AVERAGES OF ARITHMETIC FUNCTIONS OVER PRINCIPAL IDEALS

T. D. BROWNING AND E. SOFOS

ABSTRACT. For a general class of non-negative functions defined on integral ideals of number fields, upper bounds are established for their average over the values of certain principal ideals that are associated to irreducible binary forms with integer coefficients.

CONTENTS

1. INTRODUCTION

The study of averages of non-negative multiplicative arithmetic functions f over the values of polynomials has a long and venerable history in number theory. From the point of view of upper bounds, this topic goes back to work of Nair [7], which has since been substantially generalised by Nair–Tenenbaum [8] and Henriot [5]. For suitable expanding regions $\mathscr{B} \subset \mathbb{Z}^2$, several authors have extended these results to cover averages of the shape

$$\sum_{(s,t)\in\mathscr{B}} f(|F(s,t)|),$$

where $F \in \mathbb{Z}[s,t]$ is an irreducible binary form. This is the object of work by la Bretèche–Browning [1] and la Bretèche–Tenenbaum [2], for example. These estimates have since had many applications to a range of problems, most notably in the quantitative arithmetic of Châtelet surfaces [3].

Assuming for the moment that $F(x,1)$ is monic and irreducible, any root θ of the polynomial generates a number field $K = \mathbb{Q}(\theta)$ whose degree is equal to the degree of F. In this paper we instead consider a variant in which we

Date: November 7, 2018.
2010 *Mathematics Subject Classification.* 11N37 (11A25, 11N56).

take a general non-negative multiplicative function f defined on the ideals of K, and ask to bound the size of the sum

$$\sum_{(s,t)\in\mathscr{B}} f(s-\theta t),$$

where we view $(s-\theta t)$ as an ideal in the ring of integers $\mathfrak{o}_K$ of K. Our primary motivation for considering this sum is the crucial role that it plays in work of the authors [4] on Manin's conjecture for smooth quartic del Pezzo surfaces.

In order to present our main result we require some notation and definitions. Let $K/\mathbb{Q}$ be a number field with ring of integers $\mathfrak{o}_K$. Denote by $\mathscr{I}_K$ the set of ideals in $\mathfrak{o}_K$. We say that a function $f:\mathscr{I}_K\to\mathbb{R}_{\geqslant 0}$ is *pseudomultiplicative* if there exist strictly positive constants A,B,ε such that

$$f(\mathfrak{a}\mathfrak{b})\leqslant f(\mathfrak{a})\min\left\{A^{\Omega_K(\mathfrak{b})}, B(\mathrm{N}_K\,\mathfrak{b})^{\varepsilon}\right\},$$

for all coprime ideals $\mathfrak{a},\mathfrak{b}\in\mathscr{I}_K$, where $\Omega_K(\mathfrak{b})=\sum_{\mathfrak{p}\mid\mathfrak{b}}\nu_{\mathfrak{p}}(\mathfrak{b})$ and $\mathrm{N}_K\,\mathfrak{b}=\sharp\mathfrak{o}_K/\mathfrak{b}$ is the ideal norm. We denote the class of all pseudomultiplicative functions associated to A,B and ε by $\mathscr{M}_K=\mathscr{M}_K(A,B,\varepsilon)$. When $K=\mathbb{Q}$, this class contains the class of submultiplicative functions that arose in pioneering work of Shiu [9]. Note that any $f\in\mathscr{M}_K$ satisfies $f(\mathfrak{a})\ll A^{\Omega_K(\mathfrak{a})}$ and $f(\mathfrak{a})\ll(\mathrm{N}_K\,\mathfrak{a})^{\varepsilon}$, for any $\mathfrak{a}\in\mathscr{I}_K$, where the second implied constant depends on B.

We will need to work with functions supported away from ideals of small norm. To facilitate this, for any ideal $\mathfrak{a}\in\mathscr{I}_K$ and $W\in\mathbb{N}$, we set

$$\mathfrak{a}_W=\prod_{\substack{\mathfrak{p}^{\nu}\|\mathfrak{a}\\ \gcd(\mathrm{N}_K\,\mathfrak{p},W)=1}}\mathfrak{p}^{\nu}.$$

We extend this to rational integers in the obvious way. Next, for any $f\in\mathscr{M}_K$, we define $f_W(\mathfrak{a})=f(\mathfrak{a}_W)$. We will always assume that W is of the form

$$W=\prod_{p\leqslant w}p, \tag{1.1}$$

for some $w>0$. Thus $\gcd(\mathrm{N}_K\,\mathfrak{p},W)=1$ if and only if $p>w$, if $\mathrm{N}_K\,\mathfrak{p}=p^{f_{\mathfrak{p}}}$ for some $f_{\mathfrak{p}}\in\mathbb{N}$. Let

$$\mathscr{P}_K^{\circ}=\{\mathfrak{a}\subset\mathfrak{o}_K:\mathfrak{p}\mid\mathfrak{a}\Rightarrow f_{\mathfrak{p}}=1\} \tag{1.2}$$

be the multiplicative span of all prime ideals $\mathfrak{p}\subset\mathfrak{o}_K$ with residue degree $f_{\mathfrak{p}}=1$. For any $x>0$ and $f\in\mathscr{M}_K$ we set

$$E_f(x;W)=\exp\left(\sum_{\substack{\mathfrak{p}\in\mathscr{P}_K^{\circ}\text{ prime}\\ w<\mathrm{N}_K\,\mathfrak{p}\leqslant x\\ f_{\mathfrak{p}}=1}}\frac{f(\mathfrak{p})}{\mathrm{N}_K\,\mathfrak{p}}\right),$$

if f is submultiplicative, and

$$E_f(x;W) = \sum_{\substack{\mathrm{N}_K\,\mathfrak{a}\leqslant x\\ \mathfrak{a}\in\mathscr{P}_K^\circ \text{ square-free}\\ \gcd(\mathrm{N}_K\,\mathfrak{a},W)=1}} \frac{f(\mathfrak{a})}{\mathrm{N}_K\,\mathfrak{a}},$$

otherwise.

Suppose now that we are given irreducible binary forms $F_1,\dots,F_N \in \mathbb{Z}[x,y]$, which we assume to be pairwise coprime. Let $i\in\{1,\dots,N\}$. Suppose that F_i has degree d_i and that it is not proportional to y, so that $b_i = F_i(1,0)$ is a non-zero integer. It will be convenient to form the homogeneous polynomial

$$\tilde{F}_i(x,y) = b_i^{d_i-1}F(b_i^{-1}x,y). \tag{1.3}$$

This has integer coefficients and satisfies $\tilde{F}_i(1,0)=1$. We let θ_i be a root of the monic polynomial $\tilde{F}(x,1)$. Then θ_i is an algebraic integer and we denote the associated number field of degree d_i by $K_i=\mathbb{Q}(\theta_i)$. Moreover,

$$N_{K_i/\mathbb{Q}}(b_is-\theta_it) = \tilde{F}_i(b_is,t) = b_i^{d_i-1}F_i(s,t), \tag{1.4}$$

for any $(s,t)\in\mathbb{Z}^2$. (If $b_i=0$, so that $F_i(x,y)=cy$ for some non-zero $c\in\mathbb{Z}$, we take $\theta_i=-c$ and $K_i=\mathbb{Q}$ in this discussion.) Our main result is a tight upper bound for averages of $f_{1,W}((b_1s-\theta_1t))\dots f_{N,W}((b_Ns-\theta_Nt))$, over primitive vectors $(s,t)\in\mathbb{Z}^2$, for general pseudomultiplicative functions $f_i\in\mathscr{M}_{K_i}$ and suitably large values of w.

Next, for any $k\in\mathbb{N}$ and any polynomial $P\in\mathbb{Z}[x]$, we set

$$\rho_P(k) = \sharp\{x \pmod k : P(x)\equiv 0 \pmod k\}.$$

We put

$$\overline{\rho}_i(k) = \begin{cases}\rho_{F_i(x,1)}(k) & \text{if } F_i(1,0)\neq 0,\\ 1 & \text{if } F_i(1,0)=0,\end{cases} \tag{1.5}$$

and

$$h^*(k) = \prod_{p\mid k}\left(1-\frac{\overline{\rho}_1(p)+\dots+\overline{\rho}_N(p)}{p+1}\right)^{-1}. \tag{1.6}$$

To any non-empty bounded measurable region $\mathscr{R}\subset\mathbb{R}^2$, we associate

$$K_{\mathscr{R}} = 1+\|\mathscr{R}\|_\infty + \partial(\mathscr{R})\log(1+\|\mathscr{R}\|_\infty) + \frac{\mathrm{vol}(\mathscr{R})}{1+\|\mathscr{R}\|_\infty}, \tag{1.7}$$

where $\|\mathscr{R}\|_\infty = \sup_{(x,y)\in\mathscr{R}}\{|x|,|y|\}$. We say that such a region $\mathscr{R}$ is *regular* if its boundary is piecewise differentiable, $\mathscr{R}$ contains no zeros of $F_1\cdots F_N$ and there exists $c_1>0$ such that $\mathrm{vol}(\mathscr{R})\geqslant K_{\mathscr{R}}^{c_1}$. Note that we then have

$$\mathrm{vol}(\mathscr{R})\leqslant 4\|\mathscr{R}\|_\infty^2 \leqslant (1+\|\mathscr{R}\|_\infty)^4 \leqslant K_{\mathscr{R}}^4.$$

Bearing all of this in mind, we may now record our main result.

Theorem 1.1. *Let $\mathscr{R} \subset \mathbb{R}^2$ be a regular region, let $V = \mathrm{vol}(\mathscr{R})$ and let $G \subset \mathbb{Z}^2$ be a lattice of full rank, with determinant q_G and first successive minimum λ_G. Assume that $q_G \leqslant V^{c_2}$ for some $c_2 > 0$. Let $f_i \in \mathscr{M}_{K_i}(A_i, B_i, \varepsilon_i)$, for $1 \leqslant i \leqslant N$ and let*

$$\varepsilon_0 = \max\left\{1 + \frac{4}{c_1}, \frac{4(5 + 3\max\{\varepsilon_1, \dots, \varepsilon_N\})}{c_1}\right\}\Big(\sum_{i=1}^N d_i \varepsilon_i\Big).$$

Then, for any $\varepsilon > 0$ and $w > w_0(f_i, F_i, N)$, we have

$$\sum_{(s,t)\in\mathbb{Z}^2_{prim}\cap\mathscr{R}\cap G} \prod_{i=1}^N f_{i,q_G W}((b_i s - \theta_i t)) \ll \frac{V}{(\log V)^N} \frac{h^*_W(q_G)}{q_G} \prod_{i=1}^N E_{f_i}(V;W) + \frac{K_{\mathscr{R}}^{1+\varepsilon_0+\varepsilon}}{\lambda_G},$$

where the implied constant depends at most on $c_1, c_2, A_i, B_i, F_i, \varepsilon, \varepsilon_i, N, W$.

