Equivariance in the Theory of Higher Genus $\wp$-Functions

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Abstract. We discuss a synthetic use of symmetry in two constructions of relations between functions on curves and their Jacobians.

Introduction

As motivation for the material of this talk we note the role of multiperiodic functions in the understanding and solution of integrable systems. Most simply, the Weierstraß $\wp$-function arises as a solution to a symmetry reduction of the KdV equation,

$$u_{,t} + u_{xxx} + 6uu_{,x} = 0 \stackrel{u=u(x+ct)}{\longrightarrow} u'^2 = a_0u^3 + a_1u^2 + a_2u + a_3,$$

and other integrable equations (KP, Boussinesq, ...) have solutions coming from $\wp$-functions associated with higher genus curves. These are the "finite gap" solutions and their restriction to real variables are quasiperiodic.

In the example above a suitable translation and scaling of u will put the reduced equation into Weierstraß canonical form,

$$u'^2 = 4u^3 + g_2u + g_3,$$

but, in general, we need to deal with curves and equations in "general position". Thus the cubic curve associated with the Weierstraß form has a branch point at infinity but we would like to allow all branch points to be at finite places in the complex plane. These cases are related by Möbius transformations. We will see how such transformations allow us to use simple representation theory to abbreviate calculations and exhibit the underlying geometry of the differential equations satisfied by $\wp$-functions associated with higher degree curves. We will deal as far as possible with equivariant objects.

The role of "symmetry" in this work is thus *synthetic* rather than analytic. We aim to illustrate the ideas using simple cases subject to certain caveats.

Transformations

Algebras of symmetries will appear in two guises.

Firstly, consider the transformations of the genus g hyperelliptic model:

$$y^2 = \sum_{i=0}^{2g+2} \binom{2g+2}{i} a_i x^{2g+2-i} \to Y^2 = \sum_{i=0}^{2g+2} \binom{2g+2}{i} A_i X^{2g+2-i}$$

$$y = \frac{Y}{(\gamma X + \delta)^{g+1}}, \quad x = \frac{\alpha X + \beta}{\gamma X + \delta},$$

α, β, γ and δ being complex constants. The infinitesimal action on the variables x and y *and* the coefficients a_i is given by the $\mathfrak{sl}_2(\mathbb{C})$ generators,

$$\mathbf{e} = \partial_x - \sum_{i=1}^{2g+2} i a_{i-1} \partial_{a_i}$$

$$\mathbf{f} = (g+1)xy\partial_y + x^2\partial_x - \sum_{i=0}^{2g+1}(2g+2-i)ia_{i+1}\partial_{a_i},$$

under which the $\{a_0, a_1, \ldots, a_{2g+2}\}$ constitute a $2g+3$ dimensional irreducible representation.

Secondly, consider an abstract compact Riemann surface X of genus g. Divisors measure poles and zeros of meromorphic functions on X,

$$D = \sum_{P\in X}^{\text{finite}} n_P P,\ n_P \in \mathbb{Z},$$

n_P being the order of the pole (-ve) or zero (+ve) at the place P. For an *effective* divisor D (all $n_P \geq 0$) let $L(D)$ be the space of functions with poles at most $-D$. If $x_{n_1,n_2,\ldots n_r}$ is a function in $L(\sum_i n_i P_i)$ there is an $\mathfrak{sl}_N$ action

$$\mathbf{e}_{ij}(x_{n_1,\ldots,n_i,\ldots n_j,\ldots n_r}) = n_i x_{n_1,\ldots,n_i-1,\ldots n_j+1,\ldots n_r}$$

where $N = n_1 + n_2 + \ldots + n_r$ is the *degree*.

To clarify the relation between curves, $\wp$-functions and ϑ-functions note that X has a $\mathbb{C}$ basis, $\{\xi_1, \ldots, \xi_g\}$, of holomorphic 1-forms. For a fixed $P_0 \in X$ and general $P \in X$, define

$$\phi(P) = \pi \circ (\int_{P_0}^{P} \xi_1, \ldots, \int_{P_0}^{P} \xi_g) \in \mathbb{C}^g/L(X) = Jac(X)$$

where $L(X)$ is the lattice of periods associated with the homology of X. The extension by linearity to the general divisor, $\phi(D) = \sum_P n_P \phi(P)$ is the *Abel* map. The kernel of the Abel map on degree zero divisors characterises *principal* divisors - those which are divisors of meromorphic functions: *D is the divisor of a meromorphic function on X iff it has degree 0 and $\phi(D) = 0$.* [1]

σ- and ϑ-functions are entire functions on $\mathbb{C}^g$, modular under lattice translations. $\wp$-functions, on the other hand, are meromorphic functions on $Jac(X)$. In the case of the genus one curve it so happens that the curve and its Jacobian are isomorphic. For higher genus this cannot be the case.

