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THE REALIZABILITY OF SOME FINITE-LENGTH MODULES OVER THE STEENROD ALGEBRA BY SPACES

ANDREW BAKER AND TILMAN BAUER

ABSTRACT. The Joker is an important finite cyclic module over the mod-2 Steenrod algebra $\mathcal{A}$. We show that the Joker, its first two iterated Steenrod doubles, and their linear duals are realizable by spaces of as low a dimension as the instability condition of modules over the Steenrod algebra permits. This continues and concludes prior work by the first author and yields a complete characterization of which versions of Jokers are realizable by spaces or spectra and which are not. The constructions involve sporadic phenomena in homotopy theory (2-compact groups, topological modular forms) and may be of independent interest.

1. INTRODUCTION

Let $\mathcal{A}$ be the mod-2 Steenrod algebra, minimally generated by the Steenrod squares Sq^{2^s} $(s \geqslant 0)$, and $\mathcal{A}(n) \subseteq \mathcal{A}$, for $n \geqslant 0$, the finite sub-Hopf algebra generated as an algebra by $\{\mathrm{Sq}^{2^s} \mid s \leqslant n\}$.

A (left) $\mathcal{A}$-module M is *stably realizable* if there exists a spectrum X such that as $\mathcal{A}$-modules,

$$H^*(X) \underset{\mathrm{def}}{=} H^*(X;\mathbf{F}_2) \cong M.$$

For finite $\mathcal{A}$-modules, this is equivalent to the existence of a space Z such that $\widetilde{H}^*(Z) \cong \Sigma^s M$ for some s. This number s is bounded from below by the *unstable degree* $\sigma(M)$ of M, i.e. the minimal number t such that $\Sigma^t M$ satisfies the instability condition for modules over $\mathcal{A}$. We say that M is *optimally realizable* if there exists a space Z such that $\widetilde{H}^*(Z) \cong \Sigma^{\sigma(M)} M$.

We consider two constructions of new Steenrod modules from old. Firstly, for a left $\mathcal{A}$-module M, the linear dual $M^\vee = \mathrm{Hom}(M, \mathbf{F}_2)$ becomes a left $\mathcal{A}$-module using the antipode of $\mathcal{A}$. Secondly, the *iterated double* $\Phi^i M$ is the module which satisfies

$$\Phi^i M^n = \begin{cases} M^{n/2^i} & \text{if } 2^i \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

and for $x \in \Phi^i M^n$,

$$\mathrm{Sq}^{2^k} x = \begin{cases} 0 & \text{if } k < i, \\ \mathrm{Sq}^{2^{k-i}} x & \text{if } k \geqslant i. \end{cases}$$

We also set $\Phi^0 M = M$.

Let J be the quotient of $\mathcal{A}$ by the left ideal generated by Sq^3 and $\mathrm{Sq}^i \mathcal{A}$ for $i \geqslant 4$.

The main result of this paper is the following.

Theorem 1.1. *The modules $\Phi^i J$ and $\Phi^i J^\vee$ are optimally realizable for $i \leqslant 2$ and not stably realizable for $i > 2$.*

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The module J in this theorem is known as the *Joker*, although more colloquially than in actual written articles.

In [Bak18], the first author showed all cases of Theorem 1.1 with the exception of the optimal realizability of $\Phi^2 J$ and $\Phi^2 J^\vee$:

Theorem 1.2 ([Bak18])**.** *The module $\Phi^k J$ is stably realizable iff $k \leqslant 2$ iff $\Phi^k J^\vee$ is stably realizable.*

For $k \leqslant 1$, the modules $\Phi^k J$ and $\Phi^k J^\vee$ are optimally realizable.

The main result of this paper is the case of $k = 2$. We will, however, also give an alternative proof of the cases where $k < 2$, which may aid as an illustration of how the more complicated case of $k = 2$ works.

For the case of $\Phi^2 J$, we observe that this module appears as a quotient module of the rank-4 Dickson algebra, which is realized by the exotic 2-compact group $B\mathrm{DW}_3$ constructed by Dwyer and Wilkerson. Our approach to constructing an optimal realization is to map a suitable skeleton of $B\mathrm{DW}_3$ to the spectrum tmf/2 of topological modular forms modulo 2 so that a skeleton of the homotopy fiber of this map has cohomology $\Phi^2 J$. The existence of this map $\alpha\colon (B\mathrm{DW}_3)^{(24)} \to \underline{\mathrm{tmf}/2}_{14}$ is equivalent to the survival of a class $x_{-15} \in H^{15}((B\mathrm{DW}_3)^{24})$ in the $\mathcal{A}(2)$-based Adams spectral sequence. This is the content of Section 7.

