THE UNIVERSITY OF GLASGOW

DEPARTMENT OF AURONAUTICS AND FLUID MECHANICS

SOME RESULTS RELATING TO CERTAIN GENERAL TYPES OF NON-LINEAR SECOND ORDER DIFFERENTIAL EQUATION

by

A.W. Babister, M.A., Ph.D.

Report No. 7201

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SUMMARY

Two general types of non-linear equation are considered. In part 1, it is shown that the generalized Bernoulli equation

$$y\frac{d^2y}{dx^2} = k\left(\frac{dy}{dx}\right)^2 + P(x)y\frac{dy}{dx} + Q(x)y^2 + R(x)y^{k+1}$$

can be transformed into a linear differential equation, and thus the general solution can readily be found. Other related equations are also considered.

In part 2, non-linear equations of the form

$$\frac{d^2y}{dx^2} + f(x,y)\frac{dy}{dx} + g(x,y) = 0$$

are dealt with. Methods are given for obtaining the general first integral.

The aim throughout this report is to reduce many of the miscellaneous methods of solution at present used to a more orderly system. General results are given which apply to large classes of non-linear second order differential equations.

LIST OF CONTENTS

General Introduction

In the analysis of many dynamical systems, we are confronted with a set of non-linear differential equations. The classical method of solution is to confine our attention to small disturbances and to linearise the equations. However, the behaviour of systems undergoing large amplitude disturbances is often of interest; non-linearities in the forces and moments may well be of major importance in these cases.

Approximate procedures (e.g. those based on the method of Kryloff and Bogoliuboff) are widely used for determining the free-oscillation characteristics of non-linear systems. However, there is often a loss in generality in the resulting solution, as compared with the analytical solution of linear equations; this emphasizes the need for a systematic treatment of non-linear equations.

In this report we attempt to classify ordinary second-order non-linear differential equations according to their methods of solution. In part 1, we consider equations which can be reduced to linear differential equations (which can then be solved by classical methods). In part 2, we consider differential equations of the form

$$\frac{d^2y}{dx^2} + f(x,y)\,\frac{dy}{dx} + g(x,y) = 0$$

which have first integrals

$$\frac{dy}{dx} = F(x,y).$$

In both parts of this report we give some useful general results, which apply to many types of non-linear differential equations of the second-order.

Part 1 Non-linear Second Order Differential Equations Related to the Bernoulli Equation

1.1 Introduction

We consider the generalized Bernoulli equation

$$y \frac{d^2y}{dx^2} = k\left(\frac{dy}{dx}\right)^2 + P(x)y\frac{dy}{dx} + Q(x)y^2 + R(x)y^{k+1}, \tag{1}$$

where k is a constant. As shown in § 1.2, any non-homogeneous linear differential equation of the second order can be transformed into an equation of this form. Various transformations are given, relating (1) to other second order differential equations.

In § 1.4, the general solution of (1) is obtained, when a particular solution is known. Power series and polynomial solutions are examined. Two other second order equations related to the Bernoulli equation are given in § 1.6. Finally, in § 1.7 corresponding results are given for some nth order differential equations.

1.2 A Second Order Bernoulli Equation

As is well known, the first order Bernoulli equation

$$\frac{dy}{dx} = A(x)y + B(x)y^k, \tag{2}$$

where k is a constant ($k \neq 1$), can be reduced to the linear form by putting

$$u = y^{1-k}. \tag{3}$$

From (2) and (3), we see that u satisfies the non-homogeneous linear differential equation of the first order given by

$$\frac{du}{dx} - (1-k)A(x)u = (1-k)B(x). \tag{4}$$

Consider now the second order equation

$$\frac{d^2u}{dx^2} - P(x)\frac{du}{dx} - (1-k)Q(x)u = (1-k)R(x). \tag{5}$$

On making the substitution (3), we see that

$$\frac{du}{dx} = (1-k)y^{-k}\frac{dy}{dx}$$

and $$\frac{d^2u}{dx^2} = (1-k)(-k)y^{-k-1}\left(\frac{dy}{dx}\right)^2 + (1-k)y^{-k}\frac{d^2y}{dx^2}.$$

Thus (5) becomes

$$y\frac{d^2y}{dx^2} = k\left(\frac{dy}{dx}\right)^2 + P(x)y\frac{dy}{dx} + Q(x)y^2 + R(x)y^{k+1}, \tag{6}$$

which can thus be termed a second order Bernoulli equation.