Generalised Weierstraß $\wp$-functions

Consider first the genus one curve in general position [2],

$$y^2 = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$$

and let (x, y) and (x', y') be a pair of points thereon. Define a 3×3 matrix,

$$H = \begin{bmatrix} a_0 & 2a_1 & a_2 - 2\wp \\ 2a_1 & 4a_2 + 4\wp & 2a_3 \\ a_2 - 2\wp & 2a_3 & a_4 \end{bmatrix},$$

containing the $\wp$-function. We write down the *polar equivariant form*:

$$y'y - \mathbf{x}'H\mathbf{x}^t = 0, \quad \mathbf{x} = (x^2, x, 1) \text{ etc..}$$

This defines the $\wp$-function as a meromorphic function on the curve, regular at (x', y') and with a double pole at $(x', -y')$. The holomorphic 1-form of the cohomology is $du = \frac{dx}{y}$. Differentiating with respect to u: $\partial_u = y\partial_x$ one obtains the Weierstraß differential equation for the $\wp$-function:

$$\left(\frac{d\wp}{du}\right)^2 = -\frac{1}{4}|H| = 4\wp^3 + g_2\wp + g_3$$

g_2 and g_3 are $\mathfrak{sl}_2$ polynomial invariants in the a_i of degree two and three respectively.

Note that $\wp$ and $\frac{d\wp}{du}$ parametrise the Weierstraß form of the curve although we started with a more general form.

There is a species of ϑ-function, σ, such that $\wp = -\partial_u^2 \log \sigma$. σ is entire and modular on $\mathbb{C}$ whereas $\wp$ is doubly periodic, defined on the Jacobian $\mathbb{C}/Lattice$ isomorphic, in this case, to the elliptic curve itself.

In higher genus there is a $\mathbb{C}$ basis of holomorphic 1-forms forming a g dimensional irreducible representation for $\mathfrak{sl}_2(\mathbb{C})$:

$$du_i = \frac{x^{i-1}dx}{y}, \quad \text{for } i = 1, \dots, g.$$

There are $\frac{1}{2}g(g+1)$ $\wp$-functions depending on the Jacobian variables,

$$\wp_{ij} = 2\partial_{u_i}\partial_{u_j} \log \sigma(u_1, ..., u_g)$$

and $\frac{1}{6}g(g+1)(g+2)$ first derivatives

$$\wp_{ijk} = 2\partial_{u_i}\partial_{u_j}\partial_{u_k} \log \sigma(u_1, ..., u_g).$$

(From this point of view the Weierstraß $\wp$-function should be written $\wp_{11}$.)

These sets decompose into irreducible submodules as symmetric tensor products of degrees 2 and 3 of the g dimensional representation $\{\partial_{u_1}, ..., \partial_{u_g}\}$.

The genus two case is already complicated to write down fully so we restrict explicit attention to this. Our curve is the sextic,

$$y^2 = a_0x^6 + 6a_1x^5 + 15a_2x^4 + 20a_3x^3 + 15a_4x^2 + 6a_5x + a_6$$

and the matrix H is 4×4:

$$H = \begin{bmatrix} a_0 & 3a_1 & 3a_2 - 2\wp_{11} & a_3 - 2\wp_{12} \\ 3a_1 & 9a_2 + 4\wp_{11} & 9a_3 + 2\wp_{12} & 3a_4 - 2\wp_{22} \\ 3a_2 - 2\wp_{11} & 9a_3 + 2\wp_{12} & 9a_4 + 4\wp_{22} & 3a_4 \\ a_3 - 2\wp_{12} & 3a_4 - 2\wp_{22} & 3a_5 & a_6 \end{bmatrix}.$$

We choose points (x, y), (x_1, y_1) and (x_2, y_2) on the curve. There are two equivariant polar forms:

$$y_i y - \mathbf{x}_i H \mathbf{x}^t = 0, \quad \mathbf{x} = (x_i^3, x_i^2, x_i, 1) \text{ etc. } i = 1, 2.$$

