UNIVERSITY OF GLASGOW

DEPARTMENT OF AERONAUTICS AND FLUID MECHANICS

NON-LINEAR OSCILLATIONS OF A MIXED RAYLEIGH - VAN DER POL TYPE

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A.W. Babister, M.A., Ph.D.

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SUMMARY

Large amplitude motion of non-lifting missiles is considered. If the motion is principally in one degree of freedom, the system satisfies the differential equation

$$\ddot{x} + b\dot{x} + \lambda x^2\dot{x} + \mu x\dot{x}^2 + \nu\dot{x}^3 + cx = 0$$

This equation is first solved by the Method of the First Approximation, and a limit cycle of amplitude $2\sqrt{-b/(\lambda+3\nu c)}$ is found.

Trajectories in the $(x,\dot{x})$ phase plane are given. Stable limit cycles are shown to occur for a wide range of values of λ and ν, including cases in which λ and ν have opposite signs. The period is relatively insensitive to large changes in λ and ν.

Variation in the value of μ alters the damping of the system and changes the shape of the limit cycles. In general, for sufficiently large negative values of μ, some of the phase plane trajectories go off to infinity as t increases. Large positive values of μ lead to very large amplitudes in $\dot{x}$. It would therefore seem that systems with large values of μ of either sign are to be avoided.

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1. Introduction

Some fin-stabilised missiles and rockets develop a non-linear response when subjected to disturbances of large amplitude. The resulting free motion is highly complicated but, if the motion is principally in one degree of freedom at constant airspeed (e.g., if the missile rate of roll is very small), the differential equation of the system can be reduced to the form

$$\ddot{x} + f(x,\dot{x})\,\dot{x} + cx = 0, \qquad (1)$$

where $c > 0$ and the second term is a non-linear damping function. As shown in references 1 and 2, such an equation can occur in the pitching motion of a rocket (x then being the angle of pitch) and in the lateral snaking oscillation of an aircraft (x being the angle of yaw). The third term in (1), cx, is the stiffness term, which is kept in its linear form in the present treatment. The effects of non-linear stiffness have been discussed in references 3 and 4.

For oscillations of small amplitude f can be taken to be constant, but this is not true if x is large. It follows from physical considerations that, for non-lifting missiles and rockets with a vertical plane of symmetry, $f(x,\dot{x})$ must be an even function of x and its derivative taken compositely. For simplicity we shall discard terms of the fourth degree and higher. Equation (1) then becomes

$$\ddot{x} + b\dot{x} + \lambda x^2\dot{x} + \mu x\dot{x}^2 + \nu\dot{x}^3 + cx = 0, \qquad (2)$$

where b, c, λ, μ and ν are constants ($c > 0$).

We shall consider the nature of solutions of (2), and in particular those cases in which self-sustained oscillations or limit cycles occur.

This equation is of the Rayleigh type if $\lambda = \mu = 0$ and b and ν are of opposite sign, and it is of the Van der Pol type if $\mu = \nu = 0$ and b and λ are of opposite sign. Levinson and Smith (reference 5) have considered equations of the form of (1) for $f(x,\dot{x}) > 0$ for large $|x|$ and for all $\dot{x}$, but this condition is not satisfied by (2) if λ and ν are of opposite sign. We shall therefore consider the nature of solutions of (2), or of the equivalent system.

$$\left.\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -by - \lambda x^2 y - \mu x y^2 - \nu y^3 - cx, \end{aligned}\right\} \qquad (3)$$

for all combinations of sign of b, λ, μ and ν, with $c > 0$. Here x is the displacement of the system from its equilibrium position.

2. <u>Method of the First Approximation</u>

If c is positive ($= \omega^2$) and the damping parameters are small, the system (3) will have an oscillatory behaviour, and we can use Krylov and Bogouliubov's method of the first approximation. The solution of (2) is written in the form

$$x = A \sin \chi \qquad (4)$$

where $\chi = \omega t + \phi$, and A and ϕ are both functions of t.