One might interpret the survival of this class as (albeit weak) evidence that a faithful 15-dimensional spherical homotopy representation of DW_3 exists, a question we hope to address in a later paper. The lowest known faithful homotopy representation of DW_3 at the time of this writing has complex dimension 2^{46} [Zie09].

Our alternative proof for $\Phi^1 J$ in Section 6 follows the same line of reasoning as for $\Phi^2 J$, but using the rank-3 Dickson algebra (realized by the classifying space of the Lie group G_2) and real K-theory instead of topological modular forms. In this case, we are able to construct the analog of the map α geometrically.

Conventions. We assume that all spaces and spectra are completed at the prime 2, although our arguments can be easily modified to work globally. We will often assume that we are working with CW complexes which have been given minimal cell structures. All cohomology is with coefficients in $\mathbf{F}_2$, the field with two elements. For spaces, we use unreduced cohomology but for spectra reduced cohomology; hopefully the usage should be clear from the context.

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2. Realizability of modules over the Steenrod algebra

Recall that an $\mathcal{A}$-module is called *unstable* if

$$\mathrm{Sq}^i(x) = 0 \text{ if } i > |x|.$$

Definition. Let M be an $\mathcal{A}$-module. The *unstable degree* $\sigma(M)$ of M is the minimal $s \in \mathbf{Z}$ such that $\Sigma^s M$ is an unstable $\mathcal{A}$-module.

Obviously, $\sigma(M)$ is a finite number if M is a nontrivial finite module (but may be infinite otherwise). As the anonymous referee pointed out, using lower-indexing

$\mathrm{Sq}_i\, x = \mathrm{Sq}^{|x|-i}\, x$ of the Steenrod squares,

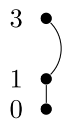
$$-\sigma(M) = \inf\{i \mid \mathrm{Sq}_i \neq 0\}$$

since a module is unstable iff $\mathrm{Sq}_i = 0$ for all $i < 0$.

If a finite module is stably realizable, Freudenthal's theorem implies that it is realizable by a space after sufficiently high suspension (cf. Proposition 4.2). If M is stably realizable by a spectrum X then M^* is stably realized by the Spanier-Whitehead dual $DX = F(X, \mathbf{S}^0)$. Only a finite number of iterated doubles of M can ever be stably realizable by the solution of the Hopf invariant 1 problem.

Example 2.1. Any $\mathcal{A}$-module M of dimension 1 over $\mathbf{F}_2$ is optimally realizable (by a point). Let M be cyclic of dimension 2 over $\mathbf{F}_2$, thus $M \cong \mathbf{F}_2\langle \iota, \mathrm{Sq}^{2^i}\, \iota\rangle$. By the solution of the Hopf invariant one problem, M is stably realizable if and only if $i = 0, 1, 2, 3$. In each case, M is optimally realizable by the projective plane over $\mathbf{R}$, $\mathbf{C}$, the quaternions, and the octonions, respectively.

Example 2.2. A simple example of a module that is not optimally realizable is the "question mark" complex



or $M = \mathcal{A}/(\mathrm{Sq}^2, \mathrm{Sq}^3, \dots)$. This picture, and others to follow, are to be read as follows. The numbers on the left denote the dimension. A dot denotes a copy of $\mathbf{F}_2$ in the corresponding dimension. A straight line up from a dot x to a dot y indicates that $\mathrm{Sq}^1 x = y$, and a curved line similarly indicates a nontrivial operation Sq^2. The unstable degree of this module is 1, but it is not optimally realizable because a hypothetical space X with $H^*(X) = \Sigma M$, $H^1(X) = \langle x\rangle$ would have $x^4 = (x^2)^2 \neq 0$ but $x^3 = 0$.

