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Twisted Laplace Maps

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Abstract

We consider general Darboux maps arising from intertwining relations on second order, linear partial differential operators, as deformations of the classical, Laplace case. We present Lax pairs for the corresponding relations on invariants and discuss the conditions for a lattice structure analogous to 2D Toda theory.

1 Introduction

We study a situation close to that of the classical Laplace maps developed in [3] and used more recently in many applications to integrable systems. Current work is (partially) summarised in the references [6, 8, 9, 15].

One approach [8, 4, 2] is via an intertwining relation of the form,

$$M^\sigma L = L^\sigma M.$$

The general solution to this relation is discussed in [10] when L and L^σ are linear, second order, hyperbolic differential operators in two variables and M and M^σ are of first order and in [11] when they are of second order. For the case of arbitrary order see [12]. The case where the operators depend on more than two independent variables is unresolved but see [5].

Here we explore the point of view that the general solution (M and M^σ) to the intertwining relation above is a deformation of the classical Laplace case and extend to it the corresponding derivation of the classical 2D Toda field theory. We will thus use the term “twisted Laplace maps” for these Darboux maps and “twisted Toda” for the consequent lattice equations. The twisted lattice is floppy, a functional deformation of the classical model to which it reduces in the untwisted Laplace limit.

The paper starts with a discussion of the classical case in order to establish notation and recall the Laplace map’s relation to a Lax pair and the Toda lattice. We then repeat the discussion for the twisted case in which a modified Lax pair can be written down and the relations between transformed and untransformed variables (Laplace invariants) are more involved. We finish with a discussion of the relations on the functional parameters that need to be satisfied in order that there be a lattice structure extending the 2D Toda lattice.

2 Classical Laplace maps and 2D Toda

In this section we recall the workings of the classical Laplace maps and formulate the derivation of the 2D Toda lattice using a purely operator based formalism borrowed from [2]. We close with a recapitulation of the Lax pair.

The classical Laplace invariants [3] were introduced in the context of second-order, linear hyperbolic partial differential equations

$$L_{12}\phi = \phi_{,12} + a_2\phi_{,1} + a_1\phi_{,2} + a_{12}\phi = 0, \quad (1)$$

where $\phi = \phi(x_1, x_2)$ and L_{12} denotes the differential operator

$$L_{12} = \partial_1\partial_2 + a_2\partial_1 + a_1\partial_2 + a_{12}, \quad (2)$$

in which the coefficients a_1 , a_2 and a_{12} are arbitrary functions of x_1 and x_2 and ∂_i denotes partial differentiation with respect to x_i . The form of the differential operator (2) is unchanged under a gauge transformation

$$L_{12} \rightarrow L_{12}^g = g^{-1}L_{12}g, \quad (3)$$

g being an arbitrary function of the independent variables.

The invariants in which we shall be interested throughout this paper are constructed by defining two first-order operators

$$L_1 = \partial_1 + a_1, \quad L_2 = \partial_2 + a_2$$

and writing down the functions

$$I_{12} = L_{12} - L_1L_2, \quad I_{21} = L_{12} - L_2L_1.$$

Thus, L_{12} can be written in two equivalent ways as

$$\begin{aligned} L_{12} &= L_1L_2 + I_{12} \\ &= L_2L_1 + I_{21}, \end{aligned}$$

where $I_{12} = a_{12} - a_1a_2 - a_{2,1}$ and $I_{21} = a_{12} - a_1a_2 - a_{1,2}$ are the classical Laplace invariants preserved by the transformation (3).

Indeed, we can see that I_{12} and I_{21} are invariants because they are 0^{th} order differential operators which commute with g . We define $\Theta_i(L) = [L, x_i]$. Then the invariance of I_{12} and I_{21} can be characterised [1] by the property that they are functions of L_{12} , L_1 and L_2 satisfying

$$\begin{aligned} \Theta_1(L_{12} - L_1L_2) &= \Theta_2(L_{12} - L_1L_2) = 0 \\ \Theta_1(L_{12} - L_2L_1) &= \Theta_2(L_{12} - L_2L_1) = 0. \end{aligned}$$

Now suppose there is a function ϕ such that $L_{12}\phi = 0$. By defining the Laplace map as $L_1\phi = \phi^\sigma$, we have the pair

$$\begin{aligned} L_1\phi &= \phi^\sigma \\ L_2\phi^\sigma + I_{21}\phi &= 0. \end{aligned} \quad (4)$$

