A Novel RPI Set Computation Method for Discrete-time LPV Systems with Bounded Uncertainties

Junbo Tan[1,2], Sorin Olaru[2*], Konstantinos Ampountolas[3], John Jairo Martinez Molina[4], Feng Xu[5]

Abstract— **Set invariance plays a fundamental role in the analysis and design of linear systems. This paper proposes a novel method for constructing *robust positively invariant* (RPI) sets for discrete-time linear parameter varying (LPV) systems. Starting from the stability assumption in the absence of disturbances, we aim to construct the RPI sets for parametric uncertain system. The existence condition of a common quadratic Lyapunov function for all vertices of the polytopic system is relaxed in the present study. Thus the proposed method enlarges the application field of RPI sets to LPV systems. A family of approximations of *minimal robust positively invariant* (mRPI) sets are obtained by using a shrinking procedure. Finally, the effect of scheduling variables on the size of the mRPI set is analyzed to obtain more accurate set characterization of the uncertain LPV system. A numerical example is used to illustrate the effectiveness of the proposed method.**

I. INTRODUCTION

Many control and fault diagnosis problems can be naturally formulated, analyzed, and solved in a set-theoretic framework [1]. Sets are naturally involved in control and fault diagnosis fields by considering system uncertainties and physical constraints. Especially, invariant sets play an important role in the solutions of many control and fault diagnosis problems in complex systems [2] [3].

LPV systems, as a bridge connecting linear and nonlinear systems, have gained a great deal of attention in recent years [4]. Many control analysis and comprehensive design problems for LPV systems are involved in the invariant set theory, which is a fundamental tool in this field. In [5], a robust ellipsoidal invariant set method was proposed with respect to maximizing the inclusion of some given reference direction by considering additive disturbances injected into the system dynamics. In [6], based on a H_∞ observer gain design and linear matrix inequality (LMI) conditions, an ellipsoidal RPI set is computed, the evolution of which is characterized to bound the estimation error at each time instant. However, regarding both methods in [5] and [6], there is a precondition that a common quadratic Lyapunov function of all vertex matrices of LPV systems must exist, which is a strict assumption hard to be satisfied for several LPV systems.

[1]Navigation and Control Research Center, Department of Automation, Tsinghua University, 100084 Beijing, P.R.China.
[2]Laboratory of Signals and Systems, Univ. Paris-Sud-CentraleSupelec-CNRS, Université Paris Saclay, 91190, Paris, France.
[3]School of Engineering, University of Glasgow, Glasgow G12 8QQ, United Kingdom.
[4]Univ. Grenoble Alpes, CNRS, GIPSA-lab, Grenoble, France.
[5]Center for Artificial Intelligence and Robotics, Graduate School at Shenzhen, Tsinghua University, 518055 Shenzhen, P.R.China.
E-mail: tjb16@mails.tsinghua.edu.cn, Sorin.Olaru@centralesupelec.fr

For linear discrete-time systems affected by bounded uncertainties, in [7], an interesting method was proposed to compute an mRPI set by using a contractive procedure starting from an initial RPI set. Based on the work in [7], we propose a novel RPI set computation method for perturbed discrete-time LPV systems. The concept of poly-quadratic stability is employed to construct the RPI sets. Poly-quadratic stability aims to check asymptotic stability of a polytopic system by mean of polytopic quadratic Lyapunov functions [8]. In this case, the existence of a common quadratic Lyapunov function for all vertices of the polytopic system is absent. Compared with the existing methods computing RPI sets for discrete-time LPV systems, the proposed method does not need to satisfy any other requirements only requires that the system is poly-quadratically stable. Furthermore, a family of outer-approximations of the mRPI set are obtained by using a shrinking process, which are also positive invariant at each step of iteration.

The remainder of this paper is organized as follows. Section II presents the discrete-time LPV dynamics affected by bounded uncertainties and stability analysis of the system based on LMIs. In Section III, a novel initial RPI set construction method is proposed and a sequence of outer-approximations of the mRPI set are obtained with a shrinking index established to evaluate the approximation precision by using the shrinking process. Some extensions regarding the mRPI set of discrete-time LPV dynamics are further analyzed in Section IV. A numerical example is used to illustrate the effectiveness of the proposed method in Section V. Some conclusions are summarized in Section VI.

II. SYSTEM MODEL

This section mainly introduces a general model of discrete-time LPV systems with bounded uncertainties and analyzes the stability of the system.

A. System Model

Consider the following discrete-time LPV system:

$$x_{k+1} = A(\rho_k)x_k + w_k, \tag{1}$$

with $k = 0, 1, 2, \ldots$ the discrete time index, $A(\rho_k) \in \mathbb{R}^{n_x \times n_x}$ is the system matrix dependent on a scheduling vector $\rho_k \in \mathbb{R}^{n_\rho}$, and $x_k \in \mathbb{R}^{n_x}$ is the system state at time instant k. The uncertain input signal $w_k \in \mathbb{R}^{n_w}$ (including process disturbances, modeling errors, etc.) is bounded by a known closed, convex set $\mathbf{W}$ containing the origin, i.e., $w_k \in \mathbf{W}$.