These are extensions of the definitions given in [3] which ultimately go back to Klein and Baker [4]. Here we have further incorporated the equivariant property and adapted to the curve in general position in analogy with the genus one case. For a vector $\mathbf{l} = (l_0, l_1, l_2, l_3)$ of arbitrary parameters define $\mathbb{P}(\mathbf{l}) = l_0\wp_{222} - l_1\wp_{122} + l_2\wp_{112} - l_3\wp_{111}$. Then

$$\mathbb{P}(\mathbf{l})\mathbb{P}(\mathbf{l}') = -\frac{1}{4} \begin{vmatrix} H & \mathbf{l}^t \\ \mathbf{l}' & 0 \end{vmatrix}$$

Since $Symm\, \mathbf{4} \otimes \mathbf{4} = Symm\, \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{7} \oplus \mathbf{3}$ we obtain this formula by finding just the highest weight relations for each representation:

$$\mathbf{7}: \quad \wp_{111}^2 = 4\wp_{11}^3 + \text{ linear in } a_i \text{ and } \wp_{ij},$$

$$\mathbf{3}: \quad \wp_{112}^2 - \wp_{111}\wp_{122} = 2(\wp_{12}^2 - \wp_{11}\wp_{22})\wp_{11} + \text{ linear in } a_i \text{ and } \wp_{ij}$$

The differential equations for the genus two $\wp$ functions are attractive written in this form as they have a striking simplicity and are analogous to the single Weierstraß differential equation.

A similar structure appears in higher genus hyperelliptic cases:

$$y_i y - \mathbf{x}_i H_{g+2} \mathbf{x}^t = 0, \quad \mathbf{x}_i = (x_i^{g+1}, \dots, 1) \text{ etc. } i = 1, \dots g.$$

$$\mathbb{P}(\mathbf{l}_1, ..., \mathbf{l}_{g-1})\mathbb{P}(\mathbf{l}'_1, ..., \mathbf{l}'_{g-1}) = -\frac{1}{4} \begin{vmatrix} H_{g+2} & \mathbf{l}_1^t & \dots & \mathbf{l}_{g-1}^t \\ \mathbf{l}'_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \mathbf{l}'_{g-1} & 0 & \dots & 0 \end{vmatrix} \qquad (*)$$

where $\mathbf{l}_i$ and $\mathbf{l}'_i$ are $g+2$ dimensional vector parameters.

The object $\mathbb{P}$ is formed from the pairing:

$$\left(Symm \bigotimes^{3} V_g \right) \times \left(\bigwedge^{g-1} V_{g+2} \right) \to \mathbb{C}.$$

The symmetric arguments correspond to the $\wp_{ijk}$ and the antisymmetric to Plücker coordinates on the Grassannian of $g-1$ planes in $\mathbb{C}^{g+2}$ which arise in a Laplace type expansion of the determinant.

Equation (*) is proven for hyperelliptic curves when $g = 1, 2, 3$ and remains conjectural for higher genus hyperelliptic curves [2].

The situation for non-hyperelliptic curves is more complex [5]. From the equivariant point of view a promising approach is to start with a curve in canonical form, i.e. $y^n = x^s + \dots$ and "fatten" it out to a family of curves by adding parameters so that the branch points are in generic places in $\mathbb{C}$ and the whole family is permuted under $\mathfrak{s}l_2$ transformations. One has then to write down the equivariant polarization in order to define the $\wp_{ij}$ functions but the analysis is harder.

Relations on abstract curves

A second manifestation of symmetry transformations is the $\mathfrak{sl}_N$ action on divisor spaces of degree N [6].

Let X be a compact (nonsingular) Riemann surface g and D an effective divisor. If $l(D)$ is the $\mathbb{C}$-dimension of the space of functions with poles at most $-D$ and $r(D)$ that of the space of holomorphic differentials with zeros at least D, then the Riemann-Roch theorem states that

$$l(D) - r(D) = N - g + 1, \quad N = \sum_P n_P.$$

As N increases by one so either $l(D)$ increases by one or $r(D)$ decreases by one.