On differentiating (4) twice with respect to t, we see that (2) can be written in the form

$$\{\ddot{A} - 2A\omega\dot{\phi} - A\dot{\phi}^2\} \sin \chi + \{2\dot{A}(\omega+\dot{\phi}) + A\ddot{\phi}\} \cos \chi$$
$$= -b\dot{x} - \lambda x^2\dot{x} - \mu x\dot{x}^2 - \nu\dot{x}^3. \qquad (5)$$

In this method the changes in the amplitude A and phase ϕ during any cycle are taken to be small, and the dominant terms in the { } are $-2A\omega\dot{\phi}$ and $2\dot{A}\omega$.

On multiplying (5) by cos χ and integrating from 0 to 2π we obtain

$$2\dot{A}\omega = -\frac{1}{\pi}\int_0^{2\pi} (b\dot{x} + \lambda x^2\dot{x} + \mu x\dot{x}^2 + \nu\dot{x}^3)\cos\chi\, d\chi, \quad (6)$$

where only the contributions from the dominant terms have been retained on the l.h.s. of the equation. On substituting for x from (4) and neglecting the variation of A over the cycle, (6) reduces to

$$2\dot{A}\omega = -bA\omega - \tfrac{1}{4}\lambda A^3\omega - \tfrac{3}{4}\nu A^3\omega^3 ,$$

that is, $\dot{A} = -\tfrac{1}{2}A\{b + \tfrac{1}{4}(\lambda + 3\nu c)A^2\}$. (7)

When the steady state is reached, $\dot{A} = 0$ and thus $A = 0$ or $2\sqrt{-b/(\lambda + 3\nu c)}$. Thus, if b and $(\lambda + 3\nu c)$ are of opposite sign (and small compared with c), the system (3) will have a limit cycle of amplitude A given by

$$A = 2\sqrt{-b/(\lambda + 3\nu c)}. \quad (8)$$

If $\nu = 0$, this reduces to the familiar result for Van der Pol's equation, and if $\lambda = 0$, it reduces to that for Rayleigh's equation. However, we have shown that, provided the damping parameters are small, the amplitude of the limit cycle is independent of μ. From (7) we see that the limit cycle is stable if $b < 0$ and $\lambda + 3\nu c > 0$, and unstable if $b > 0$ and $\lambda + 3\nu c < 0$.

On multiplying (5) by sin χ and integrating from 0 to 2π, we obtain, with the same assumptions as above,

$$2A\omega\dot{\phi} = \frac{1}{\pi}\int_0^{2\pi} \mu x\dot{x}^2 \quad \sin\chi \quad d\chi.$$

Thus $\dot{\chi} = \omega + \dot{\phi} = \sqrt{c} \quad (1 + \frac{\mu A^2}{8}).$ (9)

From (4) and (9) we see that the rate of change of phase angle, and hence the frequency of the oscillation, depends only on c and μ (and the amplitude A) and is independent of the other damping parameters provided that they are small.

3. The general nature of the solutions

We now consider the general nature of the solutions of (2) and (3). The system (3) has only one singular point, at the origin O, this point being a focus (or node) if $c > 0$. The focus at O will be stable or unstable according as $b >$ or < 0. As shown by Loud (reference 6) the focus becomes a centre if both $b = 0$ and $\lambda + 3\nu c = 0$ (with $c > 0$).

Consider next the limit cycles of (3). It is well known that limit cycles cannot occur if $\lambda = \mu = 0$ or if $\mu = \nu = 0$ and the remaining parameters are of the same sign. More generally, for the system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y) \tag{10}$$

using Dulac's extension of Bendixson's theorem (reference 7), we find that, if B(x,y) is a continuous function such that in a single-connected region of the (x,y) plane

$$\Delta = - \{\partial(BP)/\partial x + \partial(BQ)/\partial y\}$$

is of constant sign, then no closed contours exist in the region for the given system.

We take $B = \exp(\mu x^2)$. Then, for the system (3),

$$\Delta = B\{b + \lambda x^2 + 3\nu y^2\}. \tag{11}$$

Now the r.h.s. of (11) will be of constant sign over the whole (x,y) plane if b, λ and ν are all of the same sign, irrespective of the value of μ. Thus limit cycles cannot occur for the system (3), if b, λ and ν are all of the same sign. More generally, for the system

$$\left.\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -by - \lambda x^2 y - F(x)y^2 - \nu y^3 - g(x)\,, \end{aligned}\right\} \quad (12)$$

on putting $B = \exp\{2\int F(x)dx\}$, we can show that limit cycles cannot occur if b, λ and ν are all of the same sign.