Conversely any equation of the form of (6) can be reduced to a non-homogeneous linear differential equation of the second order (or lower). The solutions of equations of this type are discussed by Babister (1967) and Murphy (1960). If $k \neq 1$, we use the substitution (3). If $k = 1$, we can take $R = 0$ without loss of generality. Put

$$v = \frac{1}{y}\frac{dy}{dx}. \tag{7}$$

Then $$\frac{dy}{dx} = vy,$$

$$\frac{d^2y}{dx^2} = \left(\frac{dv}{dx} + v^2\right)y$$

and (6) becomes (with $k=1$ and $R=0$)

$$\frac{dv}{dx} = P(x)v + Q(x).$$

If $k=0$, equation (6) reduces to

$$\frac{d^2y}{dx^2} = P(x)\frac{dy}{dx} + Q(x)y + R(x).$$

If $k \neq 1$, the substitution (7) reduces (6) to

$$\frac{dv}{dx} = Q(x) + P(x)v + (k-1)v^2 + R(x)\exp\left[(k-1)\int v\,dx\right], \tag{8}$$

which reduces to a Riccati equation if $R(x) \equiv 0$.

Differentiating (8), we obtain

$$\frac{d^2v}{dx^2} = \frac{dQ}{dx} + \frac{dP}{dx}v + P\frac{dv}{dx} + 2(k-1)v\frac{dv}{dx} + \left[\frac{dR}{dx} + (k-1)Rv\right] \exp\left[(k-1)\int v\,dx\right]$$

Using (8) to eliminate the exponential term, we find that v satisfies the second order equation

$$\frac{d^2v}{dx^2} = 3(k-1)v\frac{dv}{dx} - (k-1)^2v^3 + \left(P+\frac{1}{R}\frac{dR}{dx}\right)\left[\frac{dv}{dx} - (k-1)v^2\right] +$$

$$+\left[\frac{dP}{dx} - (k-1)Q - \frac{P}{R}\frac{dR}{dx}\right]v + \frac{dQ}{dx} - \frac{Q}{R}\frac{dR}{dx}. \tag{9}$$

Putting $q_0 = Q(R'/R) - Q'$,

$q_1 = P(R'/R) - P' + (k-1)Q$,

$q_2 = -(R'/R) - P$

and $q_3 = 1-k$,

where primes denote differentiation w.r.t. x, we see that (9) is a particular case of the equation

$$v'' + (q_2+3q_3v)v' + q_0+q_1v+(q_2q_3+q_3')v^2 + q_3^2v^3 = 0, \tag{10}$$

which is the form of the Riccati equation of the second order (Wallenberg, 1900).

The corresponding result for the first order Bernoulli equation (2) is that the substitution (7) leads to the equation

$$\frac{dv}{dx} = \left(\frac{dA}{dx} - \frac{A}{B}\frac{dB}{dx}\right) + \left[\frac{1}{B}\frac{dB}{dx} - (k-1)A\right]v + (k-1)v^2,$$

which is a particular case of the generalized Riccati equation of the first order.

If $Q(x) \equiv 0$, on putting

$$w = y^{-k}\frac{dy}{dx}, \tag{11}$$

we see that (6) reduces to the non-homogeneous first order equation

$$\frac{dw}{dx} = P(x)w + R(x). \tag{12}$$

Finally, if

$$\frac{dP}{dx} = (1-k)Q,$$

equation (6) can be put in the form

$$\frac{d}{dx}\left(y^{-k}\,\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{P(x)}{1-k}\;y^{1-k}\right] + R(x).$$

On integrating, we obtain

$$\frac{dy}{dx} = \frac{1}{1-k}\,P(x)y \;+ y^{k}\int R(x)dx,$$

which is a first order Bernoulli equation.

1.3 Related Second Order Equations

The transformation $y = H(x)z$ reduces (6) to an equation of the same form,

$$z\frac{d^2z}{dx^2} = k\left(\frac{dz}{dx}\right)^2 + \left[P + \frac{2(k-1)}{H}\,\frac{dH}{dx}\right] z\frac{dz}{dx} + $$

$$+\left[Q + \frac{P}{H}\,\frac{dH}{dx} + \frac{k}{H^2}\left(\frac{dH}{dx}\right)^2 - \frac{1}{H}\,\frac{d^2H}{dx^2}\right] z^2 \;+\; RH^{k-1}\;\; z^{k+1}. \quad (13)$$

Similarly, putting $y=z^{\mu}/\mu$ (where μ is a constant) in (6), we obtain another related second order Bernoulli equation,

$$z\frac{d^2z}{dx^2} = (\mu k-\mu+1)\left(\frac{dz}{dx}\right)^2 + Pz\frac{dz}{dx} + \frac{Q}{\mu}\,z^2 + \frac{R}{\mu^k}\,z^{\mu k-\mu+2}\;. \quad (14)$$

More generally, if $y = f(z)$, (6) can be put in the form

$$\frac{d^2z}{dx^2} = S(z)\left(\frac{dz}{dx}\right)^2 + P(x)\frac{dz}{dx} + Q(x)\,\frac{f(z)}{df/dz} + R(x)\,\frac{\left[f(z)\right]^k}{df/dz}, \quad (15)$$

where $\qquad S(z) = \dfrac{k}{f}\,\dfrac{df}{dz} - \dfrac{d^2f/dz^2}{df/dz}\,.$

In particular, with $y=e^z$, we see that any differential equation of the form

$$\frac{d^2z}{dx^2} = h\left(\frac{dz}{dx}\right)^2 + P(x)\frac{dz}{dx} + Q(x) + R(x)e^{hz}, \quad (16)$$

in which we have replaced $k-1$ by h, can be reduced to the second order Bernoulli equation.

