

# Helly Property for Subtrees<sup>1</sup>

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**Summary.** We prove, following [5, p. 92], that any family of subtrees of a finite tree satisfies the Helly property.

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The articles [12], [4], [10], [3], [2], [1], [11], [9], [8], [7], and [6] provide the notation and terminology for this paper.

## 1. GENERAL PRELIMINARIES

One can prove the following proposition

- (1) For every non empty finite sequence  $p$  holds  $\langle p(1) \rangle \curvearrowright p = p$ .

Let  $p, q$  be finite sequences. The functor  $\text{maxPrefix}(p, q)$  yields a finite sequence and is defined by:

- (Def. 1)  $\text{maxPrefix}(p, q) \preceq p$  and  $\text{maxPrefix}(p, q) \preceq q$  and for every finite sequence  $r$  such that  $r \preceq p$  and  $r \preceq q$  holds  $r \preceq \text{maxPrefix}(p, q)$ .

Let us observe that the functor  $\text{maxPrefix}(p, q)$  is commutative.

Next we state several propositions:

- (2) For all finite sequences  $p, q$  holds  $p \preceq q$  iff  $\text{maxPrefix}(p, q) = p$ .  
(3) For all finite sequences  $p, q$  holds  $\text{len maxPrefix}(p, q) \leq \text{len } p$ .  
(4) For every non empty finite sequence  $p$  holds  $\langle p(1) \rangle \preceq p$ .  
(5) For all non empty finite sequences  $p, q$  such that  $p(1) = q(1)$  holds  $1 \leq \text{len maxPrefix}(p, q)$ .

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- (6) For all finite sequences  $p, q$  and for every natural number  $j$  such that  $j \leq \text{len maxPrefix}(p, q)$  holds  $(\text{maxPrefix}(p, q))(j) = p(j)$ .
- (7) For all finite sequences  $p, q$  and for every natural number  $j$  such that  $j \leq \text{len maxPrefix}(p, q)$  holds  $p(j) = q(j)$ .
- (8) For all finite sequences  $p, q$  holds  $p \not\leq q$  iff  $\text{len maxPrefix}(p, q) < \text{len } p$ .
- (9) For all finite sequences  $p, q$  such that  $p \not\leq q$  and  $q \not\leq p$  holds  $p(\text{len maxPrefix}(p, q) + 1) \neq q(\text{len maxPrefix}(p, q) + 1)$ .

## 2. GRAPH PRELIMINARIES

Next we state three propositions:

- (10) For every graph  $G$  and for every walk  $W$  of  $G$  and for all natural numbers  $m, n$  holds  $\text{len}(W.\text{cut}(m, n)) \leq \text{len } W$ .
- (11) Let  $G$  be a graph,  $W$  be a walk of  $G$ , and  $m, n$  be natural numbers. If  $W.\text{cut}(m, n)$  is non trivial, then  $W$  is non trivial.
- (12) Let  $G$  be a graph,  $W$  be a walk of  $G$ , and  $m, n, i$  be odd natural numbers. Suppose  $m \leq n \leq \text{len } W$  and  $i \leq \text{len}(W.\text{cut}(m, n))$ . Then there exists an odd natural number  $j$  such that  $(W.\text{cut}(m, n))(i) = W(j)$  and  $j = (m + i) - 1$  and  $j \leq \text{len } W$ .

Let  $G$  be a graph. One can verify that every walk of  $G$  is non empty.

The following propositions are true:

