# A DIVERGENT HOROCYCLE IN THE HOROFUNCTION COMPACTIFICATION OF THE TEICHMÜLLER METRIC

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Abstract. We give an example of a horocycle in the Teichmüller space of the five-times-punctured sphere that does not converge in the Gardiner–Masur compactification, or equivalently in the horofunction compactification of the Teichmüller metric. As an intermediate step, we exhibit a simple closed curve whose extremal length is periodic but not constant along the horocycle. The example lifts to any Teichmüller space of complex dimension greater than one via covering constructions.

## 1. Introduction

In [GM91], Gardiner and Masur defined a compactification of Teichmüller space which mimics Thurston's compactification [Thu88], but uses extremal length instead of hyperbolic length. Since the Teichmüller distance can be computed in terms of extremal lengths of simple closed curves [Ker80], one expects the Gardiner–Masur compactification to interact nicely with this metric. This is indeed the case, for it turns out that the Gardiner–Masur compactification is isomorphic to the horofunction compactification of the Teichmüller metric [LS14]. In particular, all Teichmüller geodesic rays converge in the Gardiner–Masur compactification. There is even an explicit formula for their limits [Wal19]. In contrast, Teichmüller rays can accumulate onto intervals [Len08, LLR18, CMW19], circles [BLMR], and even higher-dimensional sets [LMR18] in the Thurston boundary.

Besides Teichmüller geodesics, another family of paths that are used extensively in Teichmüller dynamics are the horocycles obtained by shearing half-translation structures (coming from quadratic differentials) with the matrices

$$h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

for $t \in \mathbb{R}$. These are called horocycles because the Teichmüller disk generated by a quadratic differential $q$ is isometric to the hyperbolic plane, and the path $\{h_t\, q \,\big|\, t \in \mathbb{R}\}$ traces a horocycle (i.e., a limit of circles whose centers go off to infinity) in this plane. Since horocycles converge in the horofunction compactification of the hyperbolic plane (which is the same as the visual compactification) as $t$ tends to $\pm\infty$, it is natural to ask whether they converge in the horofunction compactification of Teichmüller space. That is the case whenever the horizontal foliation of $q$ consists of a single cylinder [Alb16, Theorem 17] or is uniquely ergodic [Alb16, Theorem 20] [JS16, Theorem 1.3]. However, the goal of this note is to give an example of a horocycle that does not converge in the horofunction (or Gardiner–Masur) compactification. The example is then lifted to all Teichmüller spaces of Riemann surfaces of genus $g$ with $p$ punctures such that $3g + p > 4$ via covering constructions.

**Theorem 1.1.** *In every Teichmüller space of complex dimension greater than one, there exists a horocycle which does not converge in the horofunction compactification of the Teichmüller metric.*

This has the following immediate consequence, which was first observed by Miyachi in genus two [Miy14b, Section 8.1] (see also [Alb16, Section 4.3]).

**Corollary 1.2.** *In every Teichmüller space of complex dimension greater than one, there exists a Teichmüller disk whose isometric inclusion does not extend continuously to the horoboundaries.*

The example underlying Theorem 1.1 is a horocycle generated by a Jenkins–Strebel quadratic differential $q$ with two cylinders on the five-times-punctured sphere. The proof that this horocycle diverges consists in two parts. First, we show that for any $s \in \mathbb{R}$, the sequence $(h_{s+n}\, q)_{n=1}^{\infty}$ (obtained by applying successive powers of a Dehn multitwist to $h_s\, q$) converges in the Gardiner–Masur boundary and we describe its limit. This is deduced from a more general criterion for convergence along mapping class group orbits (Lemma 3.1), which we also use to reprove that every horocycle or earthquake directed by a simple closed curve converges ([Alb16, Theorem 20] and [JS16, Corollary 3.2]). The second step is to show that the limit of this sequence depends on $s$. To prove this, it suffices to check that the extremal length of a certain simple closed curve $\alpha$ is not constant along the horocycle $h_s\, q$. In Lemma 4.2, we show that the extremal length of $\alpha$ attains a strict local maximum at $s = 0$ (and hence at all the integers by periodicity). This contrasts with the convexity of hyperbolic length along earthquakes [Ker83, Theorem 1] and complements the existence of local maxima for extremal length along Teichmüller geodesics [FBR18].

As argued in [Wal19], the Gardiner–Masur compactification of Teichmüller space is best suited for problems concerning the conformal structure of surfaces whereas the Thurston compactification is tailored for doing hyperbolic geometry. There is a well-known dictionary between the two worlds, partially given in Table 1 below (see [PS15, p.33] for an extended version).

The analogy between the two points of view is reinforced by the fact that the Thurston compactification is isomorphic to the horofunction compactification of the Thurston metric [Wal14]. Since Teichmüller rays, stretch paths, and earthquakes all converge in their respective 'compatible' compactification, it is somewhat surprising that horocycles do not.

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TABLE 1. A dictionary between the conformal and hyperbolic aspects of Teichmüller theory

| Conformal | Hyperbolic |
|---|---|
| Quasiconformal homeomorpisms | Lipschitz maps |
| Teichmüller metric | Thurston metric |
| Extremal length | Hyperbolic length |
| Measured foliations | Geodesic laminations |
| Gardiner–Masur compactification | Thurston compactification |
| Teichmüller rays | Stretch paths |
| Horocycles | Earthquakes |

## 2. DEFINITIONS

We begin by recalling some definitions and results needed in the sequel.

**Teichmüller space.** Let $S$ be an oriented surface with finitely generated fundamental group. The *Teichmüller space* $\mathcal{T}(S)$ is the set of equivalence classes $[(X, f)]$ of pairs $(X, f)$ where $X$ is a closed Riemann surface minus a finite set, the *marking* $f : S \to X$ is an orientation-preserving homeomorphism, and two pairs $(X, f)$ and $(Y, g)$ are equivalent if the change of marking $g \circ f^{-1} : X \to Y$ is homotopic to a biholomorphism.

The *Teichmüller distance* between two points $[(X, f)]$ and $[(Y, g)]$ is $\frac{1}{2} \log K$ where $K \geq 1$ is the smallest real number such that $g \circ f^{-1}$ is homotopic to a $K$-quasiconformal homeomorphism.

We will usually suppress the marking and the equivalence class from the notation and write $X \in \mathcal{T}(S)$ to mean $[(X, f)]$ for some marking $f$.