By eliminating ϕ from the above system, we will obtain the σ -Laplace transformed equation

$$L_{12}^\sigma\phi^\sigma = 0$$

where L_{12}^σ can be written as

$$L_{12}^\sigma = L_1^\sigma L_2^\sigma + I_{12}^\sigma$$

and as

$$L_{12}^\sigma = L_2^\sigma L_1^\sigma + I_{21}^\sigma.$$

From (4) we have

$$\begin{aligned} L_{12}^\sigma \phi^\sigma &= I_{21} L_1 I_{21}^{-1} L_2 \phi^\sigma + I_{21} \phi^\sigma \\ &= (L_1^\sigma L_2 + I_{21}) \phi^\sigma \\ &= (L_2 L_1^\sigma + I_{21} + [L_1^\sigma, L_2]) \phi^\sigma = 0, \end{aligned}$$

so that $L_1^\sigma = I_{21} L_1 I_{21}^{-1}$, $L_2^\sigma = L_2$ and the Laplace transformations of the invariants are

$$\begin{aligned} I_{12}^\sigma &= I_{21}, \\ I_{21}^\sigma &= I_{21} + [L_1^\sigma, L_2] \\ &= 2I_{21} - I_{12} + (\log I_{21})_{,12}. \end{aligned}$$

Similarly, if we define the Laplace map as $L_2 \phi = \phi^\Sigma$, we have the pair

$$\begin{aligned} L_2 \phi &= \phi^\Sigma \\ L_1 \phi^\Sigma + I_{12} \phi &= 0. \end{aligned}$$

By eliminating ϕ from the above pair, we obtain the Σ -Laplace transformed equation which is satisfied by ϕ^Σ as follows

$$(L_2^\Sigma L_1 + I_{12}) \phi^\Sigma = (L_1 L_2^\Sigma + I_{12} + [L_2^\Sigma, L_1]) \phi^\Sigma = 0,$$

where $L_2^\Sigma = I_{12} L_2 I_{12}^{-1}$. This implies $L_1^\Sigma = L_1$ and the Σ -Laplace transformations of the invariants

$$\begin{aligned} I_{21}^\Sigma &= I_{12}, \\ I_{12}^\Sigma &= I_{12} + [L_2^\Sigma, L_1] \\ &= 2I_{12} - I_{21} + (\log I_{12})_{,12}. \end{aligned}$$

These can be combined into the three term recurrence relations:

$$\begin{aligned} I_{12}^\sigma - 2I_{12} + I_{12}^\Sigma &= (\log I_{12})_{,12} \\ I_{21}^\sigma - 2I_{21} + I_{21}^\Sigma &= (\log I_{21})_{,12}. \end{aligned} \tag{5}$$

Further, the σ and Σ are inverse in the sense that $(I_{ij}^\sigma)^\Sigma = (I_{ij}^\Sigma)^\sigma = I_{ij}$. For example,

$$\begin{aligned} (I_{12}^\sigma)^\Sigma &= 2I_{12}^\sigma - I_{21}^\sigma + (\log I_{12}^\sigma)_{,12} \\ &= 2I_{21} - 2I_{21} + I_{12} - (\log I_{21})_{,12} + (\log I_{21})_{,12} \\ &= I_{12}. \end{aligned}$$

Finally, it is possible to summarise the above relations in the simple intertwining relations

$$\begin{aligned} L_1^\sigma L_{12} &= L_{12}^\sigma L_1 \\ L_2^\Sigma L_{12} &= L_{12}^\Sigma L_2. \end{aligned} \tag{6}$$

which we can analyse using the Θ map to repeat the above derivation without introducing ϕ^σ [2].

To see this, for the σ case, apply Θ_1 successively to the relation (6). We obtain the following additional formulae,

$$\begin{aligned} L_{12} + L_1^\sigma L_2 &= L_{12}^\sigma + L_2^\sigma L_1 \\ L_2 &= L_2^\sigma. \end{aligned}$$

From the full set of relations we deduce, essentially by rearrangements, that

$$\begin{aligned} I_{12}^\sigma &= I_{21} \\ L_1^\sigma I_{21} &= I_{12}^\sigma L_1 \\ I_{21}^\sigma - I_{12}^\sigma &= I_{21} - I_{12} + (\log I_{21})_{,12}, \end{aligned}$$

so that

$$\begin{aligned} L_1^\sigma &= I_{12}^\sigma L_1 I_{21}^{-1} \\ &= I_{21} L_1 I_{21}^{-1} \\ &= L_1 - (\log I_{21})_{,1}. \end{aligned}$$

Likewise

$$L_2^\Sigma L_{12} = L_{12}^\Sigma L_2$$

implies identities:

$$\begin{aligned} L_{12} + L_2^\Sigma L_1 &= L_{12}^\Sigma + L_1^\Sigma L_2 \\ L_1 &= L_1^\Sigma \end{aligned}$$

from which we deduce

$$\begin{aligned} I_{21}^\Sigma &= I_{12} \\ L_2^\Sigma I_{12} &= I_{21}^\Sigma L_2 \\ I_{12}^\Sigma - I_{21}^\Sigma &= I_{12} - I_{21} + (\log I_{12})_{,12}. \end{aligned}$$

