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A para-universal relation for orthotropic materials

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ABSTRACT

A universal relation is an algebraic relation between stress and strain that holds for any material within a certain class, irrespective of the exact form of the material response function and parameter values. Classical universal relations, such as Rivlin's famous relation for simple shear, apply to stress components produced by *one and the same* deformation. We present a family of relations that connect stress components under *different* deformations, which we call *para-universal relations* to highlight this difference. The proposed para-universal relations hold for any orthotropic material whose response function is additively decomposed into terms, each of which possesses a symmetry with respect to one of the axes of orthotropy. Using basic properties of the permutation group S_3, we demonstrate that such an additive decomposition implies the proposed identities. The established para-universal relations hold for an arbitrary local deformation and, like classical universal relations, are linked to material symmetry and apply to a wide class of materials. Since the proposed para-universal relations do not hold for all orthotropic material models, they present a convenient way to test for the suitability of additively split strain-energy functions, which are often used to model the nonlinearly elastic response of soft tissues. Such a test can be performed on collected experimental data prior to choosing an exact form of the response function and fitting its parameters. We use published experimental data for human myocardium and also synthetic data to illustrate this.

1. Introduction

An algebraic relation that connects components of stress and strain tensors is called a *universal relation* if it holds for any material within a certain class, independently of the choice of the constitutive function and parameter values. The universal relations are useful in directing modelling and experimental studies of material response; in particular, they allow rejection of candidate material models based on the analysis of experimental data [1–3]. A well-known example is Rivlin's relation for simple shear [4],

$$\sigma_{11} - \sigma_{22} = \gamma\sigma_{12}, \tag{1}$$

which holds in any isotropic elastic material (σ_{ij} are components of the Cauchy stress tensor, and γ is the amount of shear). According to Pucci and Saccomandi [2], a class of materials to which universal relations apply is defined by a material symmetry group, such as the group of orthogonal transformations $O(3)$ in the case of relation (1). The coaxiality of the left Cauchy–Green deformation tensor $\mathbf{b}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$, which is a hallmark of isotropy, was used by Beatty [1] to describe a class of universal relations by considering the tensor equation $\boldsymbol{\sigma}\mathbf{b} = \mathbf{b}\boldsymbol{\sigma}$, from which (1) may be derived. This motivated a method for deriving universal relations that hold for any universal solution [2]. Interestingly, this method does not allow universal relations that hold for materials characterised solely by orthotropic symmetry to be obtained [5]. Furthermore, any general nonlinear universal relation is reducible to one or several linear relations [2,3]. Hence, narrower material classes than those defined by symmetry groups are considered in this context [3,6]. See [7] for a review on universal solutions and relations, and also [3,8–11] for related results.

The universal relations mentioned above link components of the Cauchy stress tensor evaluated at one arbitrarily chosen deformation. We describe a family of relations between components of stress produced in distinct but related deformations. We consider the term *para-universal relations* appropriate to highlight this difference, as *"para-"* stands for *"besides"* and *"distinct from, but analogous to"* [12]. We will show that, like universal relations, the proposed para-universal relations are linked to material symmetry and can direct constitutive modelling. To our knowledge, these relations had not been previously studied.

Latorre and Montans [13] observed that the shear response curves corresponding to 6 shear modes are linearly dependent in a spline-based material model. Specifically, if σ_{ij}, $i \neq j$, are the shear

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stress components corresponding to *different* simple shear deformations at the same amount of shear, then

$$\sigma_{12}+\sigma_{23}+\sigma_{31}-(\sigma_{21}+\sigma_{32}+\sigma_{13})=0 \tag{2}$$

holds, for a strain-energy function of the form $\Psi_{\rm LM}=\sum_{i\geq j=1}^{3}\omega_{ij}(\mathcal{E}_{ij})$, where $\boldsymbol{\mathcal{E}}$ is the Lagrangian logarithmic strain tensor. The terms $\omega_{ij}(\mathcal{E}_{ij})$ are arbitrary spline functions of the respective scalar components $\mathcal{E}_{ij}$, which are fitted to experimental data, following the "What You Get Is What You Prescribe" approach. Even though the numbers of data points and unknown coefficients match, the linear dependence of the predicted shear stresses precludes fitting the model to shear data alone, and other test protocols are required, as discussed in [13]. This observation motivated our study.

We demonstrate that condition (2) is a consequence of two factors: (*i*) the additive decomposition of the strain-energy function and (*ii*) the symmetries exhibited by the additive terms. While this condition is not satisfied by an arbitrary orthotropic material, it does hold for additively split response functions, whose terms are invariant with respect to a permutation of two axes of orthotropy. Any orthotropic response composed of orthogonal transversely isotropic components falls into this category, including many nonlinearly elastic models for soft biological tissues. In Section 2 we use basic properties of the permutation group S_3 to prove a para-universal relation for such materials in a general tensorial form, from which condition (2) is recovered as a special case. In Section 3 we use real and synthetic experimental data to illustrate how the para-universal relation indicates a suitability of the aforementioned class of constitutive models.