In the case that $g = 1$ we get the following table of basis elements:

D	$0.P$	$1.P$	$2.P$	$3.P$	$4.P$	$5.P$	$6.P$	$7.P$	$8.P$	$9.P$	$\dots$
dim	1	1	2	3	4	5	6	7	8	9	
	1	0	x_2	y_3	x_2^2	$x_2 y_3$	x_2^3	$x_2^2 y_3$	x_2^4	y_3^3	$\dots$
							y_2^2		$x_2 y_3^2$	$x_2^3 y_3$	$\dots$

There is no function with only a single pole. We get new functions x_2 and y_3 at divisors $2.P$ and $3.P$ and thereafter we may construct (at least) one new function at $n.P$, $n > 3$, as monomials in x_2 and y_3. At $6.P$ we get two such functions and hence there must be a relation (over $\mathbb{C}$) of the form $\Delta = y_3^2 - x_2^3 \sim 0$ (where $\sim$ denotes equivalence up to a linear combination of terms in $0.P, 1.P, \dots, 5.P$.) At $8.P$ and so on we have $x_2 y_3^2 - x_2^4 = x_2 \Delta \sim 0$ etc. In this case one relation, $\Delta \sim 0$, suffices to satisfy the Riemann-Roch constraint at all degrees. We have one new function at each degree: x_2^n at $2n.P$ and $y_3 x_2^{n-1}$ at $(2n+1).P$.

The description is more complex as soon as we consider two point divisors: $D = n.P + m.Q$, $n, m \in \mathbb{N}$. The corresponding divisor table is:

	$0.P$	$1.P$	$2.P$	$3.P$	$4.P$	$5.P$	$6.P$	$7.P$
$0.Q$	1	0	x_{20}	y_{30}	x_{20}^2	$x_{20}y_{30}$	x_{20}^3 y_{30}^2	$x_{20}^2y_{30}$
$1.Q$	0	x_{11}	y_{21}	$x_{11}x_{20}$	$x_{11}y_{30}$ $x_{20}y_{21}$	$x_{11}x_{20}^2$ $y_{30}y_{21}$		
$2.Q$	x_{02}	y_{12}	x_{11}^2 $x_{20}x_{02}$	$x_{02}y_{30}$ $x_{11}y_{21}$ $x_{20}y_{12}$	$x_{02}x_{20}^2$ $x_{20}x_{11}^2$ y_{21}^2 $y_{12}y_{30}$			
$3.Q$	y_{03}	$x_{11}x_{02}$	$x_{02}y_{21}$ $x_{11}y_{12}$ $x_{20}y_{03}$	$x_{20}x_{11}x_{02}$ x_{11}^3 $y_{12}y_{21}$ $y_{30}y_{03}$				
$4.Q$	x_{02}^2	$x_{11}y_{03}$ $x_{02}y_{12}$	$x_{02}^2x_{20}$ $x_{11}^2x_{02}$ y_{12}^2 $y_{21}y_{03}$					
$5.Q$	$x_{02}y_{03}$	$x_{11}x_{02}^2$ $x_{12}yx_{03}$						
$6.Q$	x_{02}^3 y_{03}^2							
$7.Q$	$x_{02}^2y_{03}$							

Recall that e.g. x_{ij} denotes a function with a pole $-(i.P + j.Q)$. Two kinds of relation arise at degrees 4, 5 and 6, those quadratic in x and y and those quadratic in y, cubic in x. We want to show that these suffice to exhaust all relations.

The x's and y's can be normalised so that the quadratic relations are all the 2×2 minors of

$$\begin{bmatrix} x_{20} & x_{11} & y_{30} & y_{21} & y_{12} \\ x_{11} & x_{02} & y_{21} & y_{12} & y_{03} \end{bmatrix}$$

Under the Lie algebra operations $\mathbf{e}x_{i,j} = ix_{i-1,j+1}$ and $\mathbf{f}x_{i,j} = jx_{i+1,j-1}$ (similarly for the y_{ij}) these ten relations decompose into invariant subspaces of dimensions 1, 2, 3 and 4 with highest weight elements

$$\mathbf{1} : x_{20}x_{02} - x_{11}^2, \quad \mathbf{2} : x_{20}y_{12} - 2x_{11}y_{21} + x_{02}y_{30}, \quad \mathbf{3} : y_{30}y_{12} - y_{21}^2 \quad \text{and} \quad \mathbf{4} : x_{20}y_{21} - x_{11}y_{30}.$$

Let I be the ideal inside $R = \mathbb{C}[x_{20}, x_{11}, x_{02}, y_{30}, y_{21}, y_{12}, y_{03}]$ generated by these ten relations. It can be factored out, using an exact sequence of syzygy modules, the issue being that the relations have relations, and those relations relations, and so on so that accounting for all relations is not quite straightforward [7].