Because, as we have seen, σ and Σ act as inverses on the invariants it is possible to attach a superscript, $i \in \mathbb{Z}$, labelling the successive applications of σ writing e.g. $(L_1^i)^\sigma = L_1^{i+1}$ etc. Hence the σ -Laplace map satisfies $L_1^{i+1} L_{12}^i = L_{12}^{i+1} L_1^i$ and we obtain the 2D Toda lattice equations [14, 16],

$$\begin{aligned} I_{12}^{i+1} - 2I_{12}^i + I_{12}^{i-1} &= (\log I_{12})_{,12} \\ I_{21}^{i+1} - 2I_{21}^i + I_{21}^{i-1} &= (\log I_{21})_{,12}. \end{aligned}$$

In addition it is instructive, for the sake of later analogy, to write the Laplace maps we have described the above in system form,

$$\begin{pmatrix} L_1 & -1 \\ I_{21} & L_2 \end{pmatrix} \begin{pmatrix} \phi \\ \phi^\sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} L_1 & I_{12} \\ -1 & L_2 \end{pmatrix} \begin{pmatrix} \phi^\Sigma \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

in which case the intertwining relations take matrix form

$$\begin{pmatrix} L_1^\sigma & 0 \\ [L_1^\sigma, L_2] & L_1^\sigma \end{pmatrix} \begin{pmatrix} L_1 & -1 \\ I_{21} & L_2 \end{pmatrix} = \begin{pmatrix} L_1^\sigma & -1 \\ I_{21}^\sigma & L_2^\sigma \end{pmatrix} \begin{pmatrix} L_1 & 0 \\ 0 & L_1^\sigma \end{pmatrix}$$

and

$$\begin{pmatrix} L_2^\Sigma & [L_2^\Sigma, L_1] \\ 0 & L_2^\Sigma \end{pmatrix} \begin{pmatrix} L_1 & I_{12} \\ -1 & L_2 \end{pmatrix} = \begin{pmatrix} L_1^\Sigma & I_{12}^\Sigma \\ -1 & L_2^\Sigma \end{pmatrix} \begin{pmatrix} L_2^\Sigma & 0 \\ 0 & L_2 \end{pmatrix}.$$

In each instance off diagonal entries are precisely the intertwining relations.

2.1 Classical Lax pair

We review the derivation of the Lax pair for the 2D Toda chain from the Laplace maps in this classical case. This treatment parallels [13].

Define a chain of Laplace maps: $\phi^{i+1} = \phi^\sigma$ with $L_{12}^i \phi^i = 0$. We can reframe these maps in terms of the L_1^i and L_2^i operators as

$$\begin{aligned} L_1^i \phi^i &= \phi^{i+1} \\ L_2^i \phi^i &= -I_{21}^{i-1} \phi^{i-1} \end{aligned}$$

where $L_1^i = \partial_1 + a_1^i$ and $L_2^{i+1} = L_2^i = \partial_2 + a_2$, the shift $i \rightarrow i+1$ denoting the σ map.

Then we have

$$(L_2^{i+1} L_1^i - L_1^{i-1} L_2^i) \phi^i = I_{21}^{i-1} \phi^i - I_{21}^i \phi^i - (\log I_{21}^{i-1})_{,1} L_2^i \phi^i.$$

Evaluating the left-hand side of the above equation, we obtain

$$(a_{1,2}^i - a_{2,1}) \phi^i + (a_1^i - a_1^{i-1}) L_2^i \phi^i = (I_{21}^{i-1} - I_{21}^i) \phi^i - (\log I_{21}^{i-1})_{,1} L_2^i \phi^i.$$

Equating the coefficients of ϕ^i and $L_2^i \phi^i$ gives us the following pair

$$\begin{aligned} a_{1,2}^i - a_{2,1} &= I_{21}^{i-1} - I_{21}^i \\ a_1^i - a_1^{i-1} &= -(\log I_{21}^{i-1})_{,1} \end{aligned}$$

and hence, since $a_{1,2} - a_{2,1} = I_{12} - I_{21}$, we obtain the 2D-Toda lattice equations [14]

$$I_{21}^{i+1} - 2I_{21}^i + I_{21}^{i-1} = (\log I_{21}^i)_{,12}$$

and

$$I_{12}^{i+1} - 2I_{12}^i + I_{12}^{i-1} = (\log I_{12}^i)_{,12}. \quad (7)$$

3 Twisted Laplace maps

Here we allow deformation (called “twisted”) of the Laplace maps recovering the results of [10] and exploring the consequences for relations between the transformed invariants under these twisted maps.

We define twisted Laplace maps via the modified intertwining relations,

$$L_1^{\prime\sigma} L_{12} = L_{12}^\sigma L_1' \quad (8)$$

$$L_2^{\prime\Sigma} L_{12} = L_{12}^\Sigma L_2', \quad (9)$$

where the primed operators are monic in ∂_1 and ∂_2 still but with coefficients distinct from the unprimed operators, L_1 and L_2 which appear in the expressions $L_{12} = L_1 L_2 + I_{12}$ etc.

As in the classical case in the previous section, we can use the Θ_i maps to analyse these twisted intertwining relations.