1.1. Basic definitions

Let a deformation of a 3-dimensional body $\mathcal{B}$ be given by $\mathbf{x}=\boldsymbol{\chi}(\mathbf{X})$. We consider unconstrained and incompressible Cauchy elastic materials, in which the Cauchy stress tensors are given by $\boldsymbol{\sigma}=\mathfrak{g}(\mathbf{F})$ and $\boldsymbol{\sigma}=-p\mathbf{1}+\mathfrak{g}(\mathbf{F})$, respectively, where $\mathbf{F}=\partial\mathbf{x}/\partial\mathbf{X}$ is the deformation gradient, p is the incompressibility-related Lagrange multiplier, and $\mathbf{1}$ is the identity tensor. In particular, we are interested in hyperelastic response $\mathfrak{g}(\mathbf{F})=2\mathbf{F}(\partial W(\mathbf{C})/\partial\mathbf{C})\mathbf{F}^T$, where $W(\mathbf{C})$ is the strain-energy function, and $\mathbf{C}=\mathbf{F}^T\mathbf{F}$ is the right Cauchy–Green deformation tensor. For brevity, we will write $\boldsymbol{\sigma}(\mathbf{F})$ and imply that the Cauchy stress is computed in one of the above ways. The objectivity requirement (also known as frame indifference), reads

$$\boldsymbol{\sigma}(\mathbf{QF})=\mathbf{Q}\boldsymbol{\sigma}(\mathbf{F})\mathbf{Q}^T,\quad \forall\mathbf{Q}\in SO(3),\ \forall\mathbf{F}, \tag{3}$$

where $SO(3)=\{\mathbf{Q}\in\mathcal{L}(\mathbb{R}^3,\mathbb{R}^3)|\mathbf{Q}^T\mathbf{Q}=\mathbf{1},\det\mathbf{Q}=1\}$ is the group of proper orthogonal transformations. Further, $\mathbf{Q}\in SO(3)$ is called a *material symmetry* of a given material model, if $\boldsymbol{\sigma}(\mathbf{FQ}^T)=\boldsymbol{\sigma}(\mathbf{F})$, $\forall\mathbf{F}$; hence, a material symmetry group $\mathbb{Q}\subseteq SO(3)$ is formed. By objectivity (3), we have,

$$\boldsymbol{\sigma}(\mathbf{QFQ}^T)=\mathbf{Q}\boldsymbol{\sigma}(\mathbf{F})\mathbf{Q}^T,\quad \forall\mathbf{Q}\in\mathbb{Q},\ \forall\mathbf{F}. \tag{4}$$

For example, the symmetry group of a transversely isotropic material consists of all rotations that do not affect the alignment of some given axis $\mathbf{m}_0$,

$$\mathbb{Q}_{\mathbf{m}_0}=\{\mathbf{Q}\in SO(3)|\mathbf{Qm}_0=\pm\mathbf{m}_0\}. \tag{5}$$

The symmetry group of an orthotropic material is

$$\mathbb{Q}_O=\{\mathbf{1},\mathbf{Q}_1,\mathbf{Q}_2,\mathbf{Q}_3\},\quad \text{with}\quad \mathbf{Q}_i=2\mathbf{e}_i\otimes\mathbf{e}_i-\mathbf{1}, \tag{6}$$

where $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ are mutually orthogonal material axes, and $\mathbf{Q}_i$ is a rotation about $\mathbf{e}_i$ by π. Axes $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ must be distinguished, as an orthotropic material may behave differently along these directions. Transverse isotropy is a special case of orthotropy: this can be checked directly by taking $\mathbf{e}_1=\mathbf{m}_0$, in which case all directions spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$ are equivalent.

2. The para-universal relation for additively decomposed orthotropic materials

In order to formulate and prove the proposed para-universal relation in its general form, we introduce additional notation. Consider the symmetric group $S_3=\{a,b,c,1,p,n\}$, which consists of all possible permutations of the three-element set {1, 2, 3}:

$$\begin{aligned}&a:\{1,2,3\}\mapsto\{2,1,3\},\ b:\{1,2,3\}\mapsto\{1,3,2\},\\ &c:\{1,2,3\}\mapsto\{3,2,1\},\ 1:\{1,2,3\}\mapsto\{1,2,3\},\\ &p:\{1,2,3\}\mapsto\{2,3,1\},\ n:\{1,2,3\}\mapsto\{3,1,2\}.\end{aligned} \tag{7}$$

The group operation is the superposition of permutations, and the following identities hold:

$$\begin{aligned}&aa=bb=cc=pn=np=1,\quad n=ab=bc=ca,\\ &p=ba=cb=ac,\quad a=nb=bp=cn=pc,\\ &b=nc=cp=an=pa,\quad c=na=ap=bn=pb.\end{aligned} \tag{8}$$

Every permutation $x\in S_3$ can be identified with a tensor $\mathbf{Q}_x\in O(3)$, which permutes coordinate axes $\{\mathbf{e}_i\}$ in the reference or current configuration, that is, $\mathbf{Q}_1=\mathbf{1}$, $\mathbf{Q}_a=\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1+\mathbf{e}_3\otimes\mathbf{e}_3$, and so on. The superposition of permutations naturally corresponds to the tensor multiplication, $\forall x,\,y\in S_3$, $\mathbf{Q}_x\mathbf{Q}_y\equiv\mathbf{Q}_{xy}$. The action of a permutation operator on a stress response function $\boldsymbol{\sigma}$ can now be defined via the rule

$$\mathbf{Q}[\boldsymbol{\sigma}(\mathbf{F})]:=\boldsymbol{\sigma}(\mathbf{FQ}),\quad \text{or,}\quad x[\boldsymbol{\sigma}(\mathbf{F})]:=\boldsymbol{\sigma}(\mathbf{FQ}_x). \tag{9}$$

