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RESEARCH ARTICLE

Partially-honest Nash implementation: a full characterization

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Abstract
A partially-honest individual is a person who follows the maxim, "Do not lie if you do not have to", to serve your material interest. By assuming that the mechanism designer knows that there is at least one partially-honest individual in a society of $n \geq 3$ individuals, a social choice rule that can be Nash implemented is termed partially-honestly Nash implementable. The paper offers a complete characterization of the (unanimous) social choice rules that are partially-honestly Nash implementable. When all individuals are partially-honest, then any (unanimous) rule is partially-honestly Nash implementable. An account of the welfare implications of partially-honest Nash implementation is provided in a variety of environments.

Keywords Nash implementation · Pure strategy Nash equilibrium · Partial honesty · Condition μ^*(ii)

JEL Classification C72 · D71

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1 Introduction

The implementation problem is the problem of designing a mechanism or game form with the property that, for each state of the world, the equilibrium outcomes of the mechanism played in that state coincide with the recommendations that a given social choice rule (SCR) F would prescribe for that state. If that mechanism design exercise can be accomplished, the SCR is said to be implementable. The fundamental paper on implementation in Nash equilibrium is thanks to Maskin (1999; circulated since 1977), who proves that any SCR that can be Nash implemented satisfies a remarkably strong invariance condition, now widely referred to as Maskin monotonicity. Moreover, he shows that when the mechanism designer faces $n \geq 3$ individuals, a SCR is Nash implementable if it is Maskin monotonic and satisfies the condition of no veto-power, subsequently, *Maskin's theorem*.[1]

Since the introduction of Maskin's theorem, economists have been interested in understanding how to circumvent the limitations imposed by Maskin monotonicity by exploring the possibilities offered by approximate (as opposed to exact) implementation (Matsushima 1988; Abreu and Sen 1991), as well as by implementation in refinements of Nash equilibrium (Moore and Repullo 1988; Abreu and Sen 1990; Palfrey and Srivastava 1991; Jackson 1992; Vartiainen 2007a) and by repeated implementation (Kalai and Ledyard 1998; Lee and Sabourian 2011; Mezzetti and Renou 2012). One additional way around those limitations is offered by implementation with partially-honest individuals.

A *partially-honest individual* is an individual who deceives the mechanism designer when the truth poses some obstacle to her material well-being. Thus, she does not deceive when the truth is equally efficacious. Simply put, a partially-honest individual is an individual who follows the maxim, "Do not lie if you do not have to", to serve your material interest.

In a general environment, a seminal paper on Nash implementation problems involving partially-honest individuals is Dutta and Sen (2012), which shows that for implementation problems involving $n \geq 3$ individuals and in which there is at least one partially-honest individual, the Nash implementability is assured by no veto-power. No veto-power means that if an outcome is at the top of all but one agent's ranking, then it must be selected by the SCR. Similar positive results are uncovered in other environments by Matsushima (2008a, b), Kartik and Tercieux (2012), Kartik et al. (2014), Saporiti (2014), Ortner (2015) and Mukherjee et al. (2017). Thus, there are far fewer limitations for Nash implementation when there are partially-honest individuals.[2]

[1] Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991) and Lombardi and Yoshihara (2013) refined Maskin's theorem by providing necessary and sufficient conditions for a SCR to be implementable in (pure strategies) Nash equilibrium. For two recent excellent surveys on the subject of implementation, see Jackson (2001) and Maskin and Sjöström (2002). For a concise and elegant collection of seminal results on the subject of mechanism design, the reader should consult Dasgupta et al. (1979).

[2] A pioneering work on the impact of decency constraints on Nash implementation problems is Corchón and Herrero (2004). These authors propose restrictions on sets of strategies available to agents that depend on the state of the world. They refer to these strategies as *decent* strategies and study Nash implementation problems in them. For a particular formulation of decent strategies, they are also able to circumvent the limitations imposed by Maskin monotonicity.

A natural question, then, is: where do the exact boundaries of those limitations lie? This question is relevant to test whether a SCR is Nash implementable or not in a society with partially-honest individuals. Indeed, though the no veto-power condition is trivially satisfied in economic environments with three or more non-satiated agents, it is not always an innocuous condition in other environments (as we discuss below). This means that Dutta and Sen's (2012) result does not offer any guidance about the Nash implementability of SCRs violating the no veto-power condition. The paper offers this guidance by providing a complete characterization for Nash implementation with partially-honest individuals when there are three or more individuals. Thus, it provides the counterpart to Moore and Repullo's (1990) conditions—which are summarized as Condition μ by Moore and Repullo (1990)—for a many-person setting with partially-honest individuals.

The necessary and sufficient conditions are derived by using the approach developed by Moore and Repullo (1990). They are derived under the following informational requirement: Although the mechanism designer knows that there are partially-honest agents, he knows neither their identity nor how many agents are partially-honest. In Sect. 3, we present the first part of our conditions, which is named *Condition* μ^**(ii)*. To understand its content, suppose that F can be Nash implemented with partially-honest individuals. Suppose that x belongs to $F(\theta)$. Then, there exists an equilibrium of the implementing mechanism that induces x under the state θ. Condition μ^*(ii) identifies situations when a deviation from the equilibrium under θ is not followed by further deviations when the state changes from θ to θ'. Let us suppose that the deviation of agent i from the equilibrium under θ results into an outcome y. Moreover, suppose that we know that y is θ'-maximal for agent i over the range of outcomes that i can generate by deviating from the equilibrium under θ, and that y is θ'-maximal in the range of the implementing mechanism for every other individual. Let us distinguish two situations. (a) If we know that agent i has deviated to a strategy that is truthful at the new state θ' and he is the unique partially-honest agent in society, then y must be an equilibrium under θ'. (b) If we know that $\theta' = \theta$, that agent i is not a partially-honest individual and that every partially-honest individual is playing a truthful strategy under the new state θ', then y must be an equilibrium under θ'. Note that since Condition μ^*(ii) applies to situations in which not all agents are partially-honest, this condition does not have any bite when all agents are partially-honest and the mechanism designer knows this. A more detailed discussion about Condition μ^*(ii) is provided in Sect. 3.

Condition μ^*(ii) is reasonably weak (albeit somewhat complex). From Dutta and Sen's (2012) work, we know that no veto-power is sufficient for Nash implementation with partially-honest individuals. However, our condition is much weaker than no veto-power. This is important because no veto-power is not always a weak requirement, such as in bargaining environments and in marriage problems. Moreover, Condition μ^* (ii) does not include any Maskin monotonicity-type conditions. Remarkably, we have found that when there are three or more individuals, and when the admissible domain of SCRs is restricted to that of unanimous ones, then Condition μ^*(ii) is not only necessary but is also sufficient for any unanimous SCR to be Nash implemented (Theorem 1). A SCR is said to be unanimous if it satisfies the following property: if every agent prefers x to any other outcome in state θ, then $x \in F(\theta)$. For the sufficiency part, a constructive proof is provided. Indeed, we construct a mechanism in which each participant chooses the information about a state as part of her strategy choice and in which a participant's play is honest if she plays a strategy choice which is veracious in its state announcement component.[3] Theorem 2 shows that every unanimous SCR is implementable when all agents are partially-honest and the mechanism designer knows this.

By means of Condition μ^{*}(ii), we can easily check what SCR is implementable within the domain of unanimous SCRs. For instance, in the coalitional game environment, we show that the *core* solution is not Nash implementable with partially-honest individuals when the mechanism designer knows the coalitional function of the games, who, however, does not know the prevailing state. As we already noted, we also show that every unanimous SCR is Nash implementable in a society in which all individuals are partially-honest and the designer knows it. This means that the core solution is Nash implementable when all individuals are partially-honest. These examples clarify that the common assumption in the present literature on implementation problems with partial honesty that the designer knows that all individuals are partially-honest is *not* innocuous.

Finally, we also analyze interesting and well-known SCRs in bargaining environments as well as in marriage problems. Indeed, we show that the *Nash bargaining solution* and the *man-optimal-stable solution* are partially-honestly Nash implementable. Note that each of these SCRs is unanimous. However, they both violate the no veto-power condition.

The remainder of this paper is divided into 4 sections. Section 2 sets out the theoretical framework and outlines the basic model. Section 3 completely characterizes the class of Nash implementable SCRs satisfying the unanimity condition and assesses its implications in a variety of environments. Section 4 offers a brief discussion of the second part of the necessary and sufficient conditions for Nash implementation with partially-honest individuals in connection with the implementability of the egalitarian bargaining solution.[4] Section 5 concludes. Appendix includes proofs not in the main body.

2 Preliminaries

2.1 Basic framework

We consider a finite set of individuals indexed by $i \in N = \{1, \ldots, n\}$, which we will refer to as a society. The set of outcomes available to individuals is X. The information held by the individuals is summarized in the concept of a state, which is a complete description of the variable characterizing the world. Write Θ for the domain of possible states, with θ as a typical state. In the usual fashion, individual i's preferences in state θ are given by a complete and transitive binary relation, subsequently an ordering,

[3] We construct a canonical mechanism, which is subject to standard criticisms (see, Jackson (1992; 2001), for discussion). The usual counterargument to these criticisms is that general results need to rely on canonical mechanisms (see, again Jackson (1992), for a discussion).

[4] We have provided only a short discussion for the sake of brevity. However, a detailed analysis is provided in Lombardi and Yoshihara (2019).

$R_i(\theta)$ over the set X. The corresponding strict and indifference relations are denoted by $P_i(\theta)$ and $I_i(\theta)$, respectively. The statement $x R_i(\theta) y$ means that individual i judges x to be at least as good as y. The statement $x P_i(\theta) y$ means that individual i judges x better than y. Finally, the statement $x I_i(\theta) y$ means that individual i judges x and y as equally good, that is, she is indifferent between them.

We assume that the mechanism designer does not know the true state, that there is complete information among the individuals in N and that the mechanism designer knows the preference domain consistent with the domain Θ. We shall sometimes identify states with preference profiles.

The goal of the mechanism designer is to implement a SCR F, which is a correspondence $F : \Theta \twoheadrightarrow X$ such that $F(\theta)$ is non-empty for every $\theta \in \Theta$. We shall refer to $x \in F(\theta)$ as an F-optimal outcome at θ. The image or range of the SCR F is the set $F(\Theta) \equiv \{x \in X | x \in F(\theta) \text{ for some } \theta \in \Theta\}$.

Given that individuals will have to be given the necessary incentives to reveal the state truthfully, the mechanism designer delegates the choice to individuals according to a mechanism $\Gamma \equiv \left(\prod_{i \in N} M_i, g\right)$, where M_i is the strategy space of individual i and $g : M \rightarrow X$, the outcome function, assigns to every strategy profile $m \in M \equiv \prod_{i \in N} M_i$ a unique outcome in X. The strategy profile m_{-i} is obtained from m by omitting the ith component, that is, $m_{-i} = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n)$, and we identify (m_i, m_{-i}) with m.

2.2 Intrinsic preferences for honesty

An individual who has an intrinsic preference for truth-telling can be thought of as an individual who is torn by a fundamental conflict between her deeply and ingrained propensity to respond to material incentives and the desire to think of herself as an honest person. In this paper, the theoretical construct of the balancing act between those contradictory desires is based on two ideas.

First, the pair (Γ, θ) acts as a " context" for individuals' conflicts. The reason for this is that an individual who has an intrinsic preference for honesty can categorize her strategy choices as truthful or untruthful relative to the state θ and the mechanism Γ designed by the mechanism designer to govern the communication with individuals. That categorization can be captured by the following notion of truth-telling correspondence:

Definition 1 For each Γ and each individual $i \in N$, individual i's *truth-telling correspondence* is a (non-empty) correspondence $T_i^{\Gamma} : \Theta \twoheadrightarrow M_i$ such that, for each $\theta \in \Theta$, $T_i^{\Gamma}(\theta) \subseteq M_i$. Strategy choices in $T_i^{\Gamma}(\theta)$ will be referred to as truthful strategy choices for θ.

What messages can be regarded as truthful ones depends on the message space M_i specified by Γ. For instance, if Γ is a canonical mechanism where $M_i = \Theta \times X \times N$ for each individual i, then agent i's truth-telling corresponding T^{Γ} can be defined by $T_i^{\Gamma}(\theta) = \{\theta\} \times X \times N$ for each state $\theta \in \Theta$.[5] This means that a message is truthful if

[5] Even within the class of canonical mechanisms, there are different ways in which one can define the truth-telling correspondence—see Lombardi and Yoshihara (2018).

it is truthful in its state announcement component. Indeed, Dutta and Sen (2012) used this type of truth-telling correspondence in their characterization result. It is important to emphasize that Condition μ^*(ii) is a necessary condition for implementation with partially-honest individuals for every notion of truth-telling. Like Dutta and Sen (2012), our sufficiency result is derived by assuming that $T_i^{\Gamma}(\theta) = \{\theta\} \times X \times N$ for each state $\theta \in \Theta$ and each individual $i \in N$.

Second, in modeling intrinsic preferences for honesty, we endorse the notion of partially-honest individuals introduced by Dutta and Sen (2012). First, a partially-honest individual is an individual who responds primarily to material incentives. Second, she strictly prefers to tell the truth whenever lying has no effect on her material well-being. That behavioral choice of a partially-honest individual can be modeled by extending an individual's ordering over X to an ordering over the strategy space M because individual's preference between being truthful and being untruthful is contingent upon announcements made by other individuals as well as the outcome(s) obtained from them. By following standard conventions of orderings, write $\succcurlyeq_i^{\Gamma,\theta}$ for individual i's ordering over M in state θ whenever she is confronted with the mechanism Γ. Formally, our notion of a partially-honest individual is as follows:

Definition 2 For each Γ, individual $i \in N$ is *partially-honest* if for all $\theta \in \Theta$ individual i's intrinsic preference for honesty $\succcurlyeq_i^{\Gamma,\theta}$ on M satisfies the following properties: for all m_{-i} and all $m_i, m_i' \in M_i$ it holds that:

(i) If $m_i \in T_i^{\Gamma}(\theta)$, $m_i' \notin T_i^{\Gamma}(\theta)$ and $g(m)\, R_i(\theta)\, g\left(m_i', m_{-i}\right)$, then $m \succ_i^{\Gamma,\theta} \left(m_i', m_{-i}\right)$.
(ii) In all other cases, $m \succcurlyeq_i^{\Gamma,\theta} \left(m_i', m_{-i}\right)$ if and only if $g(m)\, R_i(\theta)\, g\left(m_i', m_{-i}\right)$.

An intrinsic preference for honesty of individual i is captured by the first part of the above definition, in that, for a given mechanism Γ and state θ, individual i strictly prefers the strategy profile (m_i, m_{-i}) to $\left(m_i', m_{-i}\right)$ provided that the outcome $g(m_i, m_{-i})$ is at least as good as $g\left(m_i', m_{-i}\right)$ according to her ordering $R_i(\theta)$ and that m_i is truthful for θ and m_i' is not truthful for θ.