In the σ case, by successive application of Θ_1 and Θ_2 on the relation (8), we obtain the following additional relations

$$\begin{aligned} L_1^{\prime\sigma} L_1 &= L_1^\sigma L_1' \\ L_{12} + L_1^{\prime\sigma} L_2 &= L_{12}^\sigma + L_2^\sigma L_1' \\ L_1 + L_1^{\prime\sigma} &= L_1^\sigma + L_1' \\ L_2 &= L_2^\sigma \end{aligned}$$

from which we deduce that there is a function α satisfying

$$\begin{aligned} L_1' - L_1 &= \alpha \\ L_1^{\prime\sigma} - L_1^\sigma &= \alpha \\ L_1^\sigma \alpha &= \alpha L_1 \end{aligned} \quad (10)$$

$$I_{12}^\sigma = I_{21} - \alpha_{,2} \quad (11)$$

$$I_{21}^\sigma - I_{12}^\sigma = I_{21} - I_{12} + (\log \alpha)_{,12} \quad (12)$$

$$\alpha_{,1} I_{12}^\sigma - \alpha I_{12,1}^\sigma = \alpha^2 (I_{12} - I_{21}) + \alpha^2 \alpha_{,2}. \quad (13)$$

In particular

$$\begin{aligned} L_1^\sigma &= \alpha L_1 \alpha^{-1} \\ &= L_1 - (\log \alpha)_{,1}. \end{aligned} \quad (14)$$

Of these we can regard (11) and (12) as defining the Laplace transformed invariants in terms of the untransformed and (13) as a differential relation that α must satisfy. Written in terms of the untransformed invariants it is:

$$\alpha \alpha_{,12} - \alpha_{,1} \alpha_{,2} - \alpha^2 \alpha_{,2} = \alpha I_{21,1} - \alpha_{,1} I_{21} + \alpha^2 (I_{12} - I_{21}); \quad (15)$$

or in terms of the transformed invariants:

$$\alpha \alpha_{,12} - \alpha_{,1} \alpha_{,2} + \alpha^2 \alpha_{,2} = -\alpha I_{12,1}^\sigma + \alpha_{,1} I_{12}^\sigma + \alpha^2 (I_{21}^\sigma - I_{12}^\sigma). \quad (16)$$

The equations (15) and (16) differ by the interchanges: $\alpha \leftrightarrow -\alpha$ and $I_{ij} \leftrightarrow I_{ji}^\sigma$.

By rearranging the equation (15), we have

$$-(\alpha + a_1)_{,2} = \left(\frac{I_{21}}{\alpha} - a_2 - \frac{\alpha_{,2}}{\alpha} \right)_{,1}.$$

Let us choose $\alpha = -z^{-1}z_{,1} - a_1$. Then

$$\alpha z_{,2} = (I_{21} - a_2\alpha - \alpha_{,2})z$$

and z satisfies the hyperbolic differential equation

$$z_{,12} + a_2z_{,1} + a_1z_{,2} + a_{12}z = 0.$$

We can regard this element $z \in \ker L_{12}$ (or equivalently α) as a functional parameter relating the transformed to the untransformed invariants.

We may also eliminate α from (15) and (16). This is not a trivial identity but a relation between the transformed and untransformed invariants that is independent of α or z resulting in a possible analogue of the classical 2D Toda equation. To do this we rewrite the equations (15) and (16) in mixed terms as,

$$\begin{aligned}\alpha_{,1}I_{12}^\sigma - \alpha I_{12,1}^\sigma &= \alpha^2(I_{12} - I_{12}^\sigma) \\ \alpha_{,1}I_{21} - \alpha I_{21,1} &= \alpha^2(I_{21} - I_{21}^\sigma).\end{aligned}$$

This pair can be linearised by the substitution $\alpha = \frac{1}{\beta}$,

$$\begin{aligned}\beta_{,1}I_{12}^\sigma + \beta I_{12,1}^\sigma &= I_{12}^\sigma - I_{12} \\ \beta_{,1}I_{21} + \beta I_{21,1} &= I_{21}^\sigma - I_{21},\end{aligned}$$

from which we obtain

$$\begin{aligned}\Delta\beta &= I_{12}I_{21} + I_{12}^\sigma I_{21}^\sigma - 2I_{21}I_{12}^\sigma \\ \Delta\beta_{,1} &= -I_{12}I_{21,1} - I_{21}^\sigma I_{12,1}^\sigma + (I_{21}I_{12}^\sigma)_{,1},\end{aligned}$$

where $\Delta = I_{21,1}I_{12}^\sigma - I_{21}I_{12,1}^\sigma$. It is clear that the classical Laplace map $I_{12}^\sigma = I_{21}$, corresponds to the case that $\Delta = 0$.