3. The family of Jokers

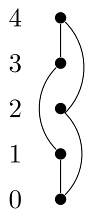
The finite cyclic $\mathcal{A}$-module

$$J = \mathcal{A}/(\mathcal{A}\,\mathrm{Sq}^3 + \mathcal{A}\,\mathrm{Sq}^4\,\mathcal{A} + \mathcal{A}\,\mathrm{Sq}^8\,\mathcal{A} + \cdots)$$

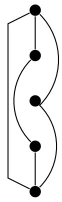
is called the Joker. Its dimension over $\mathbf{F}_2$ is 5, having dimension 1 in each degree $0 \leqslant d \leqslant 4$; a basis is given by

$$\{1, \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^2\,\mathrm{Sq}^1, \mathrm{Sq}^2\,\mathrm{Sq}^2 = \mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1\},$$

or pictorially,



The Joker appears in several contexts in homotopy theory. In [AP76], Adams and Priddy showed that J generates the torsion on the Picard group of $\mathcal{A}(1)$-modules. The Joker also appears regularly in projective resolutions of cohomologies of common spaces (such as real projective spaces) over $\mathcal{A}$ or $\mathcal{A}(1)$. Its linear dual $J^\vee = \mathrm{Hom}(J, \mathbf{F}_2)$ is also a cyclic left module by the antipode χ of $\mathcal{A}$. It is not isomorphic to J, even by a shift, since $\chi(\mathrm{Sq}^4) = \mathrm{Sq}^4 + \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1$, which means that $\mathrm{Sq}^4 \neq 0$ on $J^\vee$. Pictorially,

$J^\vee =$ 

Here and in what follows, the slanted square brackets denote nontrivial operations Sq^4 or Sq^8. Note that $J^\vee \cong J$ as $\mathcal{A}(1)$-modules.

The k-fold iterated doubles of these are $\Phi^k J$ and $\Phi^k J^\vee$, where $\Phi^1 J$ has basis vectors in even dimensions 0, 2, 4, 8, 10, $\Phi^2 J$ has basis vectors in dimensions divisible by 4, and so on.

Clearly, the unstable degrees are given by $\sigma(\Phi^k J) = 2 \cdot 2^k$ (the bottom cohomology class supports a nontrivial operation $\mathrm{Sq}^{2 \cdot 2^k}$), and $\sigma(\Phi^k J^\vee) = 4 \cdot 2^k$ (the bottom cohomology class supports also a nontrivial operation $\mathrm{Sq}^{4 \cdot 2^k}$).

Note that if $\Phi^k J$ is optimally realized by a space, then that space is weakly equivalent to a CW complex X with cells in dimensions $i \cdot 2^k$, where $i = 2, \dots, 6$, hence X has dimension $6 \cdot 2^k$. The ring structure of the cohomology is implied by the instability condition for $\mathcal{A}$-algebras, namely, $\mathrm{Sq}^i(x) = x^2$ when $|x| = i$:

$$H^*(X) = \mathbf{F}_2[x_2, x_3]/(x_2, x_3)^3 \quad (|x_i| = i \cdot 2^k).$$

If $\Phi^k J^\vee$ is optimally realizable by a space, then that space is weakly equivalent to a CW complex Y with cells in dimensions $j \cdot 2^k$, where $j = 4, 5, 6, 7, 8$. For dimensional reasons, the ring structure of the cohomology has to be

$$H^*(Y) = \mathbf{F}_2[x_4, x_5, x_6, x_7]/(x_4^3) + x_4(x_5, x_6, x_7) + (x_5, x_6, x_7)^2; \quad (|x_i| = i \cdot 2^k).$$

4. Dual Jokers

If X is a spectrum with $H^*(X) \cong \Phi^k J$ then it is obvious that the Spanier-Whitehead dual DX realizes $\Phi^k J^\vee$, up to a degree shift, i.e.,

$$\Sigma^{4 \cdot 2^k} H^*(DX) \cong \Phi^k J^\vee.$$

Unstably, the situation is a bit more complicated, but follows from a more general consideration.

Lemma 4.1. *Let M be a finite $\mathcal{A}$-module with top nonvanishing degree n and Y a space with an injective $\mathcal{A}$-module map $f\colon M \to H^*(Y)$ whose cokernel is $n-1$-connected. Then there is a space Z such that $H^*(Z) \cong M$ as $\mathcal{A}$-modules.*

Proof. Let V be a complement of $\mathrm{im}(f)$ in $H^n(Y)$ and denote by $\alpha\colon Y \to K(V, n)$ its representing map. Let Z be the n-skeleton of the homotopy fiber of α. Then $H^*(Z) \cong M$. □

Proposition 4.2. *Let M be a finite, stably realizable, nonnegatively graded $\mathcal{A}$-module with top nonvanishing degree n. Then $\Sigma^n M = H^*(Z)$ for some CW complex Z.*

Proof. Let X be a spectrum such that $M = H^*(X)$ and consider the space $Y = \Omega^\infty \Sigma^n X$.