The corresponding result for a first order differential equation is that, by putting $y=e^z$, any equation of the form

$$\frac{dz}{dx} = A(x) + B(x)e^{hz} \tag{17}$$

can be reduced to the first order Bernoulli equation (2), with $h = k-1$.

Now $\dfrac{dy}{dx} = \dfrac{1}{dx/dy}$

and $$\frac{d^2y}{dx^2} = -\frac{1}{(dx/dy)^3}\frac{d^2x}{dy^2}.$$

Thus equation (6) can be expressed in the form

$$\frac{d^2x}{dy^2} + \frac{k}{y}\frac{dx}{dy} + P(x)\left(\frac{dx}{dy}\right)^2 + \left[Q(x)y + R(x)y^k\right]\left(\frac{dx}{dy}\right)^3 = 0$$

and equation (16), with z replaced by y, in the form

$$\frac{d^2x}{dy^2} + h\frac{dx}{dy} + P(x)\left(\frac{dx}{dy}\right)^2 + \left[Q(x) + R(x)e^{hy}\right]\left(\frac{dx}{dy}\right)^3 = 0.$$

Interchanging x and y in each of these equations, we see that the solutions of the two second order equations

$$\frac{d^2y}{dx^2} + \frac{k}{x}\frac{dy}{dx} + P(y)\left(\frac{dy}{dx}\right)^2 + \left[Q(y)x + R(y)x^k\right]\left(\frac{dy}{dx}\right)^3 = 0 \quad (18)$$

and $$\frac{d^2y}{dx^2} + h\frac{dy}{dx} + P(y)\left(\frac{dy}{dx}\right)^2 + \left[Q(y) + R(y)e^{hx}\right]\left(\frac{dy}{dx}\right)^3 = 0 \quad (19)$$

can be expressed in terms of the solutions of a second order Bernoulli equation.

1.4 General solution when a particular solution is known

As shown in § 1.2, by means of the substitution (3), the second order Bernoulli equation

$$y\frac{d^2y}{dx^2} = k\left(\frac{dy}{dx}\right)^2 + P(x)y\frac{dy}{dx} + Q(x)y^2 + R(x)y^{k+1}, \tag{20}$$

with $k \neq 1$, can be reduced to the second order linear equation

$$\frac{d^2u}{dx^2} - P(x)\frac{du}{dx} - (1-k)Q(x)u = (1-k)R(x). \tag{21}$$

If the general solution of (21) is of the form

$u = f(x) + C_1f_1(x) + C_2f_2(x)$,

where C_1 and C_2 are arbitrary constants, the general solution of (20) is

$$y = \left[f(x) + C_1 f_1(x) + C_2 f_2(x)\right]^{1/(1-k)}. \qquad (22)$$

Thus, if $P(x)$, $Q(x)$ and $R(x)$ are simple functions of x, it is possible to express the solution of (20) in terms of known transcendental functions (Babister, 1967).

It may happen that a particular solution of (20) can be found by inspection. Thus, if $R(x) = cQ(x)$, where c is a constant, a particular solution is $y^{1-k} = -c$. More generally, if y_1 is a particular solution of the general equation (20), then, on putting

$$y^{1-k} = y_1^{1-k} + v\,, \qquad (23)$$

we see that v satisfies the homogeneous linear equation

$$\frac{d^2v}{dx^2} - P(x)\frac{dv}{dx} - (1-k)Q(x)v = 0. \qquad (24)$$

Alternatively, y can be shown to satisfy a first order Bernoulli equation. Put

$$w = \frac{y^{-k}y' - y_1^{-k}y_1'}{y^{1-k} - y_1^{1-k}}\,, \qquad (25)$$

in which primes denote differentiation w.r.t. x. On substituting in (20), we find that w satisfies the Riccati equation

$$\frac{dw}{dx} = Q(x) + P(x)w + (k-1)w^2. \qquad (26)$$

Using (25) we see that

$$\frac{dy}{dx} = w\left[y - \left(\frac{y}{y_1}\right)^k y_1\right] + \left(\frac{y}{y_1}\right)^k \frac{dy_1}{dx}\,.$$

If w and y_1 are known as functions of x, this is a first order Bernoulli equation.

Again, if $H(x)$ is a particular solution of the second order Bernoulli equation with $R(x) \equiv 0$, i.e., if $H(x)$ is a particular solution of

$$y\frac{d^2y}{dx^2} = k\left(\frac{dy}{dx}\right)^2 + P(x)y\frac{dy}{dx} + Q(x)y^2 ,$$

then, writing $y = H(x)z$, we see from § 1.3 that z satisfies an equation of the form (20) with the coefficient of z^2 zero. As shown in § 1.2, in that case, on putting $w=z^{-k}dz/dx$, we find that w satisfies a non-homogeneous linear equation of the first order.