By eliminating β for $\Delta \neq 0$ from the above system, we obtain the following relation

$$I_{12}^\sigma \left(\frac{I_{21}^2(I_{12}^\sigma - I_{12})}{\Delta} \right)_{,1} = I_{21} \left(\frac{I_{12}^{\sigma 2}(I_{21}^\sigma - I_{21})}{\Delta} \right)_{,1}. \quad (17)$$

Correspondingly

$$\begin{aligned}\alpha &= \frac{\Delta}{I_{12}I_{21} + I_{12}^\sigma I_{21}^\sigma - 2I_{21}I_{12}^\sigma} \\ \alpha_{,1} &= \frac{\Delta(I_{12}I_{21,1} + I_{21}^\sigma I_{12,1}^\sigma - (I_{21}I_{12}^\sigma)_{,1})}{(I_{12}I_{21} + I_{12}^\sigma I_{21}^\sigma - 2I_{21}I_{12}^\sigma)^2}\end{aligned}$$

and so the Laplace transformations (11) and (12) are

$$I_{12}^\sigma = I_{21} - \left(\frac{\Delta}{I_{12}I_{21} + I_{12}^\sigma I_{21}^\sigma - 2I_{21}I_{12}^\sigma} \right)_{,2} \quad (18)$$

$$I_{21}^\sigma - I_{12}^\sigma = I_{21} - I_{12} + \left(\frac{I_{12}I_{21,1} + I_{21}^\sigma I_{12,1}^\sigma - (I_{21}I_{12}^\sigma)_{,1}}{I_{12}I_{21} + I_{12}^\sigma I_{21}^\sigma - 2I_{21}I_{12}^\sigma} \right)_{,2}. \quad (19)$$

We now repeat the calculation interchanging the roles of the indices for the Σ -Laplace map: $L_{12} \rightarrow L_{12}^\Sigma$. We will obtain a relation

$$I_{21}^\Sigma \left(\frac{I_{12}^2(I_{21}^\Sigma - I_{21})}{\Delta'} \right)_{,2} = I_{12} \left(\frac{I_{21}^{\Sigma 2}(I_{12}^\Sigma - I_{12})}{\Delta'} \right)_{,2}, \quad (20)$$

where $\Delta' = I_{12,2}I_{21}^\Sigma - I_{12}I_{21,2}^\Sigma$, and the Laplace maps of the invariants I_{12} and I_{21} as follows

$$I_{21}^\Sigma = I_{12} - \left(\frac{\Delta'}{I_{12}I_{21} + I_{12}^\Sigma I_{21}^\Sigma - 2I_{12}I_{21}^\Sigma} \right)_{,1} \quad (21)$$

$$I_{12}^\Sigma - I_{21}^\Sigma = I_{12} - I_{21} + \left(\frac{I_{21}I_{12,2} + I_{12}^\Sigma I_{21,2}^\Sigma - (I_{12}I_{21}^\Sigma)_{,2}}{I_{12}I_{21} + I_{12}^\Sigma I_{21}^\Sigma - 2I_{12}I_{21}^\Sigma} \right)_{,1}. \quad (22)$$

We can represent the twisted Laplace maps we have described above in system form as

$$\begin{pmatrix} L'_1 & -1 \\ I_{21} - L_2\alpha & L_2 \end{pmatrix} \begin{pmatrix} \phi \\ \phi^\sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note that, unlike the classical case, we have an off-diagonal differential part. This system form allows us to motivate the Lax pair for the twisted relations below. The matrix forms of the intertwining relations are then written,

$$\begin{pmatrix} L_1^{\prime\sigma} & 0 \\ [L_1^{\prime\sigma}, L_2] & L_1^{\prime\sigma} \end{pmatrix} \begin{pmatrix} L'_1 & -1 \\ I_{21} - L_2\alpha & L_2 \end{pmatrix} = \begin{pmatrix} L_1^{\prime\sigma} & -1 \\ I_{21}^\sigma - L_2\alpha & L_2^\sigma \end{pmatrix} \begin{pmatrix} L'_1 & 0 \\ 0 & L_1^{\prime\sigma} \end{pmatrix},$$

where again an off-diagonal entry represents the scalar intertwining relation.

A point it is important to note is that the twisted maps are not simply gauge transformations of the classical ones. Were we to gauge transform L'_1 to L_1 , i.e. $L_1 = g^{-1}L'_1g$, the intertwining relations would force a compensating transformation of L_{12} to $g^{-1}L_{12}g$ and hence the relations would still be twisted.

Indeed the form of α means that

$$L'_1 = \partial_1 + a_1 + \alpha = \partial - z^{-1}z_{,1} = z\partial_1z^{-1}$$

where $L_{12}(z) = 0$. This reduces the twisted relation to

$$(z^{-1}L_1^{\prime\sigma}z)(z^{-1}L_{12}z) = (z^{-1}L_{12}^\sigma z)\partial_1$$

amounting to a special choice of gauge: the invariants will have the same values.

4 The untwisted limit

We recover the classical Laplace map and the 2D Toda equations in the limit that $\alpha \rightarrow 0$ but the limit is somewhat singular as far as z is concerned.

