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A Summary of Turbulence Modelling Approaches in CFD

Aerospace Engineering Report 0206

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Contents

1 Introduction

The Navier-Stokes equations of viscous flow have been known for more than 100 years. The system consists of Partial Differential Equations (PDEs) describing the laws of conservation for:

- Mass (continuity equation).
- Momentum (Newton's 2nd Law).
- Energy (1st Law of Thermodynamics).

The continuity equation simply states that the mass must be conserved. In Cartesian coordinates, x_i, this is written as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0 \tag{1}$$

where	ρ	is the density of the fluid,
	t	is the time,
	u_i	is the velocity vector.

In the above, tensor notation is used, which implies summation for repeated indices.

The second conservation principle states that momentum must be conserved. It is written in Cartesian coordinates as:

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \tag{2}$$

where	f_i	represents body forces,
	p	is the pressure,
	τ_{ij}	is the viscous stress tensor, which is defined as:

$$\tau_{ij} = \mu \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right] \tag{3}$$

where μ is the molecular viscosity,
δ_{ij} is the Kronecker delta, which is defined as:

$$\delta_{ij} = \begin{cases} 1 & if \text{i=j} \\ 0 & otherwise \end{cases} \tag{4}$$

The third principle can be written in Cartesian coordinates as

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}\left[u_\gamma\left(E+p\right)\right] - \frac{\partial}{\partial x_j}\left(u_i \tau_{ij} - q_j\right) = 0. \tag{5}$$

where E is the total energy of the fluid, defined as:

$$E = \rho\left[e + \frac{1}{2}u_i u_i\right] \tag{6}$$

where e is the specific internal energy,
u is the kinetic energy.

The heat flux vector q_i is calculated using Fourier's Law as

$$q_i = -k\frac{\partial T}{\partial x_i} \tag{7}$$

where k is the heat transfer coefficient,
T temperature of the fluid.

2 Reynolds Averaging

In a turbulent flow, the fields of pressure, velocity, temperature and density vary randomly in time. Reynold's approach is to separate the flow quantities into stationary and random parts. The quantities are then presented as a sum of the mean flow value and the fluctuating part:

$$\phi = \overline{\phi} + \phi' \tag{8}$$

This formulation is then inserted into the conservation equations and a process known as Reynolds averaging is performed.

There are three forms of averaging performed in the turbulence modelling research:

- Time averaging.
- Spatial averaging.
- Ensemble averaging.

2.1 Time Averaging

Time averaging is the most common. It can be used only for stationary turbulent flows, i. e. flows not varying with time on the average. For such flows, the mean flow value is defined as:

$$\overline{u_i} = \lim_{T\to\infty} \frac{1}{T} \int_i^{i+T} u_i(t)\, dt \tag{9}$$

In practice, $T \to \infty$ means that the integration time T needs to be long enough relative to the maximum period of the assumed fluctuations.

Taking the time average of the mass, momentum and energy equations, the Reynolds-Averaged Navier-Stokes (RANS) equations can be obtained. The continuity equation remains the same since it is linear with respect to velocity. However, extra terms appear in the momentum and energy equations due to the non-linearity of the convection term. The extra term is called **Reynolds Stress**, $\tau_{ij}^R = \overline{\rho u_i' u_j'}$ and the result is

$$\frac{\partial\left(\rho u_i\right)}{\partial t} + \frac{\partial\left(\rho u_i u_j\right)}{\partial x_i} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\tau_{ij} - \tau_{ij}^R\right) \tag{10}$$

where the overbar has been dropped from the mean values.

The main problem in turbulence modelling is to actually calculate the Reynolds stresses out of the known mean quantities — same happens in the energy equation.

2.2 General Description of Turbulence

- Turbulent flows are irregular in the sense that various quantities show a random variation in time and space.
- Turbulent flows have a wide range of scales.
- Turbulence develops as an instability of the laminar flow solution.

Starting with the laminar flow, fluid layers slide smoothly past each other and the molecular viscosity dampens any high-frequency small-scale instability. At high Reynolds number, the flow reaches a chaotic and non-repeating state. The fluid particles now move in a more random way! The character of the flow also changes and becomes strangely diffusive and dissipative. This flow has increased mixing friction, heat transfer rate and spreading rate. Boundary layers become thicker and harder to separate.

The non-linearity of the Navier-Stokes equations leads to various interactions between turbulent fluctuations of different wavelengths and directions. The wavelengths of motion extend from a maximum comparable to the width of the flow to a minimum fixed by viscous dissipation of energy. A key process that spreads the motion over wide range of wavelengths is called vortex stretching. The turbulent structures in the flow gain energy if the vortex elements are primarily oriented in a direction in which the mean velocity gradients can stretch them. This mechanism is called **production of turbulence**. The kinetic energy of the turbulent structures is then convected, diffused and dissipated.

Most of the energy is carried by the large scale structures, the orientation of which is sensitive to the mean flow. The large eddies cascade energy to the smaller ones via stretching. The small eddies have less pronounced preference in their orientation and statistically appear to be isotropic. In the shortest wavelengths, energy is dissipated by viscosity.

The above description corresponds to what is known as **isotropic turbulence**. For this flow, the ratio of the largest to smaller scale increases with Reynolds number.

If the unsteady Navier-Stokes equations are used to calculate the flow, a vast range of length and time scales would have to be computed. This would require a very fine grid and a very high resolution in time. This approach known as **Direct Numerical Simulation of turbulence (DNS)** is by today's computing speeds applicable only to flows at very low Reynolds number. A turbulence model needs to account for some part of the fluctuating motion in order to keep the computing cost down. The optimum model should be:

- Simple.
- General.
- Derived out of the flow physics.
- Computationally stable.
- Co-ordinate invariant.

3 Boussinesq-Based Models

The Boussinesq approximation or better the Boussinesq hypothesis states that:

$$-\rho\overline{u_i'u_j'} = \mu_T\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] - \frac{2}{3}\rho\delta_{ij}k \tag{11}$$

where k represents the specific kinetic energy of the fluctuations and is given by:

$$k \equiv \frac{u_i'u_i'}{2} \tag{12}$$

The key idea behind the Boussinesq hypothesis is that the Reynolds stresses can be calculated as a product of the kinematic eddy-viscosity, μ_T, and the strain-rate tensor of the mean flow. So

$$-\rho\overline{u_i'u_j'} = 2\mu_T S_{ij} - \frac{2}{3}\delta_{ij}k \tag{13}$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{14}$$

μ_T is a scalar and consequently the Reynolds stress components are linearly proportional to the mean strain-rate tensor. What is implied here is that compressibility plans a secondary rate in the development of the turbulent flow field. Morkovin's hypothesis suggests that compressibility affects turbulence only at hypersonic speeds.

Further modelling is required to actually compute μ_T and this is the point where turbulence models come into play.

According to the number of transport equations one has to use in order to calculate μ_T, models are classified as:

- Algebraic or zero-equation models, such as the Cebeci-Smith model.
- One-equation models, such as the Spalart-Allmaras and Baldwin-Lomax models.
- Two-equation models, such as the k-ω, k-ε, SST and k-g models.
- Multi-equation models: Three-equation and multiple time-scale models.

An additional family of models solves equations for all components of the Reynolds stress tensor. These are also known as second moment closures.

3.1 Model Equations

For linear eddy-viscosity models, the stress-strain relationship is as defined in equation (11) above and can be equivalently represented as

$$\overline{u_i u_j} = \frac{2}{3} k \delta_{ij} - \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3} \frac{\partial U_k}{\partial x_k} \delta_{ij} \right) \tag{15}$$

where the value of the eddy-viscosity, denoted as $\nu_T \equiv \frac{\mu_T}{\rho}$, is dependent on the turbulence model used:

$$\nu_T \equiv \frac{\mu_T}{\rho} = \left\{ \begin{array}{lll} C_\mu f_\mu \frac{k^2}{\varepsilon} & , & k \text{ or } k\text{-}\varepsilon \text{ models} \\ \alpha^* \frac{k}{\omega} & , & k\text{-}\omega \text{ models} \\ \beta^* k g^2 & , & k\text{-}g \text{ models} \end{array} \right\} \tag{16}$$

There is an approximate high-Re correspondance between ω, g and ε:

$$\omega \sim \frac{\varepsilon}{C_\mu k}, \qquad g \sim \left(\frac{k}{\varepsilon} \right)^{1/2} \sim (\beta * \omega)^{-1/2}. \tag{17}$$

Alternatively, in terms of coefficients,

$$C_\mu f_\mu \sim C_\mu \alpha^* \sim \beta^*. \tag{18}$$

For non-linear models, however, the eddy-viscosity is defined differently. For non-linear models up to cubic order, the stress-strain relationship in incompressible flow may be written in the following (non-unique) canonical form:

$$\begin{aligned} \mathbf{a} = &- 2 f_\mu C_\mu \mathbf{s} \\ &+ \beta_1 \left(\mathbf{s}^2 - s_2 \mathbf{I}\right) + \beta_2 \left(\mathbf{ws} - \mathbf{sw}\right) \beta_3 \left(\mathbf{w}^2 - 1/3 w_2 \mathbf{I}\right) \\ &- \gamma_1 s_2 \mathbf{s} - \gamma_2 w_2 \mathbf{s} - \gamma_3 \left(\mathbf{w}^2\mathbf{s} + \mathbf{s}\mathbf{w}^2 - w_2 \mathbf{s} - 2/3 \left\{\mathbf{wsw}\right\} \mathbf{I}\right) - \gamma_4 \left(\mathbf{w}\mathbf{s}^2 - \mathbf{s}^2\mathbf{w}\right), \end{aligned} \tag{19}$$

where the following notation is used for second-rank tensors:

$$\mathbf{T} \equiv (T_{ij}), \qquad \{\mathbf{T}\} \equiv trace\,(\mathbf{T}), \qquad T_n \equiv trace\,(\mathbf{T}^n), \qquad \mathbf{I} \equiv (\delta_{ij}). \tag{20}$$

The *dimensional* mean strain and mean vorticity tensors are denoted in upper case by

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right). \tag{21}$$

whilst *dimensionless* quantities - anisotropy $\mathbf{a}$, mean strain $\mathbf{s}$ and mean vorticity $\mathbf{w}$ - are written in lower case and defined by

$$a_{ij} = \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij}, \qquad s_{ij} = \tau S_{ij}, \qquad w_{ij} = \tau W_{ij}. \tag{22}$$

The mean shear is made non-dimensionalised using a turbulent time-scale τ, dependent on the transported turbulent scalars:

$$\tau = \left\{ \begin{array}{lll} \frac{k}{\varepsilon} & , & k \text{ or } k\text{-}\varepsilon \text{ models} \\ \frac{1}{\beta * \omega} & , & k\text{-}\omega \text{ models} \\ g^2 & , & k \text{ or } k\text{-}g \text{ models} \end{array} \right\} \tag{23}$$

For compressible flows, it is assumed that S_{ij} may be replaced in this formulation by

$$S^*_{ij} = S_{ij} - \frac{1}{3} S_{kk}\delta_{ij}, \tag{24}$$

and, for a system rotation Ω, that W_{ij} may be replaced by

$$W^*_{ij} = W_{ij} - \varepsilon_{ijk}\Omega_k. \tag{25}$$

The following shear parameters may be defined:

$$\overline{s} \equiv \sqrt{2s_{ij}s_{ij}} = \sqrt{2s_2}, \qquad \overline{w} \equiv \sqrt{2w_{ij}w_{ij}} = \sqrt{2(-w_2)}. \tag{26}$$

The rate of production of turbulent kinetic energy is

$$P = -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}. \tag{27}$$

3.2 Viscosity-Dependent Parameters

y^+ and y^* are defined by

$$y^* \equiv \frac{y_n k^{1/2}}{\nu}, \qquad y^+ \equiv \frac{y_n u_\tau}{\nu}, \tag{28}$$

where y_n is the distance from the nearest wall, $u_\tau \equiv \sqrt{\tau_w/\rho}$ and τ_w is the dynamic wall shear stress. The turbulent Reynolds numbers are

$$R_t \equiv \frac{k^2}{\nu\varepsilon}, \qquad R_\omega \equiv \frac{k}{\nu\omega}. \tag{29}$$

4 Algebraic or Zero-Equation Models

Models where the eddy-viscosity is completely determined in terms of the local mean flow variables are referred to as zero-equation or algebraic models. Zero-equation models are the simplest of all. Most of these models use Prandtl's mixing length hypothesis and compute the eddy viscosity in analogy to the molecular mixing phenomenon. There is a significant difference though: molecular viscosity is a property of the fluid while eddy-viscosity is a property of the flow. This is the reason eddy-viscosity and mixing length must be specified in terms of an algebraic relation between eddy-viscosity and length scales of the mean flow:

$$\nu_\tau = l_{\text{min}}^2 \left|\frac{\partial u_i}{\partial x_j}\right| \tag{30}$$

where l_{min} is the mixing length, which is analogous to the mean free path of the molecules in molecular mixing.

Baldwin-Lomax and Cebeci-Smith are the most common types of Algebraic or Zero-Equation Models. The main advantages of Algebraic or Zero-Equation models are:

- Easy to implement.
- Cheap in terms of CPU time.
- Stable (numerically).
- Work well for attached boundary layers (have been tuned for such cases).
- Predict well C_f and velocity profiles provided the pressure gradients are not strong.

The main problem with the Baldwin-Lomax and Cebeci-Smith models is the difficulty in predicting separated flow, jets and wakes.

5 One-Equation Models

Usually employ a transport equation, which may be theory for the kinetic energy of turbulence:

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \tau_{ij} \frac{\partial u_i}{\partial x_j} - \varepsilon + \frac{\partial}{\partial x_j} \left[\frac{\mu}{\rho} \frac{\partial k}{\partial x_j} - \frac{1}{2} \overline{u_i' u_i' u_j'} - \frac{1}{\rho} \overline{p' u_j'} \right] \tag{31}$$

where	$\tau_{ij} \frac{\partial u_i}{\partial x_j}$	represents production of turbulence,
	$\frac{\mu}{\rho} \frac{\partial k}{\partial x_j}$	is the molecular diffusion term,
	$\overline{u_i' u_i' u_j'}$	is the turbulent flux of the turbulent kinetic energy,
	$\frac{1}{\rho} \overline{p' u_j'}$	is the pressure diffusion term usually neglected due to its small contribution,
	ε	is the dissipation rate of k per unit mass of fluid, and is usually defined as:

$$\varepsilon = \frac{\mu}{\rho} \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}} \tag{32}$$

ν_T is usually calculated as

$$\mu_T = \rho C_\mu l_{mix} \sqrt{k} \tag{33}$$

where C_μ is the model coefficient.