If individual i is *not* partially-honest, this individual cares for her material well-being associated with outcomes of the mechanism and nothing else. Then, individual i's ordering over M is just the transposition into space M of individual i's relative ranking of outcomes: $m \succcurlyeq_i^{\Gamma,\theta} m' \iff g(m)\, R_i(\theta)\, g\left(m'\right)$ for all $m, m' \in M$.

2.3 Implementation problems

In formalizing the mechanism designer's problem with partially-honest individuals, we first introduce an informational assumption and discuss its implications for our analysis. It is:

Assumption 1 There exists at least one partially-honest individual in the society N.

Thus, in our setting, the mechanism designer does not know the true state and, moreover, he knows neither the identity (or identities) nor the number of the partially-honest individual(s). Indeed, the mechanism designer cannot exclude any member(s) of society from being partially-honest purely on the basis of Assumption 1. Therefore, the following considerations are in order from the viewpoint of the mechanism designer.

An environment is described by two parameters, (θ, H): a state θ and a conceivable set of partially-honest individuals H. We denote by H a typical conceivable set of partially-honest individuals in N, with h as a typical element, and by $\mathcal{H}$ the class of conceivable sets of partially-honest individuals.

A mechanism Γ and an environment (θ, H) induce a strategic game $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$, where:

$$\succcurlyeq^{\Gamma,\theta,H} \equiv \left(\succcurlyeq_i^{\Gamma,\theta}\right)_{i \in N}$$

is a profile of orderings over the strategy space M as formulated in Definition 2. Specifically, $\succcurlyeq_i^{\Gamma,\theta}$ is individual i's ordering over M as formulated in Definition 2 if individual i is in H, whereas it is the individual i's ordering over M defined as the transposition into M of individual i's ordering over X if individual i is not in H.

A (pure strategy) Nash equilibrium of the strategic game $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$ is a strategy profile m such that for all $i \in N$, it holds that

$$m \succcurlyeq_i^{\Gamma,\theta} \left(m_i', m_{-i}\right), \qquad \text{for all } m_i' \in M_i.$$

Write $NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta H}\right)$ for the set of Nash equilibrium strategies of the strategic game $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$ and $NA\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$ for its corresponding set of Nash equilibrium outcomes.

The following definition is to formulate the designer's Nash implementation problem involving partially-honest individuals.

Definition 3 Let Assumption 1 hold. A mechanism Γ *partially-honestly Nash implements* the SCR $F : \Theta \twoheadrightarrow X$ provided that for all $\theta \in \Theta$ there exists a truth-telling correspondence $T_i^{\Gamma}(\theta)$ as formulated in Definition 1 for every $i \in N$ and, moreover, it holds that

$$F(\theta) = NA\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right), \qquad \text{for every pair } (\theta, H) \in \Theta \times \mathcal{H}.$$

If such a mechanism exists, F is said to be *partially-honestly Nash implementable*.

The objective of the mechanism designer is thus to design a mechanism whose Nash equilibrium outcomes coincide with $F(\theta)$ for each state θ as well as each set H. Note that there is no distinction between the above formulation and the standard Nash implementation problem as long as Assumption 1 is discarded.

3 The characterization theorem for unanimous SCRs

In this section, we provide a full characterization of the class of unanimous SCRs that are partially-honestly Nash implementable:

Definition 4 The SCR $F : \Theta \twoheadrightarrow X$ satisfies *unanimity* provided that for all $\theta \in \Theta$ and all $x \in X$ if $x R_i(\theta) y$ for all $i \in N$ and all $y \in X$, then $x \in F(\theta)$. A SCR that satisfies this property is said to be a unanimous SCR.

In other words, it states that if an outcome is at the top of the preferences of all individuals, then it should be selected by the SCR. Unanimity is a property satisfied, for example, by the Pareto rule and, in the market contexts, by the rule which selects all core allocations.

We introduce below Condition μ^*(ii), which is necessary and sufficient for partially-honest implementation of unanimous SCRs in many-individual settings. Before doing it, let us introduce Condition μ for the sake of completeness and clarity. Moore and Repullo (1990) show that, for a society with more than two agents, Condition μ is the necessary and sufficient condition for any SCR to be Nash implementable. Let us formalize it as follows. Given a state θ, an individual i, a set of outcomes $A \subseteq X$ and an outcome $x \in X$, the *indifference set of* $R_i(\theta)$ *at* $x \in X$ *restricted to* A is $I_i(\theta, x, A) = \left\{x' \in A | x I_i(\theta) x'\right\}$; the *weak lower contour set of* $R_i(\theta)$ *at* x is $L_i(\theta, x) = \{x' \in X | x R_i(\theta) x'\}$; and the *strict lower contour set of* $R_i(\theta)$ *at* x is $SL_i(\theta, x) = \{x' \in X | x P_i(\theta) x'\}$. Therefore:

Condition μ. There exists a set $Y \subseteq X$; moreover, for all $\theta \in \Theta$ and all $x \in F(\theta)$, there is a profile of sets $(C_\ell(\theta, x))_{\ell \in N}$ such that $x \in C_\ell(\theta, x) \subseteq L_\ell(\theta, x) \cap Y$ for all $\ell \in N$; finally, for all $\theta' \in \Theta$, the following Conditions (i)-(iii) are satisfied:

(i) if $C_\ell(\theta, x) \subseteq L_\ell\left(\theta', x\right)$ for all $\ell \in N$, then $x \in F\left(\theta'\right)$;
(ii) for each $i \in N$, if $y \in C_i(\theta, x) \subseteq L_i\left(\theta', y\right)$ and $Y \subseteq L_\ell\left(\theta', y\right)$ for all $\ell \in N \setminus \{i\}$, then $y \in F\left(\theta'\right)$;
(iii) if $y \in Y \subseteq L_\ell\left(\theta', y\right)$ for all $\ell \in N$, then $y \in F\left(\theta'\right)$.[6]

Condition μ(i) is equivalent to Maskin monotonicity, while Conditions μ(ii) and μ(iii) are weaker versions of no veto-power. Note that Conditions μ(iii) is satisfied by any unanimous SCR— to see it, let consider $Y = X$. Also, note that Condition μ requires the existence of the set Y as well as the existence of the set $C_i(\theta, x)$ for each triplet (i, x, θ) with $x \in F(\theta)$.

Let us formalize Condition μ^*(ii) as follows.

Definition 5 The SCR $F : \Theta \twoheadrightarrow X$ satisfies *Condition* μ^*(ii) if there exists $Y \subseteq X$ such that $F(\Theta) \subseteq Y$ and such that the following statements hold: For every $(i, \theta, x) \in N \times \Theta \times Y$ with $x \in F(\theta)$, there exists a set $C_i(\theta, x) \subseteq Y$ with $x \in C_i(\theta, x) \subseteq L_i(\theta, x)$, such that for every pair $\left(\theta', H\right) \in \Theta \times \mathcal{H}$ we have:

(1) (a) There exists a non-empty set $S_i\left(\theta'; x, \theta\right)$ such that $S_i\left(\theta'; x, \theta\right) \subseteq C_i(\theta, x)$.
(b) For all $h \in H$, if $\theta = \theta'$ and $x \notin S_h\left(\theta'; x, \theta\right)$, then $S_h\left(\theta'; x, \theta\right) \subseteq SL_h(\theta, x)$.
(2) If $y \in C_i(\theta, x) \subseteq L_i\left(\theta', y\right)$, $Y \subseteq L_j\left(\theta', y\right)$ for all $j \in N \setminus \{i\}$, and $y \notin F\left(\theta'\right)$, then:

(a) if $H = \{i\}$, then the intersection $S_i\left(\theta'; x, \theta\right) \cap I_i\left(\theta', y, Y\right)$ is not empty and $y \notin S_i\left(\theta'; x, \theta\right)$.
(b) if $i \notin H$ and $\theta = \theta'$, then $x \notin S_j\left(\theta'; x, \theta\right)$ for some $j \in H$.

[6] We refer to the condition that requires only one of the conditions (i)–(iii) in Condition μ as Conditions μ(i)–μ(iii) respectively. Note that Condition μ implies Conditions μ(i)–μ (iii), but the converse is not true. We use similar conventions below.

Like Condition μ, Condition μ^*(ii) requires the existence of Y and $(C_i(\theta, x))_{i \in N}$ as well. Their interpretation is offered by Moore and Repullo (1990). The set Y represents the range of the mechanism by which a given SCR F is Nash implementable. The set $C_i(\theta, x)$ represents individual i's attainable set when the equilibrium outcome $x \in F(\theta)$ is selected by the outcome function. In contrast to Condition μ, part (1)(a) of Condition μ^*(ii) also requires the existence of a set $S_i(\theta'; x, \theta) \subseteq C_i(\theta, x)$ for every quadruplet (i, x, θ, θ') with $x \in F(\theta)$.

Let us present our condition from the viewpoint of necessity. To this end, suppose that F is partially-honestly implementable by a mechanism Γ. Suppose that $x \in F(\theta)$. Then, there exists a Nash equilibrium strategy profile m such that $g(m) = x$. Let us define $C_i(\theta, x)$ by $C_i(\theta, x) = g(M_i, m_{-i}) \equiv \{g(m'_i, m_{-i}) | m'_i \in M_i\}$, which represents the set of outcomes that individual i can generate by varying her own strategy, keeping the other individuals' equilibrium strategy choices fixed at m_{-i}. Let us define the set $S_i(\theta'; x, \theta)$ by $S_i(\theta'; x, \theta) = g(T_i^{\Gamma}(\theta'), m_{-i}) \equiv \{g(m'_i, m_{-i}) | m'_i \in T_i^{\Gamma}(\theta')\}$, which represents the set of outcomes that this individual can attain by playing truthful strategy choices for θ' when the state moves from θ to θ', keeping the other individuals' equilibrium strategy choices fixed at m_{-i}. Given this definition of $S_i(\theta'; x, \theta)$, we refer to elements of $S_i(\theta'; x, \theta)$ as *truthful outcomes* for individual i at the state θ' when the state moves from θ to θ' and $x \in F(\theta)$.

Part (1)(b) of Condition μ^*(ii) follows the reasoning that if $x \in F(\theta)$ but x is not a truthful outcome for the partially-honest individual $h \in H$ at this θ—that is, $x = g(m) \notin S_h(\theta; x, \theta) = g(T_i^{\Gamma}(\theta), m_{-i})$, then, in order to keep m as a Nash equilibrium strategy profile at θ it must be the case that x is strictly preferred by agent h to any truthful outcome in $S_h(\theta; x, \theta)$ according to her ordering $R_h(\theta)$.

For part (2) of Condition μ^*(ii), suppose that the state moves from θ to θ', that agent i's attainable outcome $y = g(m'_i, m_{-i}) \in C_i(\theta, x)$ is $R_i(\theta')$-maximal in the set $C_i(\theta, x)$ and that y is also $R_j(\theta')$-maximal for any other individual $j \neq i$ in the set $Y = g(M) \equiv \{g(\bar{m}) | \bar{m} \in M\}$. Also, suppose that $y \notin F(\theta')$. One can now see that only a partially-honest individual h can find it profitable to unilaterally deviate from (m'_i, m_{-i})—if no agent has incentive to unilaterally deviate, then $y \in F(\theta')$, by the implementability, which is a contradiction.

Part (2)(a) specifies that if only individual i can find a unilateral profitable deviation from (m'_i, m_{-i}), then the outcome $y = g(m'_i, m_{-i})$ is not a truthful outcome for i at θ'—that is, $y \notin S_i(\theta'; x, \theta) = g(T_i^{\Gamma}(\theta'), m_{-i})$. In addition, part (2)(a) also requires that individual i needs to find a truthful outcome $z = g(m''_i, m_{-i}) \in S_i(\theta'; x, \theta) = g(T_i^{\Gamma}(\theta'), m_{-i})$ that is equally good to y according to her ordering $R_i(\theta')$ in order to have a unilateral non-material profitable deviation—that is, $z \in S_i(\theta'; x, \theta) \cap I_i(\theta', y, Y)$.

Part (2)(b) specifies that if $\theta = \theta'$ and individual i is not a partially-honest individual—that is, $i \notin H$, then it cannot be that the deviant partially-honest individual $h \in H$ plays a truthful strategy choice at m—that is, $x \notin S_h(\theta'; x, \theta)$. Indeed, if $m_h \in T_h^{\Gamma}(\theta)$ for each $h \in H$, no partially-honest individual can find a profitable unilateral deviation in order to eliminate y from the set of Nash equilibrium outcomes at $\theta = \theta'$.

Condition μ implies Condition μ^*(ii). It is clear that Condition μ(ii) implies part (2) of Condition μ^*(ii). Then, we are left to show that part (1) of Condition μ^*(ii) is satisfied as well. To see it, let us define $\left(S_i\left(\theta'; x, \theta\right)\right)_{i \in N} \equiv \left(C_i\left(\theta, x\right)\right)_{i \in N}$ for every $\theta, \theta' \in \Theta$ and every $x \in F(\theta)$. Therefore, Condition μ implies Condition μ^*(ii).

We are now ready to present our characterization result for unanimous SCRs. However, before stating it, we assume that the structure of the family $\mathcal{H}$ satisfies the following specification:

Assumption 2 The family $\mathcal{H}$ has as elements all non-empty subsets of the set N.

This requirement is consistent with, and a natural extension of Assumption 1 since the mechanism designer cannot exclude any member(s) of the society from being partially-honest purely on the basis of that assumption. Indeed, this assumption is the natural consequence of Assumption 1. The characterization theorem can be stated as follows:

Theorem 1 *Let $n \geq 3$. Suppose that Assumptions 1–2 hold. The unanimous SCR $F : \Theta \twoheadrightarrow X$ satisfies Condition μ^*(ii) if and only if it is partially-honestly Nash implementable.*

Proof See Appendix A. □

We make several remarks below regarding Theorem 1.

Remark 1 The "if" part of the theorem continues to hold if Assumption 2 is replaced with the requirement that the family $\mathcal{H}$ *contains* N or it is *closed under union*. $\mathcal{H}$ is closed under union when the following property holds: If H is an element of $\mathcal{H}$ and if H' is another of its elements, then the union of these sets is also an element of $\mathcal{H}$. Clearly, these specifications are weaker than Assumption 2. Moreover, the specification that $\mathcal{H}$ is closed under union has an obvious expansion-consistency interpretation: If the mechanism designer views H as a conceivable set of partially-honest individuals and he also views H' as another conceivable set, then there is no reason for him to exclude their union from $\mathcal{H}$ purely on the basis of Assumption 1. The specification that $\mathcal{H}$ contains N is the minimal restriction on the family $\mathcal{H}$ that allows part (1)(b) of Condition μ^*(ii) to still be a necessary condition for partially-honest implementation. The reason is that to assure it we need to be able to select a strategy profile m that generates the F-optimal outcome x at θ as a Nash equilibrium outcome for this θ and for a set of partially-honest individuals N which contains all elements of the family $\mathcal{H}$, that is, $H \subseteq N$ for every $H \in \mathcal{H}$. This is because if N is an element of the family $\mathcal{H}$, then the strategy profile m supporting the F-optimal x at the state θ as a Nash equilibrium of $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,N}\right)$ is also a Nash equilibrium of $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$ for every other allowable set H. This allows us to show that part (1)(b) of Condition μ^*(ii) applies to whatever conceivable set of partially-honest individuals.