**Mapping class group.** The *mapping class group* $\mathrm{MCG}(S)$ of a surface $S$ is the group of homotopy classes of orientation-preserving homeomorphisms of $S$ onto itself. This group acts on the left on homotopy classes of objects on $S$ (such as closed curves) and on the right on Teichmüller space by precomposing the marking. That is, if $[(X, f)] \in \mathcal{T}(S)$ and $[\phi] \in \mathrm{MCG}(S)$ then

$$[(X, f)] \cdot [\phi] := [(X, f \circ \phi)].$$

We will write $X \cdot \phi$ in lieu of the above if the marking is implicit.

**Quadratic differentials.** A *quadratic differential* on a Riemann surface $X$ is a map $q : TX \to \mathbb{C}$ such that $q(\lambda v) = \lambda^2 q(v)$ for every $\lambda \in \mathbb{C}$ and $v \in TX$. We only consider quadratic diferentials that are holomorphic and whose area $\int_X |q|$ is finite. A *horizontal trajectory* for $q$ is a maximal smooth path $\gamma : \mathbb{R} \to X$ such that $q(\gamma'(t)) > 0$ for every $t \in \mathbb{R}$.

**Extremal length.** Let $\mathcal{C}(S)$ be the set of homotopy classes of essential (not homotopic to a point or a puncture) simple (embedded) closed curves in $S$.

We assume that $S$ is not a sphere with at most 3 punctures so that $\mathcal{C}(S)$ is non-empty.

The *extremal length* of $[\gamma] \in \mathcal{C}(S)$ on $[(X,f)] \in \mathcal{T}(S)$ is

$$\mathrm{EL}(\gamma, X) := \sup_{\rho} \frac{\inf_{\alpha \sim f(\gamma)} \ell_\rho(\alpha)^2}{\mathrm{area}(\rho)} \tag{2.1}$$

where $\ell_\rho(\alpha)$ is the length of $\alpha$ with respect to $\rho$ and the supremum is taken over all conformal metrics $\rho$ of finite positive area on $X$.

Any Riemann surface $A$ with infinite cyclic fundamental group is biholomorphic to a Euclidean cylinder $C$, and the extremal length of either generator of its fundamental group is equal to the ratio of the circumference of $C$ to its height. We denote this number by $\mathrm{EL}(A)$.

Since extremal length is monotone under conformal embeddings, we can estimate the extremal length of a curve from above using embedded annuli.

**Theorem 2.1** (Jenkins)**.** *Let $\gamma \in \mathcal{C}(S)$ and $X \in \mathcal{T}(S)$, and let $A \subset X$ be an annulus such that the generators of $\pi_1(A)$ are homotopic to $\gamma$. Then*

$$\mathrm{EL}(\gamma, X) \leq \mathrm{EL}(A)$$

*with equality if and only if the pull-back of $dz^2$ under any biholomorphism from $A$ to a Euclidean cylinder $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < m\}/\mathbb{Z}$ extends to a quadratic differential on $X$. Such an extremal annulus always exists, and is unique if $S$ is not a torus.*

If $q$ is the quadratic differential alluded to in the theorem, then the supremum in (2.1) is realized only for the conformal metric $\sqrt{|q|}$ and its scalar multiples. Both statements can be generalized in three different ways to multicurves and collections of disjoint annuli [Jen57, Str66, Ren76].

The notion of extremal length can be extended from $\mathcal{C}(S)$ to the set $\mathcal{MF}(S)$ of equivalence classes of *measured foliations* on $S$ (see [FLP12] for the definition) by setting

$$\mathrm{EL}(F, X) := \int_X |q_F|$$

for all $F \in \mathcal{MF}(S)$ and $X \in \mathcal{T}(S)$, where $q_F$ is the unique quadratic differential on $X$ whose horizontal foliation is measure-equivalent to $F$ [HM79, Ker80]. The extremal length function $\mathrm{EL} : \mathcal{MF}(S) \times \mathcal{T}(S) \to \mathbb{R}$ is continuous, as well as homogeneous of degree 2 in the first variable.

Although we will not use this here, we mention in passing that the Teichmüller distance can be recovered from extremal lengths via Kerckhoff's formula [Ker80]:

$$d(X,Y) = \frac{1}{2} \log \left( \sup_{\gamma \in \mathcal{C}(S)} \frac{\mathrm{EL}(\gamma, Y)}{\mathrm{EL}(\gamma, X)} \right) \quad \text{for all } X, Y \in \mathcal{T}(S).$$

**The Gardiner–Masur compactification.** Let $\mathbb{R}_{\geq 0} := [0,\infty)$ be the set of non-negative real numbers. The projective space $\mathbb{P}\left(\mathbb{R}_{\geq 0}^{\mathcal{C}(S)}\right)$ is the quotient of $\mathbb{R}_{\geq 0}^{\mathcal{C}(S)} \setminus \{0\}$ by the action of $\mathbb{R}_{>0}$ by multiplication. It is given the quotient topology inherited from the product topology on $\mathbb{R}_{\geq 0}^{\mathcal{C}(S)}$.

Gardiner and Masur [GM91, Section 6] showed that the map

$$\begin{array}{rcccl} \Phi & : & \mathcal{T}(S) & \to & \mathbb{P}\left(\mathbb{R}_{\geq 0}^{\mathcal{C}(S)}\right) \\ & & X & \mapsto & \left[\mathrm{EL}^{1/2}(\gamma, X)\right]_{\gamma\in\mathcal{C}(S)} \end{array}$$

is an embedding, and that the closure of its image is compact. The *Gardiner–Masur compactification* of $\mathcal{T}(S)$ is the set $\overline{\Phi(X)} \subset \mathbb{P}\left(\mathbb{R}_{\geq 0}^{\mathcal{C}(S)}\right)$, which we also denote by $\overline{\mathcal{T}}^{\mathrm{GM}}(S)$. A sequence $(X_n)_{n=1}^{\infty} \subset \mathcal{T}(S)$ *converges* to a projective vector $v \in \overline{\mathcal{T}}^{\mathrm{GM}}(S)$ if $\Phi(X_n) \to v$ as $n \to \infty$. Besides in [GM91], this compactification has been studied in [Miy08, Miy13, Miy14a, Miy14b, Miy14c, LS14, Alb16, JS16, Wal19].

**The horofunction compactification.** The *horofunction compactification* of a proper metric space $(M, d)$ is the set of all locally uniform limits of functions $M \to \mathbb{R}$ of the form

$$y \mapsto d(y, x_n) - d(x_n, b)$$

where $b \in M$ is a fixed basepoint, and $(x_n)_{n=1}^{\infty}$ ranges over all sequences in $M$. Its elements are called *horofunctions*, and their level sets are called *horospheres*.