- (13) For every graph  $G$  and for all walks  $W_1, W_2$  of  $G$  such that  $W_1 \preceq W_2$  holds  $W_1.\text{vertices}() \subseteq W_2.\text{vertices}()$ .
- (14) For every graph  $G$  and for all walks  $W_1, W_2$  of  $G$  such that  $W_1 \preceq W_2$  holds  $W_1.\text{edges}() \subseteq W_2.\text{edges}()$ .
- (15) For every graph  $G$  and for all walks  $W_1, W_2$  of  $G$  holds  $W_1 \preceq W_1.\text{append}(W_2)$ .
- (16) For every graph  $G$  and for all trails  $W_1, W_2$  of  $G$  such that  $W_1.\text{last}() = W_2.\text{first}()$  and  $W_1.\text{edges}()$  misses  $W_2.\text{edges}()$  holds  $W_1.\text{append}(W_2)$  is trail-like.
- (17) Let  $G$  be a graph and  $P_1, P_2$  be paths of  $G$ . Suppose  $P_1.\text{last}() = P_2.\text{first}()$  and  $P_1$  is open and  $P_2$  is open and  $P_1.\text{edges}()$  misses  $P_2.\text{edges}()$  and if  $P_1.\text{first}() \in P_2.\text{vertices}()$ , then  $P_1.\text{first}() = P_2.\text{last}()$  and  $P_1.\text{vertices}() \cap P_2.\text{vertices}() \subseteq \{P_1.\text{first}(), P_1.\text{last}()\}$ . Then  $P_1.\text{append}(P_2)$  is path-like.
- (18) Let  $G$  be a graph and  $P_1, P_2$  be paths of  $G$ . Suppose  $P_1.\text{last}() = P_2.\text{first}()$  and  $P_1$  is open and  $P_2$  is open and  $P_1.\text{vertices}() \cap P_2.\text{vertices}() = \{P_1.\text{last}()\}$ . Then  $P_1.\text{append}(P_2)$  is open and path-like.
- (19) Let  $G$  be a graph and  $P_1, P_2$  be paths of  $G$ . Suppose  $P_1.\text{last}() = P_2.\text{first}()$  and  $P_2.\text{last}() = P_1.\text{first}()$  and  $P_1$  is open and  $P_2$  is open and  $P_1.\text{edges}()$

misses  $P_2.\text{edges}()$  and  $P_1.\text{vertices}() \cap P_2.\text{vertices}() = \{P_1.\text{last}(), P_1.\text{first}()\}$ . Then  $P_1.\text{append}(P_2)$  is cycle-like.

- (20) Let  $G$  be a simple graph,  $W_1, W_2$  be walks of  $G$ , and  $k$  be an odd natural number. Suppose  $k \leq \text{len } W_1$  and  $k \leq \text{len } W_2$  and for every odd natural number  $j$  such that  $j \leq k$  holds  $W_1(j) = W_2(j)$ . Let  $j$  be a natural number. If  $1 \leq j \leq k$ , then  $W_1(j) = W_2(j)$ .
- (21) For every graph  $G$  and for all walks  $W_1, W_2$  of  $G$  such that  $W_1.\text{first}() = W_2.\text{first}()$  holds  $\text{len maxPrefix}(W_1, W_2)$  is odd.
- (22) For every graph  $G$  and for all walks  $W_1, W_2$  of  $G$  such that  $W_1.\text{first}() = W_2.\text{first}()$  and  $W_1 \not\subseteq W_2$  holds  $\text{len maxPrefix}(W_1, W_2) + 2 \leq \text{len } W_1$ .
- (23) For every non-multi graph  $G$  and for all walks  $W_1, W_2$  of  $G$  such that  $W_1.\text{first}() = W_2.\text{first}()$  and  $W_1 \not\subseteq W_2$  and  $W_2 \not\subseteq W_1$  holds  $W_1(\text{len maxPrefix}(W_1, W_2) + 2) \neq W_2(\text{len maxPrefix}(W_1, W_2) + 2)$ .

### 3. TREES

A tree is a tree-like graph. Let  $G$  be a graph. A subtree of  $G$  is a tree-like subgraph of  $G$ .

Let  $T$  be a tree. Observe that every walk of  $T$  which is trail-like is also path-like.

One can prove the following proposition

- (24) For every tree  $T$  and for every path  $P$  of  $T$  such that  $P$  is non trivial holds  $P$  is open.

Let  $T$  be a tree. Note that every path of  $T$  which is non trivial is also open.

The following propositions are true:

- (25) Let  $T$  be a tree,  $P$  be a path of  $T$ , and  $i, j$  be odd natural numbers. If  $i < j \leq \text{len } P$ , then  $P(i) \neq P(j)$ .
- (26) Let  $T$  be a tree,  $a, b$  be vertices of  $T$ , and  $P_1, P_2$  be paths of  $T$ . If  $P_1$  is walk from  $a$  to  $b$  and  $P_2$  is walk from  $a$  to  $b$ , then  $P_1 = P_2$ .

Let  $T$  be a tree and let  $a, b$  be vertices of  $T$ . The functor  $T.\text{pathBetween}(a, b)$  yields a path of  $T$  and is defined as follows:

(Def. 2)  $T.\text{pathBetween}(a, b)$  is walk from  $a$  to  $b$ .