It is assumed that the scheduling variable ρ_k is bounded by a known, convex and compact set $\mathbf{P}$ generating from the convex hull of its vertices, i.e., $\rho_k \in \mathbf{P} = \mathbf{Conv}\{\rho^1, ..., \rho^i, ..., \rho^{\mathcal{N}}\}$, where $\mathcal{N}$ is the number of vertices. Therefore, a linear affine function $A(\rho_k)$ of ρ_k is also bounded by a polytopic set and can be written as the sum of vertex matrices of this set:

$$A(\rho_k) = \sum\nolimits_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) A_i, \tag{2}$$

where A_i is the i-th vertex matrix of the set $A(\mathbf{P})$ and the weighting coefficients λ_i satisfy

$$\sum\nolimits_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) = 1; \quad 0 \leq \lambda_i(\rho_k) \leq 1. \tag{3}$$

B. Stability Analysis

Since w_k is an additive term in (1) and is contained in the known bounded set $\mathbf{W}$, this does not affect the bounded-input, bounded-output stability of the system (1). Moreover, let us directly consider the stability of the nominal system:

$$x_{k+1} = A(\rho_k) x_k. \tag{4}$$

We recall the following theorem related to the stability analysis of (4) and readers can refer [8] for more details.

Theorem 2.1: The dynamics (4) is poly-quadratically stable if and only if there exist symmetric positive definite matrices S_i, S_j, and matrices G_i of appropriate dimensions such that

$$\begin{bmatrix} G_i + G_i^{\mathsf{T}} - S_i & * \\ A_i G_i & S_j \end{bmatrix} \succ 0, \quad \forall\, i, j = 1, 2, ..., \mathcal{N}, \tag{5}$$

where the symbol $*$ denotes the transpose of $A_i G_i$. In this case, the time-varying parameter-dependent Lyapunov function for the stability is given as $V(x_k, \rho_k) = x_k^{\mathsf{T}} \mathcal{P}(\rho_k) x_k$, with $\mathcal{P}(\rho_k) = \sum_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) S_i^{-1}$.

For the structural point of view, the next results in [9] provides a link between stability conditions and additional structural properties of Lyapunov functions for dynamics (4). The necessary and sufficient condition regarding the poly-quadratically stability of (4) is equivalent to that there exists a scheduling-variable dependent Lyapunov function $V(x_k, \rho_k) = x_k^{\mathsf{T}} \mathcal{P}(\rho_k) x_k$ satisfying Theorem 2.1, which is considerably less conservative than the condition that there exists a common quadratic Lyapunov function for all vertex matrices of $A(\rho_k)$ [5] [6]. The subsequent computation of RPI sets assumes the fulfilment of this necessary and sufficient stability condition.

C. Robust Positively Invariant Sets

Here we first introduce the basic set invariance notions, which are the important basis of the proposed approaches in this paper.

Definition 2.1: A set X is a *positively invariant* (PI) set of the dynamics $x_{k+1} = A(\rho_k) x_k$, if $\forall \rho_k \in \mathbf{P}$, for any $x_k \in X$, one has $x_{k+1} \in X$ for all $k \geq 0$.

Definition 2.2: A set X is an RPI set of the dynamics $x_{k+1} = A(\rho_k) x_k + \omega_k$, if $\forall \rho_k \in \mathbf{P}$, for any $x_k \in X$ and any $\omega_k \in W$, one has $x_{k+1} \in X$.

Definition 2.3: The mRPI set of the dynamics is defined as an RPI set contained in any closed RPI set and the mRPI set is unique and compact.

III. COMPUTATION OF RPI SETS

This section proposes a novel invariant-set construction method by using set-theoretic notions for discrete-time LPV systems (1) and approximates the mRPI set with an arbitrarily prior given precision based on a iterative procedure.

A. Convex hull of the mRPI set

In general, the mRPI set of the dynamics (1) is not a convex set [10]. The robust positive invariance on the convex hull of the mRPI set of dynamics (1) can be guaranteed by the following theorem.

Theorem 3.1: Suppose the dynamics (4) is stable. Then, the convex hull of the mRPI set of the dynamics (1) for arbitrary $\rho_k \in \mathbf{P}$ is an RPI set.

Proof: Let $\bar{\mathbf{\Omega}}$ denote the mRPI set of the dynamics (1) and the convex hull of $\bar{\mathbf{\Omega}}$ is $\mathbf{\Omega}_\infty := \mathbf{Conv}\{\bar{\mathbf{\Omega}}\}$. Since $\bar{\mathbf{\Omega}}$ is the mRPI set of the dynamics (1), based on Definition 2.2 and Definition 2.3, we have

$$\left(\sum\nolimits_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) A_i\right) \bar{\mathbf{\Omega}} \oplus \mathbf{W} \subseteq \bar{\mathbf{\Omega}}. \tag{6}$$