Liu and Su [LS14] proved that the horofunction compactification of $\mathcal{T}(S)$ equipped with the Teichmüller metric is isomorphic to the Gardiner–Masur compactification. We will mostly work with the Gardiner–Masur formulation except in Section 5 where the horofunction point of view simplifies things.

**Horocycles lie on horospheres.** It is interesting to note that any horocycle $h_t\, q$ obtained by shearing a quadratic differential $q$ travels along some horosphere, namely, a level set of the function

$$X \mapsto -\frac{1}{2}\log \mathrm{EL}(F, X) \tag{2.2}$$

where $F$ is the horizontal foliation of $q$. Indeed, the horizontal foliation of $h_t\, q$ and its area does not depend on $t$, so that if $X_t$ denotes the underlying Riemann surface, then

$$\mathrm{EL}(F, X_t) = \int_{X_t} |h_t\, q| = \int_{X_0} |q| = \mathrm{EL}(F, X_0)$$

for all $t \in \mathbb{R}$. That (2.2) defines a horofunction (in fact a Busemann function) when $F$ is a simple closed curve follows from the proof of [FBR18, Lemma 3.3]. The density of weighted simple closed curves in $\mathcal{MF}(S)$ and the continuity of extremal length then imply that (2.2) defines a horofunction for any $F \in \mathcal{MF}(S)$, though not a Busemann function (the limit of an almost geodesic ray) in general.

The horocycle $h_t\, q$ also lies on a level set of the horofunction obtained as the forward limit of the Teichmüller geodesic $g_s\, q$ where

$$g_s = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}.$$

The resulting Busemann function can be determined by combining Liu and Su's isomorphism [LS14, Section 5] between the horofunction and Gardiner–Masur compactifications and Walsh's formula [Wal19, Corollary 1] for the limit of $g_s\, q$ in $\overline{\mathcal{T}}^{\mathrm{GM}}(S)$, though this gives a rather convoluted expression.

Our point is that the horocycle $\{h_t\, q : t \in \mathbb{R}\}$ is far from an arbitrary path—it is the intersection of one or more horospheres with a complex geodesic—yet it can still accumulate onto a non-trivial continuum in the horofunction boundary, as we will see in Section 4.

## 3. Convergence along mapping class group orbits

Our first result is a sufficient criterion for a sequence in the mapping class group orbit of a point $X \in \mathcal{T}(S)$ to converge in the Gardiner–Masur compactification. The idea behind this criterion was already exploited in [GM91, Theorem 7.2] and [JS16, Proposition 4.1].

**Lemma 3.1.** *Let $(\phi_n)_{n=1}^\infty \subset \mathrm{MCG}(S)$ be a sequence of mapping classes. Suppose that there exists a sequence $(c_n)_{n=1}^\infty$ of positive real numbers and a non-zero function $f : \mathcal{C}(S) \to \mathcal{MF}(S)$ such that $c_n \phi_n(\gamma) \to f(\gamma)$ as $n \to \infty$ for every $\gamma \in \mathcal{C}(S)$. Then for every $X \in \mathcal{T}(S)$, the sequence $X \cdot \phi_n$ converges to the projective vector*

$$\Big[\ \mathrm{EL}^{1/2}(f(\gamma), X)\ \Big]_{\gamma \in \mathcal{C}(S)}$$

*in the Gardiner–Masur compactification $\overline{\mathcal{T}}^{\mathrm{GM}}(S)$ as $n \to \infty$.*

*Proof.* Let $\gamma \in \mathcal{C}(S)$. By definition of the mapping class group action on Teichmüller space, we have $\mathrm{EL}(\gamma, X \cdot \phi_n) = \mathrm{EL}(\phi_n(\gamma), X)$ for every $n \geq 1$. It follows that

$$c_n^2\, \mathrm{EL}(\gamma, X \cdot \phi_n) = c_n^2\, \mathrm{EL}(\phi_n(\gamma), X) = \mathrm{EL}(c_n \phi_n(\gamma), X) \to \mathrm{EL}(f(\gamma), X)$$

as $n \to \infty$, by homogeneity and continuity of extremal length on $\mathcal{MF}(S)$. Thus $X \cdot \phi_n$ converges to the stipulated limit in $\overline{\mathcal{T}}^{\mathrm{GM}}(S)$ as $n \to \infty$. □

A *Dehn multitwist* is a product $\tau_1^{n_1} \circ \cdots \circ \tau_k^{n_k}$ of non-zero integer powers $n_j$ of Dehn twists $\tau_j$ about the components $\alpha_j$ of a multicurve on a surface. We will apply the above criterion when $\phi_n = \phi^n$ is a sequence of powers of a Dehn multitwist $\phi$. In order to apply the criterion, we first need to understand the effect of $\phi$ and its powers on simple closed curves. The following estimate from [Iva92, Lemma 4.2] generalizing [FLP12, Proposition

A.1] is used to determine the projective limit of $\phi^n(\gamma)$ as $n \to \infty$ for any curve $\gamma \in \mathcal{C}(S)$.

**Lemma 3.2** (Ivanov)**.** *Let $\tau = \tau_1^{n_1} \circ \cdots \circ \tau_k^{n_k}$ be a Dehn multitwist about a multicurve $\{\alpha_1, \ldots, \alpha_k\}$ in a surface $S$. Then for any two essential simple closed curves $\gamma, \beta \in \mathcal{C}(S)$ we have*

$$\sum_{j=1}^{k}(|n_j| - 2)i(\gamma, \alpha_j)i(\alpha_j, \beta) - i(\gamma, \beta) \leq i(\tau(\gamma), \beta)$$

$$\leq \sum_{j=1}^{k} |n_j| i(\gamma, \alpha_j)i(\alpha_j, \beta) + i(\gamma, \beta)$$

*where $i$ is the geometric intersection number.*

In particular, for fixed curves $\gamma$ and $\beta$, the difference between $i(\tau(\gamma), \beta)$ and $\sum_{j=1}^{k} |n_j| i(\gamma, \alpha_j)i(\alpha_j, \beta)$ is bounded independently of the powers $n_j$. We apply this to successive powers of a fixed Dehn multitwist $\phi$.