Remark 2 Condition μ^*(ii) is a necessary condition for partially-honest Nash implementation when $n \geq 2$ and when the family $\mathcal{H}$ is closed under union or it contains N.

Remark 3 The "only if" part of the theorem continues to hold if Assumption 2 is replaced with the requirement that the family $\mathcal{H}$ includes *all singletons of the set* N. This is because if m is a Nash equilibrium of some strategic game $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$ and if individual i's strategy choice m_i is a truthful one for the state θ, then this m is also a Nash equilibrium of the strategic game $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,\{i\}}\right)$ provided that the singleton $\{i\}$ is an element of $\mathcal{H}$.

Common to the literature of implementation with partially-honest individuals is also the requirement that every member of society has a taste for honesty, as per Matsushima (2008b), Dutta and Sen (2012), Saporiti (2014) and Mukherjee et al. (2017). Thus, if we follow these authors and confine our analysis to this case, we have the following characterization theorem as well:

Theorem 2 *Let $n \geq 3$ and let all individuals in N be partially-honest. Then, every unanimous SCR $F : \Theta \twoheadrightarrow X$ is partially-honestly Nash implementable.*

Proof It follows from the proof of Theorem 1, with the observation that in this case $\mathcal{H} = \{N\}$ and no Nash equilibrium strategy profile can fall into *Rule 2.2* as well as into *Rule 2.3* of the constructed mechanism. Indeed, when $\mathcal{H} = \{N\}$, every unanimous SCR satisfies Condition μ^* (ii) under the specification that the set $Y = X$ and that $S_i\left(\theta'; x, \theta\right) = C_i(\theta, x) = L_i(\theta, x)$ for every quadruplet $\left(i, \theta, \theta', x\right)$ such that x is an F-optimal outcome at θ.[7] □

In the following subsections, we propose several settings in which Theorem 1 is applied.

3.1 Applications to coalitional games

This subsection presents the *core* solution, which is the main set solution used for coalitional games, and it shows that this solution is not partially-honestly Nash implementable. It is well known that this solution is not Maskin monotonic and it violates the condition of no veto-power.

A *coalitional game* is a quadruplet $(N, X, \theta; \upsilon)$ such that:

- N is a finite set of individuals. A subset S of N is called a coalition. The class of all non-empty coalitions is denoted by $\mathcal{P}(N)$.
- X is a set of outcomes.
- θ is a state in Θ.
- $\upsilon : \mathcal{P}(N) \rightarrow 2^X$ is a function associating every element of class $\mathcal{P}(N)$ with a subset of the set X, where 2^X is a family that has as elements all subsets of X. This function is called the coalitional function of the game. $\upsilon(S)$ specifies the set of outcomes for which coalition S has the power to move to.[8]

Let $(N, X, \theta; \upsilon)$ be a coalitional game. An outcome $x \in X$ is *weakly blocked by a coalition* $S \in \mathcal{P}(N)$ if there is an outcome $y \in \upsilon(S)$ such that $yR_j(\theta)x$ for every member j of S, with $yP_j(\theta)x$ for at least one of its members.

[7] Note that part (2) of Condition μ^*(ii) is satisfied vacuously in the case in which $\mathcal{H} = \{N\}$.

[8] $\upsilon(S) = \varnothing$ means that coalition S does not have any power.

Definition 6 The *core solution* of a coalitional game $(N, X, \theta; \upsilon)$, denoted by $\mathcal{C}$, is the collection of all outcomes that are not weakly blocked by any coalition S,

$$\mathcal{C}(\theta) \equiv \left\{ \begin{array}{c} x \in X | \text{for every } S \in \mathcal{P}(N) \text{ and } y \in \upsilon(S) : x P_j(\theta) y \text{ for some } j \in S, \text{ or} \\ x R_j(\theta) y \text{ for all } j \in S \end{array} \right\}.$$

The following claim establishes the failure of partially-honestly Nash implementing the core solution when the mechanism designer knows what is feasible for every element of $\mathcal{P}(N)$, that is, he knows the coalitional function, and he does not know the true state.[9]

Claim 1 Let $n \geq 3$. Let Assumption 2 be given. Then, the core solution does not satisfy Condition μ^*(ii).

Proof Let the premises hold. Assume, to the contrary, that the core solution satisfies Condition μ^*(ii).

Since the core solution is unanimous, $Y = X$ as per Sjöström (1991), and so Y contains the range of $\mathcal{C}$.

Suppose that there are three individuals and two states θ and θ'. Individuals' preferences are represented in the table below:

θ			θ'		
1	2	3	1	2	3
y, z	x	w	y	w, x, y, z	w, x, y, z
x	w, y, z	x, z	x		
w		y	w, z		

where, as usual, $\frac{a}{b}$ for individual i means that she strictly prefers a to b, while a, b means that this i is indifferent between a and b. Suppose that the coalitional function is defined as follows:

$$\upsilon(\{1,2\}) = \{x, z\}, \upsilon(\{1,3\}) = \{w, y\}, \upsilon(\{2,3\}) = \{w, z\},$$
$$\upsilon(N) = X \text{ and } \upsilon(S) = \varnothing \text{ for every other } S \in \mathcal{P}(N).$$

In the coalitional game $(N, X, \theta; \upsilon)$, the core solution contains only the outcome x. To see this, note that w is weakly blocked by coalition $\{1, 2\}$ (via outcome x), and that y and z are both weakly blocked by coalition $\{2, 3\}$ (via outcome w). However, in the coalitional game $(N, X, \theta'; \upsilon)$, the core solution contains only the outcome y since every other outcome is weakly blocked by coalition $\{1, 3\}$.

Since $\mathcal{C}(\theta) = \{x\}$, Condition μ^*(ii) implies that there exists $(C_i(\theta, x))_{i \in N}$ such that $x \in C_i(\theta, x) \subseteq L_i(\theta, x)$ for each $i \in N$. Since $L_1(\theta', x) = \{x, w, z\} \supseteq L_1(\theta, x) = \{x, w\}$, it follows that $x \in C_1(\theta, x) \subseteq L_1(\theta, x) \subseteq L_1(\theta', x)$. Since $L_2(\theta', x) = Y = L_3(\theta', x)$ and since $x \notin \mathcal{C}(\theta')$, part (2)(a) of Condition μ^* (ii) implies for $H = \{1\}$ that $x \notin S_1(\theta'; x, \theta)$ and that the intersection $S_1(\theta'; x, \theta) \cap I_1(\theta', x, Y)$ is

[9] The following claim holds if Assumption 2 is replaced with the assumption that $\mathcal{H}$ contains N and singletons.

not empty.[10] However, since $x \notin S_1(\theta'; x, \theta)$ and since $I_1(\theta', x, Y) = \{x\}$, it follows that the intersection $S_1(\theta'; x, \theta) \cap I_1(\theta', x, Y)$ is empty, which is a contradiction.

We have proved the claim by assuming that $n = 3$. The proof will be identical for $n > 3$: just endow individual $k > 3$ with the same preferences of individual 3 considered above and just change the coalitional function as follows: $\upsilon(\{1,3\}) = \upsilon(\{1,k\})$ and $\upsilon(\{2,3\}) = \upsilon(\{2,k\})$. □

Claim 1 still holds when Assumption 2 is replaced with the assumption that the family $\mathcal{H}$ includes all singletons. As noted above in Remark 2, Condition μ^*(ii) is also a necessary condition for implementation when $n = 2$. By a reasoning like that used in the above claim, one can also show that the core solution violates Condition μ^* (ii) when $n = 2$.[11]

3.2 Applications to marriage problems

This subsection presents the basic model of matching men to women and shows that the *man-optimal stable* solution can be successfully partially-honestly Nash implemented. This result is in contrast to the literature on Nash implementation of matching solutions where no proper sub-solution of the stable solution is Nash implementable in the class of marriage games with singles—as per Kara and Sönmez (1996)—and where no single-valued sub-solution of the stable solution is Nash implementable in the class of pure marriage games, where being single is not a feasible choice or it is always the last choice of every individual—as per Tadenuma and Toda (1998).

A *marriage problem* is a quadruplet $(M, W, \theta, \mathcal{M})$ such that:

- M is a finite non-empty set of men, with m as a typical element.
- W is a finite non-empty set of women, with w as a typical element.

[10] Recall that $I_i(\theta', x, Y) = \{x' \in Y | x I_i(\theta') x'\}$.

[11] To see this, suppose that there are two individuals and two states θ and θ'. Individuals' preferences are represented in the table below:

θ		θ'	
1	2	1	2
y	z	y	x, y, z
x	x	x	
z	y	z	

Suppose that the coalitional function is defined as follows:

$$\upsilon(\{1,2\}) = X \text{ and } \upsilon(S) = \varnothing \text{ for every other } S \in \mathcal{P}(N).$$

In the coalitional game $(N, X, \theta; \upsilon)$, the core solution contains only the outcome x. In addition, in the coalitional game $(N, X, \theta'; \upsilon)$, the core solution contains only the outcome y. Thus, as in the above claim, $x \in \mathcal{C}(\theta)$ but $x \notin \mathcal{C}(\theta')$ and, moreover, $L_1(\theta, x) = L_1(\theta', x)$ and $X \subseteq L_2(\theta', x)$. Since the singleton $\{1\}$ is an element of the family $\mathcal{H}$, one can now easily check that the core solution violates part (2)(a) of Condition μ^*(ii) under the specification that $Y = X$. The reason is that there cannot exist any outcome $z \neq x$ in the set $L_1(\theta, x)$ such that individual 1 is indifferent between this z and x according to her ordering $R_1(\theta')$.

- θ is a state such that (i) every man $m \in M$'s preferences are represented by a linear ordering $P_m(\theta)$ over the set $W \cup \{m\}$ and (ii) every woman $w \in W$'s preferences are represented by a linear ordering $P_w(\theta)$ over the set $M \cup \{w\}$.
- $\mathcal{M}$ is a collection of all matchings, with μ as a typical element. $\mu : M \cup W \to M \cup W$ is a bijective function matching every individual $i \in M \cup W$ either with a partner of the opposite sex or with herself. If an individual i is matched with herself, we say that this i is *single* under μ.

Let $(M, W, \theta, \mathcal{M})$ be a marriage problem. Every man $m \in M$'s preferences over the set $W \cup \{m\}$ in the state θ can be extended to an ordering over the collection $\mathcal{M}$ in the following way:

$$\mu R_m(\theta)\mu' \Leftrightarrow \text{either } \mu(m) P_m(\theta)\mu'(m) \text{ or } \mu(m) = \mu'(m), \quad \text{for every } \mu, \mu' \in \mathcal{M}.$$

Likewise, this can be done for every woman $w \in W$.

Let $(M, W, \theta, \mathcal{M})$ be a marriage problem. A matching μ is *individually rational* in state θ if no individual $i \in M \cup W$ prefers strictly being single to being matched with the partner assigned by the matching μ; that is, for every individual i, either $\mu(i) P_i(\theta) i$ or $\mu(i) = i$. Furthermore, a matching μ is *blocked* in state θ if there are two individuals m and w of the opposite sex who would each prefer strictly to be matched with the other rather than with the partner assigned by the matching μ; that is, there is a pair (m, w) such that

$$w P_m(\theta)\mu(m) \text{ and } m P_w(\theta)\mu(w).$$

A matching μ is *stable* in state θ if it is individually rational and unblocked in state θ. A matching μ is *man-optimal stable* in state θ if it is the best stable matching from the perspective of all the men; that is, m is stable in state θ and for every man $m \in M$, $\mu R_m(\theta)\mu'$ for every other stable matching μ' in state θ. The man-optimal stable matching in state θ is denoted by μ^θ.

Definition 7 The *man-optimal stable* solution of a marriage problem $(M, W, \theta, \mathcal{M})$, denoted by $\mathcal{O}_M$, is a function associating the state θ with its man-optimal stable matching μ^θ,

$$\mathcal{O}_M(\theta) \equiv \left\{\mu^\theta\right\}, \qquad \text{for every } \theta \in \Theta.$$

The following result shows that this solution is partially-honestly Nash implementable when the mechanism designer does not know the true state. We refer to $(M, W, \Theta, \mathcal{M})$ as a *class of marriage problems*, with $(M, W, \theta, \mathcal{M})$ as typical marriage problem. Note that the man-optimal stable solution does not satisfy no veto-power (see, for instance, Kara and Sönmez 1996; Table I, p. 437).

Proposition 1 *Let* $(M, W, \Theta, \mathcal{M})$ *be a class of marriage problems with* $|M \cup W| \geq 3$. *Let Assumptions 1–2 be given. Then, the man-optimal stable solution is partially-honestly Nash implementable.*

Proof Let the premises hold. In the context of matching problems, the set X coincides with the collection $\mathcal{M}$, and N is the set $M \cup W$. We show that the man-optimal stable solution satisfies Condition μ^*(ii) with respect to $Y = X$.

Since the man-optimal stable solution is unanimous, we can set $Y = X$ as per Sjöström (1991), and so Y contains the range of $\mathcal{O}_M$. In addition, for every triplet (i, θ, θ'), let

$$C_i\left(\theta, \mu^\theta\right) \equiv L_i\left(\theta, \mu^\theta\right) \text{ and } S_i\left(\theta'; \mu^\theta, \theta\right) \equiv C_i\left(\theta, \mu^\theta\right).$$

One can check that for every state θ, it holds that $\mu^\theta \in C_i\left(\theta, \mu^\theta\right) \subseteq L_i\left(\theta, \mu^\theta\right) \subseteq Y$ for every individual i. Moreover, for every triplet (i, θ, θ'), one can also check that the set $S_i\left(\theta'; \mu^\theta, \theta\right)$ is non-empty and that $\mathcal{O}_M\left(\theta'\right) \in S_i\left(\theta'; \mu^\theta, \theta\right)$ if $\theta' = \theta$, establishing part (1) of Condition μ^*(ii). Finally, let us show that the man-optimal stable solution satisfies part (2) of Condition μ^*(ii).