1.5 Power series and polynomial solutions

In the general case, it is not possible to express the solution of (1) in terms of known transcendental functions. A solution for y as an infinite series in powers of x can most easily be obtained by using the form (16). If the solution of (16) is written as a Taylor series

$$z = \sum_{n=0}^{\infty} \left(\frac{d^n z}{dx^n}\right)_0 x^n/n! ,$$

the coefficients can be determined by successive differentiation of (16). In certain cases the series may terminate. We shall now examine some polynomial solutions of the second order Bernoulli equation (1), expressed in the form

$$x^{a+2}\left[y\frac{d^2y}{dx^2} - k\left(\frac{dy}{dx}\right)^2\right] = \bar{P}(x)y\frac{dy}{dx} + \bar{Q}(x)y^2 + \bar{R}(x)y^{k+1}, \qquad (27)$$

where a is an integer $\geqslant -2$. In (27), $\bar{P}(x)$, $\bar{Q}(x)$ and $\bar{R}(x)$ are taken to be polynomials in x of degrees $p+1$, q and r respectively.

Let $y = cx^m + c_1x^{m-1} + \dots + c_m$

be a polynomial solution of (27), and let α, β and γ be the leading coefficients of the polynomial expansions of $\bar{P}$, $\bar{Q}$ and $\bar{R}$ respectively. Writing only the leading powers of each term in (27) we have

$$m(m-1)c^2x^{a+2m} - km^2c^2x^{a+2m} + \dots$$
$$= \alpha mc^2x^{p+2m} + \beta c^2x^{q+2m} + \gamma c^{k+1}x^{r+mk+m} + \dots \qquad (28)$$

We assume firstly that α, β, γ are not zero and that $k \neq 1$. On equating the degrees of the greatest exponents in (28) we see that,

if $k < 1$, $0 \leq m(1-k) = \min\,(r-a,\ r-p,\ r-q)$, (29)

provided that $a \neq p \neq q$. If any of the coefficients α, β and γ are zero, the corresponding exponent does not occur in (29).

If any of a, p or q are equal, other values of m may be possible. Thus, if $a = p > q$, one value of m is given by $(r-a)/(1-k)$. However, it may be possible to find a larger value. Thus, if $a = p > q$ and $m(1-k) > r-a$, another relation for determining a value of m is obtained by equating the coefficients of the exponents $a + 2m$ and $p + 2m$ in (28). We find

$$m(m-km-1)c^2 = \alpha mc^2.$$

Thus, if $m \neq 0$, another value of m is given by $(1+\alpha)/(1-k)$ provided that this is a positive integer. We note that in this case the value of c is undetermined (it is, in fact, one of the arbitrary constants in the solution of the differential equation). Similar additional solutions can be found if $a = q > p$ or if $p = q > a$ or if $a = p = q$. In particular, if $a = p = q = 0$ and $\bar{P} = \alpha x$, (27) becomes

$$x^2\left[y\frac{d^2y}{dx^2} - k\left(\frac{dy}{dx}\right)^2\right] = \alpha xy\frac{dy}{dx} + \beta y^2 + \bar{R}(x)y^{k+1}.$$

On making the substitution $u = y^{1-k}$, this is transformed into the non-homogeneous Euler equation

$$x^2\frac{d^2u}{dx^2} - \alpha x\frac{du}{dx} - (1-k)\beta u = (1-k)\bar{R}(x),$$

the solution of which is readily found as a power series in x (Babister, 1967).

If $k > 1$, on putting $y = 1/v$ in (27), we obtain the equation

$$x^{a+2}\left[v\frac{d^2v}{dx^2} - k'\left(\frac{dv}{dx}\right)^2\right] = \bar{P}(x)v\frac{dv}{dx} - \bar{Q}(x)v^2 - \bar{R}(x)v^{k'+1}, \quad (30)$$

where $k' = 2-k < 1$. This equation is of precisely the same form as (27) and the above analysis holds, almost unchanged, with k replaced by k'.