**Corollary 3.3.** *Let $\phi = \tau_1^{n_1} \circ \cdots \circ \tau_k^{n_k}$ be a Dehn multitwist about a multicurve $\{\alpha_1, \ldots, \alpha_k\}$ in a surface $S$ and let $\gamma \in \mathcal{C}(S)$ be any simple closed curve. Then $\phi^n(\gamma)/n$ converges to the weighted multicurve*

$$\sum_{j=1}^{k} |n_j| i(\gamma, \alpha_j)\alpha_j$$

*in $\mathcal{MF}(S)$ as $n \to \infty$.*

*Proof.* Let $\beta \in \mathcal{C}(S)$ be any simple closed curve. We need to show that

$$i(\phi^n(\gamma)/n, \beta) = \frac{1}{n} i(\phi^n(\gamma), \beta) \quad \text{converges to} \quad \sum_{j=1}^{k} |n_j| i(\gamma, \alpha_j)i(\alpha_j, \beta)$$

as $n \to \infty$. This follows from Lemma 3.2 applied to $\tau = \phi^n$ since the error terms tend to zero after dividing by $n$. $\square$

Now that we know the projective limit of $\phi^n(\gamma)$ as $n \to \infty$, we can apply our criterion to deduce that the orbit of a point in Teichmüller space under the cyclic group generated by a Dehn multitwist converges in the Gardiner–Masur compactification (in either direction). Furthermore, we get a formula for the limit.

**Corollary 3.4.** *Let $\phi = \tau_1^{n_1} \circ \cdots \circ \tau_k^{n_k}$ be a Dehn multitwist about a multicurve $\{\alpha_1, \ldots, \alpha_k\}$ in a surface $S$ and let $X \in \mathcal{T}(S)$. Then the sequence $X \cdot \phi^n$ converges to*

$$\left[ \mathrm{EL}^{1/2} \left( \sum_{j=1}^{k} |n_j| i(\gamma, \alpha_j)\alpha_j, X \right) \right]_{\gamma \in \mathcal{C}(S)}$$

*in the Gardiner–Masur compactification $\overline{\mathcal{T}}^{\mathrm{GM}}(S)$ as $n \to \infty$.*

*Proof.* Corollary 3.3 shows that the hypotheses of Lemma 3.1 are satisfied for $\phi_n = \phi^n$, $c_n = 1/n$, and $f(\gamma) = \sum_{j=1}^k |n_j| i(\gamma, \alpha_j)\alpha_j$, from which the conclusion follows. $\square$

Observe that the limit given in Lemma 3.1 or Corollary 3.4 appears to depend on the initial point $X$. However, it may happen that for every surface $Y \in \mathcal{T}(S)$ there is a constant $c > 0$ such that

$$\mathrm{EL}(f(\gamma), Y) = c\,\mathrm{EL}(f(\gamma), X)$$

for all $\gamma \in \mathcal{C}(S)$, which results in the projective vector

$$\left[ \mathrm{EL}^{1/2}(f(\gamma), X) \right]_{\gamma \in \mathcal{C}(S)}$$

being independent of $X$. This occurs if the image of $f$ is contained in a single ray, for example if $\phi$ is a Dehn twist about a single curve $\alpha \in \mathcal{C}(S)$. In that case, $f(\gamma) = i(\gamma, \alpha)\alpha$ and $X \cdot \phi^n$ converges to

$$\left[ \mathrm{EL}^{1/2}(i(\gamma, \alpha)\alpha, X) \right]_{\gamma \in \mathcal{C}(S)} = \left[ i(\gamma, \alpha)\, \mathrm{EL}^{1/2}(\alpha, X) \right]_{\gamma \in \mathcal{C}(S)} = [\, i(\gamma, \alpha) \,]_{\gamma \in \mathcal{C}(S)}$$

as $n \to \infty$. By varying $\alpha$ and taking limits, it follows that $[\, i(\gamma, F) \,]_{\gamma \in \mathcal{C}(S)}$ belongs to $\overline{\mathcal{T}}^{\mathrm{GM}}(S)$ for any measured foliation $F \in \mathcal{MF}(S)$. That is, the Gardiner–Masur boundary contains the Thurston boundary of projective measured foliations (both are subsets of the same projective space), as was observed in [GM91, Theorem 7.1]. In the next section, we will give an example of a Dehn multitwist $\phi$ where the limit of $X \cdot \phi^n$ actually depends on $X$.

Lemma 3.1 can of course be applied to other sequences of mapping classes. For instance, if $\phi$ is a pseudo-Anosov with horizontal and vertical foliations $\mathcal{H}$ and $\mathcal{V}$ and stretch factor $\lambda > 1$, then for any $\gamma \in \mathcal{C}(S)$ the sequence $\lambda^{-n}\phi^n(\gamma)$ converges to $\frac{i(\gamma,\mathcal{V})}{i(\mathcal{H},\mathcal{V})}\,\mathcal{H}$ as $n \to \infty$. In this case, $X \cdot \phi^n$ converges to

$$\begin{aligned}\left[ \mathrm{EL}^{1/2}\left( \frac{i(\gamma,\mathcal{V})}{i(\mathcal{H},\mathcal{V})}\,\mathcal{H}, X \right) \right]_{\gamma \in \mathcal{C}(S)} &= \left[ \frac{i(\gamma,\mathcal{V})}{i(\mathcal{H},\mathcal{V})}\ \mathrm{EL}^{1/2}(\mathcal{H}, X) \right]_{\gamma \in \mathcal{C}(S)} \\ &= [\, i(\gamma, \mathcal{V}) \,]_{\gamma \in \mathcal{C}(S)}\end{aligned}$$

as $n \to \infty$, which is manifestly independent of $X$.

When this phenomenon happens, that is, when the limit in Lemma 3.1 is independent of $X$, we can promote convergence along sequences to convergence along paths.

**Proposition 3.5.** *Suppose that $\phi \in \mathrm{MCG}(S)$ is such that the sequence defined by $\phi_n := \phi^n$ satisfies the hypotheses of Lemma 3.1 and is such that the limit of $X \cdot \phi^n$ as $n \to \infty$ does not depend on $X$. Then every continuous path $\omega : \mathbb{R} \to \mathcal{T}(S)$ for which there is a $T > 0$ such that $\omega(t + T) = \omega(t) \cdot \phi$ for all $t \in \mathbb{R}$ converges to the same limit as $t \to \infty$.*

*Proof.* Lemma 3.1 generalizes easily to sequences $(X_n)_{n=1}^{\infty}$ such that $X_n \cdot \phi_n^{-1}$ converges to some $X \in \mathcal{T}(S)$ as $n \to \infty$, with the conclusion that $X_n$ converges to

$$\left[\ \mathrm{EL}^{1/2}(f(\gamma), X)\ \right]_{\gamma \in \mathcal{C}(S)}$$

as $n \to \infty$. By hypothesis, this limit $L$ is independent of $X$.