Note that the familar models for the genus one curve lurk inside the whole set of relations. For instance at $6.P$ we have the cubic identity

$$y_{30}^2 - x_{20}^3 = \lambda_{50}x_{20}y_{30} + \lambda_{40}x_{20} + \lambda_{30}y_{30} + \lambda_{20}x_{20} + \lambda_0 \sim 0,$$

for some $\lambda_{ij} \in \mathbb{C}$ which is the Weierstraß form.

At $4.P + 2.Q$ we can use the quadratic relations to find a relation of the form $y_{21}^2 = A_3(x_{11}) + y_{21}A_2(x_{11})$ where A_p is of degree p. Completing the square in y_{21} gives the quartic model

$$Y^2 = A_4(X)$$

where $Y = y_{21} - \frac{1}{2}A_2(x_{11})$ has pole divisor $-(2.P + 2.Q)$ and $X = x_{11}$ has pole divisor $-(P + Q)$.

The generators that remain once the quadratic relations have been factored out can be counted. Let $R = \bigoplus_{n=0}^{\infty} R^{[n]}$ be a grading by degree with $\deg x_{ij} = 2$ and $\deg y_{ij} = 3$. We construct a free resolution,

$$\begin{array}{ll} 0 & \rightarrow \bigoplus^{4} R^{[n-13]} \\ & \stackrel{\phi_4}{\rightarrow} \bigoplus^{3}\left(\bigoplus^{2} R^{[n-11]} \oplus \bigoplus^{3} R^{[n-10]}\right) \\ & \stackrel{\phi_3}{\rightarrow} \bigoplus^{2}\left(R^{[n-9]} \oplus \bigoplus^{6} R^{[n-8]} \oplus \bigoplus^{3} R^{[n-7]}\right) \\ & \stackrel{\phi_2}{\rightarrow} R^{[n-4]} \oplus \bigoplus^{6} R^{[n-5]} \oplus \bigoplus^{3} R^{[n-6]} \\ & \stackrel{\phi_1}{\rightarrow} R^{[n]} \stackrel{\pi}{\rightarrow} (R/I)^{[n]} \\ & \rightarrow 0. \end{array}$$

We do this by letting $\{e_0, e_1\}$ and $\{f_0, f_1, f_2\}$ be bases of two and three dimensional vector spaces respectively and defining

$$\begin{aligned} \omega_1 &= x_{20}e_0 + x_{11}e_1 + y_{30}f_0 + y_{21}f_1 + y_{12}f_2 \\ \omega_2 &= x_{11}e_0 + x_{02}e_1 + y_{21}f_0 + y_{12}f_1 + y_{03}f_2 \\ \Omega &= \omega_1 \wedge \omega_2 = \Omega^{[4]}_{e,e} + \Omega^{[5]}_{e,f} + \Omega^{[6]}_{f,f} \end{aligned}$$

Here $\Omega^{[5]}_{e,f}$, for instance, denotes a linear sum of two forms $e_i \wedge f_j$ with coefficients of degree 5.

π is the canonical projection onto the quotient by I and we define ϕ_1 so that $\operatorname{im}\phi_1 = \ker \pi$.

$$\ldots \stackrel{\phi_2}{\rightarrow} R^{[n-4]} \bigwedge^{3}_{f,f,f} \oplus R^{[n-5]} \bigwedge^{3}_{e,f,f} \oplus R^{[n-6]} \bigwedge^{3}_{e,e,f} \stackrel{\Omega\wedge}{\rightarrow} R^{[n]} \bigwedge^{5}_{e,e,f,f,f} \stackrel{\pi}{\rightarrow} \ldots$$

or

$$\ldots \stackrel{\phi_2}{\rightarrow} R^{[n-4]} \oplus \bigoplus^{6} R^{[n-5]} \oplus \bigoplus^{3} R^{[n-6]} \stackrel{\phi_1}{\rightarrow} R^{[n]} \stackrel{\pi}{\rightarrow} \ldots$$