Spalart-Almaras and Baldwin-Barth are the most common types of One-Equation Models. The main advantages of the One-Equation models are:

- Give better results than Algebraic Models.
- "History Effects" are accounted for since a differential equation is used (can be important in non-equilibrium flows).
- Work for flow regions where the mean velocity gradient is zero.
- "History Effects" not accounted for the length scale l (still algebraic in this sense).
- Tuned for aerodynamic flows with adverse pressure gradients and transonic conditions.
- Popular in aero-applications and USA.
- Modular in the sense that near-wall effects and transition can be included by adding terms.

Two common one-equation models (Baldwin-Lomax and Spalart-Almaras models) are discussed in more detail in the following sections.

5.1 Baldwin-Lomax Model

The eddy viscosity (ν_t) in the inner and outer layer viscosities are given by:

<u>Inner Layer</u>

$$\nu_{ti} = l_{mix}^2 \left|\omega\right| l_{mix} = \kappa y \left(1 - e^{-y^+/A_0^+}\right) \tag{34}$$

where l_{mix} is the mixing length,
$|\omega|$ is the magnitude of the voriticity vector,
κ is the Von Karman constant.

Outer Layer

$$\nu_{to} = \alpha C_{cp} F_{wake} F_{Kleb}\left(y; y_{max}/C_{Kleb}\right) \tag{35}$$
$$F_{wake} = \min\left(y_{max} F_{max}; C_{wk} y_{max} U^2_{dif}/F_{max}\right) \tag{36}$$
$$F_{max} = \frac{1}{\kappa}\left(y_{max} l_{mix} |\omega|\right) \tag{37}$$

where y_{max} is the value of y at which $l_{mix}|\omega|$ achieves its maximum value,
F_{Kleb} is Klebanoff's intermittency function,
C_{Kleb} is Klebanoff's boundary layer constant.

5.1.1 Closure Coefficients

κ	0.40
α	0.0168
A_0^+	26
C_{cp}	1.6
C_{Kleb}	0.3
C_{wk}	1

The function F_{Kleb} is Klebanoff's intermittency function and ω is the magnitude of the voriticity vector, i. e.

$$\omega = \left|\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right| \tag{38}$$

for 2D flows.

U_{dif} is the maximum value of U. The Baldwin-Lomax model establishes the outer-layer length scale in terms of the voriticity in the layer.

5.2 Spalart-Almaras Model

The eddy-viscosity (ν_t) is calculated by:

$$\nu_t = \tilde{\nu} \cdot f_{v1} \tag{39}$$

where

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3} \quad \text{and} \qquad \chi = \frac{\tilde{\nu}}{\nu} \tag{40}$$

In the above, and from now on, f refers to a function, c refers to a constant, ν is the molecular viscosity and $\tilde{\nu}$ is the working variable that obeys the transport equation:

$$\begin{aligned} \frac{D\tilde{\nu}}{Dt} &= c_{b1}\left(1 - f_{t2}\right) \cdot \tilde{S}\tilde{\nu} + \frac{1}{\sigma}\left(\nabla \cdot \left(\left(\nu + \tilde{\nu}\right) \nabla \tilde{\nu}\right) + c_{b2}\left(\nabla \tilde{\nu}\right)^2\right) \\ &- \left(c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2}\right)\left(\frac{\tilde{\nu}}{d}\right)^2 + f_{t1}\Delta U^2 \end{aligned} \tag{41}$$

The first term in the RHS is the production term, the second one is the diffusion term and the third one is the near-wall term. The last term stands for the tripping. The subscript b refers to the *basic*, w refers to *wall* and t stands for *trip*. σ is the turbulence Prandtl number and d is the distance to the wall.

Here S is the magnitude of the vorticity,

$$\begin{aligned} \tilde{S} &= S + \frac{\tilde{\nu}}{k^2 d^2} f_{v2} \\ f_{v2} &= 1 - \frac{\chi}{1 + \chi f_{v1}} \end{aligned} \tag{42}$$

The function f_w is

$$\begin{aligned} f_w &= g\left(\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right)^{1/6} \\ g &= r + c_{w2}\left(r^6 - r\right) \\ r &= \frac{\tilde{\nu}}{\tilde{S} k^2 d^2} \end{aligned} \tag{43}$$

For larger r, f_w reaches a constant, so large values of r can be truncated to 10 or so. The wall boundary condition is $\tilde{\nu} = 0$. In the freestream, 0 is best, provided that numerical errors do not push $\tilde{\nu}$ to negative values near the edge of the boundary layer (the exact solution cannot go negative). Values below $\nu/10$ will be acceptable. The same applies to the initial condition. The f_{t2} function is defined as:

$$f_{t2} = c_{t3} \cdot e^{-c_{t4} \cdot \chi^2} \tag{44}$$

The trip function f_{t1} is defined as follows:

$$f_{t1} = c_{t1} g_t \cdot e^{-c_{t2} \frac{\omega_t^2}{\Delta U^2} \left(d^2 + g_t^2 d_t^2\right)} \tag{45}$$

where d_t is the distance from the field point to the trip (which is on a wall),

ω_t is the wall vorticity at the trip, and

ΔU is the difference between the velocity at the field point and that at the trip.

Then

$$g_t = \min\left(0.1, \Delta U / \omega_t \Delta x\right) \tag{46}$$

where Δx is the grid spacing along the wall at the trip.

The values of the contants are given in the following section.

5.2.1 Constants

$\mathbf{c_{b1}}$	0.1355
σ	2/3
$\mathbf{c_{b2}}$	0.622
k	0.41
$\mathbf{c_{w2}}$	0.3
$\mathbf{c_{w3}}$	2
$\mathbf{c_{v1}}$	7.1
$\mathbf{c_{t1}}$	1
$\mathbf{c_{t2}}$	2
$\mathbf{c_{t3}}$	1.1
$\mathbf{c_{t4}}$	2

And

$$c_{w1} = \frac{c_{b1}}{k^2} + \frac{(1 + c_{b2})}{\sigma}. \tag{47}$$

A value of 0.9 has been used for the turbulent Prandtl number.

5.3 Model Equations

	Low-Re
Wolfshtein (1969)	Yes
Norris and Reynolds (1975)	Yes

In the one-equation models considered here, a single transport equation is solved for k:

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho U_j k) = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + \rho(P - \varepsilon), \tag{48}$$

and the turbulence length scales are specified algebraically:

$$\varepsilon = \frac{C_\mu^{3/4} k^{3/2}}{l_\varepsilon}, \tag{49}$$

$$\nu_t = C_\mu^{1/4} k^{1/2} l_\mu, \tag{50}$$

In terms of the usual k-ε damping factor,

$$f_\mu = \frac{l_\mu}{l_\varepsilon}. \tag{51}$$

The most popular models are by Wolfshtein [1] and Norris & Reynolds [2].

5.3.1 Wolfshtein

$$l_\mu = k y_n \left(1 - e^{-0.016 y^w}\right), \qquad l_\mu = k y_n \left(1 - e^{-0.263 y^w}\right). \tag{52}$$

5.3.2 Norris and Reynolds

$$l_\mu = k y_n \left(1 - e^{-0.0198 y^w}\right), \qquad l_\mu = k y_n \frac{y^*}{y^* + 2k/C_\mu^{3/4}}. \tag{53}$$

Values $C_\mu = 0.09$, $\sigma_k = 1.0$ and $k = 0.41$ are assumed.

6 Two-Equation Models

By far the most popular. Two-equation models are complete i. e. can be used to predict properties of a given flow with no prior knowledge of the turbulence structure or flow geometry. Two transport equations are used for:

1. k.
2. Turbulence length scale or function of it.

The choice of the 2nd variable is arbitrary and many proposals have been presented. The most popular are:

- ε — Dissipation rate of turbulence.

- ω — k-specific dissipation rate.
- τ — Turbulent time-scale.

Two-Equation Model	Equation	2nd Variable Used
Kolmogorov (c. 1942)	$k^{1/2}l^{-1}$	ω (Frequency Length Scale)
Rotta (c. 1950)	l	
Harlow-Nakayana (1968)	$k^{3/2}l^{-1}$	ε (Energy Dissipation Rate)
Speziale (1992)	$lk^{-1/2}$	τ (Time-Scale)

For the popular $k-\varepsilon$ model (Jones and Launder 1972):

$$\nu_T = C_\mu \frac{k^2}{\varepsilon}, \qquad \mu_T = C_\mu \rho \frac{k^2}{\varepsilon} \tag{54}$$

where C_μ is the model coefficient.

The advantage of the $k-\varepsilon$ model is that it:

- Performs well for attached flows with thin shear layers and jets.
- However, fails to predict the correct flow behaviour in many flows with adverse pressure gradients. Extended separated flow regions swirl, buoyancy, curvature secondary flows and unsteady flows.

The $k-\omega$ model uses the k-specific dissipation rate as a second variable.

$$\nu_T = \frac{k}{\omega} \qquad \text{or} \qquad \mu_T = \rho\frac{k}{\omega} \tag{55}$$

The advantages of the $k-\omega$ model are:

- Better performance in adverse pressure gradient flows.
- Same problems like $k-\varepsilon$.
- SST version performs well in separated flows.

  SST is zonal: $k-\omega$ near wall.
  $k-\omega$ transformed to $k-\varepsilon$ away from wall.
  This avoids $k-\omega$'s sensitivity to freestream conditions.

In order to take into account the effects of the Mach Number, the Sarkar/Zeman correction is implemented into the turbulence models (Wilcox [3]). For the $k-\omega$ two-equation turbulence model, the Sarkar/Zeman modification is implemented by applying a change to the closure coefficients β_0 (which is used in the ω-equation) and β_0^* (which is used in the k-equation). The new closure coefficients vary with the Mach number and are defined as follows:

$$\beta^* = \beta_0^* \left(1 + \xi^* F\left(M_t\right)\right) \tag{56}$$

$$\beta = \beta_0 - \beta_0^* \xi^* F\left(M_t\right) \tag{57}$$

where β_0^* and β_0 are the incompressible values corresponding to β^* and β, respectively, and M_t is the turbulent Mach number and is defined as:

$$M_t^2 = \frac{2k}{\alpha^2} \tag{58}$$

where k is the turbulent kinetic energy,

α is the speed of sound, and is defined by:

$$\alpha = \sqrt{\frac{\gamma p}{\rho}} \tag{59}$$

The values of ξ^* and $F\left(M_t\right)$ for both the Sarkar and Zeman models are given by:

<u>Sarkar</u>

$$\xi^* = 1 \tag{60}$$

$$F\left(M_t\right) = M_t^2 \tag{61}$$

<u>Zeman</u>

$$\xi^* = \frac{3}{4} \tag{62}$$

$$F\left(M_t\right) = \left(1 - e^{-\frac{1}{2}\frac{(\gamma+1)(M_t - M_{t0})^2}{\Lambda^2}}\right) H\left(M_t - M_{t0}\right) \tag{63}$$

where γ is the specific heat ratio,

$H(x)$ is the Heaviside step function, which is defined as:

$$H(x) = \begin{cases} 1 & , \quad x > 0 \\ 0 & , \quad x < 0 \\ 1/2 & , \quad x = 0 \end{cases} \tag{64}$$

According to Zeman [4], the values of M_{t0} and Λ are different for free shear flows and boundary layers.

Free Shear Flows

$$M_{t0} = 0.1\left(\frac{2}{\gamma+1}\right)^{1/2}, \qquad \Lambda = 0.6 \tag{65}$$

Boundary Layers

$$M_{t0} = 0.1\left(\frac{2}{\gamma+1}\right)^{1/2}, \qquad \Lambda = 0.6 \tag{66}$$

Wilcox [3] also postulated the following values for ξ^* and $F(M_t)$:

Wilcox

$$\xi^* = \frac{3}{2} \tag{67}$$

$$M_{t0} = 0.25F(M_t) \qquad = \left(M_t^2 - M_{t0}^2\right) H\left(M_t - M_{t0}\right) \tag{68}$$

6.1 Model Equations

6.1.1 k-ε Turbulence Transport Equations

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho U_j k) = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + \rho(P - \varepsilon - D) \tag{69}$$

$$\frac{\partial}{\partial t}(\rho\varepsilon) + \frac{\partial}{\partial x_j}(\rho U_j \varepsilon) = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + \rho\left(C_{\varepsilon 1} f_1 P - C_{\varepsilon 2} f_2 \varepsilon\right)\frac{\varepsilon}{k} + \rho\left(S_l + S_\varepsilon\right) \tag{70}$$

The term D is only non-zero for those models (such as Launder-Sharma [5]) that distinguish homogeneous and inhomogeneous dissipation rates. In such models, ε is often written as $\tilde{\varepsilon}$.

Additional source terms S_l and S_ε are used to control the growth of the turbulent length scale and provide the correct near-wall viscous sublayer behaviour, respectively.

6.1.2 $k-\omega$ Turbulence Transport Equations

$$\frac{\partial}{\partial t}(\rho k)+\frac{\partial}{\partial x_j}(\rho U_j k)=\frac{\partial}{\partial x_j}\left[\left(\mu+\frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]+\rho\left(P-\beta^*\omega k\right) \tag{71}$$

$$\frac{\partial}{\partial t}(\rho\omega)+\frac{\partial}{\partial x_j}(\rho U_j \omega)=\frac{\partial}{\partial x_j}\left[\left(\mu+\frac{\mu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right]+\rho\left(\frac{\alpha}{\nu_t}P-\frac{beta}{\beta\omega^2}\right)+\rho S_l \tag{72}$$

6.1.3 $k-g$ Turbulence Transport Equations

$$\frac{\partial}{\partial t}(\rho k)+\frac{\partial}{\partial x_j}(\rho U_j k)=\frac{\partial}{\partial x_j}\left[\left(\mu+\frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]+\rho\left(P-\Lambda\frac{k}{g^2}\right) \tag{73}$$

$$\frac{\partial}{\partial t}(\rho\omega)+\frac{\partial}{\partial x_j}(\rho U_j g)=\frac{\partial}{\partial x_j}\left[\left(\mu+\frac{\mu_t}{\sigma_g}\right)\frac{\partial g}{\partial x_j}\right]+\frac{\rho\Lambda}{2}\left(\alpha\frac{g}{k}P-\frac{beta}{\beta^* g}\right)+\rho\Lambda S_g \tag{74}$$

6.1.4 Linear k-ε Models

	Low-Re
Launder and Spalding (1974) — "Standard" high-*Re* k-ε	No
Yakhot et al. (1992) — RNG k-ε	No
Launder and Sharma (1974)	Yes
Lam and Bremhorst (1981)	Yes
Chien (1982)	Yes
Lien and Leschziner (1993)	Yes

High-Re Coefficients

	C_μ	$C_\varepsilon 1$	$C_\varepsilon 2$	σ_k	σ_ε	S_l
Launder and Spalding (1974)	0.09	1.44	1.92	1.0	1.3	0
Yakhot et al. (1992)	0.085	1.42	1.68	0.72	0.72	0
Launder and Sharma (1974)	0.09	1.44	1.92	1.0	1.3	0
Lam and Bremhorst (1981)	0.09	1.44	1.92	1.0	1.3	0
Chien (1982)	0.09	1.35	1.80	1.0	1.3	0
Lien and Leschziner (1993)	0.09	1.44	1.92	1.0	1.3	0

Viscous terms in low-Re models

	$\mathbf{f}_\mu$	D	$\mathbf{f}_1$
Launder-Sharma	$\exp\left[\frac{-3.4}{(1+R_t/50)^2}\right]$	$2\nu\left(\frac{\partial k^{1/2}}{\partial x_i}\right)^2$	1
Lam-Bremhorst	$\left(1-e^{-0.0165y^w}\right)^2\left(1+\frac{20.5}{R_t}\right)$	0	$1+\left(\frac{0.05}{f_\mu}\right)^3$
Chien	$1-e^{-0.0115y^w}$	$\frac{2\nu k}{y_n^2}$	1
Lien-Leschziner[1]	$l_\mu^{(l)}/l_\varepsilon^{(l)}$	0	1

Viscous terms in low-Re models (Continued)

	$\mathbf{f}_2$	$\mathbf{S}_\varepsilon$
Launder-Sharma	$1-0.3e^{-R_t^2}$	$2\nu\nu_t\left(\frac{\partial^2 U_i}{\partial x_j \partial x_k}\right)^2$
Lam-Bremhorst	$1-e^{-R_t^2}$	0
Chien	$1-0.22e^{-(R_t/6)^2}$	$-\frac{2\nu\varepsilon}{y_n^2}e^{-y^w/2}$
Lien-Leschziner[1]	$1-0.36e^{-R_t^2}$	$C_{\varepsilon 2} f_2 \frac{\varepsilon^{(1)}\varepsilon}{k} e^{-0.0022y^{w2}}$

Remarks

1. The Lien-Leschnizer model [6] asymptotes to the one-equation model of Wolfshtein [1] for the mixing and dissipation lengths. The original model description actually modifies f_1 rather than S_ε, but the two formulations are equivalent.