For any $\theta \in \mathbf{\Omega}_\infty$, there exist θ_1, $\theta_2 \in \bar{\mathbf{\Omega}}$ and $0 \leq \alpha \leq 1$, such that $\theta = \alpha\theta_1 + (1-\alpha)\theta_2$. Furthermore,

$$\begin{aligned} &\sum\nolimits_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) A_i \theta + \omega_k \\ =&\sum\nolimits_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) A_i \alpha\theta_1 + \sum\nolimits_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) A_i (1-\alpha)\theta_2 + \omega_k \\ =&\alpha \left(\sum\nolimits_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) A_i \theta_1 + \omega_k\right) \\ &\quad + (1-\alpha) \left(\sum\nolimits_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) A_i \theta_2 + \omega_k\right) \end{aligned} \tag{7}$$

Let us note that there exist $\tilde{\theta}_1, \tilde{\theta}_2 \in \bar{\mathbf{\Omega}}$ such that:

$$\begin{aligned} \tilde{\theta}_1 &= \left(\sum\nolimits_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) A_i \theta_1 + \omega_k\right), \\ \tilde{\theta}_2 &= \left(\sum\nolimits_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) A_i \theta_2 + \omega_k\right). \end{aligned}$$

Thus ultimately we have

$$\begin{aligned} \sum\nolimits_{i=1}^{\mathcal{N}} \lambda_i(\rho_k) A_i \theta + \omega_k =& \alpha\tilde{\theta}_1 + (1-\alpha)\tilde{\theta}_2 \\ &\in \mathbf{Conv}\{\bar{\mathbf{\Omega}}\} = \mathbf{\Omega}_\infty. \end{aligned} \tag{8}$$

Based on Definition 2.2, this implies that $\mathbf{\Omega}_\infty$ is an RPI set of the dynamics (1). ■

Since the convex hull of the mRPI set is the tightest convex set containing the mRPI set of the dynamics (1), its characterization will represent the objective of the remaining of the paper. In the following sections, all analyses and computations are dealing with the convex hull $\mathbf{\Omega}_\infty$ instead of the mRPI set $\bar{\mathbf{\Omega}}$ and we also denote (with an abuse of notation) $\mathbf{\Omega}_\infty$ as the mRPI set of the dynamics (1).

B. Computation of an Initial RPI Set

If the condition of Theorem 2.1 is fulfilled, then the system (4) is stable. Moreover, w_k in (1) is bounded, i.e., $w_k \in \mathbf{W}$. Therefore, there exist a family of RPI sets for the dynamics (1). More information on the relationship between system stability and set invariance can be found in [11].

Theorem 3.2: Under the condition of Theorem 2.1, consider an arbitrarily given initial convex set $\mathbf{X}_0 \supseteq (1+\delta)\mathbf{\Omega}_\infty$, where $\mathbf{\Omega}_\infty$ is the mRPI set of dynamics (1) and $\delta > 0$. By computing the following iteration:

$$\bar{\mathbf{X}}_{k+1} = \mathbf{Conv}\left\{\cup_{i=1}^{\mathcal{N}} A_i \mathbf{X}_k\right\} \oplus \mathbf{W}, \tag{9a}$$

$$\mathbf{X}_{k+1} = \mathbf{Conv}\left\{\bar{\mathbf{X}}_{k+1} \cup \mathbf{X}_k\right\}, \tag{9b}$$

there exists a finite $k^* \in \mathbb{N}^+$ such that $\mathbf{X}_{k^*+1} = \mathbf{X}_{k^*}$. Moreover, $\mathbf{X}_{k^*}$ is an initial RPI set for the dynamics (1).

Proof: Let us first consider the following sequence:

$$\tilde{\mathbf{X}}_{k+1} = \mathbf{Conv}\left\{\cup_{i=1}^{\mathcal{N}} A_i \tilde{\mathbf{X}}_k\right\} \oplus \mathbf{W}. \tag{10}$$

For a stable dynamics (1), if $(1+\delta)\mathbf{\Omega}_\infty \subseteq \tilde{\mathbf{X}}_0$, then there exists a specific positive k^* such that $\tilde{\mathbf{X}}_k \subseteq \tilde{\mathbf{X}}_0, \forall k \geq k^*$. Then let us notice that

$$\mathbf{X}_k = \mathbf{Conv}\left\{\cup_{i=0}^{k} \tilde{\mathbf{X}}_i\right\}, \tag{11}$$

with $\tilde{\mathbf{X}}_0 = \mathbf{X}_0$. For $k = k^* + 1$, we have

$$\mathbf{X}_{k^*+1} = \mathbf{Conv}\left\{\mathbf{X}_{k^*} \cup \tilde{\mathbf{X}}_{k^*+1}\right\}. \tag{12}$$

Since $\tilde{\mathbf{X}}_{k^*+1} \subseteq \tilde{\mathbf{X}}_0 \subseteq \mathbf{X}_{k^*}$, we have $\mathbf{X}_{k^*+1} = \mathbf{Conv}\left\{\mathbf{X}_{k^*} \cup \tilde{\mathbf{X}}_{k^*+1}\right\} = \mathbf{X}_{k^*}$.