The left-hand side part of definition (9) can be applied to an arbitrary $\mathbf{Q}\in\mathcal{L}(\mathbb{R}^3,\mathbb{R}^3)$, but we will only be dealing with those $\mathbf{Q}\in O(3)$ that correspond to permutations of a given triad $\{\mathbf{e}_i\}$. The response function $\boldsymbol{\sigma}$ is invariant under permutations that belong to the material's symmetry group (and only under those permutations),

$$\mathbf{Q}_x\in\mathbb{Q}\iff\mathbf{Q}_x[\boldsymbol{\sigma}]=\boldsymbol{\sigma},\quad\forall x\in S_3. \tag{10}$$

We also observe that $y[x[\boldsymbol{\sigma}]]\equiv(yx)[\boldsymbol{\sigma}],\forall x,\,y\in S_3$, from

$$\begin{aligned}\mathbf{Q}_y[\mathbf{Q}_x[\boldsymbol{\sigma}(\mathbf{F})]]&=\mathbf{Q}_y[\boldsymbol{\sigma}(\mathbf{FQ}_x)]=\boldsymbol{\sigma}(\mathbf{FQ}_y\mathbf{Q}_x)\\ &=(\mathbf{Q}_y\mathbf{Q}_x)[\boldsymbol{\sigma}(\mathbf{F})]=\mathbf{Q}_{yx}[\boldsymbol{\sigma}(\mathbf{F})].\end{aligned} \tag{11}$$

Therefore, we can drop the square brackets and write simply $yx\boldsymbol{\sigma}$ unambiguously. It will prove convenient to introduce the shortcut notation

$$\begin{aligned}(x\pm y)\boldsymbol{\sigma}(\mathbf{F})&:=x\boldsymbol{\sigma}(\mathbf{F})\pm y\boldsymbol{\sigma}(\mathbf{F})\\ &=\boldsymbol{\sigma}(\mathbf{FQ}_x)\pm\boldsymbol{\sigma}(\mathbf{FQ}_y),\ \forall a,b\in S_3.\end{aligned} \tag{12}$$

We emphasise that this notation is only used for brevity, as it is impossible to meaningfully define the "addition" of permutations (as acted upon a general response functions $\boldsymbol{\sigma}$) in such a way that $x+y$ belongs to S_3 or its extension.

The following proposition formulates the para-universal relation and proves it for a certain subset of orthotropic materials.

Proposition 1. *Let the additive components of the total material response function $\boldsymbol{\sigma}=\boldsymbol{\sigma}_a+\boldsymbol{\sigma}_b+\boldsymbol{\sigma}_c$ have symmetries $\mathbf{Q}_a,\mathbf{Q}_b,\mathbf{Q}_c$, respectively, that is, $(a-1)\boldsymbol{\sigma}_a=(b-1)\boldsymbol{\sigma}_b=(c-1)\boldsymbol{\sigma}_c=\mathbf{0}$. Then the identity*

$$(1+p+n-a-b-c)\boldsymbol{\sigma}=\mathbf{0} \tag{13}$$

holds.

Proof. We show that $(1+p+n-a-b-c)\ \boldsymbol{\sigma}=\mathbf{0}$ holds for $\boldsymbol{\sigma}=\boldsymbol{\sigma}_a,\boldsymbol{\sigma}_b,\boldsymbol{\sigma}_c$ separately. For $\boldsymbol{\sigma}=\boldsymbol{\sigma}_a$, we have

$$\begin{aligned}&(1+p+n-a-b-c)\boldsymbol{\sigma}_a\\ &\quad=(\boldsymbol{\sigma}_a-a\boldsymbol{\sigma}_a)+(n\boldsymbol{\sigma}_a-c\boldsymbol{\sigma}_a)+(p\boldsymbol{\sigma}_a-b\boldsymbol{\sigma}_a)\\ &\quad=(\boldsymbol{\sigma}_a-\boldsymbol{\sigma}_a)+(n\boldsymbol{\sigma}_a-\underbrace{ca}_{n}\boldsymbol{\sigma}_a)+(p\boldsymbol{\sigma}_a-\underbrace{ba}_{p}\boldsymbol{\sigma}_a)=\mathbf{0},\end{aligned} \tag{14}$$

where we have used $a\boldsymbol{\sigma}_a = \boldsymbol{\sigma}_a$ and the identities (8). Similarly,

$$(1+p+n-a-b-c)\boldsymbol{\sigma}_b = (\boldsymbol{\sigma}_b - \boldsymbol{\sigma}_b) + (n\boldsymbol{\sigma}_b - ab\boldsymbol{\sigma}_b) + (p\boldsymbol{\sigma}_b - cb\boldsymbol{\sigma}_b) = \mathbf{0}, \tag{15}$$

$$(1+p+n-a-b-c)\boldsymbol{\sigma}_c = (\boldsymbol{\sigma}_c - \boldsymbol{\sigma}_c) + (n\boldsymbol{\sigma}_c - bc\boldsymbol{\sigma}_c) + (p\boldsymbol{\sigma}_c - ab\boldsymbol{\sigma}_c) = \mathbf{0}. \tag{16}$$