6.1.5 Linear k-ω Models

	Low-Re
Wilcox (1988a)	Yes
Wilcox (1994)	Yes
Menter (1994) — (i) Baseline Model	Yes
Menter (1994) — (ii) SST Model	Yes

	α^*	β^*	α	β
Wilcox (1998a)	1	$\frac{9}{100}$	$\frac{5}{9}$	$\frac{3}{40}$
Wilcox (1994)	$\frac{\frac{1}{40}+\frac{R_\omega}{6}}{1+\frac{R_\omega}{6}}$	$\frac{9}{100}\frac{\frac{5}{18}+\left(\frac{R_\omega}{8}\right)^4}{1+\left(\frac{R_\omega}{8}\right)^4}$	$\frac{5}{9}\frac{\frac{1}{10}+\frac{R_\omega}{2.7}}{1+\frac{R_\omega}{2.7}}$	$\frac{3}{40}$
Menter (1994) ***(Baseline)***[1]	1	0.09	$B\begin{pmatrix}0.553\\0.440\end{pmatrix}$	$B\begin{pmatrix}0.075\\0.083\end{pmatrix}$
Menter (1994) ***(SST)***[2]	$\min\left(1, \frac{0.31}{F_2}\frac{\omega}{\overline{w}}\right)$	0.09	$B\begin{pmatrix}0.553\\0.440\end{pmatrix}$	$B\begin{pmatrix}0.075\\0.083\end{pmatrix}$

	$\sigma_{\mathbf{k}}$	σ_ω	$\mathbf{S_l}$
Wilcox (1998a)	2	2	0
Wilcox (1994)	2	2	0
Menter (1994) ***(Baseline)***[1]	$\frac{1}{B\begin{pmatrix}0.5\\1.0\end{pmatrix}}$	$\frac{1}{B\begin{pmatrix}0.5\\0.856\end{pmatrix}}$	$B\begin{pmatrix}0\\\frac{1.71}{\omega}\nabla k\cdot\nabla\omega\end{pmatrix}$
Menter (1994) ***(SST)***[2]	$\frac{1}{B\begin{pmatrix}0.85\\1.0\end{pmatrix}}$	$\frac{1}{B\begin{pmatrix}0.5\\0.856\end{pmatrix}}$	$B\begin{pmatrix}0\\\frac{1.71}{\omega}\nabla k\cdot\nabla\omega\end{pmatrix}$

Remarks

1. Menter's models [7] are constructed as a "blend" of $k-\omega/k-\varepsilon$ models, phrased in $k-\omega$ form. The blending of $k-\varepsilon$ and $k-\omega$ model values for α, β, σ_k^{-1} and σ_ω^{-1} is (with my notation) given by

$$B\begin{pmatrix}a\\b\end{pmatrix} \equiv F_1 a + (1-F_1)\, b. \tag{75}$$

The blending function is

$$F_1 = \tanh\left(arg_1^4\right), \tag{76}$$

where

$$arg_1 = min\left[\max\left(\frac{k^{1/2}}{\beta^*\omega y}, \frac{500\nu}{y_n^2\omega}\right), \frac{2k\omega}{y_n^2 \max\left(\nabla k\cdot\nabla\omega, 0.0\right)}\right]. \tag{77}$$

2. The SST model places an additional vorticity-dependent limiter on the shear stress, with

$$F_2 = \tanh\left(arg_2^2\right), \qquad arg_2 = \max\left(\frac{2k^{1/2}}{\beta^*\omega y}, \frac{500\nu}{y^2\omega}\right). \tag{78}$$

Note that this model also uses a slightly different value of σ_k.

6.1.6 Linear k-g Models

	Low-Re
Moir and Gould (1998)	Yes

	β^*	α	β	σ_k	σ_g	S_g
Moir and Gould (1998)	$\frac{9}{10}$	$\frac{5}{9}$	$\frac{3}{40}$	2	2	$-\left(\nu+\frac{\nu_t}{\sigma_g}\right)\frac{3}{g}\left(\nabla g\right)^2$

The limiter Λ is given by

$$\Lambda = \left\{\begin{array}{lll} 1 & , & \text{in near-wall sublayer} \\ \min\left(100\frac{\mu_t}{\mu}, 1\right) & , & \text{otherwise} \end{array}\right\} \tag{79}$$

For the purposes of this model, "near-wall sublayer" is defined as the nearest 8 cells.

7 Reynolds Stress Modelling

The Reynolds Stress tensor is symmetric, i.e. 6 equations in 3D and 4 equations in 2D and 1 equation length scale.

Better in cases with:

- Streamline Curvature.

- Swirl.
- Buoyancy.
- Rotation.

To obtain the transport equations one has to start from the equation for momentum and multiply the u_j term with u'_i and the u_i term with u'_j then add and time-average the results.

For constant density flows:

$$
\begin{aligned}
\frac{D\overline{u'_i u'_j}}{Dt} = &- \left(\overline{u'_i u'_k} \frac{\partial u_j}{\partial x_k} + \overline{u'_j u'_k} \frac{\partial u_i}{\partial x_k} \right) \\
&- \frac{\partial}{\partial x_k} \left[\overline{u'_i u'_j u'_k} + \frac{1}{\rho} \left(\overline{p' u'_i} \delta_{jk} + \overline{p' u'_j} \delta_{ik} \right) - \nu \frac{\partial \overline{u'_i u'_j}}{\partial x_k} \right] \\
&+ \frac{\overline{p'}}{\rho} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\
&- 2\nu \left(\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right) .
\end{aligned}
\tag{80}
$$

where $\frac{D\overline{u'_i u'_j}}{Dt}$ Convection term. Represents the rate of change of $\overline{u'_i u'_j}$ along a streamlne in "steady" flow. Equals the rate at which the Reynolds stresses are convected by the mean fluid motion.

$- \left(\overline{u'_i u'_k} \frac{\partial u_j}{\partial x_k} + \overline{u'_j u'_k} \frac{\partial u_i}{\partial x_k} \right)$ Production term. Represents the rate of production of $\overline{u'_i u'_j}$ by mean shear. The shearing is generated by interaction of transverse normal stress and shear strain. This is a large (important) term and is <u>exact</u>. Use P_{ij} hereafter.

$\frac{\partial}{\partial x_k} \left[\overline{u'_i u'_j u'_k} + \frac{1}{\rho} \left(\overline{p' u'_i} \delta_{jk} + \overline{p' u'_j} \delta_{ik} \right) - \nu \frac{\partial \overline{u'_i u'_j}}{\partial x_k} \right]$

Diffusion term. Rate of spatial transport of $\overline{u_i'u_j'}$ by the action of turbulent fluctuation, pressure fluctuations and molecular diffusion.

$\frac{p'}{\rho}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$ Redistribution or pressure-strain term. Drives turbulence towards isotropy by redistributing energy. Use ϕ_{ij} hereafter.

Redistribution of the available turbulent kinetic energy is amongst the fluctuating velocity components. It has no effect on the overall level of k since it has zero trace. If X_1 is the dominant flow direction, $\overline{u_1'u_1'}$ is generated due to shear and $\overline{u_2'u_2'}$ and $\overline{u_3'u_3'}$ are smaller. ϕ_{ij} redistributes $\overline{u_1'u_1'}$'s energy to $\overline{u_2'u_2'}$ and $\overline{u_3'u_3'}$. $\overline{u_1'u_2'}$ has an opposite sign to the shear strain of a boundary layer. ϕ_{ij} reduces $\overline{u_1'u_2'}$ since isotropic turbulence must be shear-free. The same applies for other directions.

$2\nu\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\right)$ Dissipation term. Represents the dissipation rate of $\overline{u_i'u_j'}$ due to molecular viscous action. Use ε_{ij} hereafter.

Only convection and production can be calculable out of the Reynolds stress components and the mean flow quantities. We need modelling for the following:

- Diffusion,
- Redistribution and
- Dissipation terms.

7.1 Diffusion

$$\text{Diffusion} = \text{Viscous Diffusion} + \text{Pressure Diffusion} + \text{Turbulent Diffusion}$$

1. **Viscous Diffusion** - Small contribution at high Reynolds values - usually neglected at high Reynolds numbers. It can easily be calculated.

2. **Pressure Diffusion** - As regards the pressure diffusion term, measurements cannot be conducted and estimates via indirect methods suggest that it is very small.

3. **Turbulent Diffusion** - Daly and Harlow [8] with their gradient diffusion hypothesis modelled the terms as:

$$-\overline{u_i'u_j'u_k'} = C_s \frac{k}{\varepsilon} \overline{u_k'u_l'} \frac{\partial \overline{u_i'u_j'}}{\partial x_l} \tag{81}$$

where	C_s	Empirical coefficient. Approximately 0.22 in value. Obtained via optimisation.
	$\frac{k}{\varepsilon}$	Characteristic time scale of the energy containing eddies.

Other models by Hanjalic & Launder [9] and Lumley & Khajeh-Nouri [10] are complicated and so less popular.

7.2 Redistribution Term

- Based on the Poisson equation for the instantaneous pressure.
- Start from the pressure-poisson method and subtract the mean to obtain:

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2} = -\left[\frac{\partial^2 \left(u_i'u_j' - \overline{u_i'u_j'}\right)}{\partial x_j x_i} + 2\frac{\partial u_i}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right] \tag{82}$$

The first term of the sum is for turbulent quantities while the second contains mean-velocity gradients.

- In the special case of homogeneous turbulence, Chen's integral is obtained:

$$\frac{\overline{p}}{\rho}\frac{\overline{\partial u_i}}{\partial x_j} = \frac{1}{4M}\left\{\frac{\partial^3 \overline{u_l^* u_m^* u_i}}{\partial r_l \partial r_m \partial r_j} + 2\frac{\partial u_l}{\partial x_m}\frac{\partial^2 u_m^* u_i}{\partial r_l \partial r_j}\right\}\frac{d_{vol}}{r} + \phi_{ij,w} \tag{83}$$

where $*$ denotes quantities evaluated at position x^*, $r = x^* - x$.

$\phi_{ij,w}$ is a surface integral important only if the size of the energy containing eddies is of the same order of magnitude as the distance from the wall (v. Tennekes & Lumley [11]).

Launder, Reece and Rodi proposed to model the terms surrounded in the $\{\cdots\}$ separately. This is the most common approach. Speziale, Sarkar and Gatski proposed to model the whole term. This is becoming popular since a separate model for $\phi_{ij,w}$ is not necessary.

7.3 Dissipation Term

Molecular viscosity converts the turbulent kinetic energy into heat by acting on the small-scale, high frequency motions. These motions can be assumed isotropic so no need for a tensor:

$$\varepsilon_{ij} = \frac{2}{3}\delta_{ij}\varepsilon \tag{84}$$

where ε is obtained from a transport equation.

Other alternatives like w, e exist:

$$\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_k}\left(C_\varepsilon \frac{k}{\varepsilon}\overline{u'_k u'_l}\frac{\partial \varepsilon}{\partial x_l}\right) + \frac{\varepsilon}{k}\left(C_{\epsilon 1} P_{kl} S_{lk} - C_{\epsilon 2}\varepsilon\right) \tag{85}$$

where C_ε,$C_{\epsilon 1}$,$C_{\epsilon 2}$ are model coefficients.

P_{kl} production due to strain S_{kl}.

Diffusion of the dissipation rate is also modelled using the generalised gradient diffusion model by Daly and Harlow [8].

The coefficients must be determined. Use 6 criteria:

1. Grid-generated turbulence must decay with the experimentally observed rate in the absence of shear.
2. Turbulence must adjust when a strain rate is suddenly applied to an isotropic field.
3. Homogeneous free shear data can be used to match production of dissipation.
4. The model must be able to predict the boundary layer close to a solid wall.
5. Turbulence must collapse in stabilising curvature.
6. Simple test cases can be used to optimise the coefficients.

7.4 Model Equations

	Low-Re
Gibson and Launder (1978)	No
Craft and Launder (1992)	No
Speziale et al. (1991) — SSG model	No
Jakirlic (1995)	Yes
Shima (1998)	Yes
Wilcox (1998b) — Multiscale Model	Yes

7.4.1 Transport Equations

$$\frac{\partial}{\partial t}\left(\rho\overline{u_i u_j}\right) + \frac{\partial}{\partial x_k}\left(\rho U_k \overline{u_i u_j}\right) = \frac{\partial}{\partial x_k} d_{ijk} + \rho\left(P_{ij} + \Phi_{ij} - \varepsilon_{ij}\right), \tag{86}$$

$$\frac{\partial}{\partial t}\left(\rho\varepsilon\right) + \frac{\partial}{\partial x_k}\left(\rho U_k \varepsilon\right) = \frac{\partial}{\partial x_k} d_k^{(\varepsilon)} + \rho\left(C_{\varepsilon 1} f_1 P - C_{\varepsilon 2} f_2 \varepsilon\right)\frac{\varepsilon}{k} + \rho\left(S_l + S_\varepsilon\right), \tag{87}$$

where

$$P_{ij} = -\left(\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}\right), \qquad P = 1/2 P_{ii}, \tag{88}$$

$$d_{ijk} = d_{ijk}^{(p)} + d_{ijk}^{(u)} + \mu\frac{\partial}{\partial x_k}\left(\overline{u_i u_j}\right), \tag{89}$$

$$\varepsilon = 1/2\varepsilon_{kk}. \tag{90}$$

The pressure strain correlation Φ_{ij}, dissipation ε_{ij} and non-viscous diffusion $d_{ijk}^{(p)} + d_{ijk}^{(u)}$ must be modelled. Deviatoric parts of the dissipation tensor are often absorbed into Φ_{ij}. Body forces ρG_{ij} and dilatational terms ρK_{ij} have not been implemented in the stress-transport equations.