Since for any k-connected spectrum E, the augmentation $\Sigma^\infty \Omega^\infty E \to E$ is $(2k+2)$-connected, the map $\Sigma^\infty Y \to \Sigma^n X$ is $2(n-1)+2 = 2n$-connected. Hence the induced map $H^i(\Sigma^n X) \to H^i(Y)$ is an isomorphism for $i < 2n$ and injective for

$i = 2n$. By Lemma 4.1, there exists a space Z such that $H^*(Z) \cong H^*(\Sigma^n X)$ as $\mathcal{A}$-modules. □

Corollary 4.3. *If $\Phi^k J$ is stably realizable for any k then $\Phi^k J^\vee$ is optimally realizable.*

Proof. The module $\Phi^k J$ has top nonvanishing degree $4 \cdot 2^k$, so $M = \Sigma^{4\cdot 2^k} \Phi^k J^\vee$ satisfies the condition of Proposition 4.2 for $n = 4 \cdot 2^k$. Hence there is a space Z such that $H^*(Z) \cong \Sigma^{8\cdot 2^k} \Phi^k J^\vee$. Then Z has its bottom cell in degree

$$4 \cdot 2^k = \sigma(\Phi^k J^\vee) = 4 \cdot 2^k,$$

proving the claim. □

Applying Proposition 4.2 to a stably realized $\Phi^k J$ gives a space Z such that $H^*(Z) \cong \Sigma^{4\cdot 2^k} \Phi^k J$, but since $\sigma(\Phi^k J) = 2 \cdot 2^k$, this does not suffice to prove optimal realizability of $\Phi^k J$. This is why the following sections are needed.

5. Dickson algebras and their realizations

The rank-n algebra of Dickson invariants $DI(n)$ is the ring of invariants of $\mathrm{Sym}(\mathbf{F}_2^n) = \mathbf{F}_2[t_1, \ldots, t_n]$ under the action of the general linear group $\mathrm{GL}_n(\mathbf{F}_2)$. We think of $\mathrm{Sym}(\mathbf{F}_2^n)$ as a graded commutative ring with t_i in degree 1. Dickson [Dic11] showed that

$$DI(n) \cong \mathbf{F}_2[x_{2^n-2^i} \mid 0 \leqslant i < n],$$

where subscripts denote degrees. The polynomials $x_{2^n-2^i}$ are given by the formula

$$\prod_{v\in \mathbf{F}_2^n} (X + v) = \sum_{i=0}^{n} x_{2^n-2^i} X^{2^i} \in \mathrm{Sym}(\mathbf{F}_2^n)[X],$$

where $x_0 = 1$ by convention. If we give $\mathrm{Sym}(\mathbf{F}_2^n)$ the structure of an $\mathcal{A}$-algebra with $\mathrm{Sq}(t_i) = t_i + t_i^2$ (i.e., by using the isomorphism $\mathrm{Sym}(\mathbf{F}_2^n) \cong H^*(B\mathbf{F}_2^n)$) then $DI(n)$ is an $\mathcal{A}$-subalgebra with

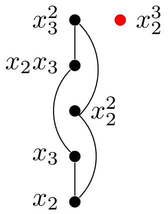
$$\mathrm{Sq}^{2^i} x_{2^n-2^{i+1}} = x_{2^n-2^i}.$$

Theorem 5.1 (Smith-Switzer, Lin-Williams, Dwyer-Wilkerson)**.** *The Dickson algebra $DI(n)$ is optimally realizable iff $n \leqslant 4$.* □

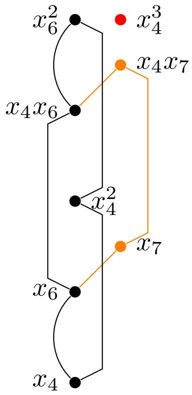
The first three Dickson algebras are realized by $\mathbf{R}P^\infty$, $BSO(3)$, and BG_2 (the classifying space of the exceptional Lie group G_2), respectively. The case $n = 4$ was settled in [DW93], where Dwyer and Wilkerson constructed a 2-complete space, the exceptional 2-compact group $B\mathrm{DW}_3$, with the required cohomology.

A graphical representation of a skeleton of the spaces realizing the Dickson algebras is given below. One observes that the Jokers $\Phi^i J$ occur as quotients of skeleta of these spaces; the kernel consists of the classes on the right of each diagram. However, realizing these quotients as fibers of certain maps is non-obvious and the purpose of the following section.