Therefore, Eq. (13) holds by additivity. □

Eq. (13) is tensorial and holds for all admissible deformation gradients $\mathbf{F}$. Therefore, it can be specialised for various forms of the deformation gradient and expanded in the components of $\boldsymbol{\sigma}$. The expansion in components is given by a double contraction with $\mathbf{e}_i \otimes \mathbf{e}_j$, $i, j = 1, 2, 3$. Using (3) and (9), we obtain

$$\begin{aligned}\left(x^{-1}\boldsymbol{\sigma}(\mathbf{F})\right) : \mathbf{e}_i \otimes \mathbf{e}_j &= \mathbf{e}_i \cdot \boldsymbol{\sigma}(\mathbf{F}\mathbf{Q}_x^T)\mathbf{e}_j \\ &= \mathbf{e}_i \cdot \mathbf{Q}_x^T\boldsymbol{\sigma}(\mathbf{Q}_x\mathbf{F}\mathbf{Q}_x^T)\mathbf{Q}_x\mathbf{e}_j = \boldsymbol{\sigma}(\mathbf{F}') : \mathbf{e}'_i \otimes \mathbf{e}'_j,\end{aligned} \tag{17}$$

where $\mathbf{e}'_i = \mathbf{Q}_x\mathbf{e}_i$, $\mathbf{F}' = \mathbf{Q}_x\mathbf{F}\mathbf{Q}_x^T$. Note that if a permutation matrix $\mathbf{Q}_x \in O(3)$ is not a proper orthogonal transformation, then it can be replaced with $-\mathbf{Q}_x$, which permutes the axes in exactly the same way.

Now consider the simple shear $\mathbf{F} = \mathbf{F}_{12}(\gamma) \equiv \mathbf{1} + \gamma\mathbf{e}_2 \otimes \mathbf{e}_1$ and choose the corresponding shear component of stress ($i = 1,\ j = 2$). In view of the above identity, $(8)_1$, and (13), we have

$$\begin{aligned}0 &= \{\boldsymbol{\sigma}(\mathbf{F}_{12}) + p\boldsymbol{\sigma}(\mathbf{F}_{12}) + n\boldsymbol{\sigma}(\mathbf{F}_{12}) \\ &\quad - a\boldsymbol{\sigma}(\mathbf{F}_{12}) - b\boldsymbol{\sigma}(\mathbf{F}_{12}) - c\boldsymbol{\sigma}(\mathbf{F}_{12})\} : \mathbf{e}_1 \otimes \mathbf{e}_2 \\ &= \boldsymbol{\sigma}(\mathbf{F}_{12}) : \mathbf{e}_1 \otimes \mathbf{e}_2 + \boldsymbol{\sigma}(\mathbf{F}_{31}) : \mathbf{e}_3 \otimes \mathbf{e}_1 + \boldsymbol{\sigma}(\mathbf{F}_{23}) : \mathbf{e}_2 \otimes \mathbf{e}_3 \\ &\quad - \boldsymbol{\sigma}(\mathbf{F}_{21}) : \mathbf{e}_2 \otimes \mathbf{e}_1 - \boldsymbol{\sigma}(\mathbf{F}_{13}) : \mathbf{e}_1 \otimes \mathbf{e}_3 - \boldsymbol{\sigma}(\mathbf{F}_{32}) : \mathbf{e}_3 \otimes \mathbf{e}_2,\end{aligned} \tag{18}$$

which we can write as

$$\sigma_{12}(\mathbf{F}_{12}) + \sigma_{23}(\mathbf{F}_{23}) + \sigma_{31}(\mathbf{F}_{31}) - \sigma_{21}(\mathbf{F}_{21}) - \sigma_{32}(\mathbf{F}_{32}) - \sigma_{13}(\mathbf{F}_{13}) = 0, \tag{19}$$

thereby recovering condition (2). Taking normal components of the stress tensor ($i = j = 1$) yields a different scalar relation,

$$\sigma_{11}(\mathbf{F}_{12}) + \sigma_{22}(\mathbf{F}_{23}) + \sigma_{33}(\mathbf{F}_{31}) - \sigma_{22}(\mathbf{F}_{21}) - \sigma_{11}(\mathbf{F}_{13}) - \sigma_{33}(\mathbf{F}_{32}) = 0, \tag{20}$$

which can be interpreted as a relation between the Poynting effect quantified in six different shear modes (see [14] for the definition and analysis of the Poynting effect).

Another example is the case of isochoric biaxial stretch, which with intentional abuse of notation we define as $\mathbf{F}_{12} = \lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + (\lambda_1\lambda_2)^{-1/2}\mathbf{e}_3 \otimes \mathbf{e}_3$, and five other deformations $\mathbf{F}_{ij}\,(i \neq j)$ are obtained from the corresponding permutations of $\{\mathbf{e}_i\}$. Scalar relations for the shear component $i = 1,\ j = 2$ and the normal component $i = j = 1$ have exactly the same form as equations (19) and (20), respectively, in which $\mathbf{F}_{ij}$ now represent biaxial stretch deformations. The relations for the remaining stress components are

$$\sigma_{13}(\mathbf{F}_{12}) + \sigma_{12}(\mathbf{F}_{23}) + \sigma_{23}(\mathbf{F}_{31}) - \sigma_{23}(\mathbf{F}_{21}) - \sigma_{12}(\mathbf{F}_{13}) - \sigma_{13}(\mathbf{F}_{32}) = 0, \tag{21}$$

$$\sigma_{22}(\mathbf{F}_{12}) + \sigma_{33}(\mathbf{F}_{23}) + \sigma_{11}(\mathbf{F}_{31}) - \sigma_{11}(\mathbf{F}_{21}) - \sigma_{22}(\mathbf{F}_{13}) - \sigma_{33}(\mathbf{F}_{32}) = 0, \tag{22}$$

$$\sigma_{23}(\mathbf{F}_{12}) + \sigma_{12}(\mathbf{F}_{23}) + \sigma_{13}(\mathbf{F}_{31}) - \sigma_{13}(\mathbf{F}_{21}) - \sigma_{12}(\mathbf{F}_{13}) - \sigma_{23}(\mathbf{F}_{32}) = 0, \tag{23}$$

$$\sigma_{33}(\mathbf{F}_{12}) + \sigma_{11}(\mathbf{F}_{23}) + \sigma_{22}(\mathbf{F}_{31}) - \sigma_{33}(\mathbf{F}_{21}) - \sigma_{22}(\mathbf{F}_{13}) - \sigma_{11}(\mathbf{F}_{32}) = 0. \tag{24}$$