7.4.2 Gibson and Launder

Diffusion

$$d_{ijk} = \left(\mu\delta_k l + C_s\frac{\rho k\overline{u_k u_l}}{\varepsilon}\right)\frac{\partial}{\partial x_l}\left(\overline{u_i u_j}\right), \tag{91}$$

$$d_k^{(\varepsilon)} = \left(\mu\delta_k l + C_\varepsilon\frac{\rho k\overline{u_k u_l}}{\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_l}, \tag{92}$$

$$C_s = 0.22, \qquad C_\varepsilon = 0.18. \tag{93}$$

Pressure-Strain

$$\Phi_{ij} = \Phi_{ij}^{(1)} + \Phi_{ij}^{(2)} + \Phi_{ij}^{(w)}, \tag{94}$$

where

$$\Phi_{ij}^{(1)} = -C_1\varepsilon a_{ij}, \tag{95}$$

$$\Phi_{ij}^{(2)} = -C_2\left(P_{ij} - \frac{1}{3}P_{kk}\delta_{ij}\right), \tag{96}$$

$$\Phi_{ij}^{(w)} = \left(\tilde{\Phi}_{kl}n_k n_l\delta_{ij} - \frac{3}{2}\tilde{Phi}_{jk}n_i n_k\right) f, \tag{97}$$

$$\tilde{\Phi}_{ij} = C_1^{(w)}\frac{\varepsilon}{k}\overline{u_i u_j} + C_2^{(w)}\phi_{ij}^{(2)}, \qquad f = \frac{k^{3/2}/\varepsilon}{C_l y_n}, \tag{98}$$

$$C_1 = 1.8, \quad C_2 = 0.6, \quad C_1(w) = 0.5, \quad C_2(w) = 0.3, \quad C_l = 2.5. \tag{99}$$

Dissipation

$$\varepsilon_{ij} = \frac{2}{3}\delta_{ij}\varepsilon. \tag{100}$$

Dissipation Equation

$$C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92, \quad S_l = 0. \tag{101}$$

Since this is a high-Re model, $f_1 = f_2 = 1$ and $S_\varepsilon = 0$.

7.4.3 Craft and Launder

This model is exactly like the one by Gibson and Launder [12], except:

$$\tilde{\Phi}_{ij} = C_1(w)\frac{\varepsilon}{k}\overline{u_i u_j} + \left(0.04P_{kk} - \frac{0.4}{3}k\frac{\partial U_n}{\partial x_n}\right)\delta_{ij} - 0.1ka_{jk}\frac{\partial U_i}{\partial x_k}. \tag{102}$$

7.4.4 Speziale, Sarkar and Gatski - SSG Model

The model developed by Speziale, Sarkar and Gatski [13] is described below.

Pressure-Strain

$$\Phi_{ij} = \Phi_{ij}^{(1)} + \Phi_{ij}^{(2)}, \tag{103}$$

where

$$\Phi^{(1)}/\varepsilon = -C_1\mathbf{a} - {}'C_1\left(\mathbf{a}^2 - \frac{1}{3}a_2\mathbf{I}\right), \tag{104}$$

$$\Phi^{(2)}/\varepsilon = C_{01}\left(\mathbf{s} - \frac{1}{3}s_1\mathbf{I}\right) + C_{11}\left(\mathbf{sa} + \mathbf{as} - \frac{2}{3}\{\mathbf{as}\}\mathbf{I}\right) + C_{12}\left(\mathbf{wa} - \mathbf{aw}\right), \tag{105}$$

$$\begin{aligned} C_1 &= 1.7 + 0.9\frac{P}{\varepsilon}, \\ C_1' &= -1.05, \\ C_{01} &= 0.8 - 0.65a_2^{1/2}, \\ C_{11} &= 0.625, \\ C_{12} &= 0.2. \end{aligned} \tag{106}$$

Note that there is no wall-reflection term. Other modelling is as for the Gibson and Launder [12] model, with the exception that $C_{\varepsilon 2} = 1.83$. Note that the original SSG paper discussed only homogeneous turbulence, leaving diffusion to be modelled independently.

7.4.5 Jakirlic

The Reynolds stress model by Jakirlic [14] employs the following.

Diffusion

$$d_{ijk} = \left(\mu\delta_{kl} + C_s \frac{\rho k \overline{u_k u_l}}{\varepsilon}\right) \frac{\partial}{\partial x_l}\left(\overline{u_i u_j}\right), \tag{107}$$

$$d_k^{(\varepsilon)} = \left(\mu\delta_{kl} + C_\varepsilon \frac{\rho k \overline{u_k u_l}}{\varepsilon}\right) \frac{\partial \varepsilon}{\partial x_l}, \tag{108}$$

$$C_s = 0.22, \qquad C_\varepsilon = 0.18. \tag{109}$$

Pressure-Strain

$$\Phi_{ij} = \Phi_{ij}^{(1)} + \Phi_{ij}^{(2)} + \Phi_{ij}^{(w)}, \tag{110}$$

where

$$\Phi_{ij}^{(1)} = -C_1 \varepsilon a_{ij}, \tag{111}$$

$$\Phi_{ij}^{(2)} = -C_2\left(P_{ij} - \frac{1}{3}P_{kk}\delta_{ij}\right), \tag{112}$$

$$\Phi_{ij}^{(w)} = \left(\tilde{\Phi}_{kl} n_k n_l \delta_{ij} - \frac{3}{2}\tilde{\Phi}_{jk} n_i n_k\right) f, \tag{113}$$

$$\tilde{\Phi}_{ij} = C_1^{(w)} \frac{\varepsilon}{k}\overline{u_i u_j} + C_2^{(w)} \phi_{ij}^{(2)}, \qquad f = \min\left(\frac{k^{3/2}/\varepsilon}{C_l y_n}, 1.4\right), \tag{114}$$

$$C_1 = C + A^{1/2}E^2, \qquad C_2 = 0.8A^{1/2}, \tag{115}$$

$$C_1^{(w)} = \max\left(1 - 0.7C, 0.3\right), \qquad C_2(w) = \min\left(A, 0.3\right), \tag{116}$$

$$C_l = 2.5, \qquad C = 2.5 A f_{a_2}^{1/4} f_{R_1}, \qquad f_{a_2} = \min\left[\left(\frac{R_t}{150}\right)^{3/2}, 1\right], \tag{117}$$

where

$$
\begin{aligned}
A &= 1 - \frac{9}{8}\left(a_2 - a_3\right), \\
E &= 1 - \frac{9}{8}\left(e_2 - e_3\right), \\
e_{ij} &= \frac{\varepsilon_{ij}}{\varepsilon} - \frac{2}{3}\delta_{ij}
\end{aligned}
\tag{118}
$$

Dissipation

$$
\varepsilon_{ij} = f_\varepsilon \frac{2}{3}\varepsilon\delta_{ij} + \left(1 + f_\varepsilon\right)\varepsilon_{ij}^*, \tag{119}
$$

where

$$
\varepsilon_{ij}^* = \frac{\varepsilon}{k} \frac{\overline{u_i u_j} + \left(\overline{u_i u_k} n_j n_k + \overline{u_j u_k} n_i n_k + \overline{u_k u_l} n_k n_l n_i n_j\right) f_d}{1 + \frac{3}{2}\left(\overline{u_p u_q}/k\right) n_p n_q f_d} \tag{120}
$$

$$
f_\varepsilon = A^{1/2} E^2, \qquad f_d = \frac{1}{1 + 0.1 R_t}. \tag{121}
$$

Dissipation Equation

$$
C_{\varepsilon 1} = 1.44, \qquad C_{\varepsilon 2} = 1.92, \qquad S_l = 0 \tag{122}
$$

$$
\begin{aligned}
f_1 = 1, \quad f_2 &= \left[1 - \frac{C_{\varepsilon 2} - 1}{C_{\varepsilon 2}} e^{-(R_t/6)^2}\right] \frac{\tilde{\varepsilon}}{\varepsilon}, \\
S_\varepsilon &= 0.25\nu \frac{k}{\varepsilon} \overline{u_j u_k} \frac{\partial^2 U_i}{\partial x_j \partial x_l} \frac{\partial^2 U_i}{\partial x_k \partial x_l},
\end{aligned}
\tag{123}
$$

where

$$
\tilde{\varepsilon} = \max\left(\varepsilon - 2\nu\left(\frac{\partial k^{1/2}}{\partial x_i}\right)^2, 0\right) \tag{124}
$$

7.4.6 Shima

Shima's model [15] employs the following terms:

Diffusion

$$d_{ijk} = \left(\mu\delta_{kl} + C_s \frac{\rho k\overline{u_k u_l}}{\varepsilon}\right) \frac{\partial}{\partial x_l}\left(\overline{u_i u_j}\right), \tag{125}$$

$$d_k^{(\varepsilon)} = \left(\mu\delta_{kl} + C_\varepsilon \frac{\rho k\overline{u_k u_l}}{\varepsilon}\right) \frac{\partial \varepsilon}{\partial x_l}, \tag{126}$$

$$C_s = 0.22, \qquad C_\varepsilon = 0.18. \tag{127}$$

Pressure-Strain

$$\Phi_{ij} - \varepsilon_{ij} = \Phi_{ij}^{(1)} + \Phi_{ij}^{(2)} - \frac{2}{3}\varepsilon\delta_{ij}, \tag{128}$$

where

$$\Phi_{ij}^{(1)} = -C_1 \varepsilon a_{ij}, \tag{129}$$

$$\begin{aligned}\Phi_{ij}^{(2)} = & - C_2\left(P_{ij} - \frac{1}{3}P_{kk}\delta_{ij}\right) - C_3\left(D_{ij} - \frac{1}{3}D_{kk}\delta_{ij}\right) \\ & - C_4 k\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right),\end{aligned} \tag{130}$$

where

$$\begin{aligned} D_{ij} &= -\left(\overline{u_i u_k}\frac{\partial U_k}{\partial x_j} + \overline{u_j u_k}\frac{\partial U_k}{\partial x_i}\right), \\ C_1 &= 1 + 2.45 a_2^{1/4} A^{3/4}\left(1 - e^{-(7A)^2}\right)\left(1 - e^{-(R_t/60)^2}\right), \quad (131) \\ C_2 &= 0.7A, C_3 = 0.3A^{1/2}, \\ C_4 &= 0.65A\left(0.23C_1 + C_2 - 1\right) + 1.3a_2^{1/4}, \end{aligned}$$

and

$$A = 1 - \frac{9}{8}\left(a_2 - a_3\right). \tag{132}$$

The fast part of the pressure-strain can also be written in the tensorial form

$$\Phi^{(2)}/\varepsilon = C_{01}\left(\mathbf{s} - \frac{1}{3}s_1\mathbf{I}\right) + C_{11}\left(\mathbf{sa} + \mathbf{as} - \frac{2}{3}\{\mathbf{as}\}\mathbf{I}\right) + C_{12}\left(\mathbf{wa} - \mathbf{aw}\right), \tag{133}$$

where

$$C_0 1 = \frac{4}{3}\left(C_2 + C_3\right) - C_4, \quad C_{11} = C_2 + C_3, \quad C_{12} = C_2 - C_3. \tag{134}$$

Dissipation Equation

$$C_{\varepsilon 1} = 1.44 + \beta_1 + \beta_2, \qquad C_{\varepsilon 2} = 1.92, \qquad S_l = 0, \tag{135}$$

where

$$\beta_1 = 0.25A \min\left(\frac{\lambda}{2.5} - 1, 0\right) - 1.4A \min\left(\frac{P}{\varepsilon} - 1, 0\right),$$

$$\beta_2 = 1.0A\lambda^2 \max\left(\frac{\lambda}{2.5} - 1, 0\right), \tag{136}$$

$$\lambda = \min\left(|\nabla\left(\frac{k^{3/2}}{\varepsilon}\right)|, 4\right).$$

Also,

$$f_1 = 1, \qquad f_2 = \frac{\tilde{\varepsilon}}{\varepsilon}, \qquad S_\varepsilon = 0, \tag{137}$$

where

$$\tilde{\varepsilon} = \max\left(\varepsilon - 2\nu\left(\frac{\partial k^{1/2}}{\partial x_i}\right)^2, 0\right). \tag{138}$$

7.4.7 Wilcox - Multiscale Model

Wilcox's multiscale model [16] is based on the premise that one can partition the turbulence spectrum into large-scale energy-bearing eddies and small-scale, isotropic, dissipative eddies. The formulation is rather different from that of momentum Reynolds-stress models and consists of transport equations for k, ω as well as equations for the upper-partition stress tensor $T_{ij} \equiv -(\overline{u_i u_j} - 2/3e\delta_{ij})$ (where e is the turbulence energy of the lower partition eddies). In addition, a tensor describing the exchange of energy between upper and lower partitions is used. Transforming Wilcox' equations into transport equations for $\overline{u_i u_j}$, $k^U = k - e$ (the upper-partition turbulence energy) and ω, we obtain:

$$\frac{\partial}{\partial t}\left(\rho\overline{u_i u_j}\right) + \frac{\partial}{\partial x_k}\left(\rho U_k \overline{u_i u_j}\right) = \rho\left(P_{ij} + \Phi_{ij}\right) + \left(F - \rho\varepsilon\right)\frac{2}{3}\delta_{ij}, \tag{139}$$

$$\frac{\partial}{\partial t}\left(\rho k^U\right) + \frac{\partial}{\partial x_i}\left(\rho U_i k^U\right) = \rho\left[\left(1 - \hat{\alpha} - \hat{\beta}\right)P - \frac{\varepsilon}{k^{3/2}}\left(k^U\right)^{3/2}\right], \tag{140}$$

$$\frac{\partial}{\partial t}\left(\rho\omega\right) + \frac{\partial}{\partial x_j}\left(\rho U_j \omega\right) = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right] + \rho\left(\frac{\alpha}{\nu_t}P - \beta\omega^2\right) + \rho S_l. \tag{141}$$

where

$$k = 1/2\overline{u_i u_i}, \qquad \varepsilon = \beta^* \omega k, \qquad \nu_t \equiv \frac{\mu_t}{\rho} = \frac{k}{\omega} \tag{142}$$

$$\begin{aligned} \Phi_{ij} = -C_1 \varepsilon a_{ij} - \hat{\alpha}\left(P_{ij} - \frac{1}{3}\delta_{ij} P_{kk}\right) - \hat{\beta}\left(D_{ij} - \frac{1}{3}\delta_{ij} D_{kk}\right) \\ - \hat{\gamma} k \left(S_{ij} - \frac{1}{3} S_{kk}\delta_{ij}\right), \end{aligned} \tag{143}$$

$$F = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right], \qquad S_l = -\beta \omega \overline{w}, \tag{144}$$

and

$$\hat{\alpha} = \frac{42}{55}, \quad \hat{\beta} = \frac{6}{55}, \quad \hat{\gamma} = \frac{1}{4}, \quad C_l = 1 + 4\left(\frac{k^U}{k}\right)^{3/2}, \tag{145}$$

$$\beta^* = \frac{9}{10}, \quad \beta = \frac{3}{40}, \quad \gamma = \frac{4}{5}, \quad \sigma_k = \sigma_\omega = 2. \tag{146}$$