To prove the result, it suffices to show that any sequence $(t_n)_{n=1}^{\infty} \subset \mathbb{R}$ tending to infinity admits a subsequence such that $\omega(t_{n_k}) \to L$ as $k \to \infty$. For each $n \geq 1$, let $m_n \in \mathbb{Z}$ be such that $s_n := t_n - m_n T$ belongs to the interval $[0, T]$. Then $\omega(t_n) \cdot \phi^{-m_n} = \omega(s_n)$ for every $n \geq 1$. Since $[0, T]$ is compact and $\omega$ is continuous, there is an $s \in [0, T]$ and a subsequence such that $s_{n_k} \to s$ and $\omega(s_{n_k}) \to \omega(s)$ as $k \to \infty$. By the previous paragraph, we get that $\omega(t_{n_k}) \to L$ as $k \to \infty$. $\square$

If $\phi$ is a Dehn twist about a curve $\alpha \in \mathcal{C}(S)$, then the above proposition implies that every horocycle directed by a Jenkins–Strebel differential with a single cylinder homotopic to $\alpha$ and every earthquake directed by $\alpha$ converges to $[\, i(\gamma, \alpha)\, ]_{\gamma \in \mathcal{C}(S)}$. This recovers [Alb16, Theorem 17] and [JS16, Corollary 3.2] respectively. If $\phi$ is a pseudo-Anosov, then we get that the axis of $\phi$ converges to $[\, i(\gamma, \mathcal{V})\, ]_{\gamma \in \mathcal{C}(S)}$ in the forward direction, where $\mathcal{V}$ is the vertical foliation. Since the foliations of a pseudo-Anosov are uniquely ergodic, this also follows from [Miy13, Corollary 2] or [Wal19, Corollary 1]. On the other hand, the result applies to all $\phi$-invariant paths.

## 4. A divergent horocycle

Let $S^1 = \mathbb{R}/\mathbb{Z}$ and let $C = S^1 \times [-1, 1]$. Seal the top and bottom of $C$ shut via the relation $(x, y) \sim (-x, y)$ for all $(x, y) \in S^1 \times \{-1, 1\}$ to create a pillowcase, and remove the four corners $(0, \pm 1)$ and $(1/2, \pm 1)$ as well as $(0, 0)$ to get a five-times-punctured sphere $X$ equipped with a quadratic differential $q$ coming from the differential $dz^2$ on $C$. Applying the horocycle flow $h_t$ to $q$ results in a twisted punctured pillowcase denoted $X_t$.

**Proposition 4.1.** *The horocycle $t \mapsto X_t$ defined above does not converge in the Gardiner–Masur compactification as $t \to \infty$.*

The proof proceeds by finding distinct limits of sequences going to infinity along the path. This is sufficient since the projective space $\mathbb{P}\left(\mathbb{R}_{\geq 0}^{\mathcal{C}(X)}\right)$ is Hausdorff, so that limits are unique when they exist.

Consider the homeomorphism $\phi$ obtained by applying a right Dehn twist about each of the horizontal curves $\alpha$ and $\beta$ at heights $-1/2$ and $1/2$ in $X$. Then $X_{s+n} = X_s \cdot \phi^n$ for every $s \in \mathbb{R}$ and $n \in \mathbb{Z}$. Indeed, $\phi$ can be realized by the matrix

$$h_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the marking for $X_{s+n}$ is given by $h_{s+n} = h_s \circ h_n = h_s \circ \phi^n$ (recall that mapping classes act on Teichmüller space on the right by pre-composing

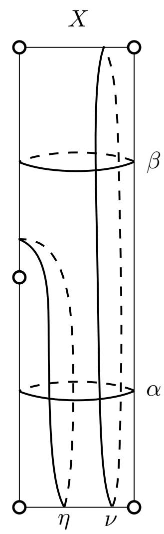


FIGURE 1. The punctured pillowcase $X$ with some curves on it.

the marking, while the horocycle flow acts on the left by post-composing charts with matrices). By Corollary 3.4, the sequence $X_{s+n}$ converges to the projective vector

$$v_s = \left[ \mathrm{EL}^{1/2}\left(i(\gamma,\alpha)\alpha + i(\gamma,\beta)\beta, X_s\right)\right]_{\gamma\in\mathcal{C}(S)} \tag{4.1}$$

as $n \to \infty$. We will show that the limit $v_s$ is not a constant function of $s$, which implies Proposition 4.1.

We first observe that $\mathrm{EL}\,(\alpha+\beta, X_s)$ is constant equal to 2 since the quadratic differential $h_s q$ has horizontal foliation $\alpha+\beta$ and area 2. Suppose on the other hand that $\mathrm{EL}\,(\alpha, X_s)$ is not constant in $s$. Then the projective vector $v_s$ depends on $s$, since we can find simple closed curves $\eta$ and $\nu$ in $X$ such that

$$i(\eta,\alpha)\alpha + i(\eta,\beta)\beta = 2\alpha \quad \text{and} \quad i(\nu,\alpha)\alpha + i(\nu,\beta)\beta = 2(\alpha+\beta)$$

(see Figure 1). The ratio of the corresponding entries in $v_s$ is then

$$\frac{\mathrm{EL}^{1/2}\,(\alpha, X_s)}{\mathrm{EL}^{1/2}\,(\alpha+\beta, X_s)} = \frac{\mathrm{EL}^{1/2}\,(\alpha, X_s)}{\sqrt{2}},$$

which is non-constant by hypothesis.

We have thus reduced Proposition 4.1 to showing that $\mathrm{EL}\,(\alpha, X_s)$ is not constant in $s$, which is our next result.