We may compare this estimate with the principal result in work of la Bretèche–Tenenbaum [2]. Take $G = \mathbb{Z}^2$ and $d_1 = \cdots = d_N = 1$. Then $b_i s - \theta_i t = F_i(s,t)$, for $1 \leqslant i \leqslant N$. In this setting Theorem 1.1 can be deduced from the special case of [2, Thm. 1.1], in which all of the binary forms are linear.

Acknowledgements. While working on this paper the first author was supported by ERC grant 306457.

2. Technical results

2.1. **Lattice point counting.** We will need general results about counting lattice points in an expanding region. Let $\mathbf{A} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ be a non-singular upper triangular matrix and consider the lattice given by $G = \{\mathbf{A}\mathbf{y} : \mathbf{y} \in \mathbb{Z}^2\}$. Recall that G is said *primitive* if the only integers m fulfilling $G \subset m\mathbb{Z}^2$ are $m = \pm 1$. We denote its *determinant* and first *successive minimum* by $\det(G)$ and λ_G, respectively. Assume that $\mathscr{R} \subset \mathbb{R}^2$ is a regular region, in the sense of Theorem 1.1. Then, for any $\mathbf{x}_0 \in \mathbb{Z}^2$ and $q \in \mathbb{N}$ such that $\gcd(\det(G)\mathbf{x}_0, q) = 1$, we will require an asymptotic estimate for the counting function

$$N(\mathscr{R}) = \sharp\{\mathbf{x} \in \mathbb{Z}^2_{\mathrm{prim}} \cap \mathscr{R} \cap G : \mathbf{x} \equiv \mathbf{x}_0 \,(\mathrm{mod}\, q)\}.$$

The following estimate follows from work of Sofos [10, Lemma 5.3].

Lemma 2.1. *Assume that G is primitive. Then*

$$N(\mathscr{R}) = \frac{\mathrm{vol}(\mathscr{R})}{\zeta(2)\det(G)q^2} \prod_{p\mid\det(G)} \left(1+\frac{1}{p}\right)^{-1} \prod_{p\mid q}\left(1-\frac{1}{p^2}\right)^{-1} + O\left(\frac{\tau(\det(G))K_{\mathscr{R}}}{\lambda_G}\right),$$

where $K_{\mathscr{R}}$ is given by (1.7) *and the implied constant is absolute.*

2.2. **Restriction to square-free support.** For a given number field $K/\mathbb{Q}$ of degree d and given $f \in \mathscr{M}_K$, it will sometimes be useful to bound sums of the shape

$$\sum_{\mathrm{N}_K\,\mathfrak{a}\leqslant x} \frac{f(\mathfrak{a})}{\mathrm{N}_K\,\mathfrak{a}},$$

by a sum restricted to square-free integral ideals supported away from ideals of small norm. This is encapsulated in the following result.

Lemma 2.2. *Let $f \in \mathscr{M}_K(A,B,\varepsilon)$. Assume that $f^\dagger$ is multiplicative and that there exists $M>0$ such that*

$$|f^\dagger(\mathfrak{p}^\nu)-1| \leqslant \frac{M}{\mathrm{N}_K\,\mathfrak{p}},$$

for all prime ideals $\mathfrak{p}$ and $\nu \in \mathbb{N}$. Assume that W is given by (1.1), *with $w > 2(A+M)$. Then*

$$\sum_{\mathrm{N}_K\,\mathfrak{a}\leqslant x} \frac{f(\mathfrak{a}_W)f^\dagger(\mathfrak{a}_W)}{\mathrm{N}_K\,\mathfrak{a}} \ll_{A,M,W} \sum_{\substack{\mathrm{N}_K\,\mathfrak{b}\leqslant x\\ \mathfrak{b}\ \text{square-free}\\ \gcd(\mathrm{N}_K\,\mathfrak{b},W)=1}} \frac{f(\mathfrak{b})}{\mathrm{N}_K\,\mathfrak{b}}.$$

If f is submultiplicative then the right hand side can be replaced by

$$\exp\left(\sum_{w<\mathrm{N}_K\,\mathfrak{p}\leqslant x} \frac{f(\mathfrak{p})}{\mathrm{N}_K\,\mathfrak{p}}\right).$$

Proof. The final part of the lemma is obvious. To see the first part we note that there is a unique factorisation $\mathfrak{a} = \mathfrak{q}\mathfrak{a}_W$, where $\mathrm{N}_K\,\mathfrak{q} \mid W^\infty$. Here, and throughout this paper, for $a,b \in \mathbb{N}$ the notation $a \mid b^\infty$ is taken to mean that every prime divisor of a is a divisor of b as well. Next, we decompose uniquely $\mathfrak{a}_W = \mathfrak{b}\mathfrak{c}$ where $\mathfrak{b},\mathfrak{c}$ are coprime integral ideals such that $\mathfrak{b}$ is square-free and $\mathfrak{c}$ is square-full. The sum in the lemma is at most

$$\sum_{\substack{\mathrm{N}_K\,\mathfrak{q}\leqslant x\\ \mathrm{N}_K\,\mathfrak{q}\mid W^\infty}} \frac{1}{\mathrm{N}_K\,\mathfrak{q}} \sum_{\substack{\mathrm{N}_K\,\mathfrak{b}\leqslant x\\ \mathfrak{b}\ \text{square-free}\\ \gcd(\mathrm{N}_K\,\mathfrak{b},W)=1}} \frac{f(\mathfrak{b})f^\dagger(\mathfrak{b})}{\mathrm{N}_K\,\mathfrak{b}} \sum_{\substack{\mathrm{N}_K\,\mathfrak{c}\leqslant x\\ \mathfrak{c}\ \text{square-full}\\ \gcd(\mathrm{N}_K\,\mathfrak{c},W)=1}} \frac{A^{\Omega_K(\mathfrak{c})}f^\dagger(\mathfrak{c})}{\mathrm{N}_K\,\mathfrak{c}}.$$

For prime ideals with $\mathrm{N}_K\,\mathfrak{p} > 2(A+M)$, we have

$$1+\sum_{d=2}^{\infty}\frac{A^{\Omega_K(\mathfrak{p}^d)}f^{\dagger}(\mathfrak{p}^d)}{(\mathrm{N}_K\,\mathfrak{p})^d} \leqslant 1+\sum_{d=2}^{\infty}\frac{A^d}{(\mathrm{N}_K\,\mathfrak{p})^d}\left(1+\frac{M}{\mathrm{N}_K\,\mathfrak{p}}\right) \leqslant \left(1-\frac{2A^2}{(\mathrm{N}_K\,\mathfrak{p})^2}\right)^{-1}.$$

Thus the sum over $\mathfrak{c}$ converges absolutely. Defining the multiplicative function $g:\mathscr{I}_K\to\mathbb{R}$ via $g(\mathfrak{p}^d)=(f^{\dagger}(\mathfrak{p})-1)\,\mathrm{N}_K\,\mathfrak{p}$, for $d\in\mathbb{N}$, we clearly have

$$f^{\dagger}(\mathfrak{b})=\sum_{\mathfrak{b}=\mathfrak{d}\mathfrak{e}}\frac{g(\mathfrak{d})}{\mathrm{N}_K\,\mathfrak{d}},$$

with $|g(\mathfrak{d})|\leqslant M^{\Omega_K(\mathfrak{d})}$, for any square-free ideal $\mathfrak{b}$. Therefore

$$\sum_{\substack{\mathrm{N}_K\,\mathfrak{b}\leqslant x\\ \mathfrak{b}\text{ square-free}\\ \gcd(\mathrm{N}_K\,\mathfrak{b},W)=1}}\frac{f(\mathfrak{b})f^{\dagger}(\mathfrak{b})}{\mathrm{N}_K\,\mathfrak{b}} = \sum_{\substack{\mathrm{N}_K\,\mathfrak{e}\leqslant x\\ \mathfrak{e}\text{ square-free}\\ \gcd(\mathrm{N}_K\,\mathfrak{e},W)=1}}\frac{f(\mathfrak{e})}{\mathrm{N}_K\,\mathfrak{e}}\sum_{\substack{\mathrm{N}_K\,\mathfrak{d}\leqslant x/(\mathrm{N}_K\,\mathfrak{e})\\ \mathfrak{d}\text{ square-free}\\ \gcd(\mathrm{N}_K\,\mathfrak{d},W)=1}}\frac{g(\mathfrak{d})}{(\mathrm{N}_K\,\mathfrak{d})^2}.$$

The sum over $\mathfrak{d}$ is absolutely convergent, whence

$$\sum_{\mathrm{N}_K\,\mathfrak{a}\leqslant x}\frac{f(\mathfrak{a}_W)f^{\dagger}(\mathfrak{a}_W)}{\mathrm{N}_K\,\mathfrak{a}} \ll_{A,M,W} \sum_{\substack{\mathrm{N}_K\,\mathfrak{e}\leqslant x\\ \mathfrak{e}\text{ square-free}\\ \gcd(\mathrm{N}_K\,\mathfrak{e},W)=1}}\frac{f(\mathfrak{e})}{\mathrm{N}_K\,\mathfrak{e}}\sum_{\substack{\mathrm{N}_K\,\mathfrak{q}\leqslant x\\ \mathrm{N}_K\,\mathfrak{q}\mid W^{\infty}}}\frac{1}{\mathrm{N}_K\,\mathfrak{q}}.$$

The inner sum over $\mathfrak{q}$ is at most $\prod_{\mathrm{N}_K\,\mathfrak{p}\mid W^d}(1-\frac{1}{\mathrm{N}_K\,\mathfrak{p}})^{-1}\ll_{A,M,W}1$, which thereby completes the proof of the lemma. ☐

2.3. **The relevance of $\mathscr{P}_K$.** Let $F\in\mathbb{Z}[x,y]$ be an irreducible non-zero binary form of degree d, which is not proportional to y. In particular $b=F(1,0)$ is a non-zero integer. We recall from (1.3) the associated binary form $\tilde{F}(x,y)=b^{d-1}F(b^{-1}x,y)$, with integer coefficients and $\tilde{F}(1,0)=1$. We let θ be a root of the polynomial $f(x)=\tilde{F}(x,1)$. Then θ is an algebraic integer and $K=\mathbb{Q}(\theta)$ is a number field of degree d over $\mathbb{Q}$. It follows that $\mathbb{Z}[\theta]$ is an order of K with discriminant $\Delta_\theta=|\det(\sigma_i(\omega^j))|^2$, where $\sigma_1,\dots,\sigma_d:K\hookrightarrow\mathbb{C}$ are the associated embeddings. As is well-known, we have

$$\Delta_\theta=[\mathfrak{o}:\mathbb{Z}[\theta]]^2D_K, \tag{2.1}$$

where D_K is the discriminant of K. Recall the definition (2.2) of $\mathscr{P}_K^{\circ}$ and define

$$\mathscr{P}_K=\{\mathfrak{a}\subset\mathscr{P}_K^{\circ}:\mathfrak{p}_1\mathfrak{p}_2\mid\mathfrak{a}\Rightarrow\mathrm{N}_K\,\mathfrak{p}_1\neq\mathrm{N}_K\,\mathfrak{p}_2\text{ or }\mathfrak{p}_1=\mathfrak{p}_2\}, \tag{2.2}$$

which is the subset of $\mathscr{P}_K^{\circ}$ that is cut out by ideals divisible by at most one prime ideal above each rational prime. The following result is crucial in our analysis and will frequently allow us to restrict attention to ideals supported on $\mathscr{P}_K$.