1.6 Other second order equations related to the Bernoulli equation

There are two other non-linear equations of the second order which it is of interest to note here. Both of them can be obtained directly from the first order Bernoulli equation

$$\frac{dy}{dx} = A(x)y + B(x)y^k. \qquad (31)$$

On dividing both sides of (31) by y, and differentiating w.r.t. x, we obtain

$$\frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = \frac{dA(x)}{dx} + \frac{dB(x)}{dx}y^{k-1} + (k-1)B(x)y^{k-2}\frac{dy}{dx},$$

that is,

$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + \frac{dA(x)}{dx}y^2 + \frac{dB(x)}{dx}y^{k+1} + (k-1)B(x)y^k\frac{dy}{dx}.$$

Conversely, any equation of the form

$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + P(x)y^2 + Q(x)y^{k+1} + R(x)y^k\frac{dy}{dx}, \qquad (32)$$

in which $\frac{d}{dx}R(x) = (k-1)Q(x)$, (33)

can be made exact on dividing by y^2. The first integral is the Bernoulli equation

$$\frac{dy}{dx} = y\int P(x)dx + R(x)\,y^k/(k-1).$$

Again, in (31), put $y = \frac{1}{z^a}\frac{dz}{dx}$. Then (31) becomes

$$\frac{1}{z^a}\frac{d^2z}{dx^2} - \frac{a}{z^{a+1}}\left(\frac{dz}{dx}\right)^2 = A(x)\frac{1}{z^a}\frac{dz}{dx} + B(x)\frac{1}{z^{ak}}\left(\frac{dz}{dx}\right)^k$$

On multiplying both sides of this equation by z^{a+1} we see that any equation of the form

$$z\frac{d^2z}{dx^2} = a\left(\frac{dz}{dx}\right)^2 + A(x)z\frac{dz}{dx} + B(x)z^{a+1-ak}\left(\frac{dz}{dx}\right)^k \qquad (34)$$

can be reduced to the first order Bernoulli equation.

1.7 Extension to differential equations of the nth order

The analysis given above can be extended to differential equations of higher order. Consider the nth order non-homogeneous linear differential equation

$$D^n u + p_1(x)D^{n-1}u + \dots + p_n(x)u = f(x), \tag{35}$$

where $D \equiv d/dx$. The general solution of (35) is of the form

$$u = F(x) + C_1 f_1(x) + \dots + C_n f_n(x)$$

where $C_1, \dots, C_n$ are arbitrary constants.

In (35), put

$$u = y^{\lambda}, \tag{36}$$

where λ is a constant. Differentiating (36) w.r.t. x we obtain

$$yDu = \lambda(Dy)u. \tag{37}$$

Repeated differentiation of (37) gives, from Leibnitz' rule,

$$yD^m u + \binom{m-1}{1} Dy.D^{m-1}u + \binom{m-1}{2} D^2y.D^{m-2}u + \dots + D^{m-1}y.Du$$
$$= \lambda\left[Dy.D^{m-1}u + \binom{m-1}{1} D^2y.D^{m-2}u + \binom{m-1}{2} D^3y.D^{m-3}u + \dots + D^m y.u\right]$$
$$(m=1 \text{ to } n). \tag{38}$$

The n equations (38) can be used to determine the ratios $D^m u/u$ ($m=1$ to n) in terms of y and its derivatives. We find

$$y^m(D^m u)/u = \begin{vmatrix} a_{11}(m)Dy & a_{12}(m)D^2y & \dots & a_{1,m-1}(m)D^{m-1}y & a_{1m}(m)D^m y \\ a_{21}(m)y & a_{22}(m)Dy & \dots & a_{2,m-1}(m)D^{m-2}y & a_{2m}(m)D^{m-1}y \\ 0 & a_{32}(m)y & \dots & a_{3,m-1}(m)D^{m-3}y & a_{3m}(m)D^{m-2}y \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{m,m-1}(m)y & a_{mm}(m)Dy \end{vmatrix} \tag{39}$$

in which the element $a_{rs}(m)$ is given, in terms of the binomial coefficients, by the formula

$$a_{rs}(m) = \lambda\binom{m-r}{s-r} - \binom{m-r}{s-r+1}, \quad (r, s = 1 \text{ to } m) \tag{40}$$

with $\binom{\mu}{\nu} = 0$ if $\nu < 0$ or if $\nu > \mu$. From (40) we see that $a_{rm}(m) = \lambda$ $(r = 1$ to $m)$. Thus we can write

$$y^m(D^m u) = \lambda U_m u \qquad (m = 1 \text{ to } n) \qquad (41)$$

where U_m is a polynomial in $y, Dy, \ldots, D^m y$ which

is homogeneous in y and its first m derivatives.

Equation (35) can then be written

$$U_n + p_1(x)yU_{n-1} + \ldots + p_{n-1}(x)y^{n-1}U_1 + p_n(x)y^n/\lambda = f(x)y^{n-\lambda}/\lambda . \quad (42)$$

The general solution of (42) is

$$y = \left[F(x) + C_1 f_1(x) + \ldots + C_n f_n(x)\right]^{1/\lambda} .$$

Such a differential equation can certainly be called a Bernoulli equation of the nth order. Put $\lambda = 1-k$. Then, if $n=2$, (42) is of the same form as (6). The corresponding third order Bernoulli equation is

$$y^2D^3y = 3kyDy.D^2y - k(k+1)(Dy)^3 - p_1(x)\left[y^2D^2y - ky(Dy)^2\right] -$$
$$- p_2(x)y^2Dy - p_3(x)y^3/(1-k) + f(x)y^{k+2}/(1-k). \quad (43)$$