One can prove the following propositions:

- (27) For every tree  $T$  and for all vertices  $a, b$  of  $T$  holds  $(T.\text{pathBetween}(a, b)).\text{first}() = a$  and  $(T.\text{pathBetween}(a, b)).\text{last}() = b$ .
- (28) For every tree  $T$  and for all vertices  $a, b$  of  $T$  holds  $a, b \in (T.\text{pathBetween}(a, b)).\text{vertices}()$ .

Let  $T$  be a tree and let  $a$  be a vertex of  $T$ . Observe that  $T.\text{pathBetween}(a, a)$  is closed.

Let  $T$  be a tree and let  $a$  be a vertex of  $T$ .

One can check that  $T.\text{pathBetween}(a, a)$  is trivial.

We now state a number of propositions:

- (29) For every tree  $T$  and for every vertex  $a$  of  $T$  holds  
 $(T.\text{pathBetween}(a, a)).\text{vertices}() = \{a\}$ .
- (30) For every tree  $T$  and for all vertices  $a, b$  of  $T$  holds  
 $(T.\text{pathBetween}(a, b)).\text{reverse}() = T.\text{pathBetween}(b, a)$ .
- (31) For every tree  $T$  and for all vertices  $a, b$  of  $T$  holds  
 $(T.\text{pathBetween}(a, b)).\text{vertices}() = (T.\text{pathBetween}(b, a)).\text{vertices}()$ .
- (32) Let  $T$  be a tree,  $a, b$  be vertices of  $T$ ,  $t$  be a subtree of  $T$ , and  $a', b'$  be vertices of  $t$ . If  $a = a'$  and  $b = b'$ , then  $T.\text{pathBetween}(a, b) = t.\text{pathBetween}(a', b')$ .
- (33) Let  $T$  be a tree,  $a, b$  be vertices of  $T$ , and  $t$  be a subtree of  $T$ . Suppose  $a \in$  the vertices of  $t$  and  $b \in$  the vertices of  $t$ . Then  $(T.\text{pathBetween}(a, b)).\text{vertices}() \subseteq$  the vertices of  $t$ .
- (34) Let  $T$  be a tree,  $P$  be a path of  $T$ ,  $a, b$  be vertices of  $T$ , and  $i, j$  be odd natural numbers. If  $i \leq j \leq \text{len } P$  and  $P(i) = a$  and  $P(j) = b$ , then  $T.\text{pathBetween}(a, b) = P.\text{cut}(i, j)$ .
- (35) For every tree  $T$  and for all vertices  $a, b, c$  of  $T$  holds  
 $c \in (T.\text{pathBetween}(a, b)).\text{vertices}()$  iff  $T.\text{pathBetween}(a, b) = (T.\text{pathBetween}(a, c)).\text{append}((T.\text{pathBetween}(c, b)))$ .
- (36) For every tree  $T$  and for all vertices  $a, b, c$  of  $T$  holds  
 $c \in (T.\text{pathBetween}(a, b)).\text{vertices}()$  iff  $T.\text{pathBetween}(a, c) \preceq T.\text{pathBetween}(a, b)$ .
- (37) For every tree  $T$  and for all paths  $P_1, P_2$  of  $T$  such that  $P_1.\text{last}() = P_2.\text{first}()$  and  $P_1.\text{vertices}() \cap P_2.\text{vertices}() = \{P_1.\text{last}()\}$  holds  $P_1.\text{append}(P_2)$  is path-like.
- (38) For every tree  $T$  and for all vertices  $a, b, c$  of  $T$  holds  
 $c \in (T.\text{pathBetween}(a, b)).\text{vertices}()$  iff  $(T.\text{pathBetween}(a, c)).\text{vertices}() \cap (T.\text{pathBetween}(c, b)).\text{vertices}() = \{c\}$ .
- (39) Let  $T$  be a tree,  $a, b, c, d$  be vertices of  $T$ , and  $P_1, P_2$  be paths of  $T$ . Suppose  $P_1 = T.\text{pathBetween}(a, b)$  and  $P_2 = T.\text{pathBetween}(a, c)$  and  $P_1 \not\subseteq P_2$  and  $P_2 \not\subseteq P_1$  and  $d = P_1(\text{len } \max\text{Prefix}(P_1, P_2))$ . Then  $(T.\text{pathBetween}(d, b)).\text{vertices}() \cap (T.\text{pathBetween}(d, c)).\text{vertices}() = \{d\}$ .