If $\mu = 0$ and $\lambda = \nu c$, it is readily seen that (2) has a particular solution given by

$$b + \lambda x^2 + \nu\dot{x}^2 = 0.$$

This is a closed curve (an ellipse) in the (x,y) plane if $c > 0$ and $b/\nu < 0$, x and t being related by the equation

$$x = A \sin \omega t$$

where $\omega = \sqrt{c}$. This corresponds to an undamped oscillation, the amplitude A being given by $\sqrt{-b/\lambda}$, in agreement with equation (8) for $\lambda = \nu c$. To determine the general nature of the solutions of (2) with $\mu = 0$ and $\lambda = \nu c$,

let $\qquad \rho = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}cx^2 + \tfrac{1}{2}b/\nu.$

Then $\qquad \dot{\rho} = \dot{x}\ddot{x} + cx\dot{x}$

and (2) becomes

$$\dot{\rho} = -2\nu\dot{x}^2\rho. \qquad (13)$$

From (13) we see that, if $\nu > 0$, ρ and $\dot{\rho}$ have opposite signs. Thus (as shown in fig. 1), with $c > 0$ and $b/\nu < 0$, all the phase trajectories approach the curve $\rho = 0$ or

$$cx^2 + y^2 = -b/\nu \quad ,$$

which is a limit cycle.

More generally, consider the energy function per unit mass

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}cx^2 \tag{14}$$

for the system (1). On multiplying (1) by $\dot{x}$ and integrating with respect to t, we obtain

$$E = E_0 - \int_0^t f(x,\dot{x})\,\dot{x}^2\,dt,$$

where E_0 is the energy at time t = 0. If f is always positive (except possibly for $x = \dot{x} = 0$), the energy E is always decreasing and the system approaches the equilibrium position as $t \to \infty$. If (1) has a periodic solution, of period T, we see that

$$\int_0^T f(x,\dot{x})\dot{x}^2\,dt = 0.$$

Thus, for (1) to have a periodic solution with $f \not\equiv 0$, $f(x,\dot{x})$ must change sign within a period.

In (2), put $x = \alpha X$, $t = \beta T$ where α and β are constants. Then

$$\frac{d^2X}{dT^2} + \beta b\frac{dX}{dT} + \alpha^2\beta\lambda X^2\frac{dX}{dT} + \alpha^2\mu X\left(\frac{dX}{dT}\right)^2 + \frac{\alpha^2\nu}{\beta}\left(\frac{dX}{dT}\right)^3 + \beta^2 cX = 0. \tag{15}$$

If (2) has the solution $x = \phi(t)$, with $x = x_0$, $y = y_0$ at t = 0, (15) has the solution $X = \alpha^{-1}\phi(\beta T)$, with $X = x_0/\alpha$ and $dX/dT = \beta y_0/\alpha$ at T = 0. Thus, if (2) has a periodic solution, (15) with any non-zero α and β will also have a periodic solution. We note that a variation in the sign of α merely affects the non-linear terms in (15). If α is replaced by $-\alpha$, (15) is unchanged but the initial values of X and dX/dT both change sign. Any solution of (15) has a corresponding one in which, at any time T, the value of X is of equal magnitude but of opposite sign;

thus, in the phase plane, if $x(t)$, $y(t)$ is a solution of (3), so is $-x(t)$, $-y(t)$. In particular, if (3) has a periodic solution which is represented by a closed curve encircling the origin and passing through the point (x,y) in the phase plane, it is readily shown by topological argument that the point $(-x,-y)$ also lies on this curve (Lefschetz, 1957). Such closed curves will have equal and opposite intercepts on the x axis; between any two consecutive points of intersection, y is a single-valued function of x. Again, if $\alpha = 1$ and $\beta = -1$, the coefficients b, λ and ν in (2) become $-b$, $-\lambda$, $-\nu$, and the variation of X with T is identical with that of x with $(-t)$. Thus the positive semi-trajectory $(T > 0)$ in the X plane is the same as the negative semi-trajectory $(t < 0)$ in the x plane.