Note that by relabelling the deformation gradients $\mathbf{F}_{ij}$, each of the Eqs. (21)–(24) can be represented as either Eqs. (19) or (20). However, this does not imply that Eqs. (19)–(24) are not independent.

3. Examples. Constitutive models for myocardium

The para-universal relations derived above can be used to test how well mechanical behaviour of a real orthotropic material can be captured by a strain-energy function that is additively decomposed, as discussed previously. This can be done prior to choosing an exact form of the response function and fitting its parameters.

We compare four hyperelastic models for myocardium, namely three variants of the Holzapfel–Ogden model [15] and the model proposed by Costa et al. [16] ("Costa Law"). All these models are incompressible, composed of exponential Fung-type terms [17], and reflect the orthotropy of myocardium response and microstructure, which is defined in terms of the local orthonormal basis $\mathbf{f}_0, \mathbf{s}_0, \mathbf{n}_0$. A general Holzapfel–Ogden model was recently studied by Guan et al. [18], as given by

$$\Psi_{\mathrm{gHO}} = \psi_{\mathrm{iso}}(I_1) + \sum_{i=\mathrm{f,s,n}} \psi_i(I_{4i}) + \sum_{ij=\mathrm{fs,fn,sn}} \psi_{ij}(\hat{I}_{8ij}), \tag{25}$$

where

$$\psi_{\mathrm{iso}}(I_1) = \frac{a}{2b}\{\exp[b(I_1 - 3)] - 1\}, \tag{26}$$

$$\psi_i(I_{4i}) = \frac{a_i}{2b_i}\left\{\exp[b_i(I_{4i} - 1)^2] - 1\right\},\ i = \mathrm{f, s, n}, \tag{27}$$

$$\psi_{ij}(\hat{I}_{8ij}) = \frac{a_{ij}}{2b_{ij}}\left\{\exp(b_{ij}\hat{I}_{8ij}) - 1\right\},\ i \neq j = \mathrm{f, s, n}, \tag{28}$$

$$I_1 = \mathrm{tr}\mathbf{C}, \quad I_{4\mathrm{f}} = \mathbf{f}_0{\cdot}\mathbf{C}\mathbf{f}_0, \quad I_{4\mathrm{s}} = \mathbf{s}_0{\cdot}\mathbf{C}\mathbf{s}_0, \quad I_{4n} = \mathbf{n}_0{\cdot}\mathbf{C}\mathbf{n}_0, \tag{29}$$

$$\hat{I}_{8\mathrm{fs}} = (\mathbf{f}_0{\cdot}\mathbf{C}\mathbf{s}_0)^2, \quad \hat{I}_{8\mathrm{fn}} = (\mathbf{f}_0{\cdot}\mathbf{C}\mathbf{n}_0)^2, \quad \hat{I}_{8\mathrm{sn}} = (\mathbf{s}_0{\cdot}\mathbf{C}\mathbf{n}_0)^2. \tag{30}$$

The strain-energy function Ψ_{gHO} is determined by 14 parameters, seven of which have the dimension of stress, and the other seven are non-dimensional. The specific Holzapfel–Ogden model introduced in [15] includes only 3 anisotropic terms (f, s, and fs), hence requiring only 8 parameters,

$$\Psi_{\mathrm{HO}} = \psi_{\mathrm{iso}}(I_1) + \psi_{\mathrm{f}}(I_{4\mathrm{f}}) + \psi_{\mathrm{s}}(I_{4\mathrm{s}}) + \psi_{\mathrm{fs}}(\hat{I}_{8\mathrm{fs}}). \tag{31}$$

This model can be modified to capture local variability of structural directions (also known as *fibre dispersion*) by means of the *Generalised Structure Tensors* [19,20], as was formulated in [21],

$$\Psi^{\star}_{\mathrm{HO}} = \psi_{\mathrm{iso}}(I_1) + \psi_{\mathrm{f}}(I^{\star}_{4\mathrm{f}}) + \psi_{\mathrm{s}}(I^{\star}_{4\mathrm{s}}) + \psi_{\mathrm{fs}}(\hat{I}^{\star}_{8\mathrm{fs}}), \tag{32}$$

where the dispersed invariants are defined as $I^{\star}_{4\mathrm{f,s}} = \mathbf{H}_{\mathrm{f,s}}{:}\mathbf{C}$ and $\hat{I}^{\star}_{8\mathrm{fs}} = 4\hat{\mathbb{H}}_{\mathrm{fs}}{::}\mathbf{E} \otimes \mathbf{E}$. Here $\mathbf{E}$ is the Green–Lagrange strain tensor, $\mathbf{H}_{\mathrm{f,s}}$ and $\hat{\mathbb{H}}_{\mathrm{fs}}$ are the second-order and fourth-order structure tensors (see [21] for details). Two additional parameters, κ_{f} and κ_{s}, are needed to describe the extent of the assumed axisymmetric orientation dispersion of structural directions around the principal material axes $\mathbf{f}_0$ and $\mathbf{n}_0$. While the structure parameters $\kappa_{\mathrm{f,s}}$ should be estimated from histological studies, we treat them here phenomenologically. The no-dispersion case $\kappa_{\mathrm{f,s}} = 0$ recovers the model (31) exactly.

