice map property if and only if G has the ITAP.

Theorem 4.18 *Let G be a discrete group. Then $(C_u^*|G|)^G \subseteq I(C_\lambda^*(G))$.*

Proof We only sketch the proof.

Since $l^\infty(G)$ is G-injective, we see that $F(\mathbb{C}, G) \cong F(\mathbb{C}, l^\infty(G) \rtimes_r G) \cong (C_u^*|G|)^G$ by Lemma 4.15 and Example 4.9. On the other hand, for any S we have $S \rtimes_r G \subseteq F(S, G) \subseteq I_G(S) \rtimes_r G \subseteq I(S \rtimes_r G)$. □

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