Other related nth order equations can be obtained from (42), for example, by putting $y = e^z$. The third order equation corresponding to (16) is found to be (with $h = k-1$)

$$D^3z = 3hDz.D^2z - h^2(Dz)^3 - p_1(x)\left[D^2z - h(Dz)^2\right] -$$
$$- p_2(x)Dz + p_3(x)/h - f(x)e^{hz}/h. \quad (44)$$

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Part 2 First Integrals of Certain Non-linear Second Order Differential Equations

2.1 Introduction

We consider those non-linear second order differential equations of the form

$$\frac{d^2y}{dx^2} + f(x,y)\frac{dy}{dx} + g(x,y) = 0 \tag{1}$$

which have first integrals

$$\frac{dy}{dx} = F(x,y). \tag{2}$$

In §§ 2.2 and 2.3, $g(x,y)$ is expressed in terms of f and F, and both particular and general first integrals of (1) are obtained. A method is given for finding the general first integral when two particular solutions are known. In § 2.5 the analysis is extended to related differential equations, and some non-linear equations having simple first integrals are considered in § 2.6.

2.2 Second order equations having a given first integral

If (2) is a first integral of the second order equation (1) we see that

$$\frac{d^2y}{dx^2} = F_x + F_y\frac{dy}{dx} = F_x + FF_y. \tag{3}$$

On using (2) and (3), we see that (1) must be of the form

$$\frac{d^2y}{dx^2} + f(x,y)\frac{dy}{dx} - F_x - (F_y + f)F = 0. \tag{4}$$

Thus $$g(x,y) = - F_x - (F_y + f)F. \tag{5}$$

Conversely, putting (4) in the equivalent form

$$\frac{d^2y}{dx^2} - F_x - F_y\frac{dy}{dx} + \phi\left(\frac{dy}{dx} - F\right) = 0, \tag{6}$$

where $$\phi = f + F_y, \tag{7}$$

and writing

$$U = \frac{dy}{dx} - F, \tag{8}$$

we see that (4) becomes

$$\frac{dU}{dx} + \phi(x,y)U = 0. \tag{9}$$

It is obvious that (4), or (9), always has a first integral given by

$$U \equiv \frac{dy}{dx} - F = 0. \tag{10}$$

This solution does not contain any arbitrary constant, and is therefore only a particular solution of (4). It may nevertheless be of interest.

2.3 <u>The general first integral</u>

Under certain circumstances, the general first integral of (4) can be obtained from (9). If ϕ is a function of x only (or a constant), we have, on integrating (9),

$$U = C \exp\left[- \int \phi(x)dx \right] \tag{11}$$

where C is an arbitrary constant. Thus, if $(f+F_y)$ <u>is a function of x only</u>, the general first integral of (4) is

$$\frac{dy}{dx} = F + C \exp\left[- \int (f+F_y)dx \right]. \tag{12}$$

If ϕ is of the form $f_1(y)dy/dx$, from (9),

$$U = C \exp\left[- \int f_1 dy \right], \tag{13}$$

where C is an arbitrary constant.

More generally, it is readily seen that, if $(f+F_y)$ <u>is a function of x only</u>, the second order equation

$$\frac{d^2y}{dx^2} + \left[f(x,y) + f_1(y)\frac{dy}{dx} \right] \frac{dy}{dx} - F_x - \left[F_y + f(x,y) + f_1(y)\frac{dy}{dx} \right] F = 0 \tag{14}$$

has the general first integral

$$\frac{dy}{dx} = F + C \exp\left[- \int (f+F_y)dx - \int f_1 dy \right]. \tag{15}$$

In a similar manner, if ϕ and ψ are <u>functions of x only</u>, and k is a constant, we see that the second order equation

$$\frac{d^2y}{dx^2} + (\phi - F_y)\frac{dy}{dx} - F_x - \phi F = \psi\left(\frac{dy}{dx} - F\right)^k \tag{16}$$

becomes

$$\frac{dU}{dx} + \phi U = \psi U^k,$$

on using (8). This is the Bernoulli equation and can readily be integrated. In particular, if $F = 0$, (16) becomes

$$\frac{d^2y}{dx^2} + \phi(x)\frac{dy}{dx} = \psi(x)\left(\frac{dy}{dx}\right)^k. \tag{17}$$

Put $p = dy/dx$. Then

$$\frac{dp}{dx} + \phi(x)p = \psi(x)p^k$$

which is the Bernoulli equation.