8 Non-Linear Models

8.1 2D Formulation

Reynolds Stress anisotropy

$$b_{ij} = \frac{\overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij}}{2k} \tag{147}$$

Dissipation tensor *is still isotropic*:

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon \delta_{ij} \tag{148}$$

The pressure-strain term follows the quasi-linear approach of all linear models.

$$
\begin{aligned}
\frac{\phi_{ij}}{\varepsilon} = & -\left(C_1 + C_1^* \frac{P}{\varepsilon}\right) b_{ij} \\
& + \left(c_2 - c_2^* \sqrt{\mathrm{II}_b}\right) S_{ij} \\
& + C_3 \left(b_{ik} S_{kj} + S_{ik} b_{kj} - \frac{2}{3} b_{kl} S_{kl} \delta_{ij}\right) \\
& - C_4 \left(b_{ik} \underline{O}_{kj} - \underline{O}_{ik} b_{kj}\right)
\end{aligned}
\tag{149}
$$

Define the dimensionless strain rate tensor as:

$$S_{ij} = \frac{1}{2}\tau \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{150}$$

Define the dimensionless voriticity tensor as:

$$\underline{O}_{ij} = \frac{1}{2}\tau \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{151}$$

τ is the maximum of turbulent and viscous time scales, e. g.

$$\tau = \max\left(\frac{k}{\varepsilon}; C_\tau \sqrt{\frac{\mu}{\rho\varepsilon}}\right) \tag{152}$$

or

$$\tau = \max\left(\frac{1}{\beta^*\omega}; C_\tau \sqrt{\frac{\mu}{\rho\beta^*\omega}}\right) \tag{153}$$

Note $C_\tau = 6.0$.

We still have an option of a linear (formulation 1) or quasi-linear (formulation 2) perssure-strain model for the pressure-strain term. Since the pressure-strain term for formulation 2 depends on $\mathrm{II}_b = b_{kl}b_{kl}$ (where b_{kl} is the second invariant of the Reynolds-stress anisotropy tensor), formulation 2 is not an Explicit Algebraic Reynolds Stress Model (EARSM).

	C_1	C_1^*	C_2	C_2^*	C_3	C_4
Formulation 1	3.60	0	0.80	0	2.00	$\frac{10}{9}$
Formulation 2	3.40	1.80	0.80	1.30	1.25	0.40

Both formulations are used for b_{ij}:

$$\begin{aligned} Nb_{ij} &= -A_1 S_{ij} + \left(b_{ik}\underline{O}_{kj} - \underline{O}_{ik} b_{kj}\right) \\ &- A_2 \left(b_{ik} S_{kj} + S_{ik} b_{kj} - \frac{2}{3} b_{kl} S_{kl} \delta_{ij}\right) \end{aligned} \tag{154}$$

$$N = A_3 + A_4 \frac{P}{\varepsilon} \tag{155}$$

For 2D:

$$\begin{aligned} b_{ij} &= \beta_1 S_{ij} \\ &+ \beta_2 \left(S_{ik} S_{kj} - \frac{1}{3} \mathrm{II}_s \delta_{ij}\right) \\ &+ \beta_4 \left(S_{ik}\underline{O}_{kj} - \underline{O}_{ik} \delta_{ij}\right) \end{aligned} \tag{156}$$

where β denote functions of invariants of S_{ij} and $\underline{O}_{ij}$.

For 2D only II_s and $\mathrm{II}_{\underline{O}}$ are independent:

$$\mathrm{II}_s = S_{kl} S_{kl} \tag{157}$$

$$\mathrm{II}_{\underline{O}} = \underline{O}_{kl} \underline{O}_{kl} \tag{158}$$

Use Caley-Hamilton Theorem as referenced by Wallin and Johansson [17]. The C-H theorem is used to seek a solution with $\mu = f\left(\frac{P}{\varepsilon}\right)$ and an equation for N is derived.

For 2D, this is a cubic equation and the solution can be found in closed form:

$$\beta_1 = -\frac{A_1 N}{2Q} \qquad \beta_2 = \frac{A_1 A_2}{Q} \qquad \beta_4 = -\frac{A_1}{2Q} \tag{159}$$

$$Q = N^2 - 2\mathrm{II}_{\underline{O}} - \frac{2}{3} A_2^2 \mathrm{II}_s \tag{160}$$

$$N = \begin{cases} N_1 = \frac{A_3}{3} + \left(P_1 + \sqrt{P_2}\right)^{1/3} + sign\left(P_1 - \sqrt{P_2}\right) \left|\left(P_1 - \sqrt{P_2}\right|^{1/3} \right. \\ N_2 = \frac{A_3}{3} + 2\left(P_1^2 + P_2^2\right)^{1/6} \cdot \cos\left(\frac{1}{3} \arccos\left(\frac{P_1}{\sqrt{P_1^2 - P_2^2}}\right)\right) \end{cases} \tag{161}$$

where

$$P_1 = \left(\frac{A_3^2}{27} + \left(\frac{A_1 A_4}{6} - \frac{2}{9}A_2^2\right)\mathrm{II}_s - \frac{2}{3}\mathrm{II}_{\underline{\Omega}}\right) A_3 \tag{162}$$

$$P_2 = P_1^2 - \left(\frac{A_3^2}{9} + \left(\frac{A_1 A_4}{3} - \frac{2}{9}A_2^2\right)\mathrm{II}_s - \frac{2}{3}\mathrm{II}_{\underline{\Omega}}\right)^3 \tag{163}$$

$$A_1 = \frac{8/3 - 2C_1C_2 - C_2^*\sqrt{\mathrm{II}_b}}{2 - C_4} \tag{164}$$

$$A_2 = \frac{2 - C_3}{2 - C_4} \tag{165}$$

$$A_3 = \frac{C_1 - 2}{2 - C_4} \tag{166}$$

$$A_4 = \frac{C_1^* + 2}{2 - C_4} \tag{167}$$

<u>**Formulation 1**</u>:

If $\mathbf{C_3 = 0} \quad \Rightarrow \mathbf{A_2 = 0} \quad \Rightarrow \beta_2 = \mathbf{0} \quad \Rightarrow \mathbf{b_{ij}} = \mathbf{b_{ij}}\left(S_{ij}, \underline{\Omega}_{ij}\right)$ only
Not a function of II_b (second anisotropy invariant, where $\mathrm{II}_b = b_{kl}b_{kl}$).

8.2 3D Formulation

Starting from the transport equation for the α_{ij} tensor:

$$\alpha_{ij} = \frac{\overline{u_i' u_j'}}{k} - \frac{2}{3}\delta_{ij} \tag{168}$$

$$\left[\frac{k}{\varepsilon}\frac{D\alpha_{ij}}{Dt}\right]\left\{-\frac{1}{\varepsilon}\left(\frac{\partial \overline{u_i' u_j' u_k'}}{\partial xl} - \frac{\overline{u_i' u_j'}}{k}\frac{\partial u_{kl}}{\partial xl}\right)\right\} = -\frac{\overline{u_i' u_j'}}{k}\left(\frac{P}{\varepsilon} - 1\right) + \frac{P_{ij}}{\varepsilon} - \frac{\varepsilon_{ij}}{\varepsilon} + \frac{\phi_{ij}}{\varepsilon} \tag{169}$$

$$\frac{P_{ij}}{2} = P \tag{170}$$

The terms surrounded by the square brackets $[\cdots]$ vary slowly so the terms surrounded by the curly brackets equals zero, i. e. $\{\cdots\} = 0$, and the equation for the α_{ij} tensor is:

$$\frac{\overline{u_i' u_j'}}{k}\left(\frac{P}{\varepsilon} - 1\right) = \frac{P_{ij}}{\varepsilon} - \frac{\varepsilon_{ij}}{\varepsilon} + \frac{\phi_{ij}}{\varepsilon} \tag{171}$$

where $\frac{\varepsilon_{ij}}{\varepsilon}$ denotes the Dissipation term.
$\frac{\phi_{ij}}{\varepsilon}$ denotes the Redistribution term.

Note both the dissipation and redistribution terms mentioned above require modelling. Also P_{ij} is usually represented as:

$$P_{ij} = -\overline{u_j' u_k'}\frac{\partial u_j}{\partial x_k} - \overline{u_j' u_k'}\frac{\partial u_i}{\partial x_k} \tag{172}$$

And as before:

$$S_{ij} = \frac{\tau}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{173}$$

$$\underline{O}_{ij} = \frac{\tau}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{174}$$

$$\tau \equiv \frac{k}{\varepsilon} \tag{175}$$

Rewrite the Production term $\frac{P_{ij}}{\varepsilon}$ as:

$$\frac{P_{ij}}{\varepsilon} = -\frac{4}{3}S_{ij} - (\alpha_{ik}S_{kj} + S_{ik}\alpha_{kj}) + (\alpha_{ik}\underline{O}_{kj} + \underline{O}_{kj}\alpha_{kj}) \tag{176}$$

$$\frac{\varepsilon_{ij}}{\varepsilon} = \frac{2}{3}\delta_{ij} \tag{177}$$

The $\frac{\phi_{ij}}{\varepsilon}$ term is modelled in 2 parts corresponding to slow (denoted as 's') and fast (denoted as 'r' for rapid) redistribution rates:

$$\frac{\phi_{ij}^{(s)}}{\varepsilon} = -C_1\alpha_{ij} \qquad \text{where } C_1 \text{ is a constant} \tag{178}$$

$$\frac{\phi_{ij}^{(r)}}{\varepsilon} = -\left(\frac{C_2+8}{11}\right)\left(P_{ij} - \frac{2}{3}\delta_{ij}P\right) - \left(\frac{30C_2-2}{55}\right)k\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \left(\frac{8C_2-2}{11}\right)\left(D_{ij} - \frac{2}{3}\delta_{ij}P\right) \tag{179}$$

[Refer to Launder, Reece and Rodi [18] for above equation.]

where C_2 is a constant and D_{ij} is defined as:

$$D_{ij} = -\overline{u_i'u_k'}\frac{\partial u_k}{\partial x_j} - \overline{u_j'u_k'}\frac{\partial u_k}{\partial x_i} \tag{180}$$

Use S_{ij}, $\underline{O}_{ij}$ and α_{ij} in $\frac{\phi_{ij}^{(r)}}{\varepsilon}$ like for the $\frac{P_{ij}}{\varepsilon}$ case.

$$\frac{\phi_{ij}^{(r)}}{\varepsilon} = \frac{4}{5}S_{ij} + \frac{(9C_2+6)}{11}\left(\alpha_{ij}S_{kj} + S_{ik}\alpha_{kj} - \frac{2}{3}\alpha_{km}S_{mk}\delta_{ij}\right) + \frac{(7C_2-10)}{11}\left(\alpha_{ik}\underline{O}_{kj} - \underline{O}_{ik}\alpha_{kj}\right) \tag{181}$$

Going back to the equation for the anisotropy tensor and using equations 176, 177 and 181 we get:

$$\left(C_1 - 1 + \frac{P}{\varepsilon}\right)\alpha_{ij} = -\frac{8}{15}S_{ij} + \frac{(7C_2+1)}{11}\left(\alpha_{ik}\underline{O}_{kj}^R - \underline{O}_{ik}^R\alpha_{kj}\right) - \frac{(5-9C_2)}{11}\left(\alpha_{ik}S_{kj} + S_{ik}\alpha_{kj} - \frac{2}{3}\delta_{ij}\alpha_{ik}S_{ki}\right) \tag{182}$$

where $\underline{O}_{ij}^R$ is the effective mean voriticity tensor. $\underline{O}_{ij}^R = \underline{O}_{ij}$ in the absence of rotation of the co-ordinate system.

Equally, for ε use a transport equation like the one for the 2-equation linear Eddy Viscosity Model (EVM).

Since by definition:

$$\frac{P}{\varepsilon} \equiv -\alpha_{ik}S_{ki} \tag{183}$$

The result of equation (182) is a non-linear equation for α_{ij} provided by S_{ij}, $\underline{O}_{ij}^R$ are known. The idea behind the Explicit Algebraic Reynolds Stress Models (EARSMs) is to simplify equation (182) so that α_{ij} can be calculated explicitly.

One may see equation (182) as:

$$L_{ij}\left(\alpha_{kl}, S_{kl}, \underline{O}_{kl}^R, \frac{P}{\varepsilon}\right) = 0 \tag{184}$$

considering the ratio $\frac{P}{\varepsilon}$ as a separate parameter.

The formal solution uses the Caley-Hamilton theorem and gives α_{ij} as a function of 10 tensorially independent groups. Among others, Wallin and Johansson [17] have presented a form of the solution as:

$$\begin{aligned}
\alpha = {} & \beta_1 S \\
& + \beta_2\left(S^2 - \frac{1}{3}\mathrm{II}_s I\right) \\
& + \beta_3\left(\underline{O}^2 - \frac{1}{3}\mathrm{II}_{\underline{O}} I\right) \\
& + \beta_4\left(S\underline{O} - \underline{O}S\right) \\
& + \beta_5\left(S^2\underline{O} - \underline{O}S^2\right) \\
& + \beta_6\left(S\underline{O}^2 + \underline{O}^2 S - \frac{2}{3}\mathrm{IV} I\right) \\
& + \beta_7\left(S^2\underline{O}^2 + \underline{O}^2 S^2 - \frac{2}{3}\mathrm{IV} I\right) \\
& + \beta_8\left(S\underline{O}S^2 - S^2\underline{O}S\right) \\
& + \beta_9\left(\underline{O}S\underline{O}^2 - \underline{O}^2 S\underline{O}\right) \\
& + \beta_{10}\left(\underline{O}S^2\underline{O}^2 - \underline{O}^2 S^2\underline{O}\right)
\end{aligned} \tag{185}$$

where	$\beta_1(\cdots)$	is the Linear term,
	$\beta_2(\cdots)$, $\beta_3(\cdots)$	are the Quadratic terms,
	$\beta_4(\cdots)$, $\beta_5(\cdots)$, $\beta_6(\cdots)$	are the Cubic terms,
	$\beta_7(\cdots)$, $\beta_8(\cdots)$, $\beta_9(\cdots)$, $\beta_{10}(\cdots)$	are the Higher Order terms.