Lemma 2.3. *Let $(s,t) \in \mathbb{Z}^2_{prim}$ such that $F(s,t) \neq 0$. Then $\mathfrak{a} \in \mathscr{P}_K$ for any integral ideal $\mathfrak{a} \mid (bs - \theta t)$ such that* $\gcd(\mathrm{N}_K \mathfrak{a}, 2b\Delta_\theta) = 1$.

Proof. Let $D = 2b\Delta_\theta$ and let $(s,t) \in \mathbb{Z}^2_{\mathrm{prim}}$ such that $F(s,t) \neq 0$. We form the integral ideal $\mathfrak{n} = (bs - \theta t)$. This has norm $\mathrm{N}_K \mathfrak{n} = |\tilde{F}(bs,t)|$. Let $k \mid \tilde{F}(bs,t)$ with $\gcd(k, D) = 1$. In particular $\gcd(k, \Delta_\theta) = 1$.

Now let $p \mid k$. Then $p \nmid t$ since $\gcd(s,t) = 1$ and $p \nmid b$. We choose $\bar{t} \in \mathbb{Z}$ such that $t\bar{t} \equiv 1 \pmod{p}$. Let $\mathfrak{p}$ be any prime ideal such that $\mathfrak{p} \mid (p)$ and $\mathfrak{p} \mid \mathfrak{n}$. Consider the group homomorphism

$$\pi : \mathbb{Z}/p\mathbb{Z} \to (\mathbb{Z}[\theta] + \mathfrak{p})/\mathfrak{p},$$

given by $m \mapsto m + \mathfrak{p}$. Suppose that $\pi(m_1) = \pi(m_2)$ for $m_1, m_2 \in \mathbb{Z}/p\mathbb{Z}$. Then $m_1 - m_2 \in \mathfrak{p}$, whence $\mathrm{N}_K \mathfrak{p} \mid (m_1 - m_2)^d$. But this implies that $p \mid m_1 - m_2$, since $\mathfrak{p} \mid (p)$, and so π is injective. Next suppose that $P(\theta) + \mathfrak{p} \in (\mathbb{Z}[\theta] + \mathfrak{p})/\mathfrak{p}$, where $P(\theta) = \sum_i c_i \theta^i$ for $c_i \in \mathbb{Z}$. Since $\mathfrak{p} \mid \mathfrak{n}$ and $\mathfrak{p} \nmid \bar{t}$, we have $bs\bar{t} - \theta \in \mathfrak{p}$. Thus $P(\theta) - P(bs\bar{t}) \in \mathfrak{p}$. Now choose $m \in \mathbb{Z}/p\mathbb{Z}$ such that $m \equiv P(bs\bar{t}) \pmod{p}$. It then follows that $\pi(m) = P(\theta) + \mathfrak{p}$. Thus π is surjective and so it is an isomorphism. Hence $[\mathbb{Z}[\theta] + \mathfrak{p} : \mathfrak{p}] = p$. In view of (2.1), we also have

$$D_K[\mathfrak{o} : \mathbb{Z}[\theta] + \mathfrak{p}]^2[\mathbb{Z}[\theta] + \mathfrak{p} : \mathbb{Z}[\theta]]^2 = \Delta_\theta.$$

This implies that $p \nmid [\mathfrak{o} : \mathbb{Z}[\theta] + \mathfrak{p}]$. Since $\mathrm{N}\,\mathfrak{p}$ is power of p, we readily conclude that

$$\mathrm{N}\,\mathfrak{p} = [\mathfrak{o} : \mathbb{Z}[\theta] + \mathfrak{p}][\mathbb{Z}[\theta] + \mathfrak{p} : \mathfrak{p}] = p.$$

This therefore establishes that $\mathfrak{a} \in \mathscr{P}_K^\circ$.

To finish the proof it remains to show that there are no distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ with $\mathfrak{p}_1\mathfrak{p}_2 \mid \mathfrak{a}$ and $\mathrm{N}\,\mathfrak{p}_1 = \mathrm{N}\,\mathfrak{p}_2$. Suppose for a contradiction that there exist such primes $\mathfrak{p}_1 \neq \mathfrak{p}_2$. Letting $p = \mathrm{N}\,\mathfrak{p}_1 = \mathrm{N}\,\mathfrak{p}_2$ and noting that $p \nmid \Delta_\theta$, an application of Dedekind's theorem on factorisation of prime ideals supplies us with distinct $n_1, n_2 \in \mathbb{Z}/p\mathbb{Z}$ such that

$$f(x) \equiv (x - n_1)(x - n_2)\Upsilon(x) \pmod{p},$$

for a polynomial $\Upsilon \in (\mathbb{Z}/p\mathbb{Z})[x]$ of degree $[K : \mathbb{Q}] - 2$, with $\mathfrak{p}_1 = (p, \theta - n_1)$ and $\mathfrak{p}_2 = (p, \theta - n_2)$. We conclude from this that $bs\bar{t} - \theta \in \mathfrak{p}_1$ and $\theta - n_1 \in \mathfrak{p}_1$, whence $bs\bar{t} - n_1 \in \mathfrak{p}_1$. Similarly, we have $bs\bar{t} - n_2 \in \mathfrak{p}_2$. But then it follows that $p = \mathrm{N}\,\mathfrak{p}_1 \mid bs\bar{t} - n_1$ and $p = \mathrm{N}\,\mathfrak{p}_2 \mid bs\bar{t} - n_2$. This implies that $n_1 \equiv n_2 \pmod{p}$, which is a contradiction. □

We close this section with an observation about the condition $\mathfrak{a} \mid (bs - \theta t)$ that appears in Lemma 2.3.

Lemma 2.4. *Let $\mathfrak{a} \in \mathscr{P}_K$ such that* $\gcd(\mathrm{N}_K \mathfrak{a}, D_K) = 1$. *Then there exists $k = k(\mathfrak{a}) \in \mathbb{Z}$ such that*

$$\mathfrak{a} \mid (bs - \theta t) \Leftrightarrow bs \equiv kt \pmod{\mathrm{N}_K \mathfrak{a}}$$

for all $(s,t)\in\mathbb{Z}^2$*,*

Proof. It suffices to check this when $\mathfrak{a}=\mathfrak{p}^a$, for some $\mathfrak{p}\in\mathscr{P}_K$ such that $p=\mathrm{N}_K\,\mathfrak{p}$ is unramified, by the Chinese remainder theorem. This is because the definition of $\mathscr{P}_K$ implies that for every rational unramified prime p there is at most one prime ideal $\mathfrak{p}$ above p such that $\mathfrak{p}\mid\mathfrak{a}$. To continue with the proof we note that since $\mathfrak{p}\in\mathscr{P}_K$, there exists $k'\in\mathbb{Z}$ satisfying $k'\equiv\theta\,(\mathrm{mod}\,\mathfrak{p})$, whence there exists $k\in\mathbb{Z}$ such that $k\equiv\theta\,(\mathrm{mod}\,\mathfrak{p}^a)$. Therefore

$$bs\equiv\theta t\,(\mathrm{mod}\,\mathfrak{p}^a)\;\Leftrightarrow\; bs-kt\in\mathbb{Z}\cap\mathfrak{p}^a.$$

We claim that the latter condition is equivalent to $bs\equiv kt\,(\mathrm{mod}\,\mathrm{N}_K\,\mathfrak{p}^a)$. The reverse implication is obvious since $\mathrm{N}_K\,\mathfrak{a}\in\mathfrak{a}$ for any integral ideal $\mathfrak{a}$. The forward implication follows on noting that $\nu_p(bs-kt)\geqslant\nu_{\mathfrak{p}}((bs-kt))\geqslant a$. □

3. The main argument

This section is devoted to the proof of Theorem 1.1, following an approach that is inspired by work of Shiu [9]. We assume familiarity with the notation introduced in §1. Since $F_1,\dots,F_N$ are pairwise coprime it follows that the resultants $\mathrm{Res}(F_i,F_j)$ are all non-zero integers for $i\neq j$. Along the way, at certain stages of the argument, we will need to enlarge the size of W in (1.1). For now we assume that $w>\max_{i\neq j}\{|D_i|,|\mathrm{Res}(F_i,F_j)|\}$, where $D_i=2b_i\Delta_{\theta_i}$ and Res denotes the resultant of two polynomials. We let $\mathrm{N}_i=\mathrm{N}_{K_i}$ and write $F=\prod_{i=1}^N F_i$. Let

$$z=V^\omega, \tag{3.1}$$

where $V=\mathrm{vol}(\mathscr{R})$, for a small constant $\omega>0$ that will be chosen in due course. (In particular, it will need to be sufficiently small in terms of $\varepsilon_1,\dots,\varepsilon_N$.) For each $(s,t)\in\mathbb{Z}^2_{\mathrm{prim}}\cap\mathscr{R}\cap G$, it follows from (1.4) that we have a factorisation

$$\prod_{i=1}^N|\,\mathrm{N}_i(b_is-\theta_it)_{q_GW}|=\prod_{i=1}^N|F_i(s,t)|_{q_GW}=|F(s,t)|_{q_GW}=p_1^{\alpha_1}\cdots p_l^{\alpha_l},$$

with $w<p_1<\dots<p_l$. We define $a_{s,t}$ to be the greatest integer of the form $p_1^{\alpha_1}\cdots p_j^{\alpha_j}$ which is bounded by z and we define $b_{s,t}=F(s,t)_{q_GW}/a_{s,t}$. We have $\gcd(a_{s,t},b_{s,t})=1$ and $P^-(b_{s,t})>P^+(a_{s,t})$. Our lower bound for w ensures that

$$\gcd(\mathrm{N}_i\,\mathfrak{a}_i,\mathrm{N}_j\,\mathfrak{a}_j)=1,$$

for any $\mathfrak{a}_i\mid(b_is-\theta_it)_{q_GW}$ and $\mathfrak{a}_j\mid(b_js-\theta_jt)_{q_GW}$, with $i\neq j$. Lemma 2.3 implies that $\mathfrak{a}_i\in\mathscr{P}_i=\mathscr{P}_{K_i}$, for $1\leqslant i\leqslant N$.