4. Phase plane trajectories for $\mu = 0$.

Trajectories in the $(x,\dot{x})$ phase plane were found by analogue computer for

$$\ddot{x} + b\dot{x} + \lambda x^2\dot{x} + \nu\dot{x}^3 + cx = 0 \tag{16}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y\ , \\ \dot{y} &= -by - \lambda x^2 y - \nu y^3 - cx. \end{aligned}\right\} \tag{17}$$

For this system, as shown in § 3, no limit cycles occur if b, λ and ν are all of the same sign. In the subsequent discussion we shall take $b < 0$ and $c > 0$; thus (16) has a negative damping term if both x and $\dot{x}$ are small.

If λ and ν are both positive (with $b < 0$), as shown by Levinson and Smith (reference 5), (17) will have a unique stable limit cycle. The general nature of the trajectories is similar to that shown in fig. 1 (although the limit cycle is not, in general, circular). Consider next the case in which λ and ν have opposite signs. In general this case does not appear to be amenable to mathematical analysis. For small values of ν (with $\lambda > 0$ and $b < 0$), the system (17) can be considered as a perturbed Van der Pol equation, and by well known results (reference 9) the perturbed system has a unique limit cycle which is stable and which lies in the neighbourhood of that of Van der Pol's equation. By a precisely similar argument we find that, if $\nu > 0$ and $b < 0$, the system (17) will have a stable limit cycle for small values of λ .

Figure 2 shows how the limit cycle varies with λ for $b = -1$, $c = 1$, $\nu = 1$. As λ decreases, both x_{max} and y_{max} increase, the cycle almost taking on the form of a parallelogram. For these values of b, c and ν, no limit cycle occurs for $\lambda < -0.57$; for such values of λ all trajectories go off to infinity. Figure 3 shows how the limit cycle varies with ν for $b = -1$, $c = 1$, $\lambda = 1$; the curve $\nu = 0$ is , of course, that corresponding to the Van der Pol equation. Here too, as ν decreases, x_{max} and y_{max} increase, but far more rapidly than in the previous case, the limit cycle becoming very elongated in the y direction, as can be seen from the following table.

TABLE

Maximum amplitudes in x and y of the limit cycles for various values of ν ($b = -1$, $c = 1$, $\lambda = 1$)

ν	0	-0.1	-0.2	-0.3	-0.4
x_{max}	2.0	2.9	4.8	6.1	7.4
y_{max}	2.7	4.9	13.8	19.6	26

Although a limit cycle exists for these negative values of ν, its practical significance is doubtful, since additional terms in the original equation of motion (1) would be of importance at these large x and y amplitudes.

More generally, from (16), it can be shown by dimensional analysis that the equation of the limit cycle can be put in either of the following forms :-

$$\sqrt{-\frac{\lambda}{b}}\; y = bf_1\left(\sqrt{\frac{-\lambda}{b}}\, x\, ,\ \frac{c\nu}{\lambda}\, ,\ \frac{b^2}{c}\right) \tag{18}$$

or $$\sqrt{-b\nu}\; y = bf_2\left(\sqrt{-b\nu}\, x\, ,\ \frac{c\nu}{\lambda}\, ,\ \frac{b^2}{c}\right). \tag{19}$$

Eq. (18) is appropriate if $\lambda/b < 0$, and (19) if $b\nu < 0$.

From either of these equations we see that, for given values of b, c and λ/ν, the maximum x amplitude of the limit cycle is proportional to $1/\sqrt{\lambda}$ (or to $1/\sqrt{\nu}$). This is shown in figure 4, in which x_{max} is plotted in terms of both λ and ν. No limit cycles occur for values of the parameters in the region to the left of the lines OA and OB.

Analogue computer results were also obtained for $b^2/c < 1$. Figures 5 and 6 show the range of values of $\lambda/c\nu$ (or its reciprocal) for which a limit cycle occurs (figure 5 is drawn for $\lambda > 0$ and figure 6 for $\nu > 0$). As noted above, the amplitude of the cycle increases rapidly for negative values of ν ; however, both the maximum x and y amplitudes decrease slightly as b^2/c decreases. The broken line on fig. 5 corresponds to limit cycles for which $x_{max} = 3.0$.