If we suppose that $\alpha = \epsilon\phi$ then equation (13) becomes

$$\epsilon(\phi\phi_{,12} - \phi_{,1}\phi_{,2} - \phi^2(I_{21}^\sigma - I_{12}^\sigma)) + \epsilon^2(\phi^2\phi_{,2}) + \phi I_{12,1}^\sigma - \phi_{,1}I_{12}^\sigma = 0$$

and in the $\epsilon \rightarrow 0$ limit $\phi \rightarrow \lambda(x_2)I_{12}^\sigma$. Hence

$$\begin{aligned} L'_1 - L_1 &\rightarrow 0 \\ L_1'^\sigma - L_1^\sigma &\rightarrow 0 \\ L_1^\sigma I_{12}^\sigma &\rightarrow I_{12}^\sigma L_1 \\ I_{12}^\sigma &\rightarrow I_{21} \\ I_1^\sigma - I_{12}^\sigma &\rightarrow I_{21} - I_{12} + (\log I_{12}^\sigma)_{,12} \end{aligned}$$

which are the classical equations.

However we also have $L_1(z) + \epsilon\phi z = 0$ and $L_{12}(z) = L_2 L_1(z) + I_{21}z = 0$. Hence

$$\begin{aligned} L_1(z) &= -\epsilon\phi z \\ \epsilon L_2(\phi z) &= I_{21}z \end{aligned}$$

and it would appear that $I_{21} \rightarrow 0$ also. To avoid this we need $L_2(z)/z \sim O(1/\epsilon)$ even as $L_1(z) \sim O(\epsilon)$.

Applied to the twisted Toda equations themselves, note that $I_{12}^\sigma = I_{21} + O(\epsilon)$ implies

$$\begin{aligned} \Delta &= O(\epsilon) \\ I_{12}I_{21} + I_{12}^\sigma I_{21}^\sigma - 2I_{12}^\sigma I_{21} &= I_{21} (I_{12} + I_{21}^\sigma - 2I_{21}) + O(\epsilon) \\ I_{12}I_{21,1} + I_{12,1}^\sigma I_{21}^\sigma - (I_{12}^\sigma I_{21})_{,1} &= I_{21,1} (I_{12} + I_{21}^\sigma - 2I_{21}) + O(\epsilon) \end{aligned}$$

In particular

$$\begin{aligned} I_{21}^\sigma - I_{12}^\sigma &= I_{21} - I_{12} + \left(\frac{I_{21,1} (I_{12} + I_{21}^\sigma - 2I_{21})}{I_{21} (I_{12} + I_{21}^\sigma - 2I_{21})} \right)_{,2} + O(\epsilon) \\ &= I_{21} - I_{12} + (\log I_{21})_{,12} + O(\epsilon) \end{aligned}$$

5 The twisted Lax pair

We now deform the classical Lax pair in order to accommodate the twisted lattice.

Let us consider the twisted σ -intertwining relation $L_1'^\sigma L_{12} = L_{12}^\sigma L_1'$, where $L_1' = L_1 + \alpha$ and $L_{12}\phi = 0$ such that

$$\begin{aligned} L_{12} &= L_2 L_1 + I_{21} \\ &= L_2 (L_1' - \alpha) + I_{21}. \end{aligned}$$

We can write the Laplace maps as a pair

$$\begin{aligned} L_1' \phi &= \phi^\sigma \\ L_2 \phi^\sigma &= (L_2 \alpha - I_{21}) \phi, \end{aligned}$$

where $L_2^\sigma = L_2$.

We now define a chain of Laplace maps $\phi^{i+1} = (\phi^i)^\sigma$ with $L_{12}^i \phi^i = 0$ so that the above pair becomes

$$\begin{aligned} L_1'^i \phi^i &= \phi^{i+1} \\ L_2 \phi^{i+1} &= (L_2 \alpha^i - I_{21}^i) \phi^i \end{aligned}$$

which can be written as a Lax pair

$$L_1^i \phi^i = \phi^{i+1} \quad (23)$$

$$L_2 \phi^i = (L_2 \alpha^{i-1} - I_{21}^{i-1}) \phi^{i-1}, \quad (24)$$

in which $L_1^i = L_1^i + \alpha^i$ and $L_2^{i+1} = L_2^i = L_2$.

Thus, we have

$$(L_2 L_1^i - L_1^{i-1} L_2) \phi^i = L_2 \phi^{i+1} - L_1^{i-1} (L_2 \alpha^{i-1} - I_{21}^{i-1}) \phi^{i-1}, \quad (25)$$

where $L_1^i = \partial_1 + a_1^i + \alpha^i$ and $L_2 = \partial_2 + a_2$.