The fourth model is the Costa Law,

$$\Psi_{\mathrm{CL}} = c\{\exp Q - 1\}, \quad Q = \mathbf{E} : \mathbb{A} : \mathbf{E}, \tag{33}$$

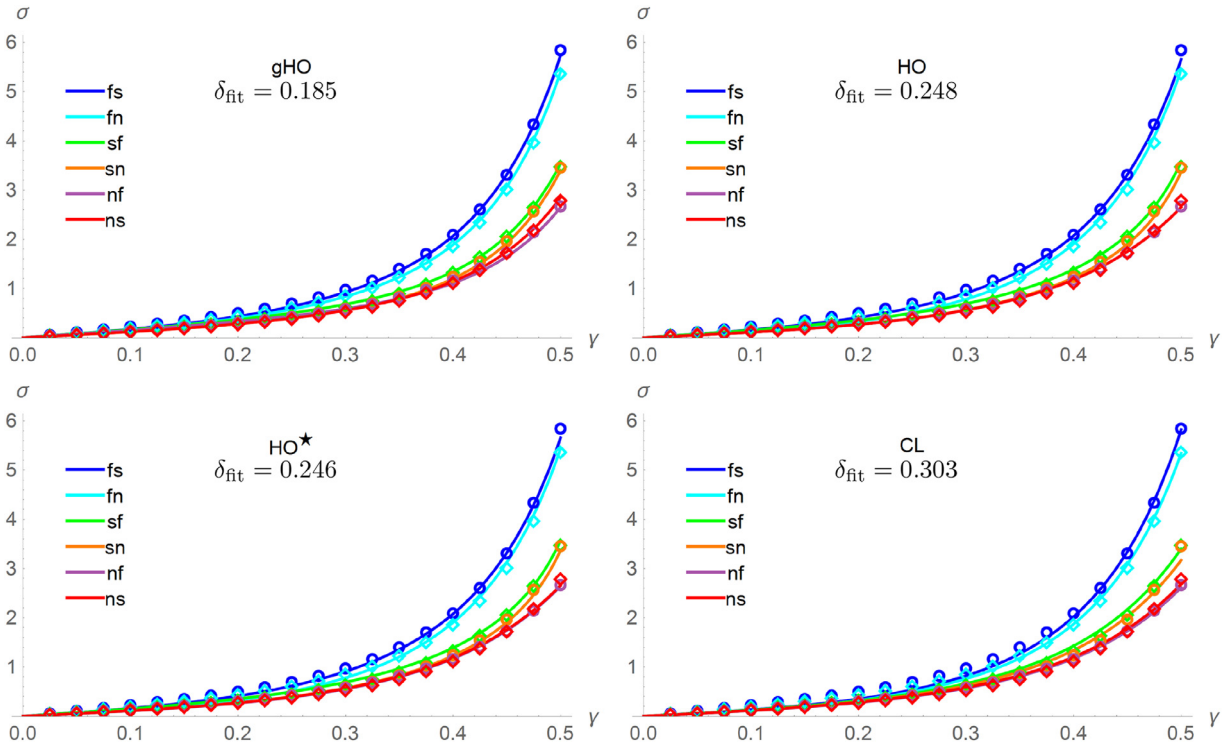

Fig. 1. Optimal fits of the shear stress versus amount of shear for the constitutive models (25), (31)–(33) to experimental data for human myocardium [23] in 6 shear modes. All models approximate the data with minor disagreement. The para-universal condition (19) is satisfied by the data up to the error of $\Delta^{\rm abs} = 0.129$, $\Delta^{\rm rel} = 0.022$. The abbreviations gHO, HO, HO⋆, CL stand for the general Holzapfel–Ogden model (25), the Holzapfel–Ogden model (31), the Holzapfel–Ogden model with dispersion (32), and the Costa Law (33), respectively.