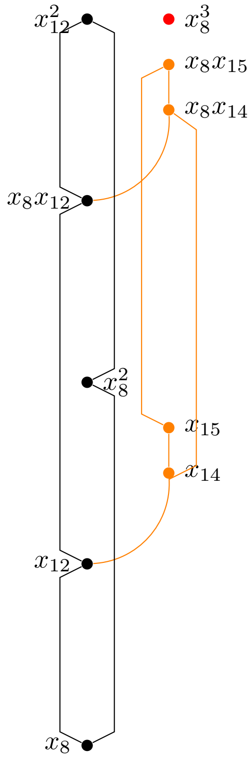
(5.2) $BSO(3)$:



(5.3) BG_2:



(5.4) BDW_3:



6. The Joker J and its double

The cohomology picture (5.2) shows that the 6-skeleton of $B\mathrm{SO}(3)$ is almost a realization of $J = \Phi^0 J$, its only defect lying in an additional class x_2^3 in the top cohomology group $H^6(B\mathrm{SO}(3))$. Let $\alpha\colon B\mathrm{SO}(3) \to K(\mathbf{F}_2, 6)$ represent this class and $X = \mathrm{hofib}(\alpha)^{(6)}$, the 6-skeleton of its homotopy fiber. Then X realizes $\Phi^0 J$ optimally.

For the double Joker $\Phi^1 J$, as seen in the cohomology picture (5.3), it does not suffice any longer to take a skeleton of $B\mathrm{G}_2$ and kill off a top-dimensional class. Since we feel that the ideas that come up here led us to the work appearing in Section 7 we feel it worth describing them in some detail.

First we recall some standard results on the exceptional Lie group G_2 and its relationship with Spin(7).

One definition of G_2 is as the group of automorphisms of the alternative division ring of Cayley numbers (octonions) $\mathbf{O}$. Since G_2 fixes the real Cayley numbers, it is a closed subgroup of $\mathrm{SO}(7) \leqslant \mathrm{SO}(8)$.

A different point of view is to consider the spinor representation of Spin(7). Recall that the Clifford algebra $Cl_6 \cong \mathrm{Mat}_8(\mathbf{R})$ is isomorphic to the even subalgebra of $Cl_7 \cong \mathrm{Mat}_8(\mathbf{R}) \times \mathrm{Mat}_8(\mathbf{R})$, so Spin(7) is naturally identified with a subgroup of $\mathrm{SO}(8) \subseteq \mathrm{Mat}_8(\mathbf{R})$, and thus acts on $\mathbf{R}^8$ with its spinor representation. Then on identifying $\mathbf{R}^8$ with $\mathbf{O}$, we find that the stabilizer subgroup in Spin(7) of a non-zero vector is isomorphic to G_2. It follows that the natural fibration

$$\mathrm{Spin}(7)/\mathrm{G}_2 \to B\mathrm{G}_2 \to B\,\mathrm{Spin}(7)$$

is the unit sphere bundle of the associated spinor vector bundle $\sigma \to B\,\mathrm{Spin}(7)$.

The mod-2 cohomologies of these spaces are related as follows. By considering the natural fibration

$$K(\mathbf{F}_2,1) \to B\,\mathrm{Spin}(7) \to B\mathrm{SO}(7)$$

we find that

$$H^*(B\,\mathrm{Spin}(7)) = \mathbf{F}_2[w_4, w_6, w_7, u_8]$$

where the w_i are the images of the universal Stiefel-Whitney classes in

$$H^*(B\mathrm{SO}(7)) = \mathbf{F}_2[w_2, w_3, w_4, w_5, w_6, w_7],$$

and $u_8 \in H^8(B\,\mathrm{Spin}(7))$ is detected by $z_1^8 \in H^8(K(\mathbf{F}_2,1))$. It is known that

$$H^*(B\mathrm{G}_2) = \mathbf{F}_2[x_4, x_6, x_7]$$

and it is easy to see that the generators can be taken to be the images of w_4, w_6, w_7 under the induced homomorphism $H^*(B\,\mathrm{Spin}(7)) \to H^*(B\mathrm{G}_2)$. As a consequence, these x_i are Stiefel-Whitney classes of the pullback $\rho_7 \to B\mathrm{G}_2$ of the natural 7-dimensional bundle $\rho \to B\mathrm{SO}(7)$ and since this lifts to a Spin bundle, it admits an orientation in real connective K-theory. This leads to the following observation.