The differential operator in the left hand side (*LHS*) of the equation (25) can be expanded as

$$\begin{aligned} L_2 L_1^i - L_1^{i-1} L_2 &= (\partial_2 + a_2)(\partial_1 + a_1^i + \alpha^i) - (\partial_1 + a_1^{i-1} + \alpha^{i-1})(\partial_2 + a_2) \\ &= (a_1^i - a_1^{i-1} + \alpha^i - \alpha^{i-1}) L_2 + I_{12}^i - I_{21}^i + \alpha_{,2}^i. \end{aligned}$$

Then, the left hand side of the above equation becomes

$$LHS = (a_1^i - a_1^{i-1} + \alpha^i - \alpha^{i-1}) L_2 \phi^i + (I_{12}^i - I_{21}^i + \alpha_{,2}^i) \phi^i.$$

The right hand side (*RHS*) of the equation (25) can be written as

$$\begin{aligned} RHS &= L_2 \phi^{i+1} - L_1^{i-1} (L_2 \alpha^{i-1} - I_{21}^{i-1}) \phi^{i-1} \\ &= (I_{21}^{i-1} - I_{21}^i + \alpha_{,2}^i - \alpha_{,2}^{i-1}) \phi^i \\ &\quad + \left(\alpha^i - \alpha^{i-1} - (\alpha^{i-1})^{-1} \alpha_{,1}^{i-1} \right) L_2 \phi^i \\ &\quad + \left(I_{21,1}^{i-1} + (I_{12}^{i-1} - I_{21}^{i-1}) \alpha^{i-1} - \alpha_{,12}^{i-1} \right. \\ &\quad \left. \alpha^{i-1} \alpha_{,2}^{i-1} - (\alpha^{i-1})^{-1} \alpha_{,1}^{i-1} (I_{21}^{i-1} - \alpha_{,2}^{i-1}) \right) \phi^{i-1}. \end{aligned}$$

Setting the LHS equal to the RHS and equating the coefficients of $L_2 \phi^i$ we have

$$a_1^i = a_1^{i-1} - (\alpha^{i-1})^{-1} \alpha_{,1}^{i-1}$$

which is

$$a_1^{i+1} = a_1^i - (\alpha^i)^{-1} \alpha_{,1}^i,$$

or

$$\begin{aligned} L_1^{i+1} &= L_1^i - \frac{\alpha_{,1}^i}{\alpha^i} \\ &= L_1^i - (\log \alpha^i)_{,1}. \end{aligned} \quad (26)$$

which is equation (14).

By equating coefficients of ϕ^i we get

$$I_{12}^i = I_{21}^{i-1} - \alpha_{,2}^{i-1}$$

which is

$$I_{12}^{i+1} = I_{21}^i - \alpha_{,2}^i, \quad (27)$$

namely (11)

Finally, equation of the coefficients of ϕ^{i-1} gives

$$\alpha_{,12}^{i-1} - \alpha^{i-1} \alpha_{,2}^{i-1} + (I_{21}^{i-1} - I_{12}^{i-1}) \alpha^{i-1} + (\alpha^{i-1})^{-1} \alpha_{,1}^{i-1} (I_{21}^{i-1} - \alpha_{,2}^{i-1}) - I_{21,1}^{i-1} = 0.$$

Multiplying both side of this equation by α^{i-1} , we get

$$\alpha^{i-1} \alpha_{,12}^{i-1} - \alpha_{,1}^{i-1} \alpha_{,2}^{i-1} - (\alpha^{i-1})^2 \alpha_{,2}^{i-1} = \alpha^{i-1} I_{21,1}^{i-1} - \alpha_{,1}^{i-1} I_{21}^{i-1} + (\alpha^{i-1})^2 (I_{12}^{i-1} - I_{21}^{i-1})$$

which is

$$\alpha^i \alpha_{,12}^i - \alpha_{,1}^i \alpha_{,2}^i - (\alpha^i)^2 \alpha_{,2}^i = \alpha^i I_{21,1}^i - \alpha_{,1}^i I_{21}^i + (\alpha^i)^2 (I_{12}^i - I_{21}^i). \quad (28)$$

This equation, again equation (13), is equivalent to $L_{12}^i(z^i) = 0$, where $\alpha^i = -a_1^i - (z^i)^{-1} z_{,1}^i$.

We already know that $L_2^{i+1} = L_2$ and $L_1^{i+1} = L_1 - (\alpha^i)^{-1} \alpha_{,1}^i$. Then by using these, we get

$$\begin{aligned} I_{21}^{i+1} - I_{12}^{i+1} &= L_1^{i+1} L_2^{i+1} - L_2^{i+1} L_1^{i+1} \\ &= \left(L_1^i - \frac{\alpha_{,1}^i}{\alpha^i} \right) L_2 - L_2 \left(L_1^i - \frac{\alpha_{,1}^i}{\alpha^i} \right) \\ &= I_{21}^i - I_{12}^i + \left(\frac{\alpha_{,1}^i}{\alpha^i} \right)_{,2}. \end{aligned}$$

Thus, we obtain

$$I_{21}^{i+1} - I_{12}^{i+1} = I_{21}^i - I_{12}^i + (\log \alpha^i)_{,12}. \quad (29)$$

So the twisted maps do provide a Lax pair, the difference from the classical case being that the pair depends on a functional parameter, α^i , related to an arbitrary element, z^i , in the kernel of L_{12}^i .