And:

$$
\begin{aligned}
\alpha &= \alpha_{ij} \\
S &= S_{ij} \\
\underline{O} &= \underline{O}_{ij} \\
(SS)_{ij} \equiv \left(S^2\right)_{ij} &= S_{ik}S_{kj} \\
I &\equiv \text{Identity Matrix} \\
\mathrm{II}_s &= \text{trace}\left(S^2\right) = S_{ij}S_{ji} \\
\mathrm{II}_{\underline{O}} &= \text{trace}\left(\underline{O}^2\right) = \underline{O}_{ij}\underline{O}_{ji} \\
\mathrm{III} &= \text{trace}\left(S^3\right) = S_{ij}S_{jk}S_{ki} \\
\mathrm{IV} &= \text{trace}\left(S\underline{O}^2\right) = S_{ij}\underline{O}_{j}\underline{O}_{ki} \\
\mathrm{V} &= \text{trace}\left(S^2\underline{O}^2\right) = S_{ij}S_{jk}\underline{O}_{kl}\underline{O}_{li}
\end{aligned}
$$

C_2 in the rapid redistribution (model) term is between 0.4 and $\frac{5}{9}$. If C_2 is set to $\frac{5}{9}$, the last term in equation (182) becomes zero and equation (182) becomes:

$$\left(C_1 - 1 + \frac{P}{\varepsilon}\right)\alpha = -\frac{8}{15}S + \frac{4}{9}\left(\alpha\underline{O} - \underline{O}\alpha\right) \tag{186}$$

Equation (186) above is much simpler than equation (182) especially for 3D cases.

To simply equation (186), multiply by 9/4:

$$
\begin{aligned}
(186) \;*\; \frac{9}{4} \Rightarrow N\alpha &= -\frac{6}{5}S + \left(\alpha\underline{O} + \underline{O}\alpha\right) \\
N &= C_1' + \frac{9}{4}\frac{P}{\varepsilon} \\
C_1' &= \frac{9}{4}\left(C_1 - 1\right)
\end{aligned}
\tag{187}
$$

where C_1 is called the Rotta Coefficient and $C_1 = 1.8$.

Equation (187) is linear in α with a slight non-linearity in N. Since equation (185) is a solution of equation (187), we can replace α from equation (185) into equation (186) and solve for the coefficients β_i. The solution uses the Caley-Hamilton theorem but β_i's are functions of N. The obtained β_i's can then be used back in equation (185) and thus results in a 6-th order equation for N for the general case of 3D flow. The β_i's are also used to replace α_{ij} in equation (185) so that finally a non-linear equation for N is obtained (again this is of order 6).

We now have 3 options:

1. Simplify for 2D mean flow.
2. Simplify for 3D mean flows.
3. Solve without simplifications.

8.2.1 Simplication for 2D Mean Flows

$$\left\{ \begin{array}{ll} \beta_1 = -\frac{6}{5}\left(\frac{N}{N^2-2\mathrm{II}_{\underline{\Omega}}}\right) & \text{are the only} \\ \beta_4 = -\frac{6}{5}\left(\frac{1}{N^2-2\mathrm{II}_{\underline{\Omega}}}\right) & \text{non-zero coefficients} \end{array} \right. \tag{188}$$

The equation for N is:

$$N^3 - C_1' N^2 - \left(\frac{27}{10}\mathrm{II}_s + 2\mathrm{II}_{\underline{\Omega}}\right) N + 2C_1'\mathrm{II}_{\underline{\Omega}} = 0 \tag{189}$$

with a solution:

$$N = \left\{ \begin{array}{l} \frac{C_1'}{3} + \left(P_1 + \sqrt{P_1}\right)^{1/3} + \left(P_1 - \sqrt{P_1}\right)^{1/3} \operatorname{sign}\left(P_1 - \sqrt{P_2}\right), P_2 \geq 0 \\ \frac{C_1'}{3} 2 + \left(P_1^2 + P_2\right) \cos\left[\frac{1}{3}\arccos\left(\frac{P_1}{\sqrt{P_1^2 - P_2}}\right)\right], P_2 \leq 0 \end{array} \right. \tag{190}$$

$$P_1 = \left(\frac{C_1^2}{27} + \frac{9}{20}\mathrm{II}_s - \frac{2}{3}\mathrm{II}_{\underline{\Omega}}\right) C_1' \tag{191}$$

$$P_2 = P_1^2 - \left(\frac{C_1^2}{9} + \frac{9}{10}\mathrm{II}_s + \frac{2}{3}\mathrm{II}_{\underline{\Omega}}\right)^3 \tag{192}$$

Note N is real and positive $\forall\ \mathrm{II}_s, \mathrm{II}_{\underline{\Omega}}$. Once N is known, $\frac{P}{\varepsilon}$ is obtained from $N = C_1' + \frac{9}{4}\frac{P}{\varepsilon}$.

8.2.2 Simplification for 3D Mean Flows

$$\beta_1 = -\frac{N}{Q}\left(2N^2 - 7\mathrm{II}_{\underline{Q}}\right) \tag{193}$$

$$\beta_3 = -\frac{\left(12N^{-1}\mathrm{IV}\right)}{Q} \tag{194}$$

$$\beta_4 = -\frac{2\left(12N^2 - 2\mathrm{II}\right)}{Q} \tag{195}$$

$$\beta_6 = -\frac{6N}{Q} \tag{196}$$

$$\beta_9 = \frac{6}{Q} \tag{197}$$

$$Q = \frac{5}{6}\left(N^2 - 2\mathrm{II}_{\underline{Q}}\right)\left(2N^2 - \mathrm{II}_{\underline{Q}}\right) \tag{198}$$

Note: The rest of β_i's coefficients are zero.

The equation for N is:

$$\begin{aligned} N^6 - C_1' N^5 - \left(\frac{27}{10}\mathrm{II}_s + \frac{5}{2}\mathrm{II}_{\underline{Q}}\right) N^4 \\ +\frac{5}{2}C_1'\mathrm{II}_{\underline{Q}}N^3 + \left(\mathrm{II}_{\underline{Q}}^2 + \frac{189}{20}\mathrm{II}_s\mathrm{II}_{\underline{Q}} - \frac{81}{5}V\right) N^2 \\ -C_1'\mathrm{II}_{\underline{Q}}^2 N - \frac{81}{5}\mathrm{IV}^2 = 0 \end{aligned} \tag{199}$$

No solution in closed form.

In 2D:

$$\mathrm{III}_s = \mathrm{IV} = 0 \tag{200}$$

and

$$\mathrm{V} \equiv \frac{\mathrm{II}_s\mathrm{II}_{\underline{Q}}}{2} \tag{201}$$

So let's use this in equation (199) to simplify and obtain:

$$N_a^3 - C_1' N_a^2 - \left(\frac{27}{10}\mathrm{II}_s - 2\mathrm{II}_{\underline{Q}}\right) + 2C_1'\mathrm{II}_P\underline{Q} = 0 \tag{202}$$

where subscript a represents an approximate value.

In addition, use:

$$\mathrm{IV} = \sqrt{\phi_1} \tag{203}$$

$$\mathrm{V} = \frac{1}{2}\mathrm{II}_s\mathrm{II}_{\underline{Q}} + \phi_2 \tag{204}$$

Both of the above are small perturbations.

Use the above two equations to improve:

$$N = N_\alpha + \frac{162\left(\phi_1 + \phi_2 N_a^2\right)}{D} + O\left(\phi_1^2, \phi_2^2, \phi_1, \phi_2\right) \tag{205}$$

$$D = 20N_\alpha^4\left(N_\alpha - \frac{1}{2}C_1'\right) - \mathrm{II}_{\underline{Q}}\left(10N_c^3 + 15C_1'N_c^2\right) + 10C_1'\mathrm{II}_{\underline{Q}}^2 \tag{206}$$

D is always greater than zero since:

$$N_a \geq C_1' \tag{207}$$

and

$$\mathrm{II}_{\underline{Q}} \leq 0 \tag{208}$$

Again use $N_1 = C_1' + \frac{9}{4}\frac{P}{\varepsilon}$ to get $\frac{P}{\varepsilon}$ once N is known.

8.2.3 General Case: No Simplifications

If $C_2 = \frac{5}{9}$ we have equation (186):

$$\left(C_1 - 1 + \frac{P}{\varepsilon}\right)\alpha = -\frac{8}{15}S + \frac{4}{9}\left(\alpha\underline{Q} - \underline{Q}\alpha\right) \tag{209}$$

but even without it we may write:

$$N_\alpha = -A_1 S + (\alpha \underline{Q} - \underline{Q}\alpha) - A_2\left(\alpha S + S\alpha - \frac{2}{3}\text{trace}\,(\alpha S)\,\mathrm{I}\right) \tag{210}$$

$$N = A_3 + A_4\frac{P}{\varepsilon} \tag{211}$$

$$A_1 = \frac{88}{11\,(7C_2+1)} \tag{212}$$

$$A_2 = \frac{5-9C_2}{7C_2+1} \tag{213}$$

$$A_3 = \frac{11\,(C_1-1)}{7C_2+1} \tag{214}$$

$$A_4 = \frac{11}{7C_2+1} \tag{215}$$

The values of the coefficients for the general Algebraic Reynolds Stress Model (ARSM) case are given below.

	New $\begin{pmatrix} C_1 = 1.8 \\ C_2 = 5/9 \end{pmatrix}$	**Original LRR** $\begin{pmatrix} C_1 = 1.5 \\ C_2 = 0.4 \end{pmatrix}$	**Linearized SSG**	**Gatski-Speziale**
$\mathbf{A_1}$	1.20	1.54	1.22	1.22
$\mathbf{A_2}$	0	0.37	0.47	0.47
$\mathbf{A_3}$	1.8	1.45	0.88	5.36
$\mathbf{A_4}$	2.25	2.89	2.37	0

According to Wallin-Johansson [17],

$$N\beta_\lambda = -A_1\delta_{1\lambda} + \sum_\gamma J_{\lambda\gamma}\beta_\gamma - A_2\sum_\gamma H_{\lambda\gamma}\beta\gamma \tag{216}$$

or

$$\left(N\delta_{\gamma\lambda} - J_{\gamma\lambda} + A_2 H_{\gamma\lambda}\right)\beta_\lambda = -A_1\delta_{1\lambda} \tag{217}$$

where

$$\mathbf{H} = \begin{bmatrix}
0 & \frac{1}{3}\mathrm{II}_s & -\frac{2}{3}\mathrm{II}_{\underline{Q}} & 0 & 0 & \frac{2}{3}\mathrm{IV} & -\frac{1}{3}\mathrm{V} & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 2\mathrm{II}_{\underline{Q}} & \mathrm{IV} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathrm{II}_s & \frac{1}{3}\mathrm{III}_s & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}\mathrm{II}_s & 0 & 0 & \frac{1}{3}\mathrm{III}_s & -\mathrm{IV} & -\frac{1}{3}V - \frac{1}{6}\mathrm{II}_s\mathrm{II}_{\underline{Q}} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \mathrm{II}_{\underline{Q}} & -\frac{2}{3}\mathrm{IV} \\
0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2}\mathrm{II}_s & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \frac{1}{3}\mathrm{II}_{\underline{Q}} \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -\frac{1}{3}\mathrm{II}_s \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0
\end{bmatrix} \tag{218}$$

$$\mathbf{J} = \begin{bmatrix}
0 & 0 & 0 & -\mathrm{II}_{\underline{Q}} & 0 & 0 & 0 & 2V - \mathrm{II}_s\mathrm{II}_{\underline{Q}} & \mathrm{II}_{\underline{Q}}^2 & 0 \\
0 & 0 & 0 & 0 & -2\mathrm{II}_{\underline{Q}} & 0 & 0 & -2\mathrm{IV} & 0 & \mathrm{II}_{\underline{Q}}^2 \\
0 & 0 & 0 & 0 & -2\mathrm{II}_s & 0 & 0 & 0 & -2\mathrm{IV} & 2\mathrm{II}_s\mathrm{II}_{\underline{Q}} - 2V \\
1 & 0 & 0 & 0 & 0 & \frac{1}{2}\mathrm{II}_{\underline{Q}} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2}\mathrm{II}_{\underline{Q}} & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & \mathrm{II}_s & -2\mathrm{II}_{\underline{Q}} & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -2\mathrm{II}_{\underline{Q}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix} \tag{219}$$

$$\beta_1 = -\frac{A_1 N}{2Q}\left(30A_2\mathrm{IV} - 21N\mathrm{II}_{\underline{Q}} - 2A_2^3\mathrm{III}_s + 6N^3 - 3A_2^2\mathrm{II}_s N\right) \tag{220}$$

$$\beta_2 = -\frac{A_1 A_2}{Q}\left(6A_2\mathrm{IV} + 12N\mathrm{II}_{\underline{Q}} + 2A_2^3\mathrm{III}_s - 6N^3 + 3A_2^2\mathrm{II}_s N\right) \tag{221}$$

$$\beta_3 = -\frac{3A_1}{Q}\left(2A_2^2\mathrm{III}_s + 3NA_2\mathrm{II}_s + 6\mathrm{IV}\right) \tag{222}$$

$$\beta_4 = -\frac{A_1}{Q}\left(2A_2^3\mathrm{III}_s + 3A_2^2\mathrm{II}_s N + 6A_2\mathrm{IV} - 6N\mathrm{II}_{\underline{Q}} + 3N^3\right) \tag{223}$$

$$\beta_5 = -\frac{9A_1A_2N^2}{Q} \tag{224}$$

$$\beta_6 = -\frac{9A_1N^2}{Q} \tag{225}$$

$$\beta_7 = \frac{18A_1A_2N}{Q} \tag{226}$$

$$\beta_8 = \frac{9A_1A_2^2N}{Q} \tag{227}$$

$$\beta_9 = \frac{9A_1N}{Q} \tag{228}$$

$$\beta_{10} = 0 \tag{229}$$

$$\begin{aligned} Q = {} & 3N^5 + \left(-\frac{15}{2}\mathrm{II}_{\underline{\Omega}} - \frac{7}{2}A_2^2\mathrm{II}_s\right)N^3 + \left(21A_2\mathrm{IV} - A_2^3\mathrm{III}_s\right)N^2 \\ & + \left(3\mathrm{II}_{\underline{\Omega}}^2 - 8\mathrm{II}_s\mathrm{II}_{\underline{\Omega}}A_2^2 + 24A_2^2V + A_2^4\mathrm{II}_s^2\right)N + \frac{2}{3}A_2^5\mathrm{II}_s\mathrm{III}_s \\ & + 2A_2^3\mathrm{II}_s\mathrm{IV} - 2A_2^3\mathrm{II}_{\underline{\Omega}}\mathrm{II}_s - 6A_2\mathrm{II}_{\underline{\Omega}}\mathrm{IV} \end{aligned} \tag{230}$$