Lemma 6.1. *There is a factorisation*

$$B\mathrm{G}_2 \to \underline{k\mathrm{O}}_7 \to K(\mathbf{F}_2, 7)$$

of a map representing $x_7 \in H^7(B\mathrm{G}_2)$.

Here

$$\underline{k\mathrm{O}}_7 = \Omega^\infty\Sigma^7 k\mathrm{O} \sim \Omega B\mathrm{O}\langle 8\rangle.$$

and $\underline{k\mathrm{O}}_7 \to K(\mathbf{F}_2,7)$ is the infinite loop map induced from the unit morphism $k\mathrm{O} \to H\mathbf{F}_2$.

The cohomology of $B\mathrm{O}\langle 8\rangle$ is a quotient of that of $B\mathrm{O}$:

$$\begin{aligned}(6.2)\quad H^*(B\mathrm{O}\langle 8\rangle) = \mathbf{F}_2[w_{2^r} : r \geqslant 3] &\otimes \mathbf{F}_2[w_{2^r+2^{r+s}} : r \geqslant 2,\ s \geqslant 1] \\ &\otimes \mathbf{F}_2[w_{2^r+2^{r+s}+2^{r+s+t}} : r \geqslant 1,\ s,t \geqslant 1] \\ &\otimes \mathbf{F}_2[w_{2^r+2^{r+s}+2^{r+s+t}} : r \geqslant 0,\ s,t \geqslant 1],\end{aligned}$$

where the w_i are images of universal Stiefel-Whitney classes in $H^*(B\mathrm{O})$. Here

$$\mathrm{Sq}^4 w_8 \equiv w_{12} \quad \text{(mod decomposables)}.$$

A routine calculation shows that $H^*(\underline{k\mathrm{O}}_7) \cong H^*(\Omega B\mathrm{O}\langle 8\rangle)$ is the exterior algebra on certain elements $e_i \in H^i(\underline{k\mathrm{O}}_7)$ where e_i suspends to the generator w_{i+1} of (6.2).

In particular, up to degree 13,

$$H^*(\underline{k\mathrm{O}}_7) = \mathbf{F}_2\{1, e_7, e_{11}\}$$

and

$$(6.3)\qquad \mathrm{Sq}^4 e_7 = e_{11}.$$

Lemma 6.4. *The module* $\Phi^1 J$ *is optimally realizable.*

Proof. Let $\alpha\colon B\mathrm{G}_2 \to \underline{k\mathrm{O}}_7$ be the factorization of Lemma 6.1. By the above computations, $H^*(B\mathrm{G}_2)$, as an algebra over $H^*(\underline{k\mathrm{O}}_7)$, is isomorphic to

$$H^*(B\mathrm{G}_2) \cong H^*(\underline{k\mathrm{O}}_7)[x_4, x_6, \text{generators in degree greater than } 12]/R$$

where the module R of relations is at least 13-connected. This means that in the Eilenberg-Moore spectral sequence for the cohomology of the fiber of α, $E_2^{s,t} = 0$ for $s+t \leqslant 12$ and $s < 0$. Thus up to degree 12,

$$H^*(\mathrm{hofib}(\alpha)) \cong \mathbf{F}_2[x_4, x_6].$$

An application of Lemma 4.1 takes care of the remaining top class x_4^3 and shows that $\Phi^1 J$ is optimally realizable. □

7. The quadruple Joker

The strategy to construct an optimal realization of $\Phi^2 J$ consists of an easy and a harder step. The easy step is to construct a space Y whose cohomology is diagram 5.4 without the topmost unattached class:

Lemma 7.1. *There exists a space Y with*

$$H^*(Y) = \mathbf{F}_2[x_8, x_{12}, x_{14}, x_{15}]/(x_8^3, \textit{polynomials of degree} > 24)$$

Proof. This follows from an application of Lemma 4.1 to the 24-skeleton of $B\mathrm{DW}_3$. □

The harder step is to realize $\Phi^2 J$ as a skeleton of the homotopy fiber of a suitable map

$$\alpha\colon Y \to \underline{\mathrm{tmf}/2}_{14}$$

into the 14th space of the spectrum of topological modular forms modulo 2. The spectrum tmf is an analog of connective real K-theory, $k\mathrm{O}$, but of chromatic level 2 [HM14, DFHH14, Beh14, Goe10] with well-known homotopy [Bau08].