Thus, according to (9b), we can further obtain

$$\mathbf{X}_{k^*} = \mathbf{Conv}\left\{\bar{\mathbf{X}}_{k^*+1} \cup \mathbf{X}_{k^*}\right\},$$

which indicates that $\bar{\mathbf{X}}_{k^*+1} \subseteq \mathbf{X}_{k^*}$ holds. By combining (9a) and (9b), we can further obtain,

$$\bar{\mathbf{X}}_{k^*+1} = \mathbf{Conv}\left\{\cup_{i=1}^{\mathcal{N}} A_i \mathbf{X}_{k^*}\right\} \oplus \mathbf{W} \subseteq \mathbf{X}_{k^*}. \tag{13}$$

If $x_k \in \mathbf{X}_{k^*}$, then

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{\mathcal{N}} \lambda_{i,k}^{\rho} A_i x_k + w_k \\ &\in \mathbf{Conv}\left\{\cup_{i=1}^{\mathcal{N}} A_i \mathbf{X}_{k^*}\right\} \oplus \mathbf{W} \subseteq \mathbf{X}_{k^*}. \end{aligned} \tag{14}$$

Therefore, $\mathbf{X}_{k^*}$ is an RPI for the dynamics of (1). ■

Corollary 3.1: If the initial set $\mathbf{X}_0$ is contained in the mRPI set $\mathbf{\Omega}_\infty$, i.e., $\mathbf{X}_0 \subseteq \mathbf{\Omega}_\infty$, then the existence of finite k^* such that $\mathbf{X}_{k^*+1} = \mathbf{X}_{k^*}$ can not be guaranteed. In this case, $\mathbf{X}_k$ is not a RPI set at any iteration and only represents an inner approximation of the mRPI set of the LPV system (1). For further details, readers can refer the works in [12].

Considering the mRPI set $\mathbf{\Omega}_\infty$ is convex, compact and unique, we can always find a proper $\mathbf{X}_0$ such that $(1+\delta)\mathbf{\Omega}_\infty \subseteq \mathbf{X}_0$. We use $\mathbf{X}_0$ as an initial set for iteration (9) and in this case, we can always find a proper $\mathbf{X}_{k^*}$ as an initial RPI set for dynamics (1). A practical method to construct a proper $\mathbf{X}_0$ will be further illustrated in Section III.D.

The alternative procedures in [5] and [6] use LMI conditions to construct an RPI set under the precondition that there exists a common quadratic Lyapunov function for all vertex matrices of system. Here we provide a more practical way to construct an RPI set based exclusively on convex operations over sets. Moreover, if $\mathbf{W}$ and $\mathbf{X}_0$ are polyhedral sets then (9a) and (9b) provide a sequence of polyhedral sets and $\mathbf{X}_{k^*}$ is polyhedral.

Next we will be concerned with the shrinking of a given RPI set in order to obtain closer outer approximations of the mRPI set and iteratively converge towards to the mRPI set by following the idea in [7].

C. Shrinking Procedure

Considering that the uncertain inputs w_k are bounded within a convex set, i.e., $w_k \in \mathbf{W}$, we can recursively build a sequence of RPI sets starting with the initial RPI set $\mathbf{X}_{k^*}$ according to the following theorem.

Theorem 3.3: Given an initial RPI $\mathbf{X}_{k^*}$ for (1), the sequence $\mathbf{\Omega}_k$:

$$\mathbf{\Omega}_{k+1} = \mathbf{Conv}\left\{\cup_{i=1}^{\mathcal{N}} A_i \mathbf{\Omega}_k\right\} \oplus \mathbf{W}, \tag{15a}$$

with $\mathbf{\Omega}_0 = \mathbf{X}_{k^*}$, ensures that at each iteration $\mathbf{\Omega}_k$ is an RPI set and

$$\mathbf{\Omega}_\infty \subseteq \mathbf{\Omega}_{k+1} \subseteq \mathbf{\Omega}_k \subseteq \mathbf{\Omega}_0 \tag{16}$$

holds for $k \geq 1$, and the set $\mathbf{\Omega}_\infty = \lim_{k \to +\infty} \mathbf{\Omega}_k$ is the exact mRPI set of the dynamics (1).

Proof: Suppose $\mathbf{\Omega}_0 = \mathbf{X}_{k^*}$ is an RPI set of the dynamics (1). $\mathbf{\Omega}_1$ can be computed as

$$\mathbf{\Omega}_1 = \mathbf{Conv}\left\{\cup_{i=1}^{\mathcal{N}} A_i \mathbf{\Omega}_0\right\} \oplus \mathbf{W}, \tag{17}$$

which characterizes the set of all possible x_1 starting from the initial $x_0 \in \mathbf{\Omega}_0$. Since $\mathbf{\Omega}_0$ is an RPI set, we have

$$\mathbf{\Omega}_1 \subseteq \mathbf{\Omega}_0. \tag{18}$$

Furthermore, by combining (18) and (15a), we can obtain

$$\begin{aligned} \mathbf{\Omega}_2 &= \mathbf{Conv}\left\{\cup_{i=1}^{\mathcal{N}} A_i \mathbf{\Omega}_1\right\} \oplus \mathbf{W} \\ &\subseteq \mathbf{Conv}\left\{\cup_{i=1}^{\mathcal{N}} A_i \mathbf{\Omega}_0\right\} \oplus \mathbf{W} = \mathbf{\Omega}_1, \end{aligned} \tag{19}$$

which means that all x_1 starting from $\mathbf{\Omega}_1$ will evolve into $\mathbf{\Omega}_2 \subseteq \mathbf{\Omega}_1$. Thus, $\mathbf{\Omega}_1$ is also an RPI set. Similarly, we can conclude that $\mathbf{\Omega}_{k+1}$ computed from (15a) is an RPI set, provided that $\mathbf{\Omega}_k$ is an RPI set and $\mathbf{\Omega}_{k+1} \subseteq \mathbf{\Omega}_k \subseteq \mathbf{\Omega}_0$.