The sum in which we are interested,

$$\sum_{(s,t)\in\mathbb{Z}^2_{\mathrm{prim}}\cap\mathscr{R}\cap G}\prod_{i=1}^N f_{i,q_GW}((b_is-\theta_it)),$$

is sorted into four distinct contributions $E^{(I)}(\mathscr{R}),\dots,E^{(IV)}(\mathscr{R})$. For an appropriate small parameter $\eta>0$, these sums are determined by which of the following attributes are satisfied by (s,t):

(I) $P^-(b_{s,t})\geqslant z^{\frac{\eta}{2}}$;
(II) $P^-(b_{s,t})<z^{\frac{\eta}{2}}$ and $a_{s,t}\leqslant z^{1-\eta}$;
(III) $P^-(b_{s,t})\leqslant \log z\log\log z$ and $z^{1-\eta}<a_{s,t}\leqslant z$;
(IV) $\log z\log\log z<P^-(b_{s,t})<z^{\frac{\eta}{2}}$ and $z^{1-\eta}<a_{s,t}\leqslant z$.

In what follows, we will allow all of our implied constants to depend on $c_1,c_2,A_i,B_i,F_i,\varepsilon,\varepsilon_i,N,W$, as in the statement of Theorem 1.1, as well as on ω and η, whose values will be indicated during the course of the proof. Any further dependence will be indicated by an explicit subscript.

We let $\Omega_i=\Omega_{K_i}$ be the number of prime ideal divisors (counted with multiplicity) and note that $\Omega_i(\mathfrak{a})=\Omega(\mathrm{N}_i\,\mathfrak{a})$ when $\mathfrak{a}\in\mathscr{P}_i$. For given (s,t), the choice of $a_{s,t},b_{s,t}$ that we have made uniquely determines coprime ideals $\mathfrak{a}^{(i)}_{s,t},\mathfrak{b}^{(i)}_{s,t}\subset\mathfrak{o}_i$, with $(b_is-\theta_it)_{q_GW}=\mathfrak{a}^{(i)}_{s,t}\mathfrak{b}^{(i)}_{s,t}$, such that

$$\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}^{(i)}_{s,t}=a_{s,t}\quad\text{ and }\quad \Omega_i(\mathfrak{a}^{(i)}_{s,t})=\Omega(\mathrm{N}_i\,\mathfrak{a}^{(i)}_{s,t}).$$

In particular we emphasise that $\mathfrak{a}^{(i)}_{s,t},\mathfrak{b}^{(i)}_{s,t}$ are supported on prime ideals whose norms are coprime to q_GW. We now have everything in place to start estimating the various contributions. Our main tools will be the geometry of numbers and the fundamental lemma of sieve theory.

Case I. We begin by considering the case $P^-(b_{s,t})\geqslant z^{\frac{\eta}{2}}$. Recalling that $f_i\in\mathscr{M}_{K_i}$, we have $f_{i,q_GW}((b_is-\theta_it))\leqslant f_i(\mathfrak{a}^{(i)}_{s,t})A_i^{\Omega_i(\mathfrak{b}^{(i)}_{s,t})}$, by the coprimality of $\mathfrak{a}^{(i)}_{s,t},\mathfrak{b}^{(i)}_{s,t}$. Hence

$$\begin{aligned}E^{(I)}(\mathscr{R})&=\sum_{\substack{(s,t)\in\mathbb{Z}^2_{\mathrm{prim}}\cap\mathscr{R}\cap G\\ P^-(b_{s,t})\geqslant z^{\frac{\eta}{2}}}}\prod_{i=1}^N f_{i,q_GW}((b_is-\theta_it))\\ &\ll\sum_{\substack{\mathfrak{a}_i\in\mathscr{P}_i\\ \gcd(\mathrm{N}_i\,\mathfrak{a}_i,q_GW\,\mathrm{N}_j\,\mathfrak{a}_j)=1\\ \prod_{i=1}^N\mathrm{N}_i\,\mathfrak{a}_i\leqslant z}}\mathscr{U}(\mathfrak{a}_1,\dots,\mathfrak{a}_N)\prod_{i=1}^N f_i(\mathfrak{a}_i),\end{aligned}\tag{3.2}$$

where

$$\mathscr{U}(\mathfrak{a}_1,\dots,\mathfrak{a}_N)=\sum_{\substack{(s,t)\in\mathbb{Z}^2_{\mathrm{prim}}\cap\mathscr{R}\cap G\\ (b_is-\theta_it)_{q_GW}\in\mathscr{P}_i\\ \mathfrak{a}_i\mid(b_is-\theta_it)\\ (\mathfrak{a}_i,(b_is-\theta_it)/\mathfrak{a}_i)_i=1\\ p\mid F(s,t)_{q_GW}/\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i\Rightarrow p\geqslant z^{\frac{\eta}{2}}}}\prod_{i=1}^N A^{\Omega_i\left((b_is-\theta_it)_{q_GW}/\mathfrak{a}_i\right)}.$$

Here, the condition $\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i\leqslant z$ comes from the fact that $a_{s,t}\leqslant z$. Moreover, we write $(\mathfrak{a},\mathfrak{b})_i=1$ if and only if the ideals $\mathfrak{a},\mathfrak{b}\subset\mathfrak{o}_i$ are coprime.

Defining b through $b\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i=\prod_{p\nmid q_GW}p^{\nu_p(F(s,t))}$, we see that

$$(z^{\frac{\eta}{2}})^{\Omega(b)}\leqslant P^-(b)^{\Omega(b)}\leqslant|b|\leqslant|F(s,t)|\ll\|\mathscr{R}\|_\infty^{\deg(F)}\leqslant K_{\mathscr{R}}^{\deg(F)}.$$

In view of (3.1) and the inequality $K_{\mathscr{R}}^{c_1}\leqslant V$ that is assumed in Theorem 1.1, this shows that $\Omega(b)\ll 1$. Noting that

$$\sum_{i=1}^N\Omega_i\left(\frac{(b_is-\theta_it)_{q_GW}}{\mathfrak{a}_i}\right)=\Omega(b),$$

we may therefore conclude that

$$\mathscr{U}(\mathfrak{a}_1,\dots,\mathfrak{a}_N)\ll\mathscr{U}_{\frac{\eta}{2}}(\mathfrak{a}_1,\dots,\mathfrak{a}_N),\tag{3.3}$$

where for any $\gamma>0$ we define $\mathscr{U}_\gamma(\mathfrak{a}_1,\dots,\mathfrak{a}_N)$ to be the cardinality of (s,t) appearing in the definition of $\mathscr{U}(\mathfrak{a}_1,\dots,\mathfrak{a}_N)$, with the lower bound $z^{\frac{\eta}{2}}$ replaced by z^γ. Our next concern is with an upper bound for this quantity.

Before revealing our estimate for $\mathscr{U}_\gamma(\mathfrak{a}_1,\dots,\mathfrak{a}_N)$, recall the definition of h^* from (1.6) and set

$$h^\dagger(k)=\prod_{p\mid k}\left(1-\frac{d_1+\cdots+d_N}{1+p}\right)^{-1}.\tag{3.4}$$

Then we have the following result.

Lemma 3.1. *Let $\delta,\gamma>0$ and let $\mathfrak{a}_i\in\mathscr{P}_i$, for $1\leqslant i\leqslant N$, with*

$$\gcd(\mathrm{N}_i\,\mathfrak{a}_i,q_GW\,\mathrm{N}_j\,\mathfrak{a}_j)=1\quad\text{and}\quad\prod_{i=1}^N\mathrm{N}_i\,\mathfrak{a}_i\leqslant z.$$

Then

$$\mathscr{U}_\gamma(\mathfrak{a}_1,\dots,\mathfrak{a}_N)\ll_\delta\frac{V}{\gamma^N(\log z)^N}\frac{h_W^*(q_G)}{q_G}\prod_{i=1}^N\frac{h^\dagger(\mathrm{N}_i\,\mathfrak{a}_i)}{\mathrm{N}_i\,\mathfrak{a}_i}+\frac{K_{\mathscr{R}}z^{2\gamma+\delta}}{\lambda_G},$$

uniformly in γ.

We defer the proof of this result, temporarily, and show how it can be used to complete the treatment of $E^{(I)}(\mathscr{R})$, via (3.2) and (3.3). We apply Lemma 3.1 with $\gamma = \frac{\eta}{2}$. We also note that since $f_i \in \mathscr{M}_{K_i}(A_i, B_i, \varepsilon_i)$, we have

$$\prod_{i=1}^N f_i(\mathfrak{a}_i) \ll \prod_{i=1}^N \mathrm{N}_i(\mathfrak{a}_i)^{\varepsilon_i} \leqslant \prod_{i=1}^N \mathrm{N}_i(\mathfrak{a}_i)^{\hat{\varepsilon}},$$

where $\hat{\varepsilon} = \max\{\varepsilon_1, \dots, \varepsilon_N\}$. The overall contribution from the second term is therefore found to be

$$\begin{aligned}
&\ll_\delta \frac{K_{\mathscr{R}} z^{\eta+\delta+\hat{\varepsilon}}}{\lambda_G} \sharp \{\mathfrak{a}_1, \dots, \mathfrak{a}_N : \mathrm{N}_1\, \mathfrak{a}_1 \cdots \mathrm{N}_N\, \mathfrak{a}_N \leqslant z\} \\
&\ll_\delta \frac{K_{\mathscr{R}} z^{1+\eta+2\delta+\hat{\varepsilon}}}{\lambda_G} \\
&\leqslant \frac{K_{\mathscr{R}}^{1+4\omega(1+\eta+2\delta+\hat{\varepsilon})}}{\lambda_G},
\end{aligned}$$

where we used the bound $V \leqslant K_{\mathscr{R}}^4$, as well as $z = V^\omega$. In view of (3.2) and (3.3), the first term in Lemma 3.1 makes the overall contribution

$$\ll_\delta \frac{V}{(\log z)^N} \frac{h_W^*(q_G)}{q_G} \prod_{i=1}^N \sum_{\substack{\mathfrak{a}_i \in \mathscr{P}_i \\ \gcd(\mathrm{N}_i\, \mathfrak{a}_i, q_G W)=1 \\ \mathrm{N}_i\, \mathfrak{a}_i \leqslant z}} \frac{f_i(\mathfrak{a}_i) h^\dagger(\mathrm{N}_i\, \mathfrak{a}_i)}{\mathrm{N}_i\, \mathfrak{a}_i}.$$