Continuing in this manner, define ϕ_2 so that

$$\operatorname{im}\phi_2 = \ker\phi_1 = \{\alpha_1 \wedge \omega_1 + \alpha_2 \wedge \omega_2 | \alpha_1, \alpha_2 \in \bigwedge^{2}\}.$$

Then

$$\ldots \stackrel{\phi_3}{\rightarrow} \bigoplus^{2}\left(R^{[n-9]} \bigwedge^{2}_{e,e} \oplus R^{[n-8]} \bigwedge^{2}_{e,f} \oplus R^{[n-7]} \bigwedge^{2}_{f,f}\right) \stackrel{\omega_e^{[2]}+\omega_f^{[3]}}{\rightarrow} R^{[n-4]} \bigwedge^{3}_{f,f,f} \oplus R^{[n-5]} \bigwedge^{3}_{e,f,f} \oplus R^{[n-6]} \bigwedge^{3}_{e,e,f} \stackrel{\phi_1}{\rightarrow} \ldots$$

and the domain of ϕ_2 is

$$\bigoplus^{2}\left(R^{[n-9]} \oplus \bigoplus^{6} R^{[n-8]} \oplus \bigoplus^{3} R^{[n-7]}\right).$$

We continue in this way and because the degrees (both of the coefficients and the wedge products) decrease to the left, the sequence is finite. Exactness at each term in the sequence implies that the alternating sum of the dimensions (over $\mathbb{C}$) for the whole sequence vanishes.

The Hilbert series, $H(t)$, for R in t is defined to have coefficient $d_n = \dim_{\mathbb{C}} R^{[n]}$ at degree n ,

$$H(t) = \sum_{0}^{\infty} d_n t^n = (1-t^2)^{-3}(1-t^3)^{-4}$$

The exactness of our sequence implies a relation for the dimension, $\tilde{d}_n$ of $(R/I)^{[n]}$

$$\tilde{d}_n = d_n - (d_{n-4} + 6d_{n-5} + 3d_{n-6}) + 2(d_{n-9} + 6d_{n-8} + 3d_{n-7}) - 3(2d_{n-11} + 3d_{n-10}) + 4d_{n-13}$$

which gives corresponding Hilbert series,

$$\begin{aligned} \tilde{H}(t) &= \frac{1 + t + 2t^2 + 4t^3 + 4t^4}{(1-t)^3(1+t+t^2)(1+t)^2} \\ &= 1 + 3t^2 + 4t^3 + 5t^4 + 6t^5 + \mathbf{14}t^6 + 8t^7 + \mathbf{18}t^8 + \ldots \end{aligned}$$

The coefficients are the number of entries remaining on the anti-diagonals of the $n.P + m.Q$ diagram once all quadratic relations have been factored out.

The $\mathfrak{sl}_2$ action on the divisor diagram adds and subtracts poles, leaving the degree unaltered. In addition the ideal I is invariant so that the seqence we have described is equivariant in the sense that the following diagram commutes for appropriate representations $\mathfrak{g}_p$ of $\mathfrak{sl}_2$.

$$\begin{array}{ccccc} \ldots \to & A_{n+1} & \stackrel{\phi_{n+1}}{\to} & A_n & \to \ldots \\ & \downarrow \mathfrak{g}_{n+1} & & \downarrow \mathfrak{g}_n & \\ \ldots \to & A_{n+1} & \stackrel{\phi_{n+1}}{\to} & A_n & \to \ldots \end{array}$$

This means that irreducible representations in A_{n+1} map to irreducibles in A_n.

Each antidiagonal in the divisor diagram is a sum of irreducible representations, including the relations, and once the quadratic ones are factored out we are left only with the irreps in $\bigotimes^n \mathbf{x}$ of dimension $2n+1$ and in $\bigotimes^n \mathbf{y}$ of dimension $3n+1$. equivariance allows us to argue that highest weight elements of the form $y_{30}^2 - x_{20}^3$ determine the remaining identities. This leaves us with a Hilbert series

$$1 + 3t^2 + 4t^3 + 5t^4 + 6t^5 + 7t^6 + 8t^7 + 9t^8 + \ldots$$

consistent with the Riemann-Roch theorem in genus one: there is one "new" function at each place in the divisor diagram excepting degree 1.