For every quadruplet $(i, \theta, \theta', \mu)$ with $\mu \in C_i\left(\theta, \mu^\theta\right)$, suppose that $C_i\left(\theta, \mu^\theta\right) \subseteq L_i\left(\theta', \mu\right)$ and that $Y \subseteq L_j\left(\theta', \mu\right)$ for every individual $j \neq i$. By construction, the man-optimal stable solution satisfies part (2) of Condition μ^* (ii) if we show that μ is the man-optimal matching in state θ'; that is, $\mu = \mu^{\theta'}$.

Assume, to the contrary, that $\mu \neq \mu^{\theta'}$. Note that the matching μ is stable in state θ'. So, by Theorem 2.13 in Roth and Sotomayor (1990; p. 33), which is due to Knuth (1976), it follows that $\mu^{\theta'} R_m\left(\theta'\right) \mu$ for every man $m \in M$ and that $\mu R_w\left(\theta'\right) \mu^{\theta'}$ for every woman $w \in W$. From this and the fact that the matching μ is also $R_j\left(\theta'\right)$-maximal for every individual $j \neq i$ in the set Y, it follows that $\mu(j) = \mu^{\theta'}(j)$ if individual j is a man. Therefore, it must be the case that individual i is a man and the mate of the man i under $\mu^{\theta'}$ differs from that under μ, that is, $\mu(i) \neq \mu^{\theta'}(i)$; otherwise, $\mu = \mu^{\theta'}$, which is a contradiction.

Since $\mu(i) \neq \mu^{\theta'}(i)$ and since, moreover, $\mu^{\theta'} R_i\left(\theta'\right) \mu$, it follows from the definition of $R_i\left(\theta'\right)$ that $\mu^{\theta'} P_i\left(\theta'\right) \mu$. From this and the fact that the matching μ is stable in state θ', we have that the man i must be matched with a partner of the opposite sex under $\mu^{\theta'}$; that is, $\mu^{\theta'}(i) = w$. Moreover, it must be the case that the mate of the woman w under $\mu^{\theta'}$ differs from that under μ, that is, $\mu(w) \neq \mu^{\theta'}(w) = i$; otherwise, the man i is matched with the same mate under μ and under $\mu^{\theta'}$, which contradicts that $\mu(i) \neq \mu^{\theta'}(i)$.

Since $\mu(w) \neq \mu^{\theta'}(w) = i$ and the matching μ is $R_w\left(\theta'\right)$-maximal in the set Y for the woman w and since, moreover, $\mu^{\theta'}$ is stable in state θ', it follows that $\mu P_w\left(\theta'\right) \mu^{\theta'}$ and that the mate of the woman w under μ is a man $m \neq i$. However, since the matching μ is $R_m\left(\theta'\right)$-maximal in the set Y for the man $m \neq i$ and since, moreover, $\mu^{\theta'} R_m\left(\theta'\right) \mu$, it must be the case that the man m is matched with the same woman w under μ and under $\mu^{\theta'}$, that is, $\mu(m) = \mu^{\theta'}(m) = w$. This implies that the woman w is matched with the same mate under μ and under $\mu^{\theta'}$, that is, $\mu(w) = \mu^{\theta'}(w)$, which is a contradiction. Thus, we conclude that $\mu = \mu^{\theta'}$.

Since the man-optimal stable solution satisfies Condition μ^*(ii) with respect to $Y \subseteq X$, Theorem 1 implies that this solution is partially-honestly Nash implementable. □

3.3 Applications to bargaining games

Last but not least, we look at the Nash implementability of the Nash (bargaining) solution. In the classical cooperative bargaining theory, initiated in Nash (1950), a number of individuals face the task of finding a unanimous agreement over the (expected) utility allocations resulting from the lotteries over a set of physical objects. The *Nash* solution, due to Nash (1950), selects the utility allocation that maximizes the product of the utilities over the feasible utility allocations. This allocation is now widely referred to as the Nash point.

The normative evaluation of the Nash solution is thus done entirely in utility space, based on the expected utility functions of the individuals. On the other hand, the objective of the abstract theory of Nash implementation is to help a uninformed mechanism designer to Nash implement outcomes satisfying certain desirable welfare criteria. This means that the shape of the utility space is unknown to the mechanism designer. One way to get these two classic areas of study closer has recently been suggested by Vartiainen (2007b) in the canonical cake sharing setting, which we follow in this last application.

We consider a situation where individuals bargain over the partition of one unit of a perfectly divisible commodity. Additionally, we assume that at each state every individual's preference over the set of possible agreements is represented by a continuous and increasing expected utility function. With these specifications, and when lotteries are feasible, every state generates a classic (non-empty, convex, compact and comprehensive) utility space. We thus require that the Nash solution associates, with each state, the set of all lotteries that generate the Nash point of the utility space generated by the state.

When both individuals and the mechanism designer know the size of the commodity and the space of lotteries but only individuals know the prevailing state, it is shown that the Nash solution can be Nash implemented in a setting with partially-honest individuals, though it violates the condition of no veto-power. This is a rather significant permissive result because several attempts have been made to give a non-cooperative foundation to the Nash solution since Nash (1953). With the exception of Naeve (1999), [12] reconstructions of the Nash point as an equilibrium point of a mechanism are based on refinements of Nash equilibrium as solution concepts. See, e.g., Howard (1992) and Miyagawa (2002).[13]

Formally, we assume that the set of possible divisions—*allocations*—of one unit of a perfectly divisible commodity among the n individuals is given by $A \equiv \{a \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} a_i \leq 1\}$, with a as a typical allocation and with a_i as a typical fraction obtained by individual i at a. This set A is kept fixed throughout. In addition, we take the complete waste of the commodity as the *disagreement point* $d = 0$, which will also be the origin of the individual utilities.

A *bargaining game* is a triplet (N, Δ, θ) such that:

- N is a finite set of individuals, with $n \geq 2$.
- Δ is the set of *outcomes*, which consists of all probability measures on the Borel σ-algebra of the space A, with p as a typical element.

[12] In a variant of the model of Serrano (1997), Naeve (1999) shows that the Nash bargaining solution can be Nash implemented. However, this could be purchased at the cost of a strong domain restriction of individuals' preferences. For instance, the set of states cannot take the structure of the Cartesian product of allowable independent characteristics for individuals (see Naeve 1999; p. 24).

[13] Moulin (1984) constructs a mechanism that implements the so-called Kalai–Smorodinsky bargaining solution in subgame perfect Nash equilibrium.

- θ is a state in Θ, at which every individual j's preferences over [0, 1] are identified by a continuous and monotonic von Neumann-Morgenstern ordering.[14] Thus, individual j's preferences in state θ can be represented by a continuous, increasing and von Neumann-Morgenstern utility function $u_j(\cdot;\theta) : [0,1] \to \mathbb{R}$ such that individual j's expected utility of a probability measure p in Δ is:

$$U_j(p;\theta) \equiv \int_A u_j\left(a_j;\theta\right) \mathrm{d}p(a), \qquad \text{for every } p \in \Delta.$$

  In addition, this utility function is uniquely determined up to a positive affine transformation.[15] Therefore, for the sake of simplicity, we also assume that $u_j(0;\theta) = 0$ and that $u_j(1;\theta) = 1$ in state θ.

Write (N, Δ, Θ) for the class of bargaining games, with (N, Δ, θ) as a typical element, where the set Θ consists of all representations of continuous and monotonic orderings over [0, 1] that are consistent with the von Neumann-Morgenstern axioms; that is, the domain Θ is *unrestricted*. To save writing, write $U(p;\theta)$ for the utility allocation $(U_1(p;\theta), \ldots, U_1(p;\theta))$ generated by the outcome p in state θ.

Let (N, Δ, θ) be a bargaining game. Define the utility possibility set associated with this bargaining game as:

$$U(\Delta;\theta) \equiv \left\{\left(U_j(p;\theta)\right)_{j\in N} | p \in \Delta\right\},$$

which is a non-empty, compact and convex set in $\mathbb{R}^n$.[16] In addition, since the utility functions representing individuals' preferences are increasing, this set $U(\Delta;\theta)$ is also comprehensive, which amounts to free disposal of utility.[17]

As already noted in Vartiainen (2007b), for every non-empty, convex and compact subset S of $\mathbb{R}^n_+$ there is a bargaining game (N, Δ, θ) in the family (N, Δ, Θ) for which the utility possibility set $U(\Delta;\theta)$ is S; that is, $U(\Delta;\theta) = S$. Therefore, in the actual setting, every element of the class of standard bargaining problems in $\mathbb{R}^n_+$ is the image of some element of the family $\{U(\Delta;\theta)\}_{\theta\in\Theta}$ of utility possibility sets generated by the class (N, Δ, Θ) of bargaining games; that is, there is an onto function from the family $\{U(\Delta;\theta)\}_{\theta\in\Theta}$ of utility possibility sets to the class of standard bargaining problems in $\mathbb{R}^n_+$. Indeed, from the welfaristic viewpoint, that is, from the point of view where only utility allocations matter, these two classes are basically equivalent.

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14 An ordering $R_j(\theta)$ on [0, 1] is *monotonic* if $a_j \geq b_j \implies a_j R_j(\theta) b_j$, for every $a_j, b_j \in [0,1]$.

15 A function $v : [0,1] \to \mathbb{R}$ is a *positive affine transformation* of $u_j(\cdot;\theta)$ if there exists a positive real number $\beta > 0$ and a real number γ such that $v\left(a_j\right) = \beta u_j\left(a_j;\theta\right) + \gamma$, for every $a_j \in [0,1]$.

16 Its convexity follows from the Lyapunov's theorem for nonatomic vector measures, whereas its compactness follows from the fact the set $U(\Delta;\theta)$ is the image of the compact set Δ under the profile of continuous functions $U(\cdot;\theta) \equiv \left(U_j(\cdot;\theta)\right)_{j\in N}$ (in the topology of weak convergence).

17 In symbols, a non-empty set $S \subseteq \mathbb{R}^n$ is said to be *comprehensive* if $x \in S$ and $0 \leq y \leq x$ together imply $y \in S$, where it is understood that for every two n-dimensional Euclidean vectors a and b, $a \geq b$ means that $a_i \geq b_i$ for every individual i, $a > b$ means that $a \neq b$ and $a_i \geq b_i$ for every individual i.

Definition 8 The *Nash* solution of a bargaining game (N, Δ, θ), denoted by ν, is the collection of all outcomes p and q of Δ that generate the same utility allocations $U(p;\theta) = U(q;\theta)$ and that maximize the product of utilities over the utility possibility set $U(\Delta, \theta)$,

$$\nu(\theta) \equiv \arg\max_{m\in\Delta} \left\{ \prod_{j\in N} U_j(m;\theta) \,|U(m;\theta) \in U(\Delta;\theta) \right\}.$$

Thus, this solution is derived under the so-called welfaristic assumption: The solution depends only on the Nash property of the utility allocations.

Since the Nash solution is a risk sensitive bargaining solution, it follows from Vartiainen (2007b; Corollary 1, p. 343) that this solution fails Maskin monotonicity.[18] The following claim establishes that the Nash solution does not satisfy the no veto-power condition either: In the abstract Arrovian domain, the condition of *no veto-power* says that if an outcome is at the top of the preferences of all individuals but possibly one, then it should be chosen irrespective of the preferences of the remaining individual: that individual cannot veto it.

Claim 2 Let $n = 3$. Then, the Nash solution does not satisfy the condition of no veto-power.[19]

Proof Since this solution is unanimous, we can set $X = \Delta$ as per Sjöström (1991). Assume, to the contrary, that the Nash solution satisfies the condition of no veto-power.

Suppose that there are three individuals and a state θ, at which each individual j's ordering over the interval $[0, 1]$ is represented by the following utility function: $u_j(a_j;\theta) = \min\{a_j, 0.5\}$ for every $a_j \in [0, 1]$ and every $j = 1, 2$, and $u_3(a_3;\theta) = a_3$ for every $a_3 \in [0, 1]$. Therefore, the triplet (N, Δ, θ) is a bargaining game with a utility possibility set $U(\Delta;\theta)$, which is equal to the convex three-dimensional polyhedron with vertices at the following elements of the space A:

$$a^0 \equiv (0, 0, 0), a^1 \equiv (0.5, 0, 0), a^2 \equiv (0.5, 0.5, 0),$$
$$a^3 \equiv (0, 0.5, 0) \text{ and } a^4 \equiv (0, 0, 1).$$

By abuse of notation, write a for the degenerate probability measure in Δ that picks the allocation a in A with certainty.

In the bargaining game (N, Δ, θ), the utility allocation generated by the probability measure a^2 in state θ is $U(a^2;\theta) \equiv (0.5, 0.5, 0)$, which is an element of $U(\Delta;\theta)$. Since the probability measure a^2 is an outcome for which $U_j(\cdot;\theta)$ attains its largest value over the set X for individual $j = 1, 2$, no veto-power implies that this outcome is an element of the Nash solution at θ, that is, $a^2 \in \nu(\theta)$. By definition of the Nash

[18] A bargaining solution is risk sensitive when an increase in one's opponent's risk aversion is advantageous to other bargainers. For a recent study on the effects on bargaining solutions when bargainers become more risk averse and when they become more uncertainty averse, see Driesen et al. (2015).

[19] If expected utility functions representing individuals' preferences are strictly increasing, it follows from the result of Dutta and Sen (2012) that the Nash solution is Nash implementable with partially-honest individuals.

solution, this means that the degenerate lottery a^2 maximize the product of utilities over the utility possibility set $U(\Delta, \theta)$. Note that the Nash product of the utility allocation of the degenerated lottery a^2 is zero.

To derive a contradiction of the definition of the Nash solution, observe that, by construction, the Nash solution is not empty for the bargaining game (N, Δ, θ). Moreover, let us consider the probability measure $p' \in \Delta$ defined by

$$p'(a) = \begin{cases} \frac{1}{2} & \text{if } a = a^2 \\ \frac{1}{2} & \text{if } a = a^4 \\ 0 & \text{otherwise} \end{cases}.$$

One can see that this probability measure generates a utility allocation equal to $U\left(p'; \theta\right) = (0.25, 0.25, 0.5)$, and so its Nash product is larger than zero, which is a contradiction. □

In contrast with the above negative results, the Nash solution is partially-honestly Nash implementable when there are $n \geq 3$ individuals:

Proposition 2 *Let* (N, Δ, Θ) *be a class of bargaining games with* $n \geq 3$. *Let Assumption 1 and Assumption 2 be given. Then, the Nash solution is partially-honestly Nash implementable.*

Proof Let the premises hold. In the context of bargaining games, the set X coincides with the space Δ. We show that the Nash solution satisfies Condition μ^*(ii) with respect to $Y = X$. A typical Nash-optimal outcome at state θ is denoted by p^θ.