**Lemma 4.2.** *The function* $s \mapsto \mathrm{EL}\,(\alpha, X_s)$ *attains a strict local maximum at zero.*

*Proof.* The curve $\alpha$ is invariant under the anti-conformal involution of $X$ given by $(x,y)\mapsto(-x,y)$ in cylinder coordinates. The embedded annulus

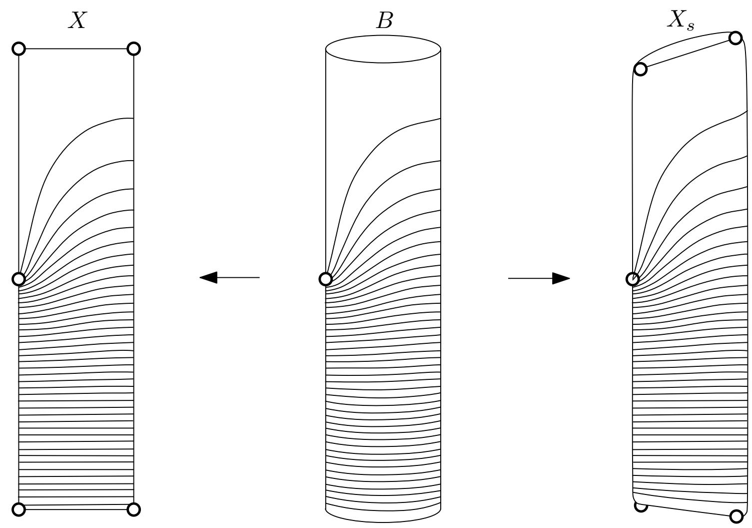


FIGURE 2. The quadratic differential realizing the extremal length of $\alpha$ on the punctured cylinder $B$ extends to a quadratic differential on $X$ but not on $X_s$ for any small $s \neq 0$.

$A \subset X$ realizing the extremal length of $\alpha$ is also invariant under that symmetry since it is unique. It follows that $A$ is disjoint from from the top and bottom edges of the punctured pillowcase $X$. In other words, $A$ is contained in the punctured cyclinder $B = S^1 \times (-1,1) \setminus \{(0,0)\}$. The latter embeds conformally in $X_s$ for every $s \in \mathbb{R}$. Indeed, $B$ is clearly invariant under the horocycle flow. It is the identifications along its boundary that change to $(x,1) \sim (2s-x,1)$ and $(x,-1) \sim (-2s-x,-1)$ in order to obtain $X_s$ (after puncturing at the folding points).

Since $A$ embeds conformally in $B$ and then in $X_s$ (in the same homotopy class as $\alpha$), Theorem 2.1 tells us that

$$\mathrm{EL}\,(\alpha, X_s) \leq \mathrm{EL}(A) = \mathrm{EL}\,(\alpha, X)$$

with equality if and only if the standard quadratic differential $\psi$ on $A$ (which pulls back to $dz^2$ in cylindrical coordinates) extends to a quadratic differential on $X_s$. In turn, this happens if and only if the gluing used to obtain $X_s$ from $B$ is isometric with respect to $\psi$.

The identifications $(x,1) \sim (2s-x,1)$ and $(x,-1) \sim (-2s-x,-1)$ on the top and bottom boundaries are of course isometries with respect to the quadratic differential $dz^2$ on $B$, but we claim that there is a neighborhood $N$ of 0 in $\mathbb{R}$ such that they are not isometries with respect to $\psi$ for any $s \in N \setminus \{0\}$. The main reason for this is that $\psi$ is not rotationally symmetric.

We focus on the bottom boundary. Let $h : S^1 \times (0,m) \to A$ be a biholomorphism between a Euclidean cylinder and $A$ chosen to be equivariant under the symmetry $(x,y) \mapsto (-x,y)$, where the $(x,y)$-coordinates in the target are those from $B$. This map $h$ followed by the inclusion $\iota : A \hookrightarrow B$

extends to an odd analytic diffeomorphism $g$ between the bottom circles of $S^1 \times (0,m)$ and $B$. Since $g$ is odd, we have $g''(0) = 0$. On the other hand, $g''$ is not constant equal to zero since $\iota \circ h$ is not affine. By the identity principle, there is a neighborhood $U$ of 0 in $S^1 \times \{0\}$ such that $g''(x) \neq 0$ for every $x \in U \setminus \{0\}$. We may assume that $U$ is connected, in which case we deduce that $g'(u) \neq g'(v)$ for every $u, v \in U$ such that $u < v < 0$ or $0 < u < v$ by Rolle's theorem.

In the cylinder coordinate $S^1 \times (0,m)$, the pull-back $h^*\psi$ becomes $dz^2$. Thus the line element $\sqrt{|\psi|}$ induced by $\psi$ on the bottom circle of $B$ is the push-forward of the Euclidean line element $|dx|$ by $g$. If the identification $x \sim -2s - x$ on the bottom of $B$ is an isometry with respect to $\psi$, then $(g^{-1})'(x) = (g^{-1})'(-2s-x)$ for every $x$. If $s > 0$ is sufficiently small and $-2s < x < 0$ then $u = g^{-1}(x)$ and $v = g^{-1}(-2s-x)$ are both in $U$, to the left of 0, and satisfy

$$g'(u) = \frac{1}{(g^{-1})'(x)} = \frac{1}{(g^{-1})'(-2s-x)} = g'(v),$$

contradicting the previous paragraph. Similarly, if $s$ is negative and sufficiently close to zero then we can find a pair of points $u, v \in U$ with $0 < u < v$ such that $g'(u) = g'(v)$. This contradiction implies our claim that there is a neighborhood $N$ of 0 in $\mathbb{R}$ such that $\psi$ does not extend to a quadratic differential on $X_s$ for any $s \in N \setminus \{0\}$. We conclude that $\mathrm{EL}\,(\alpha, X_s) < \mathrm{EL}\,(\alpha, X_0)$ for every $s \in N \setminus \{0\}$. $\square$

The above proof shows that $\mathrm{EL}\,(\alpha, X_s) \leq \mathrm{EL}\,(\alpha, X_0)$ for all $s \in \mathbb{R}$. The extremal length at $s = 0$ can be computed using Schwarz–Christoffel transformations, and is approximately 0.8196442. The horizontal trajectories of the corresponding quadratic differential are sketched in Figure 2.

As stated in the introduction, the function $s \mapsto \mathrm{EL}(\alpha, X_s)$ is periodic since

$$\mathrm{EL}\,(\alpha, X_{s+n}) = \mathrm{EL}\,(\alpha, X_s \cdot \phi^n) = \mathrm{EL}\,(\phi^n(\alpha), X_s) = \mathrm{EL}\,(\alpha, X_s)$$

for every $s \in \mathbb{R}$ and $n \in \mathbb{Z}$. Hence it has strict local maxima at all the integers.

By a similar reasoning, the projective vector $v_s$ from Equation (4.1) is $\mathbb{Z}$-periodic (and in fact $\frac{1}{2}\mathbb{Z}$-periodic since $\phi$ has a square root preserving both $\alpha$ and $\beta$) and invariant under $s \mapsto -s$. We think that $s \mapsto v_s$ is injective on $[0, 1/4]$ so that the horocycle $t \mapsto X_t$ accumulates onto an interval in the Gardiner–Masur boundary. It would be interesting to find examples with larger limit sets.