If f_1 and f_2 are <u>functions of y only</u>, and k is a constant, the second order equation

$$\frac{d^2y}{dx^2} + f_1\left(\frac{dy}{dx}\right)^2 - \left(F_y + f_1F\right)\frac{dy}{dx} - F_x = f_2\frac{dy}{dx}\left(\frac{dy}{dx} - F\right)^k \tag{18}$$

becomes, from (8),

$$\frac{dU}{dx} + f_1\frac{dy}{dx}U = f_2\frac{dy}{dx}U^k$$

or

$$\frac{dU}{dy} + f_1U = f_2U^k,$$

which is again the Bernoulli equation. In particular, if $F = 0$, (18) becomes

$$\frac{d^2y}{dx^2} + f_1(y)\left(\frac{dy}{dx}\right)^2 = f_2(y)\left(\frac{dy}{dx}\right)^{k+1} \tag{19}$$

Put $p = dy/dx$. Then $d^2y/dx^2 = p\,dp/dy$ and (19) reduces to

$$\frac{dp}{dy} + f_1(y)p = f_2(y)p^k$$

which is the Bernoulli equation.

2.4 The general first integral when two particular solutions are known

It sometimes happens that it is possible to express the second order equation (1) in the form (6) in two different ways. We then have two equations

$$\frac{dU}{dx} + \phi(x,y)U = 0 \tag{20}$$

and
$$\frac{dU_1}{dx} + \phi_1(x,y)U_1 = 0. \tag{21}$$

It may then be possible to eliminate y between (20) and (21) to obtain a single equation in x. Thus if, in (1) and (2), $f = 0$ and F is a function of y only, $g = -FF_y$ and we find that

$$U = y' - F, \ \phi = F_y \tag{22}$$

and
$$U_1 = y' + F, \ \phi_1 = -F_y. \tag{23}$$

From (20)-(23),

$$\frac{1}{U}\frac{dU}{dx} + \frac{1}{U_1}\frac{dU_1}{dx} = 0,$$

which gives the first integral of (1) in this case in the form

$$UU_1 = C,$$

that is,
$$y'^2 = F^2 + C, \tag{24}$$

where C is an arbitrary constant. Since, in this case, $f = 0$ and g is a function of y only, this solution could be found by more elementary methods. Thus

$$d^2y/dx^2 = FF_y = \frac{d}{dy}\left(\tfrac{1}{2}F^2\right)$$ and (24) follows immediately,

on putting $y'' = d/dy\,(\tfrac{1}{2}y'^2)$ and integrating. However, this method is of general application.

Thus, consider the equation

$$\frac{d^2y}{dx^2} + (k+1)f(x)\frac{dy}{dx} + \left[f'(x)+kf^2(x)\right]y = kb^2y^{2k-1} \tag{25}$$

where b and k are constants. It is readily verified that this equation can be put in the forms of (20) and (21), with

$$U = \frac{dy}{dx} + f(x)y - by^k, \qquad \phi = kf(x) + kby^{k-1} \tag{26}$$

$$\text{and } U_1 = \frac{dy}{dx} + f(x)y + by^k, \qquad \phi_1 = kf(x) - kby^{k-1}. \tag{27}$$

Using (20) and (21), we see that

$$\frac{1}{U}\,\frac{dU}{dx} + kf(x) = -\frac{1}{U_1}\,\frac{dU_1}{dx} - kf(x)$$

and thus $\log(UU_1) = C - 2k\int f(x)dx$,

which gives the first integral of (25) in the form

$$\left[\frac{dy}{dx} + f(x)y\right]^2 = b^2y^{2k} + C_1 \exp\left[-2k\int f(x)dx\right], \tag{28}$$

C and C_1 being arbitrary constants.

2.5 Related differential equations

The transformation $y = h(z)$ reduces (1) to

$$h'(z)\frac{d^2z}{dx^2} + f(x,h)h'(z)\frac{dz}{dx} + h''(z)\left(\frac{dz}{dx}\right)^2 + g(x,h) = 0. \tag{29}$$

Thus any differential equation of the form

$$\frac{d^2z}{dx^2} + \bar{f}(x,z)\frac{dz}{dx} + \frac{h''(z)}{h'(z)}\left(\frac{dz}{dx}\right)^2 + \bar{g}(x,z) = 0 \tag{30}$$

can be reduced to the form of equation (1), with

$$f(x,h) = \bar{f}(x,z)$$

and $g(x,h) = \bar{g}(x,z)h'(z)$.

In particular with $y = h(z) = \exp(az)$ we see that the equation

$$\frac{d^2z}{dx^2} + \bar{f}(x,z)\frac{dz}{dx} + a\left(\frac{dz}{dx}\right)^2 + \bar{g}(x.z) = 0 \tag{31}$$

can readily be reduced to equation (1).