Proposition 7.2. *Let Y be a space as in Lemma 7.1. Then there exists a 2-torsion class*

$$\beta \in \mathrm{tmf}^{15}(Y)$$

whose classifying map induces an isomorphism of the order-2 groups

$$H^{15}(\underline{\mathrm{tmf}}_{15}) \to H^{15}(Y).$$

Proof of Thm. 1.1. Given Prop. 7.2 and the Bockstein spectral sequence, the class β has to pull back to a class $\alpha \in (\mathrm{tmf}/2)^{14}(Y)$ whose classifying map induces an isomorphism in H^{14}. This means that under $\alpha^*\colon H^*(\underline{\mathrm{tmf}/2}_{14}) \to H^*(Y)$, the unit $\iota \in H^{14}(\underline{\mathrm{tmf}/2}_{14})$ is mapped to x_{14}.

A basic property of tmf is that $H^*(\mathrm{tmf}) \cong \mathcal{A} \otimes_{\mathcal{A}(2)} \mathbf{F}_2$ and so there is a non-split extension of $\mathcal{A}$-modules

$$0 \to H^*(\mathrm{tmf}) \to H^*(\mathrm{tmf}/2) \to \Sigma H^*(\mathrm{tmf}) \to 0$$

where Sq^1 acts non-trivially on the generator of $\Sigma H^0(\mathrm{tmf})$.

This implies that $\alpha^*(\mathrm{Sq}^1 \iota) = x_{15}$, $\alpha^*(\mathrm{Sq}^8 \iota) = x_8 x_{14}$, and $\alpha^*(\mathrm{Sq}^8 \mathrm{Sq}^1 \iota) = x_8 x_{15}$. Hence as in the case of $\Phi^1 J$,

$$H^*(Y) \cong H^*(\underline{\mathrm{tmf}/2}_{14})[x_8, x_{12}, \text{generators in degree greater than } 24]/(x_8^3, R),$$

where the module R of relations is at least 25-connected. Thus the Eilenberg-Moore spectral sequence converging to hofib(α) shows that up to degree 24,

$$H^*(\mathrm{hofib}(\alpha)) \cong \mathbf{F}_2[x_8, x_{12}]/(x_8^3),$$

The 24-skeleton of hofib(α) therefore optimally realizes $\Phi^2 J$. □

It remains to prove Prop. 7.2.

Let $Y_m^n = Y^{(n)}/Y^{(m-1)}$ denote the n-skeleton of Y modulo the $(m-1)$-skeleton, thus containing the cells from dimension m to dimension n.

Lemma 7.3. *Let Y be a space as in Lemma 7.1. Then the space Y_{16}^{20} is homotopy equivalent to a suspension of the cone of $\pm 2\nu$. In particular, in the Adams spectral sequence*

$$\mathrm{Ext}_{\mathcal{A}(2)}(\mathbf{F}_2, H^{-*}(Y_{16}^{20}) \Longrightarrow \mathrm{tmf}^{-*}(Y_{16}^{20}),$$

there is a differential $d^2(x_{-16}) = x_{-20} h_0 h_2$, where x_{-16}, x_{-20} in Ext^0 are the classes corresponding to the two cells.

Here the grading is chosen such that the spectral sequence becomes a homological spectral sequence and we will display it in the Adams grading.

Proof. Consider the space Y^{20}, the 20-skeleton of Y. In the Atiyah-Hirzebruch spectral sequence

$$H^{-*}(Y^{20}, \mathrm{tmf}^{-*}) \cong H_*(D(Y^{20}), \mathrm{tmf}_*) \Longrightarrow \mathrm{tmf}^{-*}(Y^{20}),$$

the cohomology generators x_8, x_{12} represent classes $x_{-8}, x_{-12} \in H_*(DY^{20}, \mathrm{tmf}_0)$ and, since $\mathrm{Sq}^4(x_8) = x_{12}$, there is a differential $d^4(x_{-8}) = x_{-12}\nu \pmod{2\nu}$. By multiplicativity, $d^4(x_{-8}^2) = 2x_{-8}x_{-12}\nu \pmod{4\nu}$. This shows that the top cell of Y^{20} is attached to the 16-dimensional cell by $2\nu \pm 4\nu = \pm 2\nu$. □