Thus $\mathbf{\Omega}_k$ describes a monotonic sequence (in terms of set inclusions) of RPI sets. This is lower bounded by the mRPI set which is contained in any RPI set by definition. The monotonic and lower bounded sequence is thus convergent. In order to prove that the limit set $\mathbf{\Omega}_\infty$ is the mRPI and not only a RPI set, it should be noted that

$$\mathbf{\Omega}_\infty = \mathbf{Conv}\left\{\cup_{i=1}^{\mathcal{N}} A_i \mathbf{\Omega}_\infty\right\} \oplus \mathbf{W} \tag{20}$$

and $\mathbf{\Omega}_{k+1} \subsetneq \mathbf{\Omega}_k$ whenever $\mathbf{\Omega}_k \neq \mathbf{\Omega}_\infty$. Exploiting the fact that the mRPI set is known to be unique and to verify (20), the proof is complete.

■

D. Outer-approximation of the mRPI Set with Prior Given Precision

According to Theorem 3.3, the computation of the mRPI $\mathbf{\Omega}_\infty$ will be achieved at the limit of an iterative procedure, which is overwhelming and practical impossible in most of the cases. In the following, we propose an outer-approximation method of the mRPI set with arbitrarily prior given precision. Based on the finite convex set operations (15a), we can obtain

$$\mathbf{\Omega}_{k+1} = \mathcal{A}(\mathbf{\Omega}_k) \oplus \mathbf{W}, \tag{21}$$

with $\mathcal{A}(\cdot)$ is a set mapping function and defined as

$$\mathcal{A}(\mathbf{S}) := \mathbf{Conv}\left\{\cup_{i=1}^{\mathcal{N}} A_i \mathbf{S}\right\}.$$

Thus, the recursive equation (21) can be written in a more explicit way by iterating from $\mathbf{\Omega}_0$, a polyhedral RPI set as follows:

$$\mathbf{\Omega}_k = \mathcal{A}^k(\mathbf{\Omega}_0) \oplus \sum_{i=1}^{k} \mathcal{A}^{i-1}(\mathbf{W}). \tag{22}$$

By defining the set $\mathbf{\Phi}_k := \sum_{i=1}^{k} \mathcal{A}^{i-1}(\mathbf{W})$, considering the fact that $\mathbf{\Phi}_\infty = \lim\limits_{k\to\infty} \mathbf{\Phi}_k = \sum_{i=1}^{\infty} \mathcal{A}^{i-1}(\mathbf{W})$, we can obtain,

$$\mathbf{\Phi}_\infty = \left(\sum_{i=k+1}^{\infty} \mathcal{A}^{i-1}(\mathbf{W})\right) \oplus \mathbf{\Phi}_k. \tag{23}$$

Thus, we can conclude that

$$\mathbf{\Phi}_k \subseteq \mathbf{\Phi}_\infty. \tag{24}$$

In addition, by combining (22) and (23), we have

$$\mathbf{\Omega}_k = \mathcal{A}^k(\mathbf{\Omega}_0) \oplus \mathbf{\Phi}_k \subseteq \mathcal{A}^k(\mathbf{\Omega}_0) \oplus \mathbf{\Phi}_\infty. \tag{25}$$

Furthermore, considering the fact that

$$\lim_{k\to\infty} \mathcal{A}^k(\mathbf{\Omega}_0) = 0,$$

we can find that $\mathbf{\Phi}_\infty = \mathbf{\Omega}_\infty$ is the mRPI set of the dynamics (1).

The recursive set iteration computation (15a) can be terminated when there exists a $k^\dagger \in \mathbb{N}^+$ such that

$$\mathcal{A}^{k^\dagger}(\mathbf{\Omega}_0) \subseteq \mathbb{A}_p^{n_x}(\epsilon), \tag{26}$$

with $\mathbb{A}_p^{n_x}(\epsilon) := \{x \in \mathbb{R}^{n_x} : \|x\|_p \leq \epsilon\}$ is a prior given ball with arbitrary small size. Therefore, based on (25), we can conclude that the set $\mathbf{\Omega}_{k^\dagger}$ is not only an RPI set of the dynamics (1) but also an outer approximation of the mRPI set $\mathbf{\Phi}_\infty$ with the precision $\mathbb{A}_p^{n_x}(\epsilon)$. That is

$$\mathbf{\Phi}_\infty \subseteq \mathbf{\Omega}_{k^\dagger} \subseteq \mathbb{A}_p^{n_x}(\epsilon) \oplus \mathbf{\Phi}_\infty. \tag{27}$$