Since $\mathfrak{a}_i \in \mathscr{P}_i$, (3.4) implies that

$$h^\dagger(\mathrm{N}_i\, \mathfrak{a}_i) \leqslant \prod_{\mathfrak{p} | \mathfrak{a}_i} \left(1 - \frac{d_1 + \cdots + d_N}{1 + \mathrm{N}_i\, \mathfrak{p}}\right)^{-1} = h_i^\ddagger(\mathfrak{a}_i),$$

say, where we recall that $d_i = [K_i : \mathbb{Q}]$. We enlarge w in order to use Lemma 2.2, and thereby obtain the overall contribution

$$\ll_\delta \frac{V}{(\log z)^N} \frac{h_W^*(q_G)}{q_G} \prod_{i=1}^N \sum_{\substack{\mathfrak{a}_i \in \mathscr{P}_i \\ \gcd(\mathrm{N}_i\, \mathfrak{a}_i, q_G W)=1 \\ \mathfrak{a}_i \text{ square-free} \\ \mathrm{N}_i\, \mathfrak{a}_i \leqslant z}} \frac{f_i(\mathfrak{a}_i)}{\mathrm{N}_i\, \mathfrak{a}_i}.$$

We have therefore proved that for every $\delta > 0$ we have the bound

$$E^{(I)}(\mathscr{R}) \ll_\delta \frac{V}{(\log z)^N} \frac{h_W^*(q_G)}{q_G} \prod_{i=1}^N \sum_{\substack{\mathfrak{a}_i \in \mathscr{P}_i \\ \gcd(\mathrm{N}_i\, \mathfrak{a}_i, q_G W)=1 \\ \mathfrak{a}_i \text{ square-free} \\ \mathrm{N}_i\, \mathfrak{a}_i \leqslant z}} \frac{f_i(\mathfrak{a}_i)}{\mathrm{N}_i\, \mathfrak{a}_i} + \frac{K_{\mathscr{R}}^{1+4\omega(1+\eta+2\delta+\hat{\varepsilon})}}{\lambda_G}, \quad (3.5)$$

where we recall that $\hat{\varepsilon} = \max\{\varepsilon_1, \dots, \varepsilon_N\}$.

Proof of Lemma 3.1. Let $\mathfrak{c}_i \in \mathscr{P}_i$ be given, with $\gcd(\mathrm{N}_i\,\mathfrak{c}_i, q_G W\,\mathrm{N}_j\,\mathfrak{c}_j) = 1$. Define the set

$$\Lambda(\mathfrak{c}_1,\dots,\mathfrak{c}_N) = \left\{(s,t)\in\mathbb{Z}^2\cap G : b_i s \equiv \theta_i t \,(\mathrm{mod}\,\mathfrak{c}_i),\ \text{for } i=1,\dots,N\right\}.$$

Since $\gcd(q_G, \prod_i \mathrm{N}_i\,\mathfrak{c}_i) = 1$, it follows from Lemma 2.4 that this defines a lattice in $\mathbb{Z}^2$ of rank 2 and determinant

$$\det(\Lambda(\mathfrak{c}_1,\dots,\mathfrak{c}_N)) = q_G \prod_{i=1}^N \mathrm{N}_i\,\mathfrak{c}_i.$$

Write $P(z_0) = \prod_{p<z_0} p$, for any $z_0 > 0$, with the usual convention that $P(z_0) = 1$ if $z_0 < 2$. This allows us to write

$$\mathscr{U}_\gamma(\mathfrak{a}_1,\dots,\mathfrak{a}_N) \leqslant \sum_{\substack{(s,t)\in S\\ (\mathfrak{a}_i,(b_is-\theta_it)/\mathfrak{a}_i)_i=1\\ p\mid F(s,t)_{q_GW}/\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i \Rightarrow p\geqslant z^\gamma}} 1 = \sum_{\substack{(s,t)\in S\\ (\mathfrak{a}_i,(b_is-\theta_it)/\mathfrak{a}_i)_i=1}} \sum_{\substack{d\mid F(s,t)/\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i\\ \gcd(d,q_GW)=1\\ d\mid P(z^\gamma)}} \mu(d).$$

where $S = \mathbb{Z}^2_{\mathrm{prim}} \cap \mathscr{R} \cap \Lambda(\mathfrak{a}_1,\dots,\mathfrak{a}_N)$. We shall use the fundamental lemma of sieve theory, as presented by Iwaniec and Kowalski [6, § 6.4]. This provides us with a sieve sequence λ_d^+ supported on square-free integers in the interval $[1, 2z^\gamma]$, with $\lambda_1^+ = 1$ and $|\lambda_d^+| \leqslant 1$, such that

$$\mathscr{U}_\gamma(\mathfrak{a}_1,\dots,\mathfrak{a}_N) \leqslant \sum_{\substack{(s,t)\in S\\ (\mathfrak{a}_i,(b_is-\theta_it)/\mathfrak{a}_i)_i=1}} \sum_{\substack{d\mid F(s,t)/\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i\\ \gcd(d,q_GW)=1\\ d\mid P(z^\gamma)}} \lambda_d^+.$$

Since $\gcd(a_{s,t}, b_{s,t}) = 1$, we note that only d coprime to $\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i$ appear in the inner sum. Interchanging the order of summation, we find that

$$\begin{aligned}\mathscr{U}_\gamma(\mathfrak{a}_1,\dots,\mathfrak{a}_N) &\leqslant \sum_{\mathfrak{e}_i\mid\mathfrak{a}_i} \mu_1(\mathfrak{e}_1)\cdots\mu_N(\mathfrak{e}_N) \sum_{\substack{1\leqslant d\leqslant 2z^\gamma\\ \gcd(d,q_GW)=1\\ \gcd(d,\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i)=1\\ d\mid P(z^\gamma)}} \lambda_d^+ \sum_{\substack{(s,t)\in\mathbb{Z}^2_{\mathrm{prim}}\cap\mathscr{R}\cap\Lambda(\mathfrak{a}_1\mathfrak{e}_1,\dots,\mathfrak{a}_N\mathfrak{e}_N)\\ d\mid F(s,t)}} 1\\ &= \sum_{\mathfrak{e}_i\mid\mathfrak{a}_i} \mu_1(\mathfrak{e}_1)\cdots\mu_N(\mathfrak{e}_N) \sum_{\substack{d_1,\dots,d_N\in\mathbb{N}\\ \gcd(d_i,q_GW\,\mathrm{N}_i\,\mathfrak{a}_i)=\gcd(d_i,d_j\,\mathrm{N}_j\,\mathfrak{a}_j)=1\\ d_1\cdots d_N\leqslant 2z^\gamma\\ d_1\cdots d_N\mid P(z^\gamma)}} \lambda_{d_1\cdots d_N}^+ S(\mathbf{d}),\end{aligned}$$

where if $d = d_1\cdots d_N$, then

$$S(\mathbf{d}) = \sum_{\substack{(\sigma,\tau)\,(\mathrm{mod}\,d)\\ \gcd(\sigma,\tau,d)=1\\ F_i(\sigma,\tau)\equiv 0\,(\mathrm{mod}\,d_i)}} \sum_{\substack{(s,t)\in\mathbb{Z}^2_{\mathrm{prim}}\cap\mathscr{R}\cap\Lambda(\mathfrak{a}_1\mathfrak{e}_1,\dots,\mathfrak{a}_N\mathfrak{e}_N)\\ (s,t)\equiv(\sigma,\tau)\,(\mathrm{mod}\,d)}} 1.$$

If $F_i(x,y) = cy$ for some i, then the condition $b_i s \equiv \theta_i t\,(\mathrm{mod}\,\mathfrak{a}_i\mathfrak{e}_i)$ should be replaced by $t \equiv 0\,(\mathrm{mod}\,\mathfrak{a}_i\mathfrak{e}_i)$.

Recall the definition (1.5) of $\overline{\rho}_i$ and let

$$h(d)=\prod_{p\mid d}\left(1+\frac{1}{p}\right)^{-1}.$$

The number of possible $(\sigma,\tau) \pmod d$ is equal to $\varphi(d)\overline{\rho}_1(d_1)\cdots\overline{\rho}_N(d_N)$. In $S(\mathbf{d})$ the inner sum can be estimated using the geometry of numbers. Calling upon Lemma 2.1, we deduce that

$$S(\mathbf{d})=\frac{V}{\zeta(2)}\frac{h(q_G)}{q_G}\prod_{i=1}^{N}\frac{\overline{\rho}_i(d_i)h(d_i)h(\mathrm{N}_i\,\mathfrak{a}_i)}{d_i\,\mathrm{N}_i\,\mathfrak{e}_i\,\mathrm{N}_i\,\mathfrak{a}_i}+O_\delta\left(\frac{K_{\mathscr{R}}z^{\gamma+\frac{\delta}{2}}}{\lambda_G}\right),$$

for any $\delta>0$. We emphasise that the implied constant in this estimate does not depend on any of $\mathscr{R}, d_i, \mathfrak{a}_i$ or $\mathfrak{e}_i$. Since $|\lambda_d^+|\leqslant 1$ and $\tau_{K_i}(\mathfrak{a}_i)\ll_\delta (\mathrm{N}_i\,\mathfrak{a}_i)^{\frac{\delta}{2N}}$, we find that the overall contribution to $\mathscr{U}_\gamma(\mathfrak{a}_1,\dots,\mathfrak{a}_N)$ from the error term is $O_\delta(K_{\mathscr{R}}z^{2\gamma+\delta}/\lambda_G)$, on summing trivially over $\mathfrak{e}_1,\dots,\mathfrak{e}_N$ and $d_1,\dots,d_N$. This is plainly satisfactory for Lemma 3.1.