The general (nonsingular) curve (n and s are assumed coprime) $y^n = x^s + \ldots$ of genus $g = \frac{1}{2}(n-1)(s-1)$ has a meromorphic function field generated by degree n functions: $x_{i,j}, (i + j = n)$, and degree s functions $y_{i,j}, (i + j = s)$. Quadratic relations are described by the 2×2 minors of

$$\left[\begin{array}{cccccccc} x_{n,0} & x_{n-1,1} & \ldots & x_{1,n-1} & y_{s,0} & y_{s-1,1} & \ldots & y_{1,s-1} \\ x_{n-1,1} & x_{n-2,2} & \ldots & x_{0,n} & y_{s-1,1} & y_{s-2,2} & \ldots & y_{0,s} \end{array}\right]$$

An exact equivariant sequence can again be constructed and

$$\begin{aligned} \tilde{H}(t) &= \frac{d}{dt}\left(\frac{t}{(1-t^n)(1-t^s)}\right) \\ &= \sum_{m=0}^{\infty} (\#\text{ partitions of } m \text{ into } n\text{'s and } s\text{'s})(m+1)t^m \end{aligned}$$

The equivariance argument again gives a result consistent with Riemann-Roch. In this instance we are not restricted to hyperelliptic curves but neither are we dealing with the $\wp_{ij}$ functions on the Jacobian, only functions on the curve itself.

Hence we have described *all* relations on the abstract curve.

A detailed treatment of the results in this section is given in [6].

As an aside consider the genus two Jacobian and D the ϑ-divisor, the locus of zeros of the σ-function, in $Jac(X)$. Write (ij) for $2\partial_1^i \partial_2^j \log \sigma(u_1, u_2)$. The divisor diagram reads,

$$\begin{array}{ccccc} 1 & 0 & (20) & (30) & (40), (20)^2 \\ 0 & (11) & (21) & (31), (20)(11) & \\ (02) & (12) & (22), (11)^2, (02)(20) & & \\ (03) & (13), (02)(11) & & & \\ (04), (02)^2 & & & & \end{array}$$

Terms on the antidiagonals having degree $n = i + j$ should comprise a space of $\mathbb{C}$-dimension $2n - 1$ for $n \geq 2$.

At degree 3 we have one term too few. At degree 4 we have relations $(40) - (20)^2 \sim 0$ etc. and $(20)(02) - (11)^2 \sim 0$. In fact $\Delta = (20)(02) - (11)^2$ provides the missing basis function at degree 3 and a basis for the degree n divisor space is $\{(ij) | i + j = n\} \cup \{\partial_1^i \partial_2^j \Delta | i + j = n - 3\}$.

For $m \geq 2$ and ζ a primitive m^{th} root of unity, define

$$\mathcal{H}_i^{[m]} : \bigotimes^m \sigma \mapsto \sum_{j=1}^{m} \zeta^{j-1} \partial_i^{[j]} \bigotimes^m \sigma$$

where $\partial_i^{[j]}$ acts on the j^{th} entry in the tensor product. e.g.

$$\mathcal{H}_1^{[2]}\mathcal{H}_1^{[2]}(\sigma\otimes\sigma) = \sigma,_{11}\otimes\sigma - 2\sigma,_1\otimes\sigma,_1 + \sigma\otimes\sigma,_{11}\,.$$

Symmetrising yields the $\wp$-function: $2\sigma^2(\sigma\sigma_{11} - \sigma_1^2) = 2\partial_1^2 \log\sigma = \wp_{11}$.

$\mathcal{H}$ operators are equivariant under Lattice translations

$$\begin{array}{ccc} \bigotimes^m \sigma & \overset{Lattice}{\rightarrow} & \bigotimes^m \sigma \\ \mathcal{H}\downarrow & & \mathcal{H} \\ \bigotimes^m \sigma & \overset{Lattice}{\rightarrow} & \bigotimes^m \sigma \end{array}$$

so functions constructed using them are defined on $Jac(X)$. They generate bases for the ϑ-divisor spaces [8].

Further Directions

The σ-function can be defined as a solution of a heat equation as is the case for a ϑ-function. This is achieved via the symplectic structure on the cohomology basis, $\{\xi_1, ...\xi_g, \eta_1, ..., \eta_g\}$ of first and second kind differentials [9]. Differentiation with respect to the a_i generalises Picard-Fuchs theory and leads to a *Gauß-Manin* connection, a Hamiltonian structure and the heat equation. This approach can be made equivariant by constructing an equivariant cohomology basis [10].

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