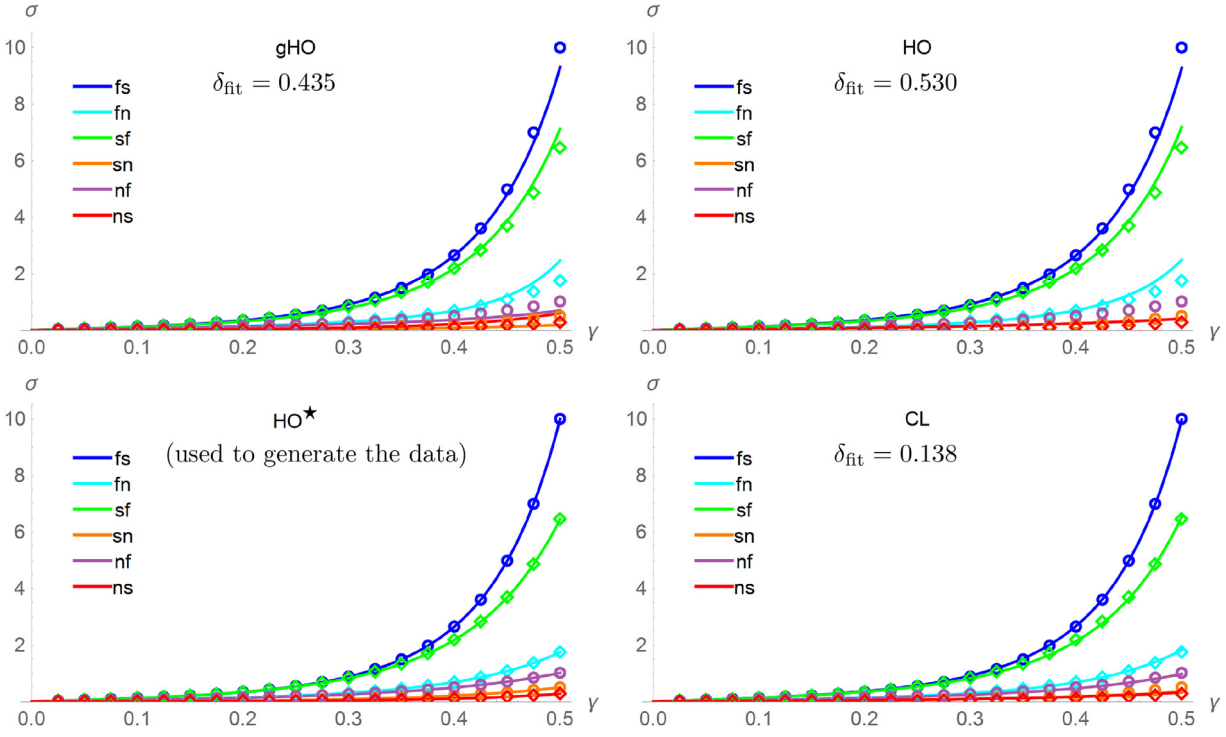

Fig. 2. Optimal fits of the shear stress versus amount of shear for the constitutive models (25), (31), and (33) to synthetic data generated using the strain-energy function (32). The models (25) and (31) (top row) provide less accurate approximation to the data, as a consequence of higher disagreement between the data and the para-universal condition (19), $\Delta^{\rm abs} = 0.299$, $\Delta^{\rm rel} = 0.029$.

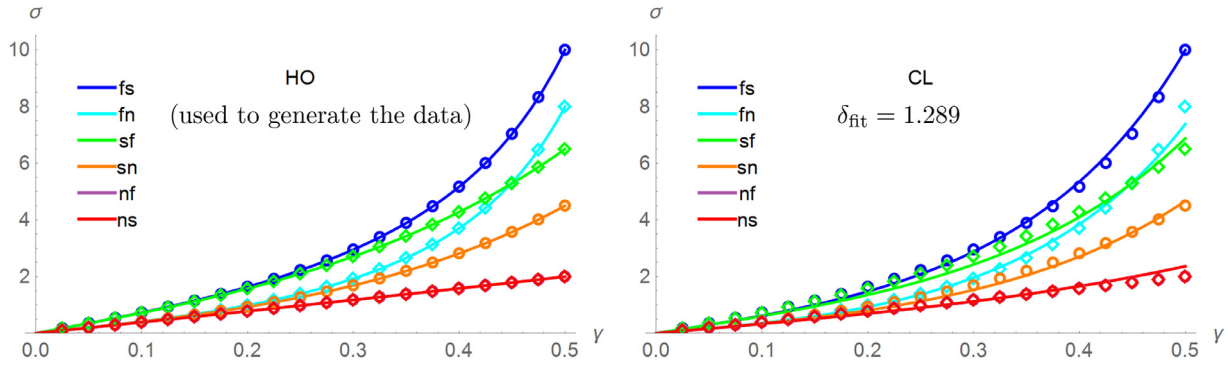

Fig. 3. Optimal fit of the shear stress versus amount of shear for the Costa Law model (33) (right) to the synthetic data generated using the Holzapfel–Ogden model (31) (left), $\Delta^{\rm abs} = \Delta^{\rm rel} = 0$. This example demonstrates that some data sets are better captured by additively split models. Models (25) and (32) are not shown, as they include model (31) as their special case.

where $\mathbb{A}$ is the fourth-order structure tensor, whose non-zero components are $A_{iiii} = b_{ii}$, $A_{ijij} = A_{ijji} = A_{jiij} = \frac{1}{2}b_{ij}$, $i \neq j$, for $i, j = \mathrm{f}, \mathrm{s}, \mathrm{n}$, that is, 7 parameters in total.

Of the four hyperelastic models introduced above, only models (25) and (31) follow the para-universal relation (13) by satisfying the prerequisites of Proposition 1: invariants $I_{4\mathrm{f}}$ and $I_{8\mathrm{sn}}$ are symmetric with respect to reflection $\mathbf{Q}_{s\leftrightarrow n} = \mathbf{f}_0 \otimes \mathbf{f}_0 + \mathbf{s}_0 \otimes \mathbf{n}_0 + \mathbf{n}_0 \otimes \mathbf{s}_0$, and so on. The same applies to the model (32) in the special case $\kappa_{\mathrm{f}} = \kappa_{\mathrm{s}}$, but not in general, due to the coupling term $\psi_{\mathrm{fs}}(\hat{I}^{\star}_{8\mathrm{fs}})$. As for the Costa Law (33), the Proposition 1 applies to it if the number of independent components of the structure tensor $\mathbb{A}$ is reduced from 6 to 2. Thus, models (33) and (32) do not adhere to the relation (13).