Turning to the contribution from the main term, we set

$$g(d)=\mathbf{1}\Big(d,q_GW\prod_{i=1}^{N}\mathrm{N}_i\,\mathfrak{a}_i\Big)\frac{h(d)}{d}\sum_{\substack{d_1\cdots d_N=d\\ \gcd(d_i,d_j)=1}}\prod_{i=1}^{N}\overline{\rho}_i(d_i),$$

where $\mathbf{1}(d,a)=1$ if $\gcd(d,a)=1$ and $\mathbf{1}(d,a)=0$, otherwise. Since $h(d)\leqslant 1$ and $\varphi_i(\mathfrak{a}_i)\leqslant \mathrm{N}_i\,\mathfrak{a}_i$, the main term contributes

$$\ll \frac{V}{q_G}\prod_{i=1}^{N}\frac{1}{\mathrm{N}_i\,\mathfrak{a}_i}\Big|\sum_{\substack{1\leqslant d\leqslant 2z^\gamma\\ d\mid P(z^\gamma)}}\lambda_d^+g(d)\Big|.$$

We may clearly assume without loss of generality that $w<2z^{\max\{\gamma,\frac{\eta}{2}\}}$. For any prime $p\nmid W$, let

$$c_p=1-\frac{h(p)}{p}\sum_{i=1}^{N}\overline{\rho}_i(p)=1-\frac{\overline{\rho}_1(p)+\cdots+\overline{\rho}_N(p)}{p+1}.$$

Recalling that $\deg(F_i)=d_i$ for $1\leqslant i\leqslant N$, we see that

$$c_p\geqslant 1-\frac{d_1+\cdots+d_N}{p+1},$$

for $p\nmid W$. Next, for $y\geqslant 0$, define

$$\Pi(y)=\prod_{\substack{p<y\\ p\nmid W}}c_p,\quad \Pi_1=\prod_{\substack{p\geqslant 2z^\gamma\\ p\mid \mathrm{N}_1\,\mathfrak{a}_1\cdots\mathrm{N}_N\,\mathfrak{a}_N}}c_p,\quad \Pi_2=\prod_{\substack{p\geqslant 2z^\gamma\\ p\mid q_G}}c_p.$$

By the fundamental lemma of sieve theory [6, Lemma 6.3], we find that

$$\sum_{\substack{1\leqslant d\leqslant 2z^\gamma\\ d|P(z^\gamma)}} \lambda_d^+ g(d) \ll \prod_{p<2z^\gamma}(1-g(p)) = \Pi(2z^\gamma)\Pi_1\Pi_2 h_W^*(q_G)\prod_{i=1}^N h^\dagger(\mathrm{N}_i\,\mathfrak{a}_i),$$

in the notation of (1.6) and (3.4). It is clear that $\Pi_i \leqslant 1$ for $i=1,2$. Noting that $\Pi(y)\ll(\log y)^{-N}$, this therefore concludes the proof of the lemma. ☐

Cases II and III. We now estimate $E^{(II)}(\mathscr{R})$ and $E^{(III)}(\mathscr{R})$. For any $(s,t)\in\mathbb{Z}^2_{\text{prim}}\cap\mathscr{R}$, we take the trivial bound

$$\prod_{i=1}^N f_{i,q_GW}((b_is-\theta_it)) \ll \prod_{i=1}^N(\mathrm{N}_i(b_is-\theta_it))^{\varepsilon_i} \ll \|\mathscr{R}\|_\infty^{\sum_i d_i\varepsilon_i} \leqslant K_{\mathscr{R}}^{\sum_i d_i\varepsilon_i}.$$

In Case II the relevant extra constraints are $P^-(b_{s,t})<z^{\frac{\eta}{2}}$ and $a_{s,t}\leqslant z^{1-\eta}$. Let $p=P^-(b_{s,t})<z^{\frac{\eta}{2}}$ and let ν be such that $p^\nu\|F(s,t)$. We must have $p^\nu\geqslant z^\eta$, since otherwise $z<p^\nu a_{s,t}<z^\eta z^{1-\eta}=z$, which is a contradiction. For each prime $p\nmid q_GW$ with $p<z^{\frac{\eta}{2}}$, we define

$$l_p=\min\{l\in\mathbb{Z}_{\geqslant 0}: p^l\geqslant z^\eta\}.$$

Clearly $z^\eta\leqslant p^{l_p}<z^{\frac{\eta}{2}l_p}$, whence $l_p\geqslant 2$ for every prime p. Therefore

$$\sum_{\substack{p<z^{\frac{\eta}{2}}\\ p\nmid q_GW}}\frac{1}{p^{l_p}} \leqslant \sum_{\substack{p<z^{\frac{\eta}{2}}\\ p\nmid q_GW}}\min\left\{\frac{1}{z^\eta},\frac{1}{p^2}\right\} \leqslant \sum_{p\leqslant z^{\frac{\eta}{2}}}\frac{1}{z^\eta}\leqslant z^{-\frac{\eta}{2}}. \tag{3.6}$$

The number of elements (s,t) satisfying the constraints of Case II is at most

$$\sum_{i=1}^N\sum_{\substack{p<z^{\frac{\eta}{2}}\\ p\nmid q_GW}}\sum_{\substack{(s,t)\in\mathbb{Z}^2_{\text{prim}}\cap\mathscr{R}\cap G\\ p^{l_p}|F_i(s,t)}}1 \ll \sum_{i=1}^N\sum_{\substack{p<z^{\frac{\eta}{2}}\\ p\nmid q_GW}}\overline{\rho}_i(p^{l_p})\left(\frac{h(q_G)}{q_G}\frac{V}{p^{l_p}}+\tau(q_G)\frac{K_{\mathscr{R}}}{\lambda_G}\right).$$

Here we have split the inner sum into $\overline{\rho}_i(p^{l_p})$ different lattices of the form $\{(s,t)\in G: s\equiv xt \pmod{p^{l_p}}\}$, where x ranges over solutions of the congruence $F_i(x,1)\equiv 0 \pmod{p^{l_p}}$, before applying Lemma 2.1 with $q=1$. Hensel's lemma implies that $\overline{\rho}_i(p^l)=\overline{\rho}_i(p)\leqslant d_i$ for each prime $p\nmid W$ and $l\in\mathbb{N}$. Let $\delta>0$ be arbitrary. Taking $h(q_G)\leqslant 1$ and $\tau(q_G)\ll_\delta q_G^{\frac{\delta}{8c_2}}\leqslant V^{\frac{\delta}{8}}\leqslant K_{\mathscr{R}}^{\frac{\delta}{2}}$, this therefore reveals that

$$\sum_{i=1}^N\sum_{\substack{p<z^{\frac{\eta}{2}}\\ p\nmid q_GW}}\sum_{\substack{(s,t)\in\mathbb{Z}^2_{\text{prim}}\cap\mathscr{R}\cap G\\ p^{l_p}|F_i(s,t)}}1 \ll_\delta \frac{V}{q_G}\sum_{\substack{p<z^{\frac{\eta}{2}}\\ p\nmid q_GW}}\frac{1}{p^{l_p}}+\frac{K_{\mathscr{R}}^{1+\frac{\delta}{2}}z^{\frac{\eta}{2}}}{\lambda_G} \ll_\delta \frac{V}{q_G}\frac{1}{z^{\frac{\eta}{2}}}+\frac{K_{\mathscr{R}}^{1+\frac{\delta}{2}}z^{\frac{\eta}{2}}}{\lambda_G},$$

by (3.6). Recalling (3.1), we see that $z^{-\frac{\eta}{2}} = V^{-\beta}$, with $\beta = \frac{\eta\omega}{2}$. Likewise, $z^{\frac{\eta}{2}} = V^{\frac{\eta\omega}{2}} \leqslant K_{\mathscr{R}}^{2\eta\omega}$. Noting that

$$K_{\mathscr{R}}^{\sum_i d_i\varepsilon_i} \leqslant V^{\frac{1}{c_1}\sum_i d_i\varepsilon_i},$$

we have therefore proved that for every $\delta > 0$ we have the bound

$$E^{(II)}(\mathscr{R}) \ll_\delta \frac{V^{1-\frac{\eta\omega}{2}+\frac{1}{c_1}\sum_i d_i\varepsilon_i}}{q_G} + \frac{K_{\mathscr{R}}^{1+\frac{\delta}{2}+2\eta\omega+\sum_i d_i\varepsilon_i}}{\lambda_G}. \tag{3.7}$$

We now turn to the contribution from Case III, for which the defining constraints are $P^-(b_{s,t}) \leqslant \log z \log\log z$ and $z^{1-\eta} < a_{s,t} \leqslant z$. We assume that $w > \max_{i\neq j} |\operatorname{Res}(F_i, F_j)|$ in the definition (1.1) of W. For any $(s,t) \in \mathbb{Z}^2_{\text{prim}}$ it follows that the integer factors of $F_i(s,t)_W$ are necessarily coprime to the factors of $F_j(s,t)_W$ for all $i \neq j$. Hence the number of elements (s,t) satisfying the constraints of Case III is at most

$$\begin{aligned}\sum_{\substack{(s,t)\in\mathbb{Z}^2_{\text{prim}}\cap\mathscr{R}\cap G\\ P^-(b_{s,t})\leqslant\log z\log\log z\\ z^{1-\eta}<a_{s,t}\leqslant z}} 1 &\leqslant \sum_{\substack{z^{1-\eta}<a\leqslant z\\ \gcd(a,q_GW)=1\\ P^+(a)\leqslant\log z\log\log z}} \sum_{\substack{(s,t)\in\mathbb{Z}^2_{\text{prim}}\cap\mathscr{R}\cap G\\ a\mid F(s,t)}} 1\\ &\leqslant \sum_{\substack{z^{1-\eta}<a_1\cdots a_N\leqslant z\\ \gcd(a_i,q_GWa_j)=1\\ P^+(a_i)\leqslant\log z\log\log z}} \sum_{\substack{(s,t)\in\mathbb{Z}^2_{\text{prim}}\cap\mathscr{R}\cap G\\ a_i\mid F_i(s,t)}} 1.\end{aligned}$$

As before, the final sum can be split into at most $\prod_{i=1}^N \overline{\rho}_i(a_i) = O_\delta(z^\delta)$ lattices, for any $\delta > 0$, each of determinant $q_G\prod_{i=1}^N a_i$. Thus the right hand side is

$$\begin{aligned}&\ll_\delta z^\delta \sum_{\substack{z^{1-\eta}<a_1\cdots a_N\leqslant z\\ P^+(a_1\cdots a_N)\leqslant\log z\log\log z}} \left(\frac{V}{q_Ga_1\cdots a_N} + \frac{K_{\mathscr{R}}}{\lambda_G}\right)\\ &\ll_\delta z^{2\delta} \sum_{\substack{z^{1-\eta}<a\leqslant z\\ P^+(a)\leqslant\log z\log\log z}} \left(\frac{V}{q_Ga} + \frac{K_{\mathscr{R}}}{\lambda_G}\right),\end{aligned}$$

whence [9, Lemma 1] yields

$$\sum_{\substack{(s,t)\in\mathbb{Z}^2_{\text{prim}}\cap\mathscr{R}\cap G\\ P^-(b_{s,t})\leqslant\log z\log\log z\\ z^{1-\eta}<a_{s,t}\leqslant z}} 1 \ll_\delta z^{3\delta}\left(\frac{V}{q_Gz^{1-\eta}} + \frac{K_{\mathscr{R}}}{\lambda_G}\right).$$