In order to further characterize the velocity of shrinking to the mRPI set, we define a relative shrinking index as follows:

$$\beta_k = \frac{\text{radius}(\mathbf{\Omega}_k) - \text{radius}(\mathbf{\Omega}_{k+1})}{\text{radius}(\mathbf{\Omega}_0)}, \tag{28}$$

where radius$(\cdot)$ computes the radius of the Chebyshev ball of a polyhedron. Note that, as a shrinking index, β_k can also characterize the achieved precision of mRPI set approximation as a function of the iteration number k. That is, we can stop the iterative approximation procedure to the mRPI set once $\beta_k \leq \beta_\epsilon$, where $k = k_\beta^\dagger$ is the number of iterations and β_ϵ is an arbitrary given positive scalar.

IV. Extensions regarding RPI sets

In the previous sections, the computation of RPI sets approximating the mRPI set for the dynamics (1) has been done by considering the convex hull of vertex matrices to guarantee the robustness of computed RPI sets but without utilizing any other information on the scheduling vector ρ_k. Practically the time varaition of ρ_k is contained in the polytopic hypercube $\mathbf{P}$, i.e., $\rho_k \in \mathbf{P}$. Thus, if we can further make use of the information of ρ_k (such as the specific value of ρ_k at each time instant, the smaller varying range, and so on), it is possible to decrease the size of computed RPI sets and obtain a more accurate state set containing the real state x_k in accordance with the information on the scheduling variable.

According to the constructing procedure of the mRPI set in Section III, we can find that if the scheduling vector ρ_k has a smaller varying range, that is

$$\mathbf{P}' \subseteq \mathbf{P}, \tag{29}$$

then the computed mRPI set based on the proposed method will be contained in the original mRPI set, i.e.,

$$\mathbf{\Omega}_\infty(\mathbf{P}') \subseteq \mathbf{\Omega}_\infty(\mathbf{P}), \tag{30}$$

where $\mathbf{\Omega}_\infty(\mathbf{P}')$ and $\mathbf{\Omega}_\infty(\mathbf{P})$ denotes the computed mRPI sets corresponding to $\rho_k \in \mathbf{P}'$ and $\rho_k \in \mathbf{P}$, respectively. Furthermore, we can obtain that if $\mathbf{P}' \subseteq \{\mathbf{P}_1 \cap \mathbf{P}_2 \cap \cdots \cap \mathbf{P}_s\}$, then

$$\mathbf{\Omega}_\infty(\mathbf{P}') \subseteq \{\mathbf{\Omega}_\infty(\mathbf{P}_1) \cap \mathbf{\Omega}_\infty(\mathbf{P}_2) \cap \cdots \cap \mathbf{\Omega}_\infty(\mathbf{P}_s)\}. \tag{31}$$

Precisely, when the varying range of the scheduling vector ρ_k degenerates to a fixed point from a polytopic hypercube $\mathbf{P}'$, the above set inclusion relations (30) and (31) still hold. In this case, the LPV system (1) degenerates into a linear time invariant (LTI) system with bounded uncertainties.

Besides the varying range of scheduling vector ρ_k, if further information related to the specific values of scheduling vector ρ_k (such as the dynamics of ρ_k) is available or if the speed of variation of ρ_k can be guaranteed, then a decrease on the size of the initial mRPI $\mathbf{\Omega}_\infty(\mathbf{P})$ can be achieved in order to obtain a more accurate set describing the impact of the disturbance around the equilibrium of the nominal dynamics (4). For instance, suppose that the scheduling vector ρ_k satisfies the following bounding on the rate of variation:

$$|\rho_{k+1} - \rho_k| \leq \bar{\delta}, \tag{32}$$

where $\bar{\delta}$ is a given positive constant vector and the inequality is considered element-wise. By considering now the bounding condition (32), we can get a smaller varying range of ρ_k at the respective time instants to improve the precision of state sets. Moreover, suppose that the variation of the scheduling variable ρ_k is driven by a dynamical process, i.e.,

$$\rho_{k+1} = F\rho_k + v_k, \tag{33}$$

where $F \in \mathbb{R}^{n_\rho \times n_\rho}$ is a Schur matrix, v_k is a disturbance vector contained in a bounded set $\mathbf{V}$ such that ρ_k is always in the polytopic hypercube $\mathbf{P}$. We can now obtain the specific value of the scheduling vector ρ_k at each time instant and compute on-line a tighter state set based on the initial mRPI $\mathbf{\Omega}_\infty(\mathbf{P})$ according to the following iterative procedure:

$$\mathcal{X}_0 = \mathbf{\Omega}_\infty(\mathbf{P}), \tag{34a}$$
$$\mathcal{X}_{k+1} = A(\rho_k)\mathcal{X}_k \oplus \mathbf{W}. \tag{34b}$$

The analysis of the estimated state sets by using the conditions (29), (32) and (33) are further illustrated in the numerical example of the next section.