6 Twisted lattices

Because of this functional parameter (i.e. z or α) in the twisted transformation, there will not be a discrete ‘‘Toda’’ like lattice on which the I_{12} and I_{21} live unless we impose some extra conditions.

We introduce parameters labelled γ to play the role in the Σ maps corresponding to that played by those denoted α in the σ maps.

Natural choices to create a lattice structure would be either to require the diagram

$$\begin{array}{ccc} (I_{12}, I_{21}) & \xrightarrow{\alpha} & (I_{12}^\sigma, I_{21}^\sigma) \\ \gamma \downarrow & & \downarrow \gamma^\sigma \\ (I_{12}^\Sigma, I_{21}^\Sigma) & \xrightarrow{\alpha^\Sigma} & (\cdot, \cdot) \end{array}$$

to commute, i.e. $(I_{ij}^\Sigma)^\sigma = (I_{ij}^\sigma)^\Sigma$, or to require that particular choices of α and γ^σ (or γ and α^Σ) should lead to $(I_{ij}^\Sigma)^\sigma = (I_{ij}^\sigma)^\Sigma = I_{ij}$,

$$(I_{12}, I_{21}) \xrightarrow{\alpha} (I_{12}^\sigma, I_{21}^\sigma) \xrightarrow{\gamma^\sigma} (I_{12}, I_{21}).$$

The former will give a \mathbb{Z}^2 lattice; the latter a \mathbb{Z} lattice.
From the maps,

$$\begin{aligned} I_{12}^\sigma &= I_{21} - \alpha_{,2} \\ I_{21}^\sigma &= 2I_{21} - I_{12} + (\log \alpha)_{,12} - \alpha_{,2} \\ I_{12}^\Sigma &= 2I_{12} - I_{21} + (\log \gamma)_{,12} - \gamma_{,1} \\ I_{21}^\Sigma &= I_{12} - \gamma_{,1} \end{aligned}$$

we get

$$\begin{aligned} (I_{12}^\Sigma)^\sigma &= I_{21}^\Sigma - \alpha_{,2}^\Sigma \\ &= I_{12} - \gamma_{,1} - \alpha_{,2}^\Sigma \\ (I_{12}^\sigma)^\Sigma &= 2I_{12}^\sigma - I_{21}^\sigma + (\log \gamma^\sigma)_{,12} - \gamma^\sigma_{,1} \\ &= I_{12} + (\log \gamma^\sigma)_{,12} - (\log \alpha)_{,12} - \gamma^\sigma_{,1} - \alpha_{,2} \end{aligned}$$

and

$$\begin{aligned} (I_{21}^\sigma)^\Sigma &= I_{12}^\sigma - \gamma^\sigma_{,1} \\ &= I_{21} - \gamma_{,1}^\sigma - \alpha_{,2} \\ (I_{21}^\Sigma)^\sigma &= 2I_{21}^\Sigma - I_{12}^\Sigma + (\log \alpha^\Sigma)_{,12} - \alpha^\Sigma_{,2} \\ &= I_{21} + (\log \alpha^\Sigma)_{,12} - (\log \gamma)_{,12} - \alpha^\Sigma_{,2} - \gamma_{,1}. \end{aligned}$$

Consistency requires

$$\left(\log \frac{\alpha^\Sigma \gamma^\sigma}{\alpha \gamma} \right)_{,12} = 0.$$

If we label the effect of $\sigma^i \Sigma^j$ by the pair (i, j) then the simplest way of satisfying this relation is to require

$$\alpha^{(i,j)} \gamma^{(i,j)} = \alpha^{(i,j+1)} \gamma^{(i+1,j)}.$$

Existence of inverses, mirroring the classical case, requires the stronger conditions,

$$\begin{aligned} \gamma_{,1} + \alpha^\Sigma_{,2} &= 0 \\ \gamma^\sigma_{,1} + \alpha_{,2} &= 0 \\ \left(\log \frac{\gamma^\sigma}{\alpha} \right)_{,12} &= 0 \\ \left(\log \frac{\alpha^\Sigma}{\gamma} \right)_{,12} &= 0. \end{aligned}$$

Equally these amount to conditions on the choices of elements belonging to the kernels of L_{12} , L_{12}^σ and L_{12}^Σ .

7 Conclusions

We have studied the possibility of a redescription of Laplace and Darboux maps by considering the general map to be a deformation of the Laplace case. The general map is seen as arising from a “twisted” intertwining relation and it

provides relations between transformed invariants of a more complex character than the classical Laplace maps. It extends the classical relation, which persists in the “untwisted” limit, and retains a Lax pair containing a functional parameter, e.g. z , which is an element of the kernel of the untransformed, linear operator. One may build families of lattices by requiring these parameters to satisfy relations which force the relevant diagrams to commute.

These relations should be seen as the fundamental description of the lattice.

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