Hence we have shown that for every $\delta > 0$ one has

$$E^{(III)}(\mathscr{R}) \ll_\delta \frac{V^{1-(1-\eta)\omega+3\delta\omega+\frac{1}{c_1}\sum_i d_i\varepsilon_i}}{q_G} + \frac{K_{\mathscr{R}}^{1+3\delta\omega+\sum_i d_i\varepsilon_i}}{\lambda_G}. \tag{3.8}$$

Case IV. The final case to consider is characterised by the constraints

$$\log z \log\log z < P^-(b_{s,t}) < z^{\frac{\eta}{2}} \quad \text{and} \quad z^{1-\eta} < a_{s,t} \leqslant z.$$

Arguing as in (3.2) in the treatment of Case I, we find that

$$E^{(IV)}(\mathscr{R}) \ll \sum_{\substack{\mathfrak{a}_i \in \mathscr{P}_i \\ \gcd(\mathrm{N}_i\,\mathfrak{a}_i, q_G W\,\mathrm{N}_j\,\mathfrak{a}_j)=1 \\ z^{1-\eta} < \prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i \leqslant z}} \mathscr{U}^\dagger(\mathfrak{a}_1,\dots,\mathfrak{a}_N)\prod_{i=1}^N f_i(\mathfrak{a}_i),$$

where $\mathscr{U}^\dagger(\mathfrak{a}_1,\dots,\mathfrak{a}_N)$ is as in the definition of $\mathscr{U}(\mathfrak{a}_1,\dots,\mathfrak{a}_N)$ after (3.2), but with the condition $P^-(F(s,t)_{q_GW}/\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i) \geqslant z^{\frac{\eta}{2}}$ replaced by

$$\log z \log\log z < P^-\left(\frac{F(s,t)_{q_GW}}{\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i}\right) < z^{\frac{\eta}{2}}.$$

In particular, in view of the coprimality of $a_{s,t}$ and $b_{s,t}$, we see that

$$P^+\left(\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i\right) < P^-\left(\frac{F(s,t)_{q_GW}}{\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i}\right).$$

We will find it convenient to enlarge the sum slightly, replacing the condition $\mathfrak{a}_i \in \mathscr{P}_i$ by the condition that each $\mathfrak{a}_i$ belongs to the multiplicative span of degree 1 prime ideals in $\mathfrak{o}_i$.

We may assume without loss of generality that $\frac{2}{\eta} \in \mathbb{Z}_{\geqslant 2}$. Thus

$$(\log z \log\log z, z^{\frac{\eta}{2}}) \subset \bigcup_{k=\frac{2}{\eta}}^{k_0+1} (z^{\frac{1}{k+1}}, z^{\frac{1}{k}}],$$

where $k_0 = [\log z/\log(\log z \log\log z)]$ satisfies $k_0 \leqslant \log z/\log\log z$. Notice that for any integer b satisfying $\log b \ll \log z$ and $z^{\frac{1}{k+1}} < P^-(b) \leqslant z^{\frac{1}{k}}$ we must have $\Omega(b) \ll k$. Applying this with $b = F(s,t)_{q_GW}/\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i$, for any $(s,t) \in \mathbb{Z}^2_{\mathrm{prim}} \cap \mathscr{R}$, we deduce that

$$\prod_{i=1}^N A_i^{\Omega_i\left(\frac{(b_is-\theta_it)_{q_GW}}{\mathfrak{a}_i}\right)} \leqslant \max_{1\leqslant i\leqslant N} A_i^{\Omega(b)} \leqslant A^k,$$

for a suitable constant $A \gg \max_{1\leqslant i\leqslant N} A_i$, where A_i is the constant appearing in the definition of $\mathscr{M}_{K_i} = \mathscr{M}_{K_i}(A_i, B_i, \varepsilon_i)$. Hence

$$E^{(IV)}(\mathscr{R}) \leqslant \sum_{k=\frac{2}{\eta}}^{k_0+1} A^k \sum_{\substack{\mathfrak{a}_i \in \mathscr{P}^\circ_{K_i} \\ \gcd(\mathrm{N}_i\,\mathfrak{a}_i, q_G W\,\mathrm{N}_j\,\mathfrak{a}_j)=1 \\ z^{1-\eta} < \prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i \leqslant z \\ P^+(\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i) < z^{\frac{1}{k}}}} \mathscr{U}_{\frac{1}{k+1}}(\mathfrak{a}_1,\dots,\mathfrak{a}_N)\prod_{i=1}^N f_i(\mathfrak{a}_i),$$

in the notation of Lemma 3.1, which we now use to estimate $\mathscr{U}_{\frac{1}{k+1}}(\mathfrak{a}_1,\dots,\mathfrak{a}_N)$.

The overall contribution from the second term is

$$\ll_\delta \frac{K_{\mathscr{R}}}{\lambda_G}\sum_{k=\frac{2}{\eta}}^{k_0+1} A^k z^{1+\frac{2}{k+1}+2\delta} \leqslant \frac{K_{\mathscr{R}} z^{\frac{5}{3}+2\delta}}{\lambda_G}\sum_{k=\frac{2}{\eta}}^{k_0+1} A^k \ll_\delta \frac{K_{\mathscr{R}} z^{\frac{5}{3}+3\delta}}{\lambda_G} \leqslant \frac{K_{\mathscr{R}}^{1+4\omega(\frac{5}{3}+3\delta)}}{\lambda_G},$$

since $2 \leqslant 2/\eta \leqslant k \leqslant k_0 \ll \log z/\log\log z$ and $z = V^\omega \leqslant K_{\mathscr{R}}^{4\omega}$.

It remains to consider the contribution from the main term in Lemma 3.1. This is

$$\ll \frac{V}{(\log z)^N}\frac{h_W^*(q_G)}{q_G}\sum_{k=\frac{2}{\eta}}^{k_0+1} A^k(k+1)^N E(z^{1-\eta}, z^{\frac{1}{k}}), \tag{3.9}$$

where

$$E(S,T) = \sum_{\substack{\mathfrak{a}_i\in\mathscr{P}_{K_i}^\circ \\ \gcd(\mathrm{N}_i\,\mathfrak{a}_i, W\,\mathrm{N}_j\,\mathfrak{a}_j)=1 \\ \prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i > S \\ P^+(\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i)<T}} \prod_{i=1}^N \frac{f_i(\mathfrak{a}_i)h_i^\dagger(\mathfrak{a}_i)}{\mathrm{N}_i\,\mathfrak{a}_i}.$$

Note that we have dropped the condition $\gcd(\prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i, q_G) = 1$.

Let us define the multiplicative function $u : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ via

$$u(a) = \sum_{\substack{\mathfrak{a}_i\in\mathscr{P}_{K_i}^\circ \\ \gcd(\mathrm{N}_i\,\mathfrak{a}_i, \mathrm{N}_j\,\mathfrak{a}_j)=1 \\ \prod_{i=1}^N \mathrm{N}_i\,\mathfrak{a}_i = a}} \prod_{i=1}^N f_i(\mathfrak{a}_i)h_i^\dagger(\mathfrak{a}_i). \tag{3.10}$$

Note that

$$u(p^k) = \sum_{i=1}^N \sum_{\substack{\mathfrak{p}_i\subset\mathfrak{o}_i \text{ prime} \\ \mathrm{N}_i\,\mathfrak{p}_i = p}} f_i(\mathfrak{p}_i^k)h_i^\dagger(\mathfrak{p}_i^k) \leqslant C^k, \tag{3.11}$$

for an appropriate constant $C > 1$ depending on A_i, d_i and N. We may therefore write

$$E(S,T) = \sum_{\substack{\gcd(a,W)=1 \\ a>S \\ P^+(a)<T}} \frac{u(a)}{a}.$$

Drawing inspiration from the proof of [8, Lemma 2], we shall find an upper bound for $E(S,T)$ in terms of partial sums involving $u(a)$. This is the object of the following result.

Lemma 3.2. *Assume that* $T > \mathrm{e}^{\frac{C}{10}}$ *and let* $\kappa \in (\frac{1}{10}, C^{-1}\log T)$*. Then*

$$E(S,T) \ll_\kappa \mathrm{e}^{-\kappa\frac{\log S}{\log T}} \sum_{\substack{\gcd(b,W)=1\\ b\leqslant T}} \frac{u(b)}{b}.$$

Taking this result on faith for the moment, we return to (3.9) and apply it with κ satisfying $\mathrm{e}^{\kappa(1-\eta)} > 2A$. (Note that the implied constant in Lemma 3.2 depends on κ and so the choice $\kappa = (\log 2A)/(1-\eta)+1$ is acceptable.) This produces the overall contribution

$$\begin{aligned}
&\ll \frac{V}{(\log z)^N}\frac{h_W^*(q_G)}{q_G}\sum_{k=\frac{2}{\eta}}^{k_0+1}\frac{A^k(k+1)^N}{\mathrm{e}^{\kappa k(1-\eta)}}\sum_{\substack{\gcd(b,W)=1\\ b\leqslant z^{\frac{1}{k}}}}\frac{u(b)}{b}\\
&\ll \frac{V}{(\log z)^N}\frac{h_W^*(q_G)}{q_G}\sum_{k=\frac{2}{\eta}}^{k_0+1}\frac{(k+1)^N}{2^k}\sum_{\substack{\gcd(b,W)=1\\ b\leqslant z}}\frac{u(b)}{b}\\
&\ll \frac{V}{(\log z)^N}\frac{h_W^*(q_G)}{q_G}\sum_{\substack{\gcd(b,W)=1\\ b\leqslant z}}\frac{u(b)}{b}.
\end{aligned}$$