Proof of Prop. 7.2. The claim boils down to showing that in the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}(H^{-*}(\mathrm{tmf}), H^{-*}(Y)) \cong \mathrm{Ext}_{\mathcal{A}(2)}(\mathbf{F}_2, H^{-*}(Y)) \Longrightarrow \mathrm{tmf}^{-*}(Y),$$

the unique nontrivial class

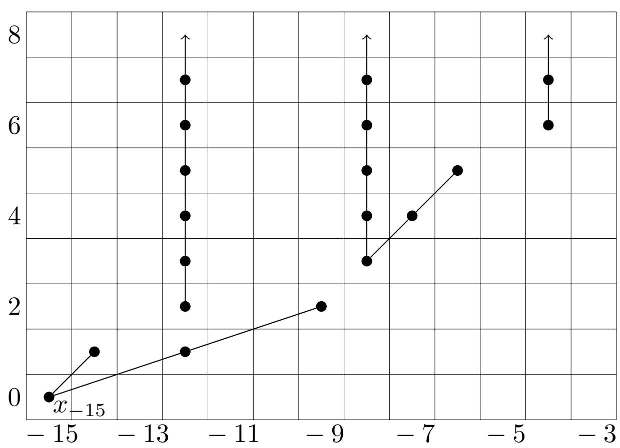
$$x_{-15} \in E_2^{0,-15} = \mathrm{Hom}_{\mathcal{A}}(H^0(\mathrm{tmf}), H^{15}(Y))$$

is an infinite cycle.

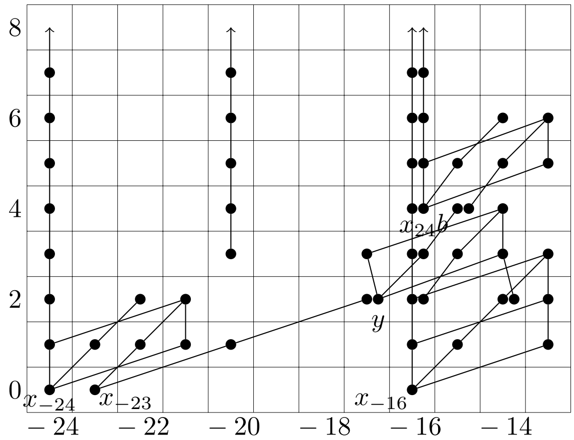
As modules over $\mathcal{A}(2)$,

$$H^*(Y) \cong H^*(Y_0^{15}) \oplus H^*(Y_{16}^{24}),$$

hence the E_2-term above splits as a sum as well. The following is the E_2-term of the Adams spectral sequence converging to $\mathrm{tmf}^{-*}(Y^{(15)})$, determined with Bob Bruner's program [Bru]:

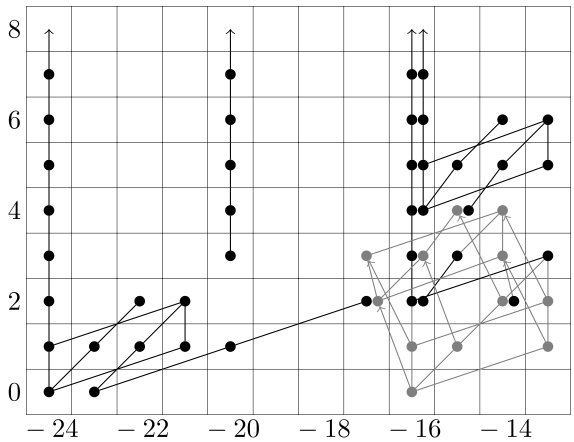


Similarly, the E_2-term of the spectral sequence computing $\mathrm{tmf}^*(Y_{16}^{24})$ is the following.



We will identify the possible targets of differentials on ι. Since $h_0\iota = 0$ and $h_1^2\iota = 0$, only classes in the kernel of h_0 and the kernel of h_1^2 can be targets. This means that there is no possible target for a d_2 in bidegree $(-16, 2)$ in the displayed Adams grading.

It is easy to see that under the inclusion $Y_{16}^{20} \to Y_{16}^{24}$, the class $x_{-20}h_0h_2$ maps to the displayed class y. By Lemma 7.3, there is thus a differential $d_2(x_{-16}) = y$. This implies that the E_3-term for Y_{16}^{24} is given by the following chart.



The only remaining possible target of a d_3 in bidegree $(-16, 3)$ is $x_{-16}h_0^3$, which is impossible for the same reason as before (h_0 on it is nontrivial.)

There are no longer differentials possible on ι either because no classes in filtration 4 or higher are h_0-torsion in any E_n-term. □

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