Since this solution is unanimous, we can set $Y = X$ as per Sjöström (1991), and so Y contains the range of ν. In addition, let

$$C_i\left(\theta, p^\theta\right) \equiv L_i\left(\theta, p^\theta\right) \text{ and } S_i\left(\theta; p^\theta, \theta\right) \equiv \nu(\theta), \quad \text{for every pair } (i, \theta) \in N \times \Theta.$$

One can check that for every state θ, it holds that $p^\theta \in C_i\left(\theta, p^\theta\right) \subseteq L_i\left(\theta, p^\theta\right) \subseteq Y$ for every individual i. Moreover, one can also check that $p^\theta \in S_i\left(\theta; p^\theta, \theta\right)$, establishing part (1) of Condition μ^*(ii) when $\theta' = \theta$. Next, let us show that the Nash solution satisfies part (2) of Condition μ^*(ii) when $\theta' = \theta$. We do it by showing that the outcome q is a Nash-optimal outcome at state θ provided that this $q \in L_i\left(\theta, p^\theta\right)$ is an outcome for which $U_i(\cdot; \theta)$ attains its largest value on the set $L_i\left(\theta, p^\theta\right)$ for some individual i and that this q is also an outcome for which $U_j(\cdot; \theta)$ attains its largest value on the set Y for every other individual j. To see this, note that $U_i\left(p^\theta; \theta\right) = U_i(q; \theta)$ and that $U_j(q; \theta) \geq U_j\left(p^\theta; \theta\right)$ for every individual $j \neq i$. By the efficiency of the Nash solution, it must be the case that $U_j(q; \theta) = U_j\left(p^\theta; \theta\right)$ for every individual $j \neq i$. Thus, by the definition of the Nash solution it follows that q is an element of $\nu(\theta)$, as was to be shown. In summary, the Nash solution satisfies Condition μ^*(ii) when $\theta' = \theta$.[20]

[20] The *Pareto optimal* set of Δ at θ is:

$$P(\theta) \equiv \{q \in \Delta | \text{ there is no } p \in \Delta : U(p; \theta) > U(q; \theta)\}, \qquad \text{for every } \theta \in \Theta.$$

We next turn to deal with the case where $\theta \neq \theta'$. Let us then first provide a construction of the set $S_i\left(\theta'; p^{\theta}, \theta\right)$ for every individual i when $\theta \neq \theta'$. To this end, for every triplet $\left(i, \theta, \theta'\right)$ with $\theta \neq \theta'$, define the set $S_i\left(\theta'; p^{\theta}, \theta\right)$ as follows:

- For all $q \in Y$, if $q \in C_i\left(\theta, p^{\theta}\right) \subseteq L_i\left(\theta', q\right)$ and $Y \subseteq L_j\left(\theta', q\right)$ for every other individual j and if $q \notin \nu\left(\theta'\right)$, then:

$$S_i\left(\theta'; p^{\theta}, \theta\right) \equiv \left\{r \in C_i\left(\theta, p^{\theta}\right) | U_i\left(r; \theta'\right) = U_i\left(q; \theta'\right) \text{ and } U_j\left(r; \theta'\right) = 0 \text{ for every } j \neq i\right\}.$$

- In all other cases, $S_i\left(\theta'; p^{\theta}, \theta\right) \equiv C_i\left(\theta, p^{\theta}\right)$.

Firstly, suppose that the premises of part (2)(a) of Condition μ^* (ii) never apply to outcomes in $C_i\left(\theta, p^{\theta}\right)$. Then, $S_i\left(\theta'; p^{\theta}, \theta\right)$ coincides with the non-empty set $C_i\left(\theta, x^{\theta}\right)$, which shows that part (1)(a) as well as part (2)(a) of Condition μ^* (ii) are satisfied for this i.

Secondly, suppose that the premises of part (2)(a) of the condition apply to at least one outcome $q \in C_i\left(\theta, p^{\theta}\right)$. Then, to satisfy part (2)(a) of Condition μ^*(ii) we need to have that this q is not an element of $S_i\left(\theta'; p^{\theta}, \theta\right)$ and, moreover, that the intersection $S_i\left(\theta'; p^{\theta}, \theta\right) \cap I_i\left(\theta', q, Y\right)$ is not empty. This is the case by construction of the set $S_i\left(\theta'; p^{\theta}, \theta\right)$ provided that this set is not empty. Indeed, if the set $S_i\left(\theta'; p^{\theta}, \theta\right)$ is not empty, then Condition μ^* (ii) is satisfied because there would exist an outcome r in $S_i\left(\theta'; p^{\theta}, \theta\right)$ such that the expected utility of individual i at r and at q in state θ' is the same, that is, $U_i\left(r; \theta'\right) = U_i\left(q; \theta'\right)$, establishing that the intersection $S_i\left(\theta'; p^{\theta}, \theta\right) \cap I_i\left(\theta', q, Y\right)$ is not empty, as well as because every element of $S_i\left(\theta'; p^{\theta}, \theta\right)$ is an outcome of $C_i\left(\theta, p^{\theta}\right)$ which results in a zero expected utility in state θ' for every individual $j \neq i$, establishing that the outcome q cannot be an element of this $S_i\left(\theta'; p^{\theta}, \theta\right)$.

Thus, to show that the set $S_i\left(\theta'; p^{\theta}, \theta\right)$ is not empty, it suffices to show that this set is not empty for every triplet $\left(i, \theta, \theta'\right)$ for which the premises of part (2)(a) of Condition μ^*(ii) apply to some $q \in C_i\left(\theta, p^{\theta}\right)$. To this end, take any of these triplets and denote it by $\left(i, \theta, \theta'\right)$.

Given that the utility allocation which assigns $U_i\left(q; \theta'\right)$ to individual i and zero to every other individual j is an element of the utility possibility set $U\left(\Delta; \theta'\right)$, it follows from this that there is a probability measure s in Δ which generates this utility allocation. From the available ones, let r denote the one for which it also holds that $U_i\left(q; \theta\right) = U_i\left(r; \theta\right)$. This r exists because the space of outcomes Δ consists of all probability measures on the Borel σ-algebra of the space A. Thus, this r is an element of $C_i\left(\theta, p^{\theta}\right)$ for which it holds that $U_i\left(q; \theta'\right) = U_i\left(r; \theta'\right)$ and that $U_j\left(r; \theta'\right) = 0$ for every individual $j \neq i$, establishing that the set $S_i\left(\theta'; p^{\theta}, \theta\right)$ is not empty. □

The Nash solution is efficient since $\nu\left(\theta\right) \subseteq P\left(\theta\right)$ for every $\theta \in \Theta$.

4 A full characterization: a brief discussion of the necessary conditions

The characterization results presented above are limited to the class of unanimous SCRs. However, some interesting SCRs are not unanimous: a typical example of such a SCR is the egalitarian bargaining solution. In the standard Nash implementation theory, as Moore and Repullo's (1990) Condition μ(iii) states—see page 9 for a definition of Condition μ(iii), a SCR F must be unanimous with respect to a subset Y of X, with $F(\Theta) \subseteq Y$, if it is Nash implementable. Unfortunately, this condition is not a necessary one for partially-honest Nash implementation. Thus, we establish a new necessary condition, called Condition μ^*(iii), which is sufficient for partially-honest Nash implementation when combined with Condition μ^*(ii) and with another necessary condition, named Condition μ^*(i). The conditions we have obtained are reasonably weak, albeit somewhat complex. For this reason, in this section, we provide an intuition of these conditions by focusing on the implementability of the egalitarian bargaining solution. A complete discussion of the full characterization is presented in Lombardi and Yoshihara (2019).

Consider the set of allocations $A \equiv \{a \in \mathbb{R}^n_+ \mid \sum_{i=1}^n a_i \leq 1\}$ as in Sect. 3.3. Assume that each agent is endowed with the class of the standard continuous, increasing, and concave utility functions defined over [0, 1]. Then, as in Sect. 3.3, for each state θ, one bargaining problem is specified by a utility possibility set $U(\theta)$. As the egalitarian bargaining solution is non-unanimous, we can specify a bargaining problem in which a unanimously best outcome does not support the egalitarian solution.

Consider a three-individual society and a bargaining problem in state θ which is defined by the comprehensive hull of a utility allocation $(1, 1, 2)$, $U(\theta) = comp\{(1, 1, 2)\}$. Thus, one can easily see that the utility allocation $(1, 1, 2)$ is unanimously most preferred, which is derived from an outcome $x \in X$ in state θ, whereas the allocation $(1, 1, 1)$ is the egalitarian utility allocation in state θ. Suppose that Assumptions 1–2 hold. Under such a bargaining problem, and given that x does not generate the egalitarian utility allocation but it generates the utility allocation $(1, 1, 2)$, Condition μ^*(iii) requires that, for each potential partially-honest individual i, that is, for each $\{i\} \in \mathcal{H}$, there exists an outcome, $y^{(i)}$, which is indifferent to the unanimously best outcome x for i. For instance, if $i = 1$, then the utility allocation corresponding to the outcome $y^{(1)}$ should be $(1, 0, 0)$.[21] This is the case because for any implementing mechanism and for any message profile m supporting $x \notin F(\theta)$ as an outcome of the outcome function, individual i's play m_i is not a truthful strategy choice, that is, $m \notin T_i^{\Gamma}(\theta)$, and this individual has non-material incentives to play a truthful strategy choice $m_i' \in T_i^{\Gamma}(\theta)$ such that $g\left(m_i', m_{-i}\right) = y^{(i)}$. This requirement applies to each individual i, by our Assumption 2.

This is one of the requirements of Condition μ^*(iii). However, this part of the condition open the following problem: What outcome should be selected by the outcome function when the outcome $y^{(i)}$ is not a top outcome according to each agent's ranking and when it does not generate the egalitarian utility allocation? The other requirements of Condition μ^*(iii) provide an answer to this question. They do so by requiring for

[21] By the definition of the bargaining problem as a utility possibility set, such an outcome indeed exists.

each $H \in \mathcal{H}$, the existence of a common truthful feasible outcome, which is characterized by the fact that every partially-honest individual in H is playing a truthful strategy choice. Moreover, each of these requirements of Condition μ^*(iii) states that if the specified common truthful outcome is a Nash equilibrium with partially-honest individuals at θ, then it generates the egalitarian utility allocation.

A full characterization of the class of SCRs that are partially-honestly Nash implementable is obtained by requiring that F jointly satisfies Condition μ^*(iii), Condition μ^*(ii) and another condition, named *Condition* μ^*(i). Condition μ^*(i) is our third necessary condition, which is a weak variant of Maskin monotonicity. Though from Theorem 1 and from Dutta and Sen (2012)'s Theorem, we know that no monotonicity-type condition is necessary for Nash implementation of unanimous SCRs, a weak variant of Maskin monotonicity is required to obtain our characterization result of non-unanimous SCRs.

5 Conclusions

The main practical aim of adopting an axiomatic approach to implementation theory is to distinguish between implementable and non-implementable SCRs. Drawing from the recent literature on implementation with partially-honest individuals, this paper identifies necessary and sufficient conditions for the Nash implementation of unanimous SCRs in a many-person setting with partially-honest individuals. The application of the necessary and sufficient conditions to test the implementability is relatively easy in many problems, as discussed in Sects. 3.1, 3.2, and 3.3. Existing literature on the subject has thus far offered only sufficient conditions in a variety of environments.

In an environment in which knowledge is dispersed, how individuals will interact with the mechanism designer is a natural starting point when it comes to Nash implement a SCR. A particular kind of communication is, as we have done in this paper, to ask participants to report the entire state of the world. There is, however, no reason to restrict attention to such schemes.

On this issue, Lombardi and Yoshihara (2018) have recently identified conditions for Nash implementation with partially-honest individuals which, if satisfied, send us back to the limitations imposed by Maskin's theorem. In terms of mechanisms, these conditions basically result in the impossibility to structure the communication in a way that does not allow the mechanism designer to elicit enough information of individuals' characteristics from the partially-honest participants. For instance, the limitations of Maskin's theorem remain valid when participants are asked to report only their own characteristics.

However, this does not mean that there are not mechanisms that resemble real-life mechanisms and that, at the same time, allow us to escape the limitations imposed by Maskin monotonicity in a setting with partially-honest individuals. One of these mechanisms is represented by the price-quantity mechanism [studied, for example, in Dutta et al. (1995), Sjöström (1996) and Saijo et al. (1996)], in which each individual chooses prices of commodities as well as a consumption bundle as her strategy choice. This is so because the announcement of prices serves the purpose to acquire some local information about individuals' indifference curves, such as the common marginal rate of substitution at an efficient allocation. Indeed, we now know that the Walrasian solution is Nash implementable in a many-person setting with partially-honest individuals by this type of market mechanism (see Lombardi and Yoshihara 2017).[22]

Postlewaite and Schmeidler (1986), Palfrey and Srivastava (1989) and Jackson (1991) have shown that Maskin's theorem can be generalized to Bayesian environments. A necessary condition for Bayesian Nash implementation is Bayesian monotonicity. In a Bayesian environment involving at least three individuals, Bayesian monotonicity combined with no veto-power is sufficient for Bayesian Nash implementation provided that a necessary condition called closure and the Bayesian incentive compatibility condition are satisfied (Jackson 1991). Korpela (2014) studies Bayesian Nash implementation and provides sufficient conditions for implementation in a setting with partially-honest participants. This characterization result shows that Bayesian monotonicity becomes redundant in this environment, and so there are far fewer limitations for Bayesian Nash implementation when individuals have a taste for honesty. As yet, where the exact boundaries of those limitations lay for Bayesian environments is far from known. This subject is left for future research.

The same remark applies to implementation models based on the assumption that agents are maximin expected utility maximizers (Gilboa and Schmeidler 1989). Under this assumption, Guo and Yannelis (2018) are the first to study (full) implementation of social choice sets in essentially Bayesian Nash equilibrium with ambiguous beliefs, named implementation in ambiguous equilibrium. Echoing the results in Bayesian environments, Guo and Yannelis (2018) show that ambiguous incentive compatibility, ambiguous monotonicity and closure are necessary and almost sufficient for implementing any social choice set as ambiguous equilibria.[23] One would expect that ambiguous monotonicity would become redundant when individuals have intrinsic preferences for honesty. Again, this subject is left for future research.

Appendix A: Proof of Theorem 1

Let the premises hold. Suppose that SCR $F : \Theta \twoheadrightarrow X$ satisfies unanimity.

Let us first show that F satisfies Condition μ^{*}(ii) with respect to $Y \subseteq X$ if it is partially-honestly Nash implemented by the mechanism $\Gamma = (M, g)$. Let Γ be the mechanism that partially-honestly Nash implements F. Then, $T_i^{\Gamma}\left(\bar{\theta}\right) \neq \varnothing$ for every

[22] The provided characterization does not rely on any sort of "tail-chasing" construction.