8.3 Model Equations

8.3.1 Non-Linear k-ε Models

	Low-Re
Speziale (1987)	No
Rubinstein and Barton (1992)	No
Shih et al. (1993)	No
Shih et al. (1995)	No
Gatski and Speziale (1993)	No
Suga (1995)	Yes
Lien et al. (1996)	Yes
Apsley and Leschziner (1998)	Yes

Non-linear stress-strain relationship

	$\mathbf{C}_\mu$	$(\beta_1, \beta_2, \beta_3)$
Speziale (1987)[1]	0.09	(0.054,0.054,0)
Rubinstein and Barton (1992)	0.085	(0.230,0.047,0.189)
Shih *et al* (1993)	$\frac{2/3}{1.25+\overline{s}+0.9\overline{w}}$	$\frac{1}{1000+\overline{s}^3}(3, 15, -19)$
Shih *et al* (1995)[2]	$\frac{1}{6.5+A_s^*\sigma^*}$	$\frac{\left(1-9C_\mu^2 s_2\right)^{1/2}}{1+6[s_2(-w_2)]^{1/2}}(0, 2, 0)$
Gatski and Speziale (1993)[3]	$1/2\alpha_1 C_1^*$	$(\alpha_3 C_3^*, \alpha_2 C_2^*, 0)$
Suga (1995)[4]	$\frac{0.3\left(1-e^{-.036e^{0.75\eta}}\right)}{1+0.35\eta^{3/2}}$	$C_\mu f_m u\,(-0.4, 0.4, -1.04)$
Lien *et al* (1996b)	$\frac{2/3}{1.25+\overline{s}+0.9\overline{w}}$	$\frac{f_\mu}{1000+\overline{s}^3}(3, 15, -19)$
Apsley and Leschnizer (1998)[6]	$\frac{\left(-a_{12}^*\right)}{1+\overline{\beta}^2/3-\overline{\gamma}^2}\frac{f_P}{\sigma^*}$	$\left(\frac{f_P}{\sigma^*}\right)^2\left(6\left(a_{11}^*+a_{22}^*\right), a_{11}^*-a_{22}^*, 0\right)$

Non-linear stress-strain relationship (Continued)

	$(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$
Speziale (1987)[1]	(0,0,0,0)
Rubinstein and Barton (1992)	(0,0,0,0)
Shih *et al* (1993)	(0,0,0,0)
Shih *et al* (1995)[2]	(0,0,0,0)
Gatski and Speziale (1993)[3]	(0,0,0,0)
Suga (1995)[4]	$C_\mu^3 f_m u\,(40, 40, 0, -80)$
Lien *et al* (1996b)	$C_\mu^3 f_m u\,(16, 16, 0, -80)$
Apsley and Leschnizer (1998)[6]	$4C_\mu\left(\frac{f_P}{\sigma^*}\right)^2\left(\frac{1}{3}\overline{\beta}^2, \overline{\gamma}^2, \frac{3}{2}\overline{\gamma}^2, \frac{3}{2}\overline{\beta}\overline{\gamma}\right)$

Turbulence Transport Equations

	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	σ_k	σ_ε	S_l
Speziale	1.44	1.92	1.0	1.3	0
Rubinstein and Barton	1.42	1.68	0.72	0.72	$C_\mu \frac{\bar{s}^3(\bar{s}/4.38-1)}{1+0.012\bar{s}^3}\frac{\varepsilon^2}{k}$
Shih *et al* (1993)	1.44	1.92	1.0	1.3	0
Shih *et al* (1995)	1.44	1.92	1.0	1.3	0
Gatski and Speziale	1.44	1.83	(1.0)	(1.3)	0
Lien *et al*	1.44	1.92	1.0	1.3	0
Suga[4]	1.44	1.92	1.0	1.3	*YAP*
Apsley and Leschziner[6]	1.44	1.83	$\frac{1.0}{1+\bar{\beta}^2/3-\bar{\gamma}^2}$	$\frac{1.37}{1+\bar{\beta}^2/3-\bar{\gamma}^2}$	0

Viscous Terms in Low-Re Models

	f_μ	D	f_1	f_2
Suga[4]	$1-e^{-(R_t/90)^{1/2}-(R_t/400)^2}$	$2\nu\left(\frac{\partial k^{1/2}}{\partial x_i}\right)^2$	1	$1-0.3e^{-R_t^2}$
Lien *et al*[5]	$l_\mu^{(1)}/l_\varepsilon^{(1)}$	0	1	$1-0.3e^{-R_t^2}$
Apsley and Leschziner[6]	1	0	1	1

Viscous Terms in Low-Re Models(Continued)

	S_ε
Suga[4]	$0.0022\frac{\bar{s}\nu_t k^2}{varepsilon}\left(\frac{\partial^2 U_i}{\partial x_j \partial x_k}\right)^2 (R_t \leq 250)$
Lien *et al*[5]	$C_{\varepsilon 2} f_2 \frac{\varepsilon^{(1)}\varepsilon}{k} e^{-0.00375 y^{*2}}$
Apsley and Leschziner[6]	$C_{\varepsilon 2}\frac{\varepsilon^{(1)}\varepsilon}{k} e^{-0.0038 y^{*2}}$

Remarks

1. The original Speziale [19] model included terms involving DS_{ij}/Dt. These have been found to provoke serious numerical instability and have, therefore, been omitted from the stress-strain relationship.

2. In the Shih *et al* model [20], the constant A_s^* is derived as the positive root of a cubic equation and is given by

$$A_s^* = \sqrt{6}\cos\phi, \qquad \phi = \frac{1}{3}\arccos\left(\sqrt{6}\frac{s_3}{s_2^{3/2}}\right). \tag{231}$$

The shear parameter is (allowing for system rotation)

$$\sigma^* = \left(s_{ij}s_{ij} + w^*_{ij}w^*_{ij}\right)^1/2, \qquad w^*_{ij} = \frac{k}{\varepsilon}\left[W_{ij} - 2\varepsilon_{ijk}\Omega_k\right] = w_{ij} - \frac{k}{\varepsilon}\varepsilon_{ijk}\Omega_k. \tag{232}$$

3. In the Gatski and Speziale model [21], C_i^* are shear-dependent terms based on the *regularisation of the 2D solution* as given by Speziale and Xu [22]:

$$C_1^* = \frac{(1+2\zeta^2)(1+6\eta^5) + \frac{5}{3}\eta^2}{(1+2\zeta^2)(1+2\zeta^2+6b_1\eta^6)},$$

$$C_{2,3}^* = \frac{(1+2\zeta^2)(1+\eta^4) + \frac{2}{3}\eta^2}{(1+2\zeta^2)(1+2\zeta^2+6b_{2,3}\eta^6)}, \tag{233}$$

$$(b_1, b_2, b_3) = (7.0, 6.3, 4.0)$$

where

$$\eta = \frac{1}{2}\frac{\alpha_3}{\alpha_1}(s_2)^{1/2}, \qquad \zeta = \frac{\alpha_2}{\alpha_1}(-w_2)^{1/2} \tag{234}$$

and

$$\alpha_1 = \left(\frac{4}{3} - C_2\right)g,$$

$$\alpha_2 = \frac{1}{2}\alpha_1(2 - C_4)g, \tag{235}$$

$$\alpha_3 = \alpha_1(2 - C_3)g,$$

where

$$g = \frac{1}{1/2C_1 + \left(\frac{P}{\varepsilon}\right)_{eq} - 1}, \qquad \left(\frac{P}{\varepsilon}\right)_{eq} = \frac{C_{\varepsilon 2} - 1}{C_{\varepsilon 1} - 1}. \tag{236}$$

The constants, which come from the SSG pressure-strain model, are

$$C_1 = 6.8, \quad C_2 = 0.36, \quad C_3 = 1.25, \quad C_4 = 0.40. \tag{237}$$

Note also that, for systems rotation in this model only, w_{ij} is alternatively defined as

$$w_{ij} = \tau\left[W_{ij} - \left(\frac{4 - C_4}{2 - C_4}\right)\varepsilon_{ijk}\Omega_{ijk}\right]. \tag{238}$$

4. In the Suga model [23],

$$\eta = \max\left(\overline{s}, \overline{w}\right). \tag{239}$$

The Yap correction (Yap [24]) in the dissipation equation is given by

$$YAP = \max\left(0.83\left(\gamma - 1\right)\gamma^2 \frac{\varepsilon^2}{k}, 0\right), \quad \gamma = \frac{k^{3/2}}{c_l \varepsilon y_n}, \quad c_l = 2.5. \tag{240}$$

5. The low-*Re* terms in the Lien *et al* [25] model are based on the one-equation model of Norris and Reynolds [2] for the mixing and dissipation lengths near the wall. The original model actually modifies f_1 rather than S_ε, but the two formulations are equivalent.

6. In the Apsley and Leschziner [26] model, f_μ is incorporated naturally into C_μ. $a^*_{\alpha\beta}$ and σ^* are curve fits to the three independent anisotropy components and shear flow parameters, respectively, from DNS data for plane channel flow:

$$\begin{aligned} a^*_{11} &= 1 + 0.42\exp\left(0.296 y^{*1/2} - 0.040 y^*\right) & -\frac{2}{3}, \\ a^*_{22} &= 0.404\left[1 - \exp\left(-0.001 y^* - 0.000147 y^{*2}\right)\right] & -\frac{2}{3}, \\ a^*_{12} &= -0.3\left[1 - \exp\left(-0.00443 y^{*1/2} - 0.0189 y^*\right)\right], & \end{aligned} \tag{241}$$

and

$$\sigma^* = 3.33\left[1 - \exp\left(-0.45 y^*\right)\right]\left[1 + 0.277 y^{*3/2}\exp\left(-0.088 y^*\right)\right]. \tag{242}$$

The constants $\overline{\beta}$ and $\overline{\gamma}$ are based on the values of the anisotropy components and shear parameter in the log-law region and are given by

$$\overline{\beta} = 0.222, \qquad \overline{\gamma} = 0.623. \tag{243}$$

Modifications to C_μ, σ_k and σ_ε arise because, unlike the other models, the first two cubic terms do not cancel in simple shear. The non-equilibrium parameter f_P, which accounts for departures of the *local* shear parameter $\sigma = (k/\varepsilon)\sqrt{\left(\partial U_i/\partial x_j\right)^2} = \left(s_2 - w_2\right)^{1/2}$ from the calibration value σ^*, is given by

$$f_P = \frac{2 f_0}{1 + \sqrt{1 + 4 f_0\left(f_0 - 1\right)\left(\sigma/\sigma^*\right)^2}}, \qquad f_0 = 1 + 1.25\max\left(0.09\sigma^{*2}, 1.0\right). \tag{244}$$

The additional term in the dissipation equation is based upon a curve fit to the DNS data for the dissipation length:

$$l_\varepsilon^{(1)} = 0.179 y_n \left(1 + \frac{128}{y^*}\right)\left(1 - e^{-y^{*2}/279}\right), \quad \varepsilon^{(1)} = \frac{C_\mu^{3/4} k^{3/2}}{l_\varepsilon^{(1)}}. \tag{245}$$

9 Wall Boundary Conditions

9.1 Low-*Re* Models

Subscript w denotes the value *at* the wall and subscript p the value at the near-wall node. The following boundary conditions are assumed for a direct integration to the wall.

$$\text{For } k\text{:} \quad k_w = 0, \qquad flux(k)_w = 0. \tag{246}$$

$$\text{For } \varepsilon \text{ when } D = 0\text{:} \quad \varepsilon_w = \frac{2\nu k_p}{y_p^2}, \qquad flux(\varepsilon)_w = 0. \tag{247}$$

$$\text{For } \varepsilon \text{ when } D \neq 0\text{:} \quad \varepsilon = 0, \qquad flux(\varepsilon)_w = -\nu\nabla\varepsilon. \tag{248}$$

$$\text{For } \omega\text{:} \quad \omega = 0, \qquad flux(\omega)_w = -\nu\nabla\omega. \tag{249}$$

$$\text{For } g\text{:} \quad g = 0, \qquad flux(g)_w = -\nu\nabla g. \tag{250}$$

$$\text{For } \overline{u_i u_j}\text{:} \quad \overline{u_i u_j} = 0, \qquad flux(\overline{u_i u_j})_w = 0. \tag{251}$$

9.2 Wall Functions

In the finite-volume approach, wall functions supply the following quantities when the near-wall cell is too large to resolve the flow structure close to the wall:

- Wall shear stress (and heat flux) for the mean-flow equations;
- Volume-averaged production and dissipation terms for the k equation;
- Near-wall values of ε (and, in principle, ω and g) and structure functions $\overline{u_\alpha u_\beta}/k$.

In equilibrium shear flows, a universal law of the wall is assumed such that the mean velovity profile (for a smooth wall) takes the form

$$\frac{U}{u_\tau} = \left\{ \begin{array}{lll} \frac{1}{k}\ln\left(Ey^+\right) & , & y^+ \geq y_v^+ \\ y^+ & , & y^+ \leq y_v^+ \end{array} \right\} \tag{252}$$

where $y^+ y_n u_\tau/\nu$ and k, E and y_ν^+ are constants. Continuity requires $E = \exp\left(ky_\nu^+\right)/y_\nu^+$. A typical dimensionless sublayer height is $y_\nu^+ = 11.2$. Kinematic shear stress $\tau_w = uT^2$ would be calculated from U at the near-wall node. However, this leads to an eddy viscosity $\nu_t = ku_\tau y$ which vanishes at impingement points, where U vanishes, at variance with the observed maximum heat transfer at such points.

The solution (Chieng and Launder [27]) is to adopt a turbulent velocity scale based on the turbulent kinetic energy, $u_0 \sim C_\mu^{1/4} k_P^{1/2}$, and corresponding eddy viscosity, so that the (constant) shear stress in the fully turbulent layer is

$$\tau = ku_0 y \frac{\partial U}{\partial y}, \tag{253}$$

with solution at the near-wall node (assumed to lie in the fully turbulent layer):

$$U = \frac{\tau_w}{ku_0} \ln\left(E\frac{yu_0}{\nu}\right). \tag{254}$$

Thus the wall shear stress is deduced from

$$\tau_w = \frac{k\left(C_\mu^{1/4} k_p^{1/2}\right) U_p}{\ln\left(E^* y_p^*\right)}, \tag{255}$$

where a subscript 'p' denotes a value at the near-wall node. k, E^* and $y_n^* u$ take the values 0.41, 5.4 and 20.4, respectively. Strictly, the use of wall functions requires the near-wall node to lie within the fully turbulent region; say, $y^+ \geq 30$ or $y^+ \geq 55$.If, for any reason, $y^* \leq y_n^* u$, then τ_w should be set equal to the viscous stress $\nu U_p/y_p$. To implement the scheme in the momentum equations, the coordinate projections of τ_w are used to replace the diffusive fluxes on cell faces abutting the boundary through the source terms.