The variation of x with t is shown in figure 7 for b = -1, c = 1 and λ and ν equal to 1 or 0 (the curves with λ = 0 and with

$\nu = 0$ correspond to the Rayleigh and Van der Pol equations). In all cases the initial conditions are $x = 2$, $\dot{x} = 0$. With $\lambda = \nu = 1$ (for which the phase plane curve is given in Fig. 1), the period is slightly less than that for the other systems shown in fig. 7; we see that the period is relatively insensitive to large changes in λ and ν. The general shape of the (x,t) curve is sensitive to the value of ν, as is also the shape of the limit cycle in the phase plane.

5. Phase plane trajectories for $\mu \neq 0$.

Finally we consider trajectories in the $(x,\dot{x})$ phase plane for

$$\ddot{x} + b\dot{x} + \lambda x^2\dot{x} + \mu x\dot{x}^2 + \nu\dot{x}^3 + cx = 0. \qquad (20)$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y , \\ \dot{y} &= -by - \lambda x^2 y - \mu x y^2 - \nu y^3 - cx. \end{aligned}\right\} \qquad (21)$$

Figure 8 shows the limit cycles for $b = -1$, $c = 1$, $\lambda = 1$, $\nu = 1$ and zero and positive values of μ; limit cycles for negative values of μ are shown in figure 9. For the given values of b, c, λ and ν, no limit cycles occurred for $\mu < -2.2$; for such values of μ, the phase plane trajectories went off to infinity as $t \to \infty$.

From (21), at $x = 0$, $dy/dx = -b - \nu y^2$. With the given values of the parameters, we see that, if $\mu > 0$, the limit cycle cuts the y axis at points for which $|y| > \sqrt{-b/\nu}$ and thus $dy/dx < 0$; if $\mu < 0$, the limit cycle cuts the y axis at points for which $|y| < \sqrt{-b/\nu}$ and thus $dy/dx > 0$. From figure 8 we see that, as μ increases, the limit cycle resembles a slightly tilted oval, the x amplitude decreasing and the y amplitude increasing. From Fig. 9, for negative values of μ ($\mu > -2.2$), the limit cycle becomes more elongated in the x direction.

If b, λ and ν are all zero, eq. (20) can be integrated to give

$$\mu y^2 + c = A\, e^{-\mu x^2} \tag{22}$$

where $y = \dot{x}$ and A is a constant. This gives a series of cycles (but no limit cycles) in the (x,y) phase plane, as shown in fig. 10 ($\mu > 0$) and fig. 11 ($\mu < 0$). In the latter case (with $\mu < 0$), cyclic solutions only occur for $A > 0$, that is, for $|y| < \sqrt{-c/\mu}$; all the trajectories for which $|y| \geqslant \sqrt{-c/\mu}$ go off to infinity as t increases. It can be seen that there are certain similarities between these two figures and figures 8 and 9, for values of μ having the same sign.

From (20), if $\mu > 0$, the system has increased damping in the first and third quadrants, and decreased damping in the second and fourth quadrants; the reverse is true if $\mu < 0$. In general, as shown above, for sufficiently large negative values of μ (with $c > 0$), some of the phase plane trajectories go off to infinity as t increases; however, if $b > 0$ (i.e., if the system is stable for very small disturbances), trajectories which approach sufficiently close to the origin will be 'captured' and brought into the null-position of equilibrium (as shown in fig. 12, in which $b = c = 1$ and $\mu = -1$). Large positive values of μ lead to very large amplitudes in $\dot{x}$, and thus while the system may be theoretically stable such large amplitudes would be completely unacceptable. It would therefore seem that systems with large values of μ of either sign are to be avoided.