[23] In particular, in a private values model in which agents have Wald-type maximin preferences, they also show that the set of all ambiguous Pareto efficient and individually rational social choice functions in an economy are implementable as an ambiguous equilibrium. This result extends that of de Castro et al. (2017) to full implementation and social choice sets. Another extension of de Castro et al. (2017)'s result can be found in Liu (2016).

pair $\left(i,\bar{\theta}\right)\in N\times\Theta$ and, moreover, it holds that

$$F\left(\bar{\theta}\right)=NA\left(\Gamma,\succsim^{\Gamma,\bar{\theta},\bar{H}}\right),\qquad \text{for every pair }\left(\bar{\theta},\bar{H}\right)\in\Theta\times\mathcal{H}.$$

Let

$$Y=\{z\in X|g\left(m\right)=z\text{ for some }m\in M\}.$$

Thus, Y contains the range of F.

For what follows, fix any pair $(x,\theta)\in Y\times\Theta$ with $x\in F\left(\theta\right)$.

Given that $N\in\mathcal{H}$ by Assumption 2, there exists m such that $g\left(m\right)=x$ and that $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,N}\right)$. Thus, for every $i\in N$, let

$$C_i\left(\theta,x\right)=g\left(M_i,m_{-i}\right).\tag{A1}$$

Clearly, $x\in C_i\left(\theta,x\right)\subseteq L_i\left(\theta,x\right)$ and $C_i\left(\theta,x\right)\subseteq Y$. For what follows, fix also any pair $\left(\theta',H\right)\in\Theta\times\mathcal{H}$.

Given that $g\left(m\right)=x$ and $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,N}\right)$, define $S_i\left(\theta';x,\theta\right)$ as follows:

$$S_i\left(\theta';x,\theta\right)=g\left(T_i^{\Gamma}\left(\theta'\right),m_{-i}\right).\tag{A2}$$

Clearly, $S_i\left(\theta';x,\theta\right)\neq\varnothing$ and, moreover, $S_i\left(\theta';x,\theta\right)\subseteq C_i\left(\theta,x\right)$, establishing part (1)(a) of Condition μ^*(ii).

Next, we show that F satisfies part (1)(b) of Condition μ^* (ii). Take any $h\in H$ and suppose that $\theta'=\theta$. Also, suppose that $x\notin S_h\left(\theta';x,\theta\right)$. It follows that $m_h\notin T_h^{\Gamma}\left(\theta\right)$. Suppose that there exists $z\in S_h\left(\theta';x,\theta\right)$ such that $zR_h\left(\theta'\right)x$. Given that $z\in S_h\left(\theta';x,\theta\right)$, it follows that there exists $m_h'\in T^{\Gamma}\left(\theta\right)$ such that $g\left(m_h',m_{-h}\right)=z$. Thus, $\left(m_h',m_{-h}\right)\succ_h^{\Gamma,\theta}m$, which contradicts that $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,N}\right)$. We thus conclude that $xP_h\left(\theta\right)z$ for all $z\in S_h\left(\theta;x,\theta\right)$.

Finally, we show that F satisfies part (2) of Condition μ^* (ii). Fix any pair $(i,y)\in N\times C_i\left(\theta,x\right)$. Then, given that $g\left(m\right)=x$ and $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,N}\right)$, it follows that $g\left(m_i',m_{-i}\right)=y$ for some $m_i'\in M_i$. To economize on notation, we write m' for $\left(m_i',m_{-i}\right)$.

Suppose that $C_i\left(\theta,x\right)\subseteq L_i\left(\theta',y\right)$ and that $Y\subseteq L_j\left(\theta',y\right)$ for all $j\neq i$. Moreover, suppose that $y\notin F\left(\theta'\right)$. By the partially-honest Nash implementability of F, we have that $m'\notin NE\left(\Gamma,\succsim^{\Gamma,\theta',H}\right)$. Given that $g\left(M_k,m_{-k}'\right)\subseteq L_k\left(\theta',y\right)$ for every $k\in N$, only a partially-honest individual $h\in H$ can find it profitable to unilaterally deviate from m'. Thus, it is the case that $m_h'\notin T_h^{\Gamma}\left(\theta'\right)$ and that there is $m_h''\in T_h^{\Gamma}\left(\theta'\right)$ such that $g\left(T_h^{\Gamma}\left(\theta'\right),m_{-h}'\right)\cap I_h\left(\theta',y,Y\right)\neq\varnothing$.

This shows that the intersection

$$S_i\left(\theta';x,\theta\right)\bigcap I_i\left(\theta',y,Y\right)\tag{A3}$$

is not empty if $H=\{i\}$. Suppose that $H=\{i\}$. If $g\left(m_i'',m_{-i}'\right)=y$, then $y\in NA\left(\Gamma,\succsim^{\Gamma,\theta',H}\right)$, which contradicts that $y\notin F\left(\theta'\right)$. Thus, when $H=\{i\}$, we have that $y \notin S_i(\theta'; x, \theta)$ and that the intersection in (A3) is not empty, establishing part (2)(a) of Condition μ^*(ii).

Finally, let us show that F satisfies part (2)(b) of Condition μ^*(ii) as well. Thus, suppose that $i \notin H$ and that $\theta' = \theta$. This implies that the deviant $h \in H$ is such that $i \neq h$. Recall that for this h it holds that $m'_h = m_h \notin T_h^\Gamma(\theta')$. Since $g(m) = x$ and $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$, it holds that individual h cannot break it via any unilateral deviation.[24]

Assume, to the contrary, that $x \in S_j(\theta'; x, \theta)$ for all $j \in H$. Then, individual $h \in H$ identified above can find a strategy choice $\hat{m}_h \in T_h^\Gamma(\theta')$ such that $g(\hat{m}_h, m_{-h}) = x$. Since $m'_h = m_h \notin T_h^\Gamma(\theta')$, it follows that $\hat{m}_h \neq m_h$ and so individual h can break the strategy profile m from being a Nash equilibrium of $(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$, which is a contradiction. Thus, it is the case that $x \notin S_j(\theta'; x, \theta)$ for some individual $j \in H$, establishing part (2)(b) of Condition μ^*(ii).

In what follows, we show that F is partially-honestly Nash implementable if it satisfies Condition μ^*(ii) with respect to $Y \subseteq X$. Note that $Y = X$ because F satisfies unanimity. To this end, suppose that F satisfies Condition μ^*(ii) with respect to X.

Let us construct a mechanism which will partially-honestly Nash implement F. First, agent i's strategy choice space is defined by

$$M_i = (\Theta \cup \Omega) \times X \times N,$$

where Ω is a non-empty set such that its intersection with Θ is empty and that there is a bijection ϕ from Θ to Ω. This bijection is to make a code $\phi(\theta)$ to send the information of each state θ, rather than its direct message θ . Thus, individual i's strategy consists of an outcome in X, an element of the set $\Theta \cup \Omega$ and an individual index $k \in N$. A typical strategy played by individual i is denoted by $m_i = (m_{i1}, x^i, k^i)$ with m_{i1} as a typical element of $\Theta \cup \Omega$. The strategy choice space of individuals is the product space $M = \prod_{i \in N} M_i$, with m as a typical strategy profile.

For every pair $(\bar{\theta}, x) \in \Theta \times X$ with $x \in F(\bar{\theta})$, define individual p's set $\sigma_p(\bar{\theta}, x)$ as follows:

$$\sigma_p(\bar{\theta}, x) = \begin{cases} \{\phi(\bar{\theta})\} \times \{x\} \times N & \text{if } \exists q \in N \backslash \{p\} \text{ such that } x \in S_q(\bar{\theta}; x, \bar{\theta}) \\ & \quad \& \; x \notin S_j(\bar{\theta}; x, \bar{\theta}) \;\; (\forall j \in N \backslash \{q\}); \\ \{\bar{\theta}\} \times \{x\} \times N & \text{otherwise.} \end{cases}$$

Write $\sigma(\bar{\theta}, x)$ for a typical profile of sets, that is, $\sigma(\bar{\theta}, x) = (\sigma_p(\bar{\theta}, x))_{p \in N}$; and write $\sigma_{p1}(\bar{\theta}, x)$ for a typical first coordinate of the set $\sigma_p(\bar{\theta}, x)$. As F satisfies Condition μ^* (ii), we can always specify a profile $\sigma(\bar{\theta}, x)$ corresponding to each pair $(\bar{\theta}, x) \in \Theta \times X$ with $x \in F(\bar{\theta})$. Indeed, for any of such pairs, the corresponding profile $(S_p(\bar{\theta}; x, \bar{\theta}))_{p \in N}$ is specified by part (1)(a) of Condition μ^* (ii). Then, given this profile $(S_p(\bar{\theta}; x, \bar{\theta}))_{p \in N}$, we can specify the profile $\sigma(\bar{\theta}, x)$ in the following manner: if there exists a unique individual $q \in N$ such that $x \in S_q(\bar{\theta}; x, \bar{\theta})$ and $x \notin S_j(\bar{\theta}; x, \bar{\theta})$

[24] Observe that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$ given that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,N})$ and that $H \subseteq N$.

for any other individual $j \neq q$, then $\sigma_q\left(\bar{\theta}, x\right) = \left\{\bar{\theta}\right\} \times \{x\} \times N$ and $\sigma_j\left(\bar{\theta}, x\right) = \left\{\phi\left(\bar{\theta}\right)\right\} \times \{x\} \times N$ for any $j \neq q$; otherwise, $\sigma_p\left(\bar{\theta}, x\right) = \left\{\bar{\theta}\right\} \times \{x\} \times N$ for any individual $p \in N$. Such a specification of $\sigma\left(\bar{\theta}, x\right)$ will be used in the construction of the outcome function—see below. When agent p plays a strategy $m_p \in \sigma_p\left(\bar{\theta}, x\right)$ such that $m_{p1} = \phi\left(\bar{\theta}\right)$, our interpretation is that this agent is reporting the state $\bar{\theta}$ indirectly via ϕ. The reason is that ϕ is a bijection ϕ from Θ to Ω.

By means of the profile $\sigma\left(\bar{\theta}, x\right)$, let us introduce the notion of consistent strategy profile with respect to each pair $\left(\bar{\theta}, x\right) \in \Theta \times X$ with $x \in F\left(\bar{\theta}\right)$ as follows.

Definition 9 For every pair $\left(\bar{\theta}, x\right) \in \Theta \times X$ with $x \in F\left(\bar{\theta}\right)$ and every strategy profile $m \in M$,

(a) m is *consistent with* $\sigma\left(\bar{\theta}, x\right)$ if $m_i \in \sigma_i\left(\bar{\theta}, x\right)$ for every individual $i \in N$.
(b) m is *quasi-consistent with* $\sigma\left(\bar{\theta}, x\right)$ if $m_i \notin \sigma_i\left(\bar{\theta}, x\right)$ for one and only one individual $i \in N$.

In words, m is consistent with $\sigma\left(\bar{\theta}, x\right)$ when every individual i is playing one of the strategies specified by $\sigma_i\left(\bar{\theta}, x\right)$. m is quasi-consistent with $\sigma\left(\bar{\theta}, x\right)$ if there exists only one agent i who is playing a strategy that is not included in $\sigma_i\left(\bar{\theta}, x\right)$.

The outcome function g of the mechanism is defined by the following three rules:
Rule 1: If m is consistent with $\sigma\left(\bar{\theta}, x\right)$, then $g\left(m\right) = x$.

Rule 2: If m is quasi-consistent with $\sigma\left(\bar{\theta}, x\right)$ and $m_i = \left(m_{i1}, x^i, k^i\right) \notin \sigma_i\left(\bar{\theta}, x\right)$ for some $i \in N$, then we can have three cases:

1. If $m_{i1} = \theta^i = \bar{\theta} \in \sigma_{i1}\left(\bar{\theta}, x\right)$ or $m_{i1} = \phi\left(\theta^i\right) = \phi\left(\bar{\theta}\right)$, then $g\left(m\right) = x$.
2. If $m_{i1} = \theta^i \neq \bar{\theta}$ or $m_{i1} = \phi\left(\theta^i\right) \neq \phi\left(\bar{\theta}\right)$, then given that $\theta^i = \left(\phi^{-1} \circ \phi\right)\left(\theta^i\right)$: ($a$) $g\left(m\right) = x^i$ if $x^i \in S_i\left(\theta^i; x, \bar{\theta}\right)$; ($b$) $g\left(m\right) = x^i$ if $x^i \in C_i\left(\bar{\theta}, x\right) \backslash S_i\left(\theta^i; x, \bar{\theta}\right)$ and $S_i\left(\theta^i; x, \bar{\theta}\right) \subseteq SL_i\left(\theta^i, x^i\right)$; ($c$) $g\left(m\right) = y$ if $x^i \in C_i\left(\bar{\theta}, x\right) \backslash S_i\left(\theta^i; x, \bar{\theta}\right)$ and $y \in S_i\left(\theta^i; x, \bar{\theta}\right) \cap I_i\left(\theta^i, x^i, X\right)$; ($d$) otherwise, $g\left(m\right) = z$ for some $z \in S_i\left(\theta^i; x, \bar{\theta}\right)$.
3. If $m_{i1} = \theta^i = \bar{\theta} \notin \sigma_{i1}\left(\bar{\theta}, x\right)$, then: ($a$) $g\left(m\right) = x^i$ if $x^i \in S_i\left(\theta^i; x, \bar{\theta}\right)$; ($b$) otherwise, $g\left(m\right) = z$ for some $z \in S_i\left(\theta^i; x, \bar{\theta}\right)$.

Rule 3: Otherwise, a modulo game is played: divide the sum $\sum_{i \in N} k^i$ by n and identify the remainder, which can be either 0, 1, . . ., or $n - 1$. The individual having the same index of the remainder is declared the winner of the game and the alternative implemented is the one she selects, with the convention that the winner is individual n if the remainder is 0.
Note that in *Rule 2–3*, $m_{i1} = \theta^i = \bar{\theta} \notin \sigma_{i1}\left(\bar{\theta}, x\right)$ implies that there exists one and only one individual $q \neq i$ such that $x \in S_q\left(\bar{\theta}; x, \bar{\theta}\right)$ and $x \notin S_j\left(\bar{\theta}; x, \bar{\theta}\right)$ for every $j \in N \backslash \{q\}$. In contrast, in *Rule 2–1*, if $m_{i1} = \bar{\theta} \in \sigma_{i1}\left(\bar{\theta}, x\right)$, then there can be three cases: i is the unique individual who has $x \in S_i\left(\bar{\theta}; x, \bar{\theta}\right)$ and $x \notin S_j\left(\bar{\theta}; x, \bar{\theta}\right)$ for every $j \in N \backslash \{i\}$; there are at least two individuals p and q such that $x \in S_p\left(\bar{\theta}; x, \bar{\theta}\right) \cap S_q\left(\bar{\theta}; x, \bar{\theta}\right)$; or $x \notin S_p\left(\bar{\theta}; x, \bar{\theta}\right)$ holds for any individual p.