Let  $T$  be a tree and let  $a, b, c$  be vertices of  $T$ . The functor  $\text{middleVertex}(a, b, c)$  yielding a vertex of  $T$  is defined as follows:

- (Def. 3)  $(T.\text{pathBetween}(a, b)).\text{vertices}() \cap (T.\text{pathBetween}(b, c)).\text{vertices}() \cap (T.\text{pathBetween}(c, a)).\text{vertices}() = \{\text{middleVertex}(a, b, c)\}$ .

We now state a number of propositions:

- (40) For every tree  $T$  and for all vertices  $a, b, c$  of  $T$  holds  $\text{middleVertex}(a, b, c) = \text{middleVertex}(a, c, b)$ .
- (41) For every tree  $T$  and for all vertices  $a, b, c$  of  $T$  holds  $\text{middleVertex}(a, b, c) = \text{middleVertex}(b, a, c)$ .
- (42) For every tree  $T$  and for all vertices  $a, b, c$  of  $T$  holds  $\text{middleVertex}(a, b, c) = \text{middleVertex}(b, c, a)$ .
- (43) For every tree  $T$  and for all vertices  $a, b, c$  of  $T$  holds  $\text{middleVertex}(a, b, c) = \text{middleVertex}(c, a, b)$ .
- (44) For every tree  $T$  and for all vertices  $a, b, c$  of  $T$  holds  $\text{middleVertex}(a, b, c) = \text{middleVertex}(c, b, a)$ .
- (45) For every tree  $T$  and for all vertices  $a, b, c$  of  $T$  such that  $c \in (T.\text{pathBetween}(a, b)).\text{vertices}()$  holds  $\text{middleVertex}(a, b, c) = c$ .
- (46) For every tree  $T$  and for every vertex  $a$  of  $T$  holds  $\text{middleVertex}(a, a, a) = a$ .
- (47) For every tree  $T$  and for all vertices  $a, b$  of  $T$  holds  $\text{middleVertex}(a, a, b) = a$ .
- (48) For every tree  $T$  and for all vertices  $a, b$  of  $T$  holds  $\text{middleVertex}(a, b, a) = a$ .
- (49) For every tree  $T$  and for all vertices  $a, b$  of  $T$  holds  $\text{middleVertex}(a, b, b) = b$ .
- (50) Let  $T$  be a tree,  $P_1, P_2$  be paths of  $T$ , and  $a, b, c$  be vertices of  $T$ . If  $P_1 = T.\text{pathBetween}(a, b)$  and  $P_2 = T.\text{pathBetween}(a, c)$  and  $b \notin P_2.\text{vertices}()$  and  $c \notin P_1.\text{vertices}()$ , then  $\text{middleVertex}(a, b, c) = P_1(\text{len maxPrefix}(P_1, P_2))$ .
- (51) Let  $T$  be a tree,  $P_1, P_2, P_3, P_4$  be paths of  $T$ , and  $a, b, c$  be vertices of  $T$ . Suppose  $P_1 = T.\text{pathBetween}(a, b)$  and  $P_2 = T.\text{pathBetween}(a, c)$  and  $P_3 = T.\text{pathBetween}(b, a)$  and  $P_4 = T.\text{pathBetween}(b, c)$  and  $b \notin P_2.\text{vertices}()$  and  $c \notin P_1.\text{vertices}()$  and  $a \notin P_4.\text{vertices}()$ . Then  $P_1(\text{len maxPrefix}(P_1, P_2)) = P_3(\text{len maxPrefix}(P_3, P_4))$ .
- (52) Let  $T$  be a tree,  $a, b, c$  be vertices of  $T$ , and  $S$  be a non empty set. Suppose that for every set  $s$  such that  $s \in S$  holds there exists a subtree  $t$  of  $T$  such that  $s = \text{the vertices of } t$  but  $a, b \in s$  or  $a, c \in s$  or  $b, c \in s$ . Then  $\bigcap S \neq \emptyset$ .

#### 4. THE HELLY PROPERTY

Let  $F$  be a set. We say that  $F$  has Helly property if and only if:

- (Def. 4) For every non empty set  $H$  such that  $H \subseteq F$  and for all sets  $x, y$  such that  $x, y \in H$  holds  $x$  meets  $y$  holds  $\bigcap H \neq \emptyset$ .

One can prove the following proposition

- (53) Let  $T$  be a tree and  $X$  be a finite set such that for every set  $x$  such that  $x \in X$  there exists a subtree  $t$  of  $T$  such that  $x =$  the vertices of  $t$ . Then  $X$  has Helly property.

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