## 5. Lifting the example to higher complexity

Let $S_{g,p}$ be an oriented surface of genus $g$ with $p$ punctures. The following lemma is taken from [GM, Lemma 7.1].

**Lemma 5.1** (Gekhtman–Markovic)**.** *If $3g-3+p>1$, then there is a branched cover $\overline{S_{g,p}} \to \overline{S_{0,5}}$ that branches at all pre-images of marked points that are not marked and induces an isometric embedding $\mathcal{T}(S_{0,5}) \hookrightarrow \mathcal{T}(S_{g,p})$.*

We use this to export the example from Proposition 4.1 to all Teichmüller spaces $\mathcal{T}(S_{g,p})$ of complex dimension $3g-3+p>1$. We explain how this works from the horofunction point of view as well as from the Gardiner–Masur one.

*First proof Theorem 1.1.* Let $t \mapsto X_t$ be the divergent horocycle in $S_{0,5}$ constructed in Proposition 4.1 and let $\iota : \mathcal{T}(S_{0,5}) \to \mathcal{T}(S_{g,p})$ be any isometric embedding induced by a branched cover (Lemma 5.1). Then $t \mapsto \iota(X_t)$ is a horocycle in $\mathcal{T}(S_{g,p})$ since the $\mathrm{SL}(2,\mathbb{R})$-action on quadratic differentials commutes with the pull-back by the branched cover.

Let $b \in \mathcal{T}(S_{0,5})$ be any basepoint. We take $\iota(b)$ as the basepoint for the horofunction compactification of $\mathcal{T}(S_{g,p})$ (the choice of basepoint only changes horofunctions by an additive constant). Suppose that $\iota(X_t)$ converges to a horofunction $h : \mathcal{T}(S_{g,p}) \to \mathbb{R}$ as $t \to \infty$. Then $X_t$ converges to the function $h \circ \iota$ as $t \to \infty$, a contradiction. Thus $\iota(X_t)$ diverges. □

*Second proof Theorem 1.1.* Let $\pi : \overline{S_{g,p}} \to \overline{S_{0,5}}$ be a branched cover and let $\iota : \mathcal{T}(S_{0,5}) \to \mathcal{T}(S_{g,p})$ be the induced isometric embedding. For any simple closed curve $\gamma \in \mathcal{C}(S_{0,5})$ and any $X \in \mathcal{T}(S_{0,5})$, we have the identity

$$\mathrm{EL}(\pi^{-1}(\gamma), \iota(X)) = d \cdot \mathrm{EL}(\gamma, X)$$

where $d$ is the degree of $\pi$. Indeed, if $\theta$ is the quadratic differential on $X$ whose horizontal foliation is measure-equivalent to $\gamma$, then the horizontal foliation of the pull-back differential $\pi^*\theta$ is measure-equivalent to $\pi^{-1}(\gamma)$ and the area of $\pi^*\theta$ is $d$ times that of $\theta$.

Let $t \mapsto X_t$ be the divergent horocycle from Section 4 directed by the quadratic differential $q$, with $\alpha, \beta, \eta, \nu \subset X$ the same curves as in Figure 1. Then there exists some $m \in \mathbb{N}$ such that for every $s \in \mathbb{R}$ and $n \in \mathbb{Z}$ we have $\iota(X_{s+mn}) = \iota(X_s) \cdot \psi^n$ where $\psi$ is a Dehn multitwist about $\pi^{-1}(\alpha \cup \beta)$. Indeed, each component $c$ of $\pi^{-1}(\alpha \cup \beta)$ covers either $\alpha$ or $\beta$ with some degree $d_c \in \mathbb{N}$, which may vary from one component to another. Thus, each cylindrical component of $\pi^*q$ corresponding to a curve $c$ has height 1 and circumference $d_c$. If $m$ is the least common multiple of the degrees $d_c$, then the matrix $h_m$ performs a right Dehn twist to the power $m/d_c$ about each component $c$ of $\pi^{-1}(\alpha \cup \beta)$.

In particular, the sequence $\iota(X_{s+mn})$ converges to some limit $w_s$ in the Gardiner–Masur compactification as $n \to \infty$ (Corollary 3.4), but the limit depends on $s$. Indeed, $\mathrm{EL}(\pi^{-1}(\alpha \cup \beta), \iota(X_s))$ is constant while

$$\mathrm{EL}(\pi^{-1}(\alpha), \iota(X_s)) = d \cdot \mathrm{EL}(\alpha, X_s)$$

is not by Lemma 4.2. The only difference with the proof of Proposition 4.1 is that here $\pi^{-1}(\eta)$ and $\pi^{-1}(\nu)$ are not necessarily simple closed curves in $S_{g,p}$

(they might not be connected). However, the map $f : \mathcal{C}(S_{g,p}) \to \mathcal{MF}(S_{g,p})$ defined by

$$f(\gamma) = \sum_{c \subset \pi^{-1}(\alpha \cup \beta)} \frac{m}{d_c} \, i(\gamma, c) \, c$$

(where the sum is over connected components) extends continuously to the space of measured foliations $\mathcal{MF}(S_{g,p})$. By fixing a small $s \neq 0$ such that $\mathrm{EL}(\alpha, X_s) \neq \mathrm{EL}(\alpha, X_0)$ and by approximating $\pi^{-1}(\eta)$ and $\pi^{-1}(\nu)$ with simple closed curves $\gamma_n, \delta_n \in \mathcal{C}(S_{g,p})$ we get that

$$\frac{\mathrm{EL}^{1/2}\left(f(\gamma_n), \iota(X_s)\right)}{\mathrm{EL}^{1/2}\left(f(\delta_n), \iota(X_s)\right)} \neq \frac{\mathrm{EL}^{1/2}\left(f(\gamma_n), \iota(X_0)\right)}{\mathrm{EL}^{1/2}\left(f(\delta_n), \iota(X_0)\right)}$$

if $n$ is large enough, and hence that $w_s \neq w_0$. Here we are using the fact that

$$i(\pi^{-1}(\mu), c) = d_c \cdot i(\mu, \pi(c))$$

for every $\mu \in \mathcal{C}(S_{0,5})$ and $c \in \mathcal{C}(S_{g,p})$, which implies that

$$\begin{aligned} f(\pi^{-1}(\eta)) &= \sum_{c \subset \pi^{-1}(\alpha \cup \beta)} \frac{m}{d_c} \, i(\pi^{-1}(\eta), c) \, c \\ &= m \sum_{c \subset \pi^{-1}(\alpha \cup \beta)} i(\eta, \pi(c)) \, c \\ &= m \sum_{c \subset \pi^{-1}(\alpha)} i(\eta, \alpha) \, c \\ &= 2m \, \pi^{-1}(\alpha) \end{aligned}$$

and similarly $f(\pi^{-1}(\nu)) = 2m \, \pi^{-1}(\alpha \cup \beta)$. □

## 6. Concluding remark

In [JS16, Section 6], Jiang and Su conjectured that there exist earthquakes directed by disconnected multicurves that do not converge in the Gardiner–Masur compactification. We agree with this intuition and further believe that all earthquakes and horocycles diverge except for those directed by indecomposable laminations or foliations.