By the above definitions, we have that $\Gamma \equiv (M, g)$ is a mechanism. To paraphrase this mechanism:

- *Rule 1* corresponds to the case in which individuals play a message profile m that is consistent with $\sigma\left(\bar{\theta}, x\right)$. Therefore, given our interpretation of $\phi\left(\bar{\theta}\right)$, this rule applies if each individual i plays a strategy $m_i \in \sigma_i\left(\bar{\theta}, x\right)$ via which she announces $\bar{\theta}$ as a state and $x \in F\left(\bar{\theta}\right)$ as an outcome.
- *Rule 2* applies to situations where the message profile reported by agents is quasi-consistent with $\sigma\left(\bar{\theta}, x\right)$. Then, all but one agent make the same announcement about the state $\bar{\theta}$ and the outcome $x \in F\left(\bar{\theta}\right)$, and some individual i make an announcement m_i such that $m_i \notin \sigma_i\left(\bar{\theta}, x\right)$. In that case, our outcome function distinguishes three cases.
  The outcome is x if all individuals make the same announcement about the state–see *Rule 2.1*.
  Rule 2.2 applies to cases in which agent i's announcement about the state is different from that reported by all other agents. In this case, the outcome is the x^i announced by agent i if either x^i is a truthful outcome according to the state announced by agent i–that is, $x^i \in S_i\left(\theta^i; x, \bar{\theta}\right)$, or x^i is not a truthful outcome according to agent i's announcement but it is an attainable outcome–that is, $x^i \in C_i\left(\bar{\theta}, x\right) \backslash S_i\left(\theta^i; x, \bar{\theta}\right)$—and x^i is better than any other truthful outcome in $S_i\left(\theta^i; x, \bar{\theta}\right)$—see parts ($a$)-($b$) of *Rule 2.2*. The outcome is y if x^i is not a truthful outcome according to agent i's announcement but it is an attainable outcome, and if this y is a truthful outcome according to the state announced by agent i and it is as good as x^i according to the ordering corresponding to the state announced by agent i—see parts (c) of *Rule 2.2*. The outcome is $z \in S_i\left(\theta^i; x, \bar{\theta}\right)$ if the outcome announced by agent i, x^i, is not an attainable outcome, that is, $x^i \notin C_i\left(\bar{\theta}, x\right)$—see part ($d$) of *Rule 2.2*.
  Rule 2.3 applies to cases in which all individuals make the same announcement about the state and agent i's announcement about the state is such that $m_{i1} = \theta^i = \bar{\theta} \notin \sigma_{i1}\left(\bar{\theta}, x\right)$. In this case, the outcome is the x^i announced by agent i if x^i is a truthful outcome according to the state announced by agent i. Otherwise, the outcome is z for some $z \in S_i\left(\theta^i; x, \bar{\theta}\right)$.
- *Rule 3* applies the remaining announcements that do not fall under *Rule 1* or *Rule 2*. The outcome is the alternative announced by the winner of the modulo game.

We show that this Γ partially-honestly implements F. For every individual i, define the truth-telling correspondence as follows:

$$T_i^{\Gamma}\left(\bar{\theta}\right) = \left\{\bar{\theta}\right\} \times X \times N, \qquad \text{for every state } \bar{\theta} \in \Theta.$$

It is clear that the truth-telling correspondence is not empty, as required by Definition 3.

Thus, we are left to show that

$$F\left(\bar{\theta}\right) = NA\left(\Gamma, \succsim^{\Gamma, \bar{\theta}, \bar{H}}\right), \qquad \text{for every pair } \left(\bar{\theta}, \bar{H}\right) \in \Theta \times \mathcal{H}.$$

To this end, fix any pair $(\theta, H) \in \Theta \times \mathcal{H}$.

Let us first show that if $x \in F(\theta)$, then there is a strategy profile $m \in NE\left(\Gamma, \succsim^{\Gamma, \theta, H}\right)$ with $g(m) = x$.

Suppose that $x \in F(\theta)$. Given that the profile $\sigma(\theta, x)$ is well defined, take any strategy profile m that is consistent with $\sigma(\theta, x)$. Thus, m falls into *Rule 1* and $x = g(m)$. We claim that $m \in NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$.

To see this, first observe that any deviation of j will get her to an outcome in $C_j(\theta, x)$ by *Rule 2*, and so $g\left(M_j, m_{-j}\right) \subseteq C_j(\theta, x)$. Since $C_j(\theta, x) \subseteq L_j(\theta, x)$, such deviations are not profitable if $j \notin H$. To see that such deviations are also not profitable for $j \in H$, we proceed according to whether $\sigma_{j1}(\theta, x) = \{\theta\}$ or not.

Suppose that $\sigma_{j1}(\theta, x) = \{\theta\}$. Then, given that $m_j \in T_j^{\Gamma}(\theta)$, there is no unilateral profitable deviation for this $j \in H$. Suppose that $\sigma_{j1}(\theta, x) = \{\phi(\theta)\}$. Then, $m_j \notin T_j^{\Gamma}(\theta)$, and, moreover, $x \notin S_j(\theta; x, \theta)$, by definition of $\sigma_j(\theta, x)$. Note that by definition of *Rule 2.3* any deviation to a truthful strategy choice for θ by this j will result in outcomes of $S_j(\theta; x, \theta)$. Since part (1)(b) of Condition μ^*(ii) implies that $S_j(\theta; x, \theta) \subseteq SL_j(\theta, x)$, there is no unilateral profitable deviation for this j.

In summary, j's deviations from m are not profitable, and so $m \in NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$, as we sought.

For the converse, suppose that $m \in NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$. We show that $g(m) \in F(\theta)$. To obtain a contradiction, we suppose that $g(m) \notin F(\theta)$. We proceed by cases.

Case 1 m falls into *Rule 3*

Given the richness of the strategy space, we see that $X \subseteq g\left(M_j, m_{-j}\right)$ for every j. Since $m \in NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$, it follows that $X \subseteq L_j(\theta, g(m))$ for every j. Given that the SCR F is unanimous, $g(m) \in F(\theta)$, which is a contradiction.

Case 2 m falls into *Rule 1*

Then, $g(m) = x$. Note that Condition μ^*(ii) implies that $x \in C_j\left(\bar{\theta}, x\right)$ for each $j \in N$. If $\theta = \bar{\theta}$, there is an immediate contradiction. We thus suppose that $\theta \neq \bar{\theta}$. It follows that $m_h \notin T_h^{\Gamma}(\theta)$ for every $h \in H$. Fix any $h \in H$. Suppose that there exists $y \in S_h\left(\theta; x, \bar{\theta}\right)$ such that $y I_h(\theta)\, x$. This agent can change m_h into $m_h' = \left(\theta, y, k^h\right)$. Then, by part (*a*) of *Rule 2.2*, $g\left(m_{-h}, m_h'\right) = y$, yielding a contradiction. Since m is a Nash equilibrium at $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$ and since agent h can obtain any outcome in $S_h\left(\theta; x, \bar{\theta}\right)$ by part (*a*) of *Rule 2.2*, it follows that $x P_h(\theta)\, y$ for all $y \in S_h\left(\theta; x, \bar{\theta}\right)$. By changing m_h into $m_h' = \left(\theta, x, k^h\right)$, this agent obtains $g\left(m_{-h}, m_h'\right) = x$, by part (*b*) of *Rule 2.2*, given that $x \in C_h\left(\bar{\theta}, x\right) \backslash S_h\left(\theta; x, \bar{\theta}\right)$ and $S_h\left(\theta; x, \bar{\theta}\right) \subseteq SL_h(\theta, x)$. Since $m_h' = \left(\theta, x, k^h\right) \in T_h^{\Gamma}(\theta)$ and $m_h \notin T_h^{\Gamma}(\theta)$, and since h is a partially-honest agent, we have that $\left(m_h', m_{-h}\right) \succ_h^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$.

Case 3 m falls into *Rule 2.1*

Then, $g(m) = x$. Note that Condition μ^*(ii) implies that $x \in C_j\left(\bar{\theta}, x\right)$ for each $j \in N$. Again, if $\theta = \bar{\theta}$, there is an immediate contradiction. We thus suppose that $\theta \neq \bar{\theta}$. We show that $m \notin NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$, which is a contradiction. Since $\theta \neq \bar{\theta}$, it follows that $m_h \notin T_h^{\Gamma}(\theta)$ for every $h \in H$. Fix any $h \in H$.

Suppose that $h = i$. This i can change m_i into $m_i' = \left(\theta, x, k^i\right) \in T_i^{\Gamma}(\theta)$ so as to induce either part *(a)*, part *(b)*, or part *(c)* of *Rule 2.2*. If $\left(m_i', m_{-i}\right)$ induces part (*a*), then $x \in S_i\left(\theta; x, \bar{\theta}\right)$ and $g\left(m_i', m_{-i}\right) = x$. If $\left(m_i', m_{-i}\right)$ induces part *(b)*, then $S_i\left(\theta; x, \bar{\theta}\right) \subseteq SL_i(\theta, x)$ and $g\left(m_i', m_{-i}\right) = x$. Finally, if $\left(m_i', m_{-i}\right)$ induces part *(c)*, then $x \in C_i\left(\bar{\theta}, x\right) \backslash S_i\left(\theta; x, \bar{\theta}\right)$ and there exists $y \in S_i\left(\theta; x, \bar{\theta}\right)$ such that $y I_i(\theta)\, x$, which results in $g\left(m_i', m_{-i}\right) = y$. Thus, in either case, we have that $g(m_i', m_{-i})\, I_i(\theta)\, x$. Since $m_i \notin T_i^{\Gamma}(\theta)$ and $m_i' \in T_i^{\Gamma}(\theta)$, it follows that $(m_i', m_{-i}) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$.

Suppose that $h \neq i$. This h can change m_h into $m_h' = (\theta, x, k^h) \in T_h^{\Gamma}(\theta)$ so as to induce *Rule 3*. To attain x, h has only to adjust k^h so as to win the modulo game. Since $m_h \notin T_h^{\Gamma}(\theta)$ and $m_h' \in T_h^{\Gamma}(\theta)$, it follows that $(m_h', m_{-h}) \succ_h^{\Gamma,\theta} m$, which is a contradiction.

Case 4 m falls into *Rule 2.3*

Then, $g(m) \in S_i(\theta^i; x, \bar{\theta})$. Given that in this case it must hold that $m_i \notin \sigma_i(\bar{\theta}, x)$ and that $m_{i1} = \theta^i = \bar{\theta} \notin \sigma_{i1}(\bar{\theta}, x)$, it follows from the definition of the profile $\sigma(\bar{\theta}, x)$ and the fact that m falls into *Rule 2.3* that $x \in S_q(\bar{\theta}; x, \bar{\theta})$ for one and only one individual $q \neq i$,[25] and so $g(m) \neq x$. We proceed according to whether $\theta = \bar{\theta}$ or not.

Sub-case 4.1 $\theta = \theta^i = \bar{\theta}$

Observe that $x \notin S_i(\theta; x, \theta)$ given that $\bar{\theta} \notin \sigma_{i1}(\bar{\theta}, x)$. Suppose that $i \in H$. Given that i can attain x by inducing *Rule 1*, we have that $x \in g(M_i, m_{-i})$. Given that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$, it also holds that $g(m)\, R_i(\theta)\, x$. However, since $x \in C_i(\theta, x) \subseteq L_i(\theta, x)$ and, moreover, $g(m) \in S_i(\theta; x, \theta) \subseteq C_i(\theta, x)$, it follows that $x I_i(\theta)\, g(m)$, which contradicts part (1)(b) of Condition μ^* (ii). Therefore, it must be the case that $i \notin H$.

Suppose that $H \setminus \{q\} \neq \varnothing$. Then, take any $h \in H \setminus \{q\}$. Note that $m_h \notin T_h^{\Gamma}(\theta)$ given that $\sigma_{h1}(\theta, x) = \{\phi(\theta)\}$. This h can change m_h into $m_h' = (\theta, g(m), k^h) \in T_h^{\Gamma}(\theta)$ so as to induce *Rule 3*. To attain $g(m)$, h has only to adjust k^h so as to win the modulo game. Thus, $(m_h', m_{-h}) \succ_h^{\Gamma,\theta} m$, which contradicts that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$. Otherwise, let $H \setminus \{q\} = \varnothing$. Given that $H \neq \varnothing$ and that $i \notin H$, we are left to consider the case where $H = \{q\}$. Recall that $x \in S_q(\theta; x, \theta)$ for one and only one individual $q \neq i$.

Let us show that $X \subseteq L_j(\theta, g(m))$ for every $j \neq i$. To this end, take any $z \in X \setminus \{x\}$ and any $j \neq i$. By changing m_j into $m_j' = (\phi(\theta), z, k^j)$, j can induce *Rule 3*. To attain z, this j has only to adjust k^j so as to win the modulo game. To attain x, j has only to adjust k^j so as to allow $k \in N \setminus \{i, j\}$ to win the modulo game. Thus, we have that $X \subseteq g(M_j, m_{-j})$ and so $X \subseteq L_j(\theta, g(m))$ given that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$, as was to be shown.

Next, let us show that $C_i(\theta, x) \subseteq L_i(\theta, g(m))$. To attain x, i can change m_i into any strategy choice in $\sigma_i(\theta, x)$ and induce *Rule 1*. Thus, $x \in g(M_i, m_{-i})$. Since $g(m) \in C_i(\theta, x)$ and $C_i(\theta, x) \subseteq L_i(\theta, x)$ and, moreover, since $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$, we see that $x I_i(\theta)\, g(m)$. By transitivity, it follows from $C_i(\theta, x) \subseteq L_i(\theta, x)$ that $C_i(\theta, x) \subseteq L_i(\theta, g(m))$.

Since $X \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and $C_i(\theta, x) \subseteq L_i(\theta, g(m))$ and since, moreover, $H = \{q\}$ and $g(m) \notin F(\theta)$, part (2)(b) of Condition μ^*(ii) implies that $x \notin S_q(\theta; x, \theta)$, which is a contradiction.

Sub-case 4.2 $\theta \neq \theta^i = \bar{\theta}$

We show that $m \notin NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$. Note that $m_h \notin T_h^{\Gamma}(\theta)$ for every $h \in H$. Fix any $h \in H$. Suppose that $h = i$. This i can change m_i into $m_i' =$

[25] If $q = i$, then $\sigma_{i1}(\bar{\theta}, x) = \{\bar{\theta}\}$, which is not the case.

$\left(\theta, g\left(m\right), k^{i}\right) \in T_{i}^{\Gamma}\left(\theta\right)$ so as to induce *Rule 2.2* and to obtain $g\left(m_{i}', m_{-i}\right)$ such that $g\left(m_{i}', m_{-i}\right) I_{i}\left(\theta\right) g\left(m\right)$.[26] Therefore, $\left(m_{i}', m_{-i}\right) \succ_{i}^{\Gamma,\theta} m$, yielding a contradiction. Thus, suppose that $h \neq i$. This h can change m_{h} into $m_{h}' = \left(\theta, g\left(m\right), k^{h}\right) \in T_{h}^{\Gamma}\left(\theta\right)$ so as to induce *Rule 3*. To attain $g\left(m\right)$, h has only to adjust k^{h} so as to win the modulo game. Again, $\left(m_{h}', m_{-h}\right) \succ_{h}^{\Gamma,\theta} m$, which is a contradiction.