To establish turbulence quantities, universal profiles must be adopted for production and dissipation. It is assumed that

$$P = 0, \qquad \varepsilon \sim \frac{2\nu k}{y^2}, \qquad (y \leq y_n u) \tag{256}$$

$$P = \tau_w \frac{\partial U}{\partial x}, \qquad \varepsilon \sim \frac{u_0^3}{ky}, \qquad (y \geq y_n u) \tag{257}$$

where $y_\nu = y_n u^* \nu / k_\nu^{1/2}$, and k_ν is the turbulent kinetic energy at the top of the viscous sublayer, taken equal to k_p (although more complex algorithms may be constructed). In the dissipation equation the value of ε at the near-wall node is set equal to that defined above. In the k equation, k and its flux are set to zero at the wall and *cell-averaged* values of dissipation used in the near-wall cell. Integrating the profiles above,

$$P_{av} = \frac{\tau_w^2}{C_\mu^{1/4} k_p^{1/2} k\delta} \ln(\delta / y_\nu) \tag{258}$$

$$\varepsilon_{av} = \frac{2\nu k_\nu}{y_\nu \delta} + \frac{C_\mu^{3/4} k_p^{3/2}}{k\delta} \ln(\delta / y_\nu) \tag{259}$$

where $\delta \, (= 2y_p)$ is the cell height.

For the Reynolds-stress transport equations, the values of the individual stresses at the near-wall node are fixed by the value of k and structure functions $\overline{u_\alpha u_\beta}/k$ derived from the transport equations on the assumption of local equilibrium ($P = \varepsilon$), vanishing advection and diffusion and isotropic dissipation. In local coordinates aligned such that x is parallel to the flow and y normal to the wall, we get the following with the standard linear pressure-strain model:

$$
\begin{aligned}
\frac{\overline{v^2}}{k} &= \frac{2}{3}\left(\frac{C_1 + C_2 - 2C_2C_2^{(w)} - 1}{C_1 + 2C_1^{(w)}}\right), \\
\frac{\overline{u^2}}{k} &= \frac{2}{3}\left(\frac{2 + C_1 - 2C_2 + C_2C_2^{(w)}}{C_1}\right) + \frac{C_1^{(w)}}{C_1}\frac{\overline{v^2}}{k}, \\
\frac{\overline{w^2}}{k} &= \frac{2}{3}\left(\frac{-1 + C_1 + C_2 + C_2C_2^{(w)}}{C_1}\right) + \frac{C_1^{(w)}}{C_1}\frac{\overline{v^2}}{k}, \\
\frac{\overline{-uv}}{k} &= \sqrt{\left(\frac{1 - C_2 + \frac{3}{2}C_2C_2^{(w)}}{C_1 + \frac{3}{2}C_1^{(w)}}\right)\frac{\overline{v^2}}{k}}.
\end{aligned}
\tag{260}
$$

The stress components in the fixed cartesian system may then be obtained by coordinate rotation. For the pressure-strain coefficients in the Gibson-Launder model [12] this gives, in wall-aligned coordinates,

$$
\begin{aligned}
\overline{u^2}/k &= 1.098, \\
\overline{v^2}/k &= 0.247, \\
\overline{w^2}/k &= 0.655, \\
\overline{-uv}/k &= 0.255,
\end{aligned}
\tag{261}
$$

and, for simplicity, these values have also been assumed for the other models.

10 Detached-Eddy Simulation (DES)

As described in the previous sections, the Reynolds-Averaging Navier-Stokes (RANS) method consists of time-averaged Navier-Stokes equations. Large-Eddy simulation (described in greater detail in the following section), is another approach to modelling/simulating the turbulent flows. LES, however, splits up the flow in terms of the size of the scales, with the large scale eddies being directly resolved on the grid and the smaller scales being modelled. The main difference between the RANS and LES approaches is highlighted below.

RANS	vs	**LES**
Time-averaged equations		Spatially-filtered equations

There exists another approach called Detached-Eddy Simulation, or DES for short, which acts as an intermediary between the RANS and LES methods. DES combines both RANS and LES approaches: it takes advantage of near-wall modelling from the RANS method and switches to LES in wall-resolved areas where the CFD mesh is fine.

Some additional remarks can be made about these modelling/simulating methods:

- LES has problems in resolving the near-wall turbulent stresses since the required resources approach Direct Numerical Simulation (DNS).
- Overall pure LES just gives 10 times higher Reynolds numbers than DNS even using modern super-computers so it is of limited practical application.
- Alternatives to LES include:

  DNS - (Laminar formulation in equations).

  RANS + turbulence model .

  LES + Sub-grid model .

  DES or LNS - Hybrid LES and RANS.

The original idea of using DES came from Spalart and co-workers. It involved using RANS for near-wall and boundary layer and LES everywhere outside. It is this concept that is called ***DES: Detached-Eddy Simulation***.

Spalart *et al* [28] modified the S-A model to achieve a DES equivalent. The only modification is in the dissipation term of the transport equation of $\tilde{\nu}$:

$$-C_{w1} f_{w1} \left(\frac{\tilde{\nu}}{\tilde{d}} \right)^2 \tag{262}$$

Originally,

$$\tilde{d} = d = \text{distance of the nearest wall} \tag{263}$$

whereas for DES, it is:

$$\tilde{d} = C_{DES} \Delta \tag{264}$$

where C_{DES} is a constant,
Δ is the metric of the grid size.

In practice, we use:

$$\tilde{d} = \min\left(d, C_{DES}\Delta\right) \tag{265}$$
$$\Delta = \max\left(\Delta_x, \Delta_y, \Delta_z\right) \qquad \forall \text{ cells.} \tag{266}$$

NB: Other metric relations are possible.

For closures other than S-A (like 2-equation models), a similar idea has been put forward by Batten *et al* [29]. This is called ***LNS: Limited Numerical Scales*** and has several advantages compared to the original DES including:

- LNS claims to be "automatic" by detecting the areas of application of the RANS and LES without *a priori* knowledge of the location of walls or wall-distances.
- LNS approaches DNS as $\Delta \rightarrow 0$.
- LNS will go back to RANS at the far-field of the flow if the grid there is coarse.

The implementation of LNS is performed as follows:

- Start from the space-filtered Navier-Stokes equations and assume that the resulting Leonard's stresses have a zero sum.
- Model the Reynolds Stresses via a 2-equation turbulence model (other models are also possible).
- Use Speziale's suggestion that the Reynolds Stress is:

$$\widetilde{u_i'u_j'} = \alpha \overline{u_i'u_j'}_{modelled} \tag{267}$$
$$\alpha = \left(1 - e^{-\beta \frac{L^\Delta}{L^k}}\right)^n \tag{268}$$

where $\overline{u_i'u_j'}_{modelled}$ is the Reynolds Stress predicted by the model,
β,n are parameters,
L^{Δ} is the Mesh scale,
L^k is Kolmogorov's scale and is defined by:

$$L^k = \frac{\nu^{3/4}}{\tilde{\varepsilon}^{1/4}} \tag{269}$$

For the mesh spacing, use:

$$L^{\Delta} = 4 \max\left(\overline{r}_c - \overline{r}_k\right) \qquad \forall \text{k}. \tag{270}$$

where k represents $1 \cdots$ n number of cell faces,
$\overline{r}_c$ denotes the centroid of the cell,
$\overline{r}_k$ denotes the mid-point of the face.

which for uniform grids gives:

$$L^{\Delta} = 2 \max\left(\Delta_x, \Delta_y, \Delta_y\right) \tag{271}$$

where the coefficient 2 is due to Nyquist's sampling theorem and α is known as the Latency parameter and is given by:

$$\alpha \equiv \frac{\min\left(l_{VLES}, l_{VEA}\right)}{l_{VEA}} \tag{272}$$

where subscripts $_{VLES}$ and $_{VEA}$ represent characteristic scales.

- Modify sources of the k-equation:

$$P_k - \alpha\rho\tilde{\varepsilon} \tag{273}$$

- Modify sources of the ω-equation:

$$\beta P_k - \alpha\left(\beta^* \rho\omega\right) T_t^{-1} \tag{274}$$

- In the Reynold's Stresses, multiply the term $\frac{2}{3}\rho k \delta_{ij}$ by α. This will change the viscous fluxes in the code.

- Multiply μ_T by α. This will give a term αk, which is called the unresolved turbulent kinetic energy.

- Use the following for α:

$$\alpha = \frac{\min\left[\left(C_s C L^{\Delta}\right)^2 \sqrt{\frac{S_{kl} S_{kl}}{2}}, \left(\frac{C_\mu k^2}{\bar{\varepsilon}} + \delta\right)\right]}{\left(\frac{C_\mu k^2}{\bar{\varepsilon}} + \delta\right)} \tag{275}$$

$$\delta = O\left(10^{-20}\right) \tag{276}$$

$$C_s = 0.05 \tag{277}$$

$$C_\mu = 0.05. \tag{278}$$

11 Large-Eddy Simulation (LES)

LES is another means of modelling/simulating turbulent flows but is different from the more conventional turbulent modelling approaches such as RANS and URANS, which identifies the features in a flow in terms of its statistical properties, in that the flow is defined in terms of its scale properties. In LES, the flow is instead divided into two components: large-eddy scales and smaller-eddy scales. The large scales, which are dictated by geometry and boundary conditions, are explicitly simulated, i. e. are resolved directly on the grid, whereas the smaller scales, which are less influenced by the geometry and will therefore tend to be more isotropic, are modelled using a turbulence model.

Large-Eddy Simulation itself is very closely related to Direct Numerical Simulation (DNS). In DNS, no turbulence model is employed and the flow is solely resolved on the grid. This implies that the grid must be sufficiently fine in order to be able to numerically capture flow features of all sizes. Although a tremendous amount of information can be obtained from DNS (sometimes even more than may be required), the strain on computational requirements (and hence the finite budget) becomes more and more insurmountable as the Reynolds number and also as the need for finer grids (such as for complex geometrical configurations) increases.

It is inevitable that where the demand to perform a DNS surpasses the capacity of computational overhead, the solution would be to use a coarser grid. This means that only the large-scale eddies are resolved numerically on the grid whereas the smaller ones (those that are perhaps smaller than one or two cells) are not. Pragmatically, however, it can be understood that there will exist some sort of interaction between the motions of all scales and consequently the large-scale eddies captured by the coarser grid will in general be incorrect if the influence of the finer scales on the larger ones is not taken into account and this is where a sub-grid scale (SGS) model comes into play.

LES does exactly this - it uses a coarser grid to resolve the large-scale structures of the flow and a sub-grid scale (SGS) model to model the smaller scales (although recent advances in LES have revealed that it is possible to perform LES without using an SGS model, by simply using an upwind or monotone numerical scheme instead). By doing this, LES relieves the strain on the computational overhead by overcoming the impasse imposed by the need for greater computational capacity for DNS computations and consequently is often referred to as a more cost-effective DNS approach. The clever part of LES lies in the SGS model, which compensates for the unresolved smaller scales and further generates the important yet complex link between the large and small scales. With the large-scale structures being resolved (and only the small scales being modelled), LES offers an additional advantage over RANS/URANS methods in which the entire spectrum must be modelled. In complex flows, therefore, such as in extensive vortex shedding flows, etc. LES becomes more preferred than the RANS approach. It also offers the opportunity to access the unsteady motions relevant to several applications such as aero-acoustics, fluid-structure coupling or even the control of turbulence by suitable unsteady forcing.

11.1 Compressible LES formulations

Although most of the LES work has been carried out for incompressible cases, compressible flow investigations with LES have started to appear. Starting from the application of spatial filtering:

$$\overline{f} = \int_V G f \mathrm{d}v \tag{279}$$

where G represents a grid filtering function. Each variable of f is decomposed as:

$$f = \overline{f} + f_{sg} \tag{280}$$

where	$\overline{f}$	denotes the filtered part,
	f_{sg}	denotes the sub-grid part.

The different treatment of the compressible cases starts here replacing the filtered part with its Favre-averaged component:

$$\tilde{f} = \frac{\overline{\rho f}}{\overline{\rho}} \tag{281}$$

The above formulation in conjunction with the filtering results in several additional terms to the Navier-Stokes equations. All new terms are concentrated in the viscous flux vectors.

In the transformed co-ordinate system and for the ξ-direction, the viscous flux reads:

$$F = \frac{1}{J} \begin{bmatrix} \xi_{xi}\left(G_{i1} + \tau_{i1}\right) \\ \xi_{xi}\left(G_{i2} + \tau_{i2}\right) \\ \xi_{xi}\left(G_{i3} + \tau_{i3}\right) \\ \xi_{xi}\left[u_j\left(G_{ij} + \tau_{ij}\right) - q_i - Q_i\right] \end{bmatrix} \tag{282}$$

where G_{ij} is the Favre-averaged stress tensor and is defined as:

$$G_{ij} = \mu\left(\frac{\partial u_i}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j} + \frac{\partial u_j}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial \xi_l}\frac{\partial \xi_l}{\partial x_k}\right) \tag{283}$$

q_i is the Favre-averaged heat flux vector:

$$q_i = -\left[\frac{1}{\left[\left(\gamma - 1\right)M_a^2\right]}\left(\frac{\mu}{P_r}\right)\frac{\partial \tau}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i}\right] \tag{284}$$

τ_{ij} is the sub-grid stress and is defined as:

$$\tau_{ij} = -Re\ \rho\left(\tilde{u_i u_j} - \tilde{u}_i\tilde{u}_j\right) \tag{285}$$

and Q_i is the sub-grid heat flux, which is defined as:

$$Q_i = Re\ \rho\left(\tilde{u_i T} - \tilde{u}_i\tilde{T}\right) \tag{286}$$

Note that both τ_{ij} and Q_i need to be modelled.

The simplest approach is Smagorinski's suggestion of a sub-grid turbulent viscosity:

$$\mu_t = Re\ C_s J^{-\frac{2}{3}}\rho S_m \tag{287}$$

$$S_m = \sqrt{2S_{ij}S_{ij}} \tag{288}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j} + \frac{\partial u_j}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_i}\right) \tag{289}$$

$J^{-\frac{2}{3}}$ comes from the transformation and the grid-filter. C_s is a "constant". Also,

$$\tau_{ij} = 2\mu_t \left(S_{ij} - \frac{1}{3} S_{kk}\delta_{ij} \right) + \frac{1}{3}\tau_{kk}\delta_{ij} \tag{290}$$

where $\frac{1}{3}\tau_{kk}\delta_{ij}$ is the isotropic part and is usually small for incompressible flows compared to the pressure.

For compressible flow:

$$\tau_{kk} = -2\mathrm{Re} C_I J^{-\frac{2}{3}} \rho S_m^2 \tag{291}$$

$$Q_i = \left(\frac{\mu_t}{P_{rt}} \right) \frac{\partial \tilde{T}}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i} \tag{292}$$

where C_I is a constant.

Improved predictions can be obtained using variable coefficients C_s and C_I.

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