V. Numerical Example

Consider a discrete-time LPV sytem (1) with

$$A(\rho_k) = \begin{bmatrix} 0.8134 - 0.2464\rho_k & 0.1031 - 0.0075\rho_k \\ -0.07312 + 0.3326\rho_k & 0.2867 + 0.0464\rho_k \end{bmatrix}. \tag{35}$$

The system is affected by uncertainties $\|w_k\|_\infty \leq 1$ for all $k \geq 0$. Considering the scheduling variable ρ_k is contained in the bounded set $\mathbf{P} = [-1, 1]$ ($\mathbf{P}$ is an interval hull), we can compute the vertex matrices as

$$A_1 = \begin{bmatrix} 1.0597 & 0.1106 \\ -0.7038 & 0.2403 \end{bmatrix}, A_2 = \begin{bmatrix} 0.5670 & 0.0956 \\ -0.0387 & 0.3330 \end{bmatrix}.$$

Based on Theorem 2.1, we can solve the LMI (5) and obtain the proper parametric matrices using YALMIP :

$$S_1 = \begin{bmatrix} 0.5951 & -0.4850 \\ -0.4850 & 1.5214 \end{bmatrix}, S_2 = \begin{bmatrix} 1.0729 & -0.4999 \\ -0.4999 & 1.3832 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} 0.5759 & -0.1716 \\ -0.4378 & 1.2881 \end{bmatrix}, G_2 = \begin{bmatrix} 0.9493 & -0.2171 \\ -0.3136 & 1.1867 \end{bmatrix}.$$

Under the conditions of Theorem 2.1, the stability of the LPV system (1) is guaranteed and we can compute the RPI sets for the LPV dynamics as shown in the sequel. However, we cannot find proper parametric matrices satisfying the LMI conditions proposed in [5] and [6]. In other words, there does not exist a common quadratic Lyapunov function for all vertex matrices of the above LPV system and RPI sets cannot be constructed based on the methods in [5] and [6]. Thus, we can conclude that the proposed method for computing RPI sets for LPV systems is more flexible and has a wider application scope than those in previous works.

TABLE I
Related Computation Parameters

Varying range of ρ_k	k^*	$k^\dagger$	$k^\dagger_\beta$	ϵ	β_ϵ
$\mathbf{P}_1 = [-1,\ -1]$	4	213	27	0.001	0.001
$\mathbf{P}_2 = [-1,\ 0.5]$	5	213	8	0.001	0.001
$\mathbf{P}_3 = [-0.5,\ 0]$	2	62	11	0.001	0.001
$\mathbf{P}_4 = [0,\ 0.5]$	1	35	11	0.001	0.001
$\mathbf{P}_5 = [0.5,\ 1]$	1	24	11	0.001	0.001
$\mathbf{P}_{10} = [-0.5,\ 0.5]$	2	62	15	0.001	0.001
$\mathbf{P}_{11} = \{-0.5\}$	2	62	7	0.001	0.001
$\mathbf{P}_{12} = \{0.5\}$	1	24	8	0.001	0.001

To start with, we set the initial set $\mathbf{X}_0$ with $\|\mathbf{X}_0\|_\infty \leq 40$. The iterative procedure for searching of the initial RPI

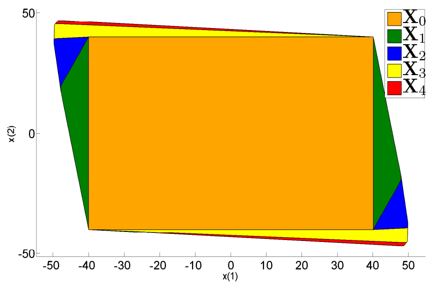

Fig. 1. Procedure of computing the initial RPI set.

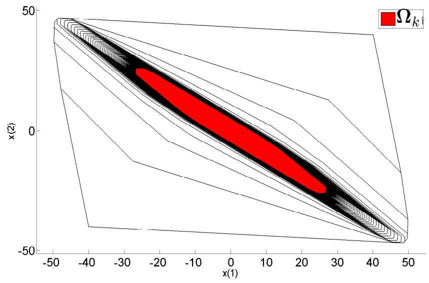

Fig. 2. Shrinking procedure to obtain a sequence of RPI sets.

set $\mathbf{X}_{k^*}$ is shown in Fig. 1. As can be seen it needs 4 steps to get the initial RPI set $\mathbf{X}_{k^*}$, i.e., $\mathbf{\Omega}_0 = \mathbf{X}_4$. Then by using the shrinking procedure proposed in Section III-C, we can compute the outer-approximation of the mRPI set starting from the initial polyhedron RPI set $\mathbf{X}_4$. The shrinking process is shown in Fig. 2. It can be observed that the mRPI set approximations are always invariant sets at each iteration. $\Omega_{k^\dagger}$ is the outer-approximation of the mRPI set with a given precision $\epsilon = 0.001$. The resulting parameters related to the iterations are displayed in the second row of Table I.