## References


[Alb16] V. Alberge, *Convergence of some horocyclic deformations to the Gardiner-Masur boundary*, Ann. Acad. Sci. Fenn. Math. **41** (2016), no. 1, 439–455.

[BLMR] J. Brock, C. Leininger, B. Modami, and K. Rafi, *Limit sets of Teichmüller geodesics with minimal nonuniquely ergodic vertical foliation, II*, to appear in *Journal für reine und angewandte Mathematik*, `https://doi.org/10.1515/crelle-2017-0024`.

[CMW19] J. Chaika, H. Masur, and M. Wolf, *Limits in $\mathcal{PMF}$ of Teichmüller geodesics*, J. Reine Angew. Math. **747** (2019), 1–44.

[FBR18] M. Fortier Bourque and K. Rafi, *Non-convex balls in the Teichmüller metric*, J. Differential Geom. **110** (2018), no. 3, 379–412.

[FLP12] A. Fathi, F. Laudenbach, and V. Poénaru, *Thurston's work on surfaces*, Mathematical Notes, vol. 48, Princeton University Press, Princeton, NJ, 2012, Translated from the 1979 French original by Djun M. Kim and Dan Margalit.

[GM] D. Gekhtman and V. Markovic, *Classifying complex geodesics for the Carathéodory metric on low-dimensional Teichmüller spaces*, to appear in *Journal d'Analyse Mathématique*, `arXiv:1711.04722`.

[GM91] F. Gardiner and H. Masur, *Extremal length geometry of Teichmüller space*, Complex Variables, Theory and Application: An International Journal **16** (1991), 209–237.

[HM79] J. Hubbard and H. Masur, *Quadratic differentials and foliations*, Acta Math. **142** (1979), no. 3-4, 221–274.

[Iva92] N.V. Ivanov, *Subgroups of Teichmüller modular groups*, Translations of Mathematical Monographs, vol. 115, American Mathematical Society, Providence, RI, 1992, Translated from the Russian by E. J. F. Primrose and revised by the author.

[Jen57] J.A. Jenkins, *On the existence of certain general extremal metrics*, Ann. of Math. (2) **66** (1957), 440–453.

[JS16] M. Jiang and W. Su, *Convergence of earthquake and horocycle paths to the boundary of Teichmüller space*, Sci. China Math. **59** (2016), no. 10, 1937–1948.

[Ker80] S.P. Kerckhoff, *The asymptotic geometry of Teichmüller space*, Topology **19** (1980), 23–41.

[Ker83] ———, *The Nielsen realization problem*, Ann. of Math. (2) **117** (1983), no. 2, 235–265.

[Len08] A. Lenzhen, *Teichmüller geodesics that do not have a limit in $\mathcal{PMF}$*, Geom. Topol. **12** (2008), no. 1, 177–197.

[LLR18] C. Leininger, A. Lenzhen, and K. Rafi, *Limit sets of Teichmüller geodesics with minimal non-uniquely ergodic vertical foliation*, J. Reine Angew. Math. **737** (2018), 1–32.

[LMR18] A. Lenzhen, B. Modami, and K. Rafi, *Teichmüller geodesics with d-dimensional limit sets*, Journal of Modern Dynamics **12** (2018), 261–283.

[LS14] L. Liu and W. Su, *The horofunction compactification of the Teichmüller metric*, Handbook of Teichmüller theory. Vol. IV, IRMA Lect. Math. Theor. Phys., vol. 19, Eur. Math. Soc., Zürich, 2014, pp. 355–374.

[Miy08] H. Miyachi, *Teichmüller rays and the Gardiner-Masur boundary of Teichmüller space*, Geom. Dedicata **137** (2008), 113–141.

[Miy13] ———, *Teichmüller rays and the Gardiner-Masur boundary of Teichmüller space II*, Geom. Dedicata **162** (2013), 283–304.

[Miy14a] ———, *Extremal length geometry*, Handbook of Teichmüller theory. Vol. IV, IRMA Lect. Math. Theor. Phys., vol. 19, Eur. Math. Soc., Zürich, 2014, pp. 197–234.

[Miy14b] ———, *Lipschitz algebras and compactifications of Teichmüller space*, Handbook of Teichmüller theory. Vol. IV, IRMA Lect. Math. Theor. Phys., vol. 19, Eur. Math. Soc., Zürich, 2014, pp. 375–413.

[Miy14c] ———, *Unification of extremal length geometry on Teichmüller space via intersection number*, Math. Z. **278** (2014), no. 3-4, 1065–1095.

[PS15] A. Papadopoulos and W. Su, *On the Finsler structure of Teichmüller's metric and Thurston's metric*, Expo. Math. **33** (2015), no. 1, 30–47.

[Ren76] H. Renelt, *Konstruktion gewisser quadratischer differentiale mit hilfe von dirichletintegralen*, Math. Nachr. **73** (1976), no. 1, 125–1142.

[Str66] K. Strebel, *Über quadratische Differentiale mit geschlossenen Trajektorien und extremale quasikonforme Abbildungen*, Festband 70. Geburtstag R. Nevanlinna, Springer, Berlin, 1966, pp. 105–127.

[Thu88] W.P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 2, 417–431.

[Wal14] C. Walsh, *The horoboundary and isometry group of Thurston's Lipschitz metric*, Handbook of Teichmüller theory, IRMA Lect. Math. Theor. Phys., 19, vol. 4, Eur. Math. Soc., Zürich, 2014, pp. 327–353.

[Wal19] ______, *The asymptotic geometry of the Teichmüller metric*, Geom. Dedicata **200** (2019), no. 1, 115–152.

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