Case 5 m falls into *Rule 2.2*

Let us show that $X \subseteq L_{j}\left(\theta, g\left(m\right)\right)$ for every $j \neq i$. To this end, take any $z \in X \backslash \{x\}$ and any $j \neq i$. By changing m_{j} into $m_{j}' = \left(\phi\left(\theta\right), z, k^{j}\right)$, j can induce *Rule 3*. To attain z, this j has only to adjust k^{j} so as to win the modulo game. To attain x, j has only to adjust k^{j} so as to allow $k \in N \backslash \{i, j\}$ to win the modulo game. Thus, we have that $X \subseteq g\left(M_{j}, m_{-j}\right)$ and so $X \subseteq L_{j}\left(\theta, g\left(m\right)\right)$ given that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. Since the choice of j is arbitrary, we have that $X \subseteq L_{j}\left(\theta, g\left(m\right)\right)$ for each $j \neq i$.

Next, let us show that $C_{i}\left(\bar{\theta}, x\right) \subseteq L_{i}\left(\theta, g\left(m\right)\right)$. To attain x, i can change m_{i} into $m_{i}' \in \sigma_{i}\left(\bar{\theta}, x\right)$ and induce *Rule 1*. Thus, $x \in g\left(M_{i}, m_{-i}\right)$. Let us proceed according to whether $\theta = \bar{\theta}$ or not.

Suppose that $\theta = \bar{\theta}$. Since $g\left(m\right) \in C_{i}\left(\bar{\theta}, x\right)$ and $C_{i}\left(\bar{\theta}, x\right) \subseteq L_{i}\left(\bar{\theta}, x\right)$ and, moreover, since $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, we see that $x I_{i}\left(\theta\right) g\left(m\right)$. By transitivity, it follows from $C_{i}\left(\theta, x\right) \subseteq L_{i}\left(\theta, x\right)$ that $C_{i}\left(\theta, x\right) \subseteq L_{i}\left(\theta, g\left(m\right)\right)$.

Suppose thus that $\theta \neq \bar{\theta}$. By changing m_{i} into $m_{i}' = \left(\phi\left(\theta\right), z^{i}, k^{i}\right)$ with $z^{i} \in S_{i}\left(\theta; x, \bar{\theta}\right)$, i can induce part (a) of *Rule 2.2* and obtain this z^{i}. It follows that i can also attain any outcome in $S_{i}\left(\theta; x, \bar{\theta}\right)$, establishing that $S_{i}\left(\theta; x, \bar{\theta}\right) \cup \{x\} \subseteq L_{i}\left(\theta, g\left(m\right)\right)$. Assume, to the contrary, that there exists $w \in C_{i}\left(\bar{\theta}, x\right) \backslash S_{i}\left(\theta; x, \bar{\theta}\right)$ such that $w P_{i}\left(\theta\right) g\left(m\right)$. By transitivity, we see that $S_{i}\left(\theta; x, \bar{\theta}\right) \cup \{x\} \subseteq SL_{i}\left(\theta, w\right)$. Individual i can change m_{i} into $m_{i}' = \left(\phi\left(\theta\right), w, k^{i}\right)$ so as to obtain $g\left(m_{i}', m_{-i}\right) = w$ by part (b) of *Rule 2.2*, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. Thus, we conclude that $C_{i}\left(\bar{\theta}, x\right) \subseteq L_{i}\left(\theta, g\left(m\right)\right)$.

(+) Suppose that $\sigma_{h1}\left(\bar{\theta}, x\right) = \left\{\phi\left(\bar{\theta}\right)\right\}$ for some $h \in H \backslash \{i\}$ if the set $H \neq \{i\}$. Then, $m_{h} \notin T_{h}^{\Gamma}\left(\theta\right)$. By changing m_{h} into $m_{h}' = \left(\theta, g\left(m\right), k^{h}\right) \in T_{h}^{\Gamma}\left(\theta\right)$, h can induce *Rule 3*. To attain $g\left(m\right)$, this h has only to adjust k^{h} so as to win the modulo game. It follows that $\left(m_{h}', m_{-h}\right) \succ_{h}^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. Therefore, it must be the case that $\sigma_{h1}\left(\bar{\theta}, x\right) = \left\{\bar{\theta}\right\}$ for every $h \in H \backslash \{i\}$ if the set $H \neq \{i\}$.

We distinguish the following cases: (1) $i \notin H$, (2) $H = \{i\}$ and (3) $i \in H$ and $H \cap \left(N \backslash \{i\}\right) \neq \varnothing$.

Sub-case 5.1 $i \notin H$

Suppose that $\bar{\theta} \neq \theta$. Then, $m_{h} \notin T_{h}^{\Gamma}\left(\theta\right)$ for each $h \in H$. Fix any h. The contradiction that $m \notin NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ follows from the argument used above in (+). Thus, in what follows, we assume that $\bar{\theta} = \theta$. We distinguish whether $x \in S_{h}\left(\theta; x, \theta\right)$ for some $h \in H$ or not.

[26] Note that if $g\left(m\right) \notin S_{i}\left(\theta; x, \bar{\theta}\right)$ and there does not exist any outcome $y \in C_{i}\left(\bar{\theta}, x\right)$ such that $y \in S_{i}\left(\theta; x, \bar{\theta}\right) \cap I_{i}\left(\theta, g\left(m\right), Y\right)$, then by part (b) of *Rule 2.2* it follows that $g\left(m_{i}', m_{-i}\right) = g\left(m\right)$.

Suppose that $x \in S_h(\theta; x, \theta)$ for some $h \in H$. By Assumption 2, $\{h\} \in \mathcal{H}$ holds. Then, since $m \in NE(\Gamma, \succsim^{\Gamma,\theta,H})$, it follows that $m \in NE(\Gamma, \succsim^{\Gamma,\theta,\{h\}})$.[27] Since $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, g(m))$ and $X \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and since, moreover, $x \in S_h(\theta; x, \theta)$ and $\{h\} \in \mathcal{H}$, part (2)(b) of Condition μ^*(ii) implies that $g(m) \in F(\theta)$, which is a contradiction.

Suppose that $x \notin S_h(\theta; x, \theta)$ for every $h \in H$. Since $H \neq \{i\}$, it follows from (+) that $\sigma_{h1}(\theta, x) = \{\theta\}$ for every $h \in H$. This implies that $m_h \in T_h^\Gamma(\theta)$ for every $h \in H$. Suppose that $x \in S_p(\theta; x, \theta)$ for some $p \in N \backslash H$ with $p \neq i$. Since $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, g(m))$ and $X \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and since, moreover, $x \in S_p(\theta; x, \theta)$ and $\{p\} \in \mathcal{H}$, part (2)(b) of Condition μ^*(ii) implies that $g(m) \in F(\theta)$, which is a contradiction. Therefore, we have established that $x \notin S_j(\theta; x, \theta)$ for every $j \neq i$. Furthermore, given that $H \neq \varnothing$ and that $i \notin H$ and given that $m_h \in T_h^\Gamma(\theta)$ for every $h \in H$, it cannot be that $x \in S_q(\theta; x, \theta)$ for one and only one individual $q = i$. It follows that $x \notin S_i(\theta; x, \theta)$. By Assumption 2, it also holds that $\{i\} \in \mathcal{H}$. Since $\bar{\theta} = \theta$, part (1)(b) of Condition μ^*(ii) implies that $S_i(\theta; x, \theta) \subseteq SL_i(\theta, x)$ for i when $\{i\} \in \mathcal{H}$ is considered. Now, since $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, x)$ and since $x \in C_i(\theta, x) \subseteq L_i(\theta, g(m))$, it follows that $x I_i(\theta) g(m)$, and so $S_i(\theta; x, \theta) \subseteq SL_i(\theta, g(m))$, by transitivity. However, since $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, g(m))$ and $X \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and since, moreover, $\{i\} \in \mathcal{H}$ and $g(m) \notin F(\theta)$, part (2)(a) of Condition μ^*(ii) implies that $S_i(\theta; x, \theta) \cap I_i(\theta, g(m), X) \neq \varnothing$, which contradicts that $S_i(\theta; x, \theta) \subseteq SL_i(\theta, g(m))$.

Sub-case 5.2 $H = \{i\}$

Then, $m_i \in T_i^\Gamma(\theta)$. To see this, assume, to the contrary, that $m_i \notin T_i^\Gamma(\theta)$. We proceed according to whether $\sigma_{i1}(\bar{\theta}, x) = \{\theta\}$ or not.

Suppose that $\sigma_{i1}(\bar{\theta}, x) = \{\theta\}$. Then, $\theta = \bar{\theta}$. To attain x, i can change m_i into $m_i' = (\theta, x, k^i) \in T_i^\Gamma(\theta)$ and induce *Rule 1*. Since $H = \{i\}$, it follows that $(m_i', m_{-i}) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE(\Gamma, \succsim^{\Gamma,\theta,H})$.

Suppose that $\sigma_{i1}(\bar{\theta}, x) \neq \{\theta\}$. We proceed according to whether $\theta = \bar{\theta}$ or not. Suppose that $\theta \neq \bar{\theta}$. Then, by changing m_i into $m_i' = (\theta, g(m), k^i) \in T_i^\Gamma(\theta)$, i can induce *Rule 2.2*. Since $g(m_i', m_{-i}) I_i(\theta) g(m)$, it follows that $(m_i', m_{-i}) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE(\Gamma, \succsim^{\Gamma,\theta,H})$. Suppose that $\theta = \bar{\theta}$. Then, $\sigma_{i1}(\bar{\theta}, x) = \{\phi(\bar{\theta})\}$ given that $\sigma_{i1}(\bar{\theta}, x) \neq \{\theta\}$, and so it must be the case that $x \in S_q(\bar{\theta}; x, \theta)$ for one and only one individual $q \neq i$ and, consequently, that $\sigma_{p1}(\bar{\theta}, x) = \{\phi(\bar{\theta})\}$ for every $p \neq q$.[28] Since $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, g(m))$ and $X \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and since, moreover, $g(m) \notin F(\theta)$, part (2)(a) of Condition μ^*(ii) implies for $\{i\} = H$ that $g(m) \notin S_i(\theta; x, \theta)$ and that there is $z \in S_i(\theta; x, \theta) \cap I_i(\theta, g(m), X)$. Thus, by changing m_i into $m_i' = (\theta, z, k^i) \in T_i^\Gamma(\theta)$, i can induce

27 Note that if m_p is a best reply to m_{-p} in $(\Gamma, \succsim^{\Gamma,\theta,H})$ where $p \in H$, then m_p is still a best reply to m_{-p} in $(\Gamma, \succsim^{\Gamma,\theta,H'})$ where $p \notin H'$ and $H' \subsetneq H$. Therefore, $m \in NE(\Gamma, \succsim^{\Gamma,\theta,H})$ implies $m \in NE(\Gamma, \succsim^{\Gamma,\theta,\{h'\}})$ for every $h' \in H$.

28 Again, if $q = i$, then $\sigma_{i1}(\bar{\theta}, x) = \{\bar{\theta}\}$, which is not the case.

Rule 2.3 and obtain $g\left(m_i', m_{-i}\right) = z$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. We conclude that $m_i \in T_i^{\Gamma}(\theta)$.

Since $g(m) \in C_i\left(\bar{\theta}, x\right) \subseteq L_i(\theta, g(m))$ and $X \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and since, moreover, either $g(m) \in S_i\left(\theta; x, \bar{\theta}\right)$ or $S_i\left(\theta; x, \bar{\theta}\right) \subseteq SL_i(\theta, g(m))$, part (2)(a) of Condition μ^*(ii) implies that $g(m) \in F(\theta)$, which is a contradiction.

Sub-case 5.3 $i \in H$ and $H \cap (N \backslash \{i\}) \neq \varnothing$

Then, from the same arguments used for *Sub-case 5.1*, one can see that $\bar{\theta} = \theta$. It also follows from (+) that $\sigma_{h1}(\theta, x) = \{\theta\}$ for every $h \in H \backslash \{i\}$, and so $m_h \in T_h^{\Gamma}(\theta)$ for every $h \in H \backslash \{i\}$. Note that $m_i \notin T_i^{\Gamma}(\theta)$ given that $m_{i1} = \theta^i \neq \theta$ or $m_{i1} = \phi\left(\theta^i\right) \neq \phi(\theta)$. Also, note that given that $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, x)$ and that $C_i(\theta, x) \subseteq L_i(\theta, g(m))$ we have that $g(m)\, I_i(\theta)\, x$. We proceed according to whether $\sigma_{i1}(\theta, x) = \{\theta\}$ or not.

Suppose that $\sigma_{i1}(\theta, x) = \{\theta\}$. Then, by changing m_i into $m_i' = \left(\theta, x, k^i\right) \in T_i^{\Gamma}(\theta)$, i can induce *Rule 1* and obtain $g\left(m_i', m_{-i}\right) = x$. Given that $g(m)\, I_i(\theta)\, x$, it follows that $\left(m_i', m_{-i}\right) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$.

Suppose that $\sigma_{i1}(\theta, x) \neq \{\theta\}$. Thus, $\sigma_{i1}(\theta, x) = \{\phi(\theta)\}$, and so there exists exactly one $q \neq i$ such that $x \in S_q(\theta; x, \theta)$ and, consequently, $\sigma_{p1}(\theta, x) = \{\phi(\theta)\}$ for every $p \neq q$. Given that $i \in H$ and $H \cap (N \backslash \{i\}) \neq \varnothing$ and given that $m_h \in T_h^{\Gamma}(\theta)$ for every $h \in H \backslash \{i\}$, it needs to be the case that $H = \{q, i\}$ —otherwise, $m_h \notin T_h^{\Gamma}(\theta)$ for every $h \in H \backslash \{i, q\}$, which is a contradiction.

Since $x \notin S_q(\theta; x, \theta)$, part (1)(b) of Condition μ^*(ii) implies that $S_i(\theta; x, \theta) \subseteq SL_i(\theta, x)$. Furthermore, given that $g(m)\, I_i(\theta)\, x$, it also follows from transitivity that $S_i(\theta; x, \theta) \subseteq SL_i(\theta, g(m))$. Since $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, and since, moreover, Assumption 2 holds, $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,\{i\}}\right)$. Since the premises of part (2)(a) of Condition μ^*(ii) are met, we have that $S_i(\theta; x, \theta) \cap I_i(\theta, g(m), X) \neq \varnothing$, which is a contradiction. □

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