The para-universal relation Eqs. (19)–(24) can be used to examine experimental data only if the material response is measured in 6 different deformation modes, which are related in a precise way. Such data sets are rare and, to our knowledge, are limited to triaxial shear tests [22,23]. In order to measure quantitatively how well the experimental data satisfy condition (19), we define the absolute and the relative discrepancies

$$\Delta^{\rm abs} = \frac{1}{N}\sum_{k=1}^{N}\left|\sigma^k_{12} + \sigma^k_{23} + \sigma^k_{31} - \sigma^k_{21} - \sigma^k_{32} - \sigma^k_{13}\right|, \tag{34}$$

$$\Delta^{\rm rel} = \Delta^{\rm abs} / \max_{i,j,k}\left|\sigma^k_{ij}\right|, \tag{35}$$

where k is the data point index, and N is the total number of data points in each deformation mode. For a given material model, we denote the predicted stresses $\hat{\sigma}^k_{ij}$ and define the absolute discrepancy $\hat{\Delta}^{\rm abs}$, by analogy with (34), and the goodness of fit, $\delta_{\rm fit} = \sum_{k=1}^{N}\sum_{i,j}\left|\sigma^k_{ij} - \hat{\sigma}^k_{ij}\right|$. It can be shown that

$$\Delta^{\rm abs} \leq \delta_{\rm fit} + \hat{\Delta}^{\rm abs}. \tag{36}$$

Given a suitable data set, the evaluation of $\Delta^{\rm abs}$ and $\Delta^{\rm rel}$ is straightforward. Inequality (36) implies that $\Delta^{\rm abs}$ is a lower bound of the goodness of fit $\delta_{\rm fit}$ for models that satisfy condition (19), since $\hat{\Delta}^{\rm abs} = 0$ holds for them. The higher the discrepancies are for the actual data, the worse is the best possible fit of such models. If $\Delta^{\rm abs}$ and $\Delta^{\rm rel}$ are sufficiently high, then this class of additively split models has to be rejected. Otherwise, such models should be considered and may or may not produce good fits depending on other factors. We illustrate this below.

As the first example we take the data collected from mechanical tests of passive human myocardium [23]. To make the experimental data consistent across different shear modes, we interpolate each mode at $N = 20$ points $\gamma = 0.025 \ldots 0.5$. The values $\Delta^{\rm abs} = 0.129$, $\Delta^{\rm rel} = 0.022$ are computed as defined in (34)–(35). The parameter optimisation was performed in *Mathematica 11* using the Nelder–Mead method implemented by the built-in function *Minimize* [24]. All four models (25), (31)–(33) produce almost a perfect fit to the data, as shown in Fig. 1.

In order to consider the case of higher discrepancies $\Delta^{\rm abs}, \Delta^{\rm rel}$, we resort to generating an artificial data set using the model (32). For the specific data generated we have $\Delta^{\rm abs} = 0.299$, $\Delta^{\rm rel} = 0.0299$. We used the same optimisation procedure, as previously described, to obtain the optimal fits of models (25), (31)–(33), which are shown in Fig. 2. As expected, the additively split models produce a worse fit, as they follow the para-universal relation (13), while the data does not. The Costa Law (33), which does not have this particular constraint, replicates the material behaviour well. However, no general conclusions can be drawn from this example, which is only used to illustrate how the para-universal relations can help select the most suitable class of material models based on experimental data.

It should be emphasised that the additive decomposition of the response function should not be viewed as a definite limitation. In Fig. 3 we show that artificial data generated by the Holzapfel–Ogden model (31), which is additively split, is not approximated well by the Costa Law (33), which is not additively split.

4. Conclusion

We have proposed a new type of universal results—the para-universal relations, which hold for a wide class of materials and relate material response in different deformation modes. We have considered the para-universal relation (13) and its special forms (19)–(24). In Proposition 1 we showed that the relation holds for any additively decomposed orthotropic response function, as long as each additive term is symmetric with respect to one of the axes of orthotropy. In fact, it is only required that each component is invariant with respect to some permutation of the material axes. These prerequisites are automatically satisfied for all transversely isotropic and isotropic materials, in which the para-universal relation can be viewed as a direct consequence of material symmetry.

Additively decomposed response functions are widespread in structure-based constitutive modelling of soft biological tissues, see e.g., [19,25]. The decomposition can be justified by the correspondence between physical components of the tissue and the terms of the response function, or motivated by the convenience of handling such function, fitting its parameters and its linearised form. However, the consequences of the additive decomposition assumption in nonlinear elasticity have not been studied. Our result provides the first rigorous account of this. In addition, the proposed para-universal relation strongly distinguishes two classes of

orthotropic materials. It is well known that a material with two families of fibres is orthotropic if the families are mechanically equivalent or if the associated special directions are orthogonal [26,27]. The proposed para-universal relation only holds for the latter case. The para-universal relation breaks down, for example, in the case of models with fibre dispersion in coupling terms or models that are not additively decomposed. We also note that the para-universal relation should be distinguished from the so-called pseudo-universal relations, as discussed in [28].

Although our result is primarily of theoretical interest, its possible practical application to constitutive modelling is illustrated in Section 3. If the para-universal relation is not satisfied by experimental data collected from mechanical tests, the class of material models mentioned above should be rejected, as no model of this class will produce a satisfactory representation. Note that the available experimental data sets [22,23] fulfil the para-universal condition up to the order of experimental error. In that regard, one could make use of the converse result, which has not been presently established: if a material follows the para-universal relation, then its response function has the additively split form described above. Possible directions of future investigation include proving or disproving this statement, as well as establishing new para-universal relations for other general forms of response functions.

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