Recalling (3.10) and enlarging w to enable the use of Lemma 2.2, shows that the last quantity is

$$\ll \frac{V}{(\log z)^N}\frac{h_W^*(q_G)}{q_G}\prod_{i=1}^N \sum_{\substack{\mathrm{N}_i\,\mathfrak{a}_i\leqslant z\\ \mathfrak{a}_i\in\mathscr{P}^\circ_{K_i}\text{ square-free}\\ \gcd(\mathrm{N}_i\,\mathfrak{a}_i,W)=1}}\frac{f_i(\mathfrak{a}_i)}{\mathrm{N}_i\,\mathfrak{a}_i},$$

which shows that for every $\delta>0$ we have

$$E^{(IV)}(\mathscr{R}) \ll_\delta \frac{V}{(\log z)^N}\frac{h_W^*(q_G)}{q_G}\prod_{i=1}^N \sum_{\substack{\mathrm{N}_i\,\mathfrak{a}\leqslant z\\ \mathfrak{a}_i\in\mathscr{P}^\circ_{K_i}\text{ square-free}\\ \gcd(\mathrm{N}_i\,\mathfrak{a},W)=1}}\frac{f_i(\mathfrak{a}_i)}{\mathrm{N}_i\,\mathfrak{a}_i} + \frac{K_{\mathscr{R}}^{1+4\omega(\frac{5}{3}+3\delta)}}{\lambda_G}. \quad (3.12)$$

Proof of Lemma 3.2. Let $\beta = \frac{\kappa}{\log T}$. Then

$$E(S,T) \leqslant S^{-\beta}\sum_{\substack{\gcd(a,W)=1\\ P^+(a)<T}}\frac{u(a)}{a}a^\beta.$$

Define the multiplicative function ψ_β via $a^\beta = \sum_{c|a}\psi_\beta(c)$, for $a\in\mathbb{N}$. We observe that $\psi_\beta(p^k) = p^{\beta k} - p^{\beta(k-1)}$, for any $k\in\mathbb{N}$, whence $0<\psi_\beta(a)<a^\beta$ for

all $a \in \mathbb{N}$. We now have

$$E(S,T) \leqslant S^{-\beta} \sum_{\substack{\gcd(c,W)=1\\ P^+(c)<T}} \frac{\psi_\beta(c)}{c} \sum_{\substack{\gcd(d,W)=1\\ P^+(d)<T}} \frac{u(cd)}{d}.$$

Writing $d = jd'$, with $\gcd(d',c) = 1$ and $j \mid c^\infty$, shows that

$$E(S,T) \leqslant S^{-\beta} \sum_{\substack{\gcd(d',W)=1\\ P^+(d')<T}} \frac{u(d')}{d'} \sum_{\substack{\gcd(c,d'W)=1\\ P^+(c)<T}} \sum_{j|c^\infty} \frac{\psi_\beta(c)u(cj)}{cj}.$$

After possibly enlarging w, it follows from (3.11) that the sum over c is

$$\begin{aligned}\leqslant \prod_{\substack{p<T\\ p\nmid d'W}} \left(1 + \sum_{\substack{k\geqslant 1\\ j\geqslant 0}} \frac{\psi_\beta(p^k)u(p^{k+j})}{p^{k+j}}\right) &\leqslant \prod_{\substack{p<T\\ p\nmid d'W}} \left(1 + \sum_{\substack{k\geqslant 1\\ j\geqslant 0}} \frac{(p^{\beta k} - p^{\beta(k-1)})C^{k+j}}{p^{k+j}}\right)\\ &\leqslant \exp\left(O\left(\sum_{\substack{p<T\\ p\nmid d'W}} \frac{p^\beta - 1}{p}\right)\right).\end{aligned}$$

Writing $p^\beta = \exp(\frac{\kappa \log p}{\log T}) = 1 + O(\frac{\kappa \log p}{\log T})$, this is found to be at most

$$\exp\left(O\left(\frac{\kappa}{\log T} \sum_{p\leqslant T} \frac{\log p}{p}\right)\right) \ll_\kappa 1.$$

Our argument so far shows that

$$E(S,T) \ll_\kappa \mathrm{e}^{-\kappa\frac{\log S}{\log T}} \sum_{\substack{\gcd(d,W)=1\\ P^+(d)<T}} \frac{u(d)}{d}. \tag{3.13}$$

Let $\xi \in (0,1)$. Observe that each d with $P^+(d) < T$ can be written uniquely in the form $d = d_-d_+$ for coprime $d_-, d_+ \in \mathbb{N}$ such that $P^+(d_-) \leqslant T^\xi$ and $P^-(d_+) > T^\xi$. We clearly have $P^+(d_+) < T$. Thus

$$\sum_{\substack{\gcd(d,W)=1\\ P^+(d)<T}} \frac{u(d)}{d} \leqslant \sum_{\substack{\gcd(d_-,W)=1\\ P^+(d_-)\leqslant T^\xi}} \frac{u(d_-)}{d_-} \sum_{\substack{\gcd(d_+,W)=1\\ P^-(d_+)>T^\xi\\ P^+(d_+)<T}} \frac{u(d_+)}{d_+}.$$

By (3.11), the inner sum is at most $\prod_{T^\xi<p<T}(1+\frac{1}{p})^{2C} \ll_{C,\xi} 1$. Thus, once combined with (3.13), we deduce that

$$E(S,T) \ll_\kappa \mathrm{e}^{-\kappa\frac{\log S}{\log T}} \sum_{\substack{\gcd(d_-,W)=1\\ P^+(d_-)<T^\xi}} \frac{u(d_-)}{d_-}.$$

In order to complete the proof of the lemma, it remains to show that the

$$\sum_{\substack{\gcd(d_-,W)=1\\ P^+(d_-)\leqslant T^\xi}} \frac{u(d_-)}{d_-} \ll \sum_{\substack{\gcd(d,W)=1\\ d<T}} \frac{u(d)}{d}.$$

This is trivial when $T^\xi<2$. Suppose now that $T^\xi>2$. Taking $\kappa=2$ and (T,T^ξ) in place of (S,T), it follows from (3.13) that

$$\sum_{\substack{\gcd(d_-,W)=1\\ d_->T\\ P^+(d_-)<T^\xi}} \frac{u(d_-)}{d_-} \ll \mathrm{e}^{-\frac{2}{\xi}} \sum_{\substack{\gcd(d_-,W)=1\\ P^+(d_-)<T^\xi}} \frac{u(d_-)}{d_-}.$$

Taking ξ suitably small, we conclude that

$$\begin{aligned}\sum_{\substack{\gcd(d_-,W)=1\\ P^+(d_-)<T^\xi}} \frac{u(d_-)}{d_-} &= \sum_{\substack{\gcd(d_-,W)=1\\ P^+(d_-)<T^\xi\\ d_-\leqslant T}} \frac{u(d_-)}{d_-} + \sum_{\substack{\gcd(d_-,W)=1\\ P^+(d_-)<T^\xi\\ d_->T}} \frac{u(d_-)}{d_-}\\ &\leqslant \sum_{\substack{\gcd(d_-,W)=1\\ d_-\leqslant T}} \frac{u(d_-)}{d_-} + \frac{1}{2}\sum_{\substack{\gcd(d_-,W)=1\\ P^+(d_-)<T^\xi}} \frac{u(d_-)}{d_-},\end{aligned}$$

so that

$$\sum_{\substack{\gcd(d_-,W)=1\\ P^+(d_-)<T^\xi}} \frac{u(d_-)}{d_-} \leqslant 2 \sum_{\substack{\gcd(d_-,W)=1\\ d_-\leqslant T}} \frac{u(d_-)}{d_-},$$

as claimed. □

Proof of Theorem 1.1. Let us define

$$\eta = \frac{2}{3} \quad \text{and} \quad \omega = \frac{(3+\delta)}{c_1}\sum_{i=1}^N d_i\varepsilon_i,$$

where $\delta>0$ is to be determined. Now let ε be any positive constant. Taking δ sufficiently small compared to ε we see that the exponent of $K_{\mathscr{R}}$ in the second term of (3.5) is

$$1+4\omega(1+\eta+2\delta+\hat{\varepsilon}) \leqslant 1+\varepsilon+\frac{4}{c_1}\Big(\sum_{i=1}^{N} d_i\varepsilon_i\Big)(5+3\hat{\varepsilon}),$$

where $\hat{\varepsilon}=\max\{\varepsilon_1,\dots,\varepsilon_N\}$. Thus $E^{(I)}(\mathscr{R})$ makes a satisfactory contribution for Theorem 1.1. Taking δ sufficiently small allows us to check that

$$\frac{\eta\omega}{2} > \frac{1}{c_1}\sum_{i=1}^{N} d_i\varepsilon_i \text{ and } (1-\eta)\omega > 3\delta\omega+\frac{1}{c_1}\sum_{i=1}^{N} d_i\varepsilon_i,$$

for our choice of η and ω. This therefore shows that the first term in the right hand side of (3.7) and (3.8) is

$$\ll \frac{V}{(\log V)^{N+1}q_G},$$

which, owing to $h_W^*(q_G)\geqslant 1$ and $E_{f_i}(V;W)\geqslant 1$, is satisfactory for Theorem 1.1. A straightforward calculation now shows that for sufficiently small δ the contribution of the second terms on the right of (3.7), (3.8) and (3.12) is also satisfactory for Theorem 1.1. □

References

[1] R. de la Bretèche and T. D. Browning, Sums of arithmetic functions over values of binary forms. *Acta Arith.* **125** (2006), 291–304.

[2] R. de la Bretèche and G. Tenenbaum, Moyennes de fonctions arithmétiques de formes binaires. *Mathematika* **58** (2012), 290–304.

[3] R. de la Bretèche, T. D. Browning and E. Peyre, On Manin's conjecture for a family of Châtelet surfaces. *Annals of Math.* **175** (2012), 297–343.

[4] T.D. Browning and E. Sofos, Counting rational points on quartic del Pezzo surfaces with a rational conic. *Submitted*, 2016. (`arXiv:1609.09057`)

[5] K. Henriot, Nair-Tenenbaum bounds uniform with respect to the discriminant. *Math. Proc. Cambridge Philos. Soc.* **152** (2012), 405–424.

[6] H. Iwaniec and E. Kowalski, *Analytic number theory.* American Math. Soc. Providence, RI, 2004.

[7] M. Nair, Multiplicative functions of polynomial values in short intervals. *Acta Arith.* **62** (1992), 257–269.

[8] M. Nair and G. Tenenbaum. Short sums of certain arithmetic functions. *Acta Math.* **180** (1998), 119–144.

[9] P. Shiu, A Brun–Titchmarsh theorem for multiplicative functions. *J. reine angew. Math.* **313** (1980), 161–170.

[10] E. Sofos, Serre's problem on the density of isotropic fibres in conic bundles. *Proc. London Math. Soc.* **113** (2016), 1–28.

School of Mathematics, University of Bristol, Bristol, BS8 1TW, UK
E-mail address: t.d.browning@bristol.ac.uk

Max Planck Institute for Mathematics, Bonn, Vivatgasse 7, 53111, Germany
E-mail address: sofos@mpim-bonn.mpg.de