A. Case 1

According to Table I, we know that $\mathbf{P}_i \subseteq \mathbf{P}_1$, $\forall i = 2, 3, 4, 5$. The results in Fig. 3 show that the set inclusion relation (30) holds with $\mathbf{\Omega}_\infty(\mathbf{P}_i) \subseteq \mathbf{\Omega}_\infty(\mathbf{P}_1)$, $\forall i = 2, 3, 4, 5$. Since there is no set inclusion relation among $\mathbf{P}_i$, $\forall i = 2, 3, 4, 5$, we can obtain that $\mathbf{\Omega}_\infty(\mathbf{P}_i)$, $\forall i = 2, 3, 4, 5$ is not included in each other according to Fig. 3.

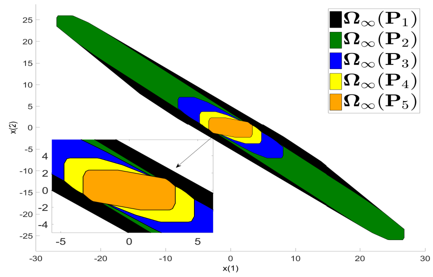

Fig. 3. mRPI sets w.r.t. different varying ranges of ρ_k.

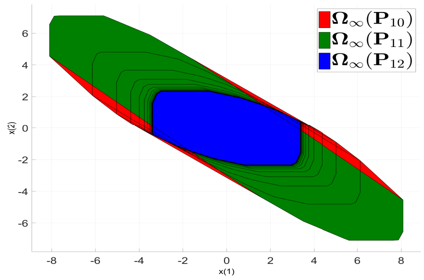

Fig. 4. mRPI sets w.r.t. ρ_k satisfying bounded varying condition.

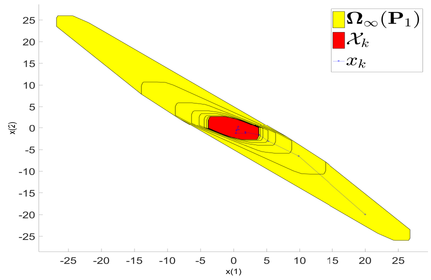

Fig. 5. Robust state set computation given the initial mRPI set.

B. Case 2: Scheduling vector satisfies a bounding condition

As we know, the shape, size and position of the mRPI set might change as the scheduling variable ρ_k varies. Consider that ρ_k varies from -0.5 to 0.5 satisfying the bounding condition (32) with $\bar{\delta} = 1$. Fig. 4 shows that the set inclusion relations hold, i.e., $\mathbf{\Omega}_\infty(\mathbf{P}_{11}) \subseteq \mathbf{\Omega}_\infty(\mathbf{P}_{10})$ and $\mathbf{\Omega}_\infty(\mathbf{P}_{12}) \subseteq \mathbf{\Omega}_\infty(\mathbf{P}_{10})$, which verifies (30). Then by using the set iteration (34b) starting from $\mathbf{\Omega}_\infty(\mathbf{P}_{11})$, the state set changes to $\mathbf{\Omega}_\infty(\mathbf{P}_{12})$ from initial $\mathbf{\Omega}_\infty(\mathbf{P}_{11})$ after 17 iterations. Moreover, we can find that during the whole iterative procedure, the state sets are included in $\mathbf{\Omega}_\infty(\mathbf{P}_{10})$, since ρ_k is always contained in $\mathbf{P}_{10}$, i.e., $\rho_k \in \mathbf{P}_{10}$.

C. Case 3: Scheduling vector dynamics are available

Suppose that the dynamics of ρ_k given by: $\rho_{k+1} = 0.9\rho_k + v_k$, with $\|v_k\|_\infty \leq 0.1$. The dynamics of ρ_k allow us to obtain the specific value of scheduling variable ρ_k at each time instant, and thus to compute a tighter state set containing the real system state x_k. By using $\mathbf{\Omega}_\infty(\mathbf{P}_1)$ to initialize and iterate the dynamics (34b), we can find from Fig. 5 that $x_k \in \mathcal{X}_k \subseteq \mathbf{\Omega}_\infty(\mathbf{P}_1)$ holds at each time instant. It is obvious that $\mathcal{X}_k$ has a smaller size than that of the mRPI set $\mathbf{\Omega}_\infty(\mathbf{P}_1)$, which characterizes a more accurate state set containing the real state x_k.

VI. CONCLUSIONS

This paper proposes a novel two-stage mRPI set computation method for discrete-time LPV systems with bounded uncertainties if and only if the system is poly-quadratically stable, which does not need to satisfy the condition that there must exist a common quadratic Lyapunov function for all the vertex matrices of LPV system. The first stage of the computational method aims to construct in a finite number of iterations and RPI set. The second stage is iterative and converges to the mRPI set in the limit.

Based on this shrinking process, we can obtain a family of approximations for the mRPI set that are also RPI sets at each step of iteration with a predetermined precision. Meanwhile, a shrinking index β_k is established to characterize the precision of the mRPI set outer-approximations and thus offer a finite-time computational method.

Finally, as expected, one can decrease the size of mRPI set and obtain a tighter and more accurate state set containing the real state on-line/off-line whenever supplementary information on the scheduling vector is made available. In the future, we will extend this method to fault diagnosis and fault-tolerant control design for discrete-time LPV systems with bounded uncertainties.

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