In (1), we can write

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

and
$$\frac{d^2y}{dx^2} = -\frac{1}{(dx/dy)^3}\frac{d^2x}{dy^2} .$$

Thus equation (1) can be expressed in the form

$$\frac{d^2x}{dy^2} - f(x,y)\left(\frac{dx}{dy}\right)^2 - g(x,y)\left(\frac{dx}{dy}\right)^3 = 0$$

and equation (31), with z replaced by y, in the form

$$\frac{d^2x}{dy^2} - a\frac{dx}{dy} - \bar{f}(x,y)\left(\frac{dx}{dy}\right)^2 - \bar{g}(x,y)\left(\frac{dx}{dy}\right)^3 = 0.$$

Interchanging x and y in each of these equations, we see that the solutions of the two second order equations

$$\frac{d^2y}{dx^2} - f(y,x)\left(\frac{dy}{dx}\right)^2 - g(y,x)\left(\frac{dy}{dx}\right)^3 = 0 \tag{32}$$

and
$$\frac{d^2y}{dx^2} - a\frac{dy}{dx} - \bar{f}(y,x)\left(\frac{dy}{dx}\right)^2 - \bar{g}(y,x)\left(\frac{dy}{dx}\right)^3 = 0 \tag{33}$$

can be simply found from those of (1) and (31). As shown in part I [eq.(19)], if $\bar{f}$ and $\bar{g}$ are functions of y only, the solution of (33) can be expressed in terms of that of a second order Bernoulli equation.

Equation (32) is of the form

$$\frac{d^2y}{dx^2} + P(x,y)\left(\frac{dy}{dx}\right)^{\alpha} + Q(x,y)\left(\frac{dy}{dx}\right)^{\beta} = 0, \tag{34}$$

where $\alpha \neq \beta$. As shown in § 2.3, if P and Q are both functions of the same variable, the first integral of (34) can be found in a simple form for certain values of α and β.

If P is a function of x only and Q is a function of y only, divide (34) by $(dy/dx)^{\alpha}$. Then (34) becomes

$$\left(\frac{dy}{dx}\right)^{-\alpha}\frac{d^2y}{dx^2} + P(x) + Q(y)\left(\frac{dy}{dx}\right)^{\beta-\alpha} = 0. \tag{35}$$

This equation is exact if $\beta = \alpha + 1$, with a first integral

$$\frac{1}{1-\alpha}\left(\frac{dy}{dx}\right)^{1-\alpha} + \int P(x)dx + \int Q(y)dy = \text{const.}, \qquad (\alpha \neq 1),$$

$$\text{or } \log(dy/dx) + \int P(x)dx + \int Q(y)dy = \text{const.} \qquad (\alpha = 1).$$

This case can be further generalized as follows.

Consider the equation

$$f\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + P(x)\left(\frac{dy}{dx}\right)^{\alpha} + Q(y)\left(\frac{dy}{dx}\right)^{\beta} = 0. \tag{36}$$

On putting $p = dy/dx$, we see that (36) can be written in the form

$$f(p)\frac{dp}{dx} + P(x)p^{\alpha} + Q(y)p^{\beta} = 0,$$

that is, $f(p)p^{-\alpha}\dfrac{dp}{dx} + P(x) + Q(y)\dfrac{dy}{dx}\, p^{\beta-\alpha-1} = 0.$

This equation is exact if $\beta = \alpha + 1$, with a first integral

$$\int f(p)p^{-\alpha}\,dp + \int P(x)dx + \int Q(y)dy = \text{const.}$$

2.6 <u>Other non-linear equations having simple first integrals</u>

We have seen that, in a number of cases, non-linear second order equations can be reduced to Bernoulli's equation, and hence the first integral can readily be found. Smith (1961) has shown that if

$$f(y) = (n+2)by^n - 2a$$

and $g(y) = y\left[c + (by^n - a)^2\right]$,

where a, b, c and n are constants, the general solution of

$$\frac{d^2y}{dx^2} + f(y)\frac{dy}{dx} + g(y) = 0 \tag{37}$$

can be expressed in terms of elementary functions.

Another family of non-linear equations which are readily integrated is

$$yy'' = y'^2 f(y^m y'^n). \tag{38}$$

Put $y' = p$, $y'' = p\,dp/dy$. Then (38) becomes

$$y\, dp/dy = p\, f(y^m p^n). \tag{39}$$

Let $u = y^m p^n$. Then, as shown by Murphy (1960), we obtain

$$y\, du/dy = u(m+nf).$$

Thus (38) has the first integral

$$\log y + C = \int \frac{du}{u(m+nf)}, \tag{40}$$

where $$u = y^m y'^n \tag{41}$$

and C is a constant.

Similarly the equation

$$xy'' = y' f(x^m y'^n) \tag{42}$$

can be shown to have the first integral

$$\log x + C = \int \frac{dv}{v(m+nf)} \tag{43}$$

where $$v = x^m y'^n. \tag{44}$$

Finally, in the equation

$$x^2 y'' = y f(xy'/y), \tag{45}$$

put $w = xy'/y$. Then the first integral of (45) can be obtained from either of the equations

$$\frac{dx}{x} = \frac{dw}{f(w)-w(w-1)} \tag{46}$$

or $$\frac{dy}{y} = \frac{wdw}{f(w)-w(w-1)}. \tag{47}$$

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