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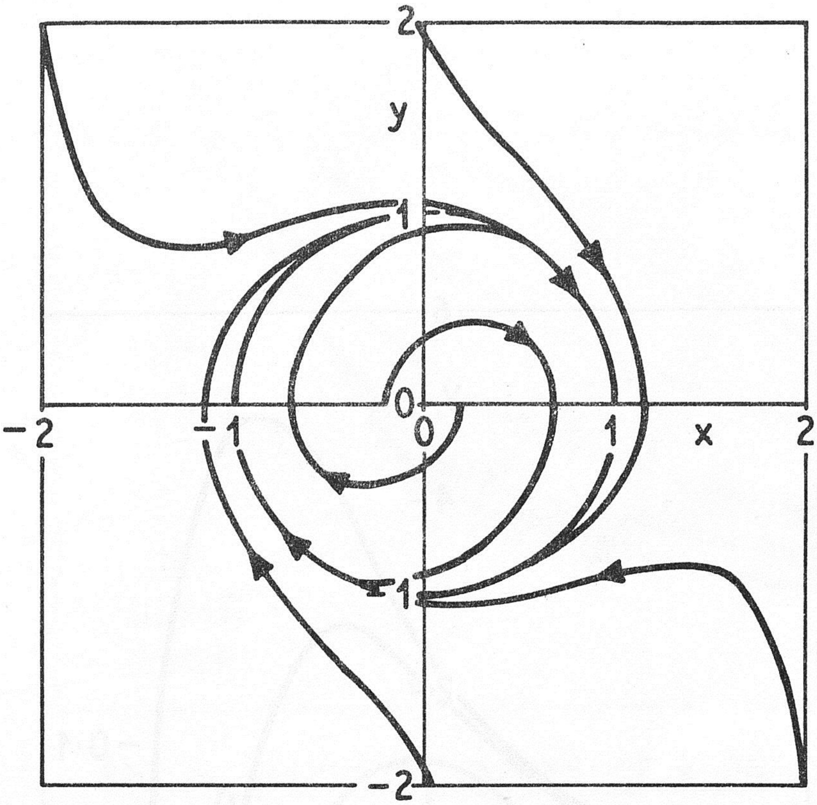

FIG. 1. TRAJECTORIES $(b=-1,\ \lambda=v=c=1)$

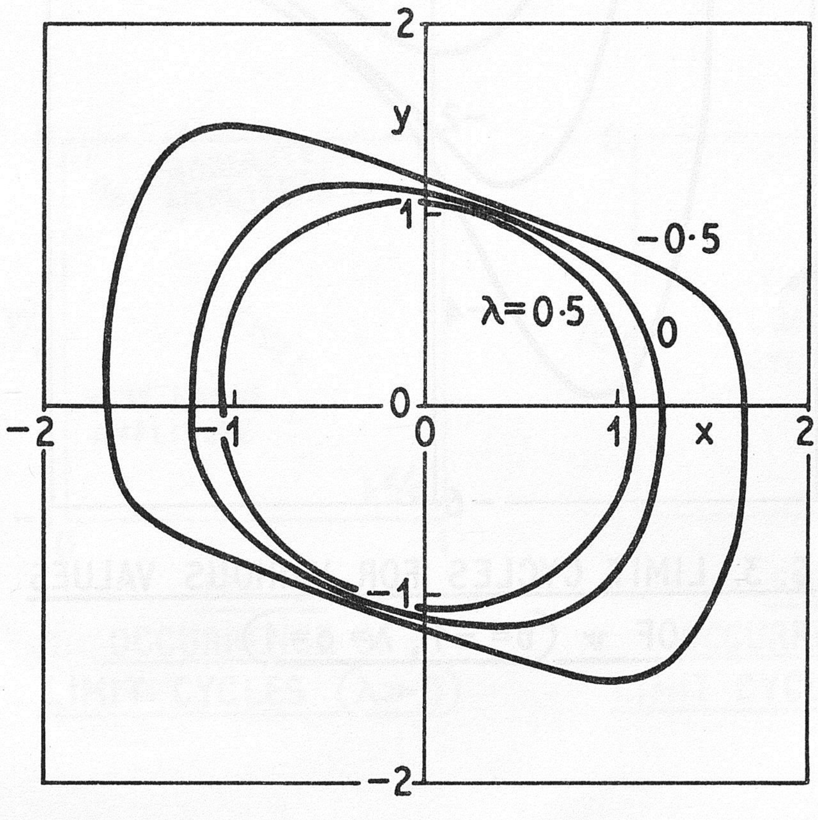

FIG. 2. LIMIT CYCLES FOR VARIOUS VALUES OF λ $(b=-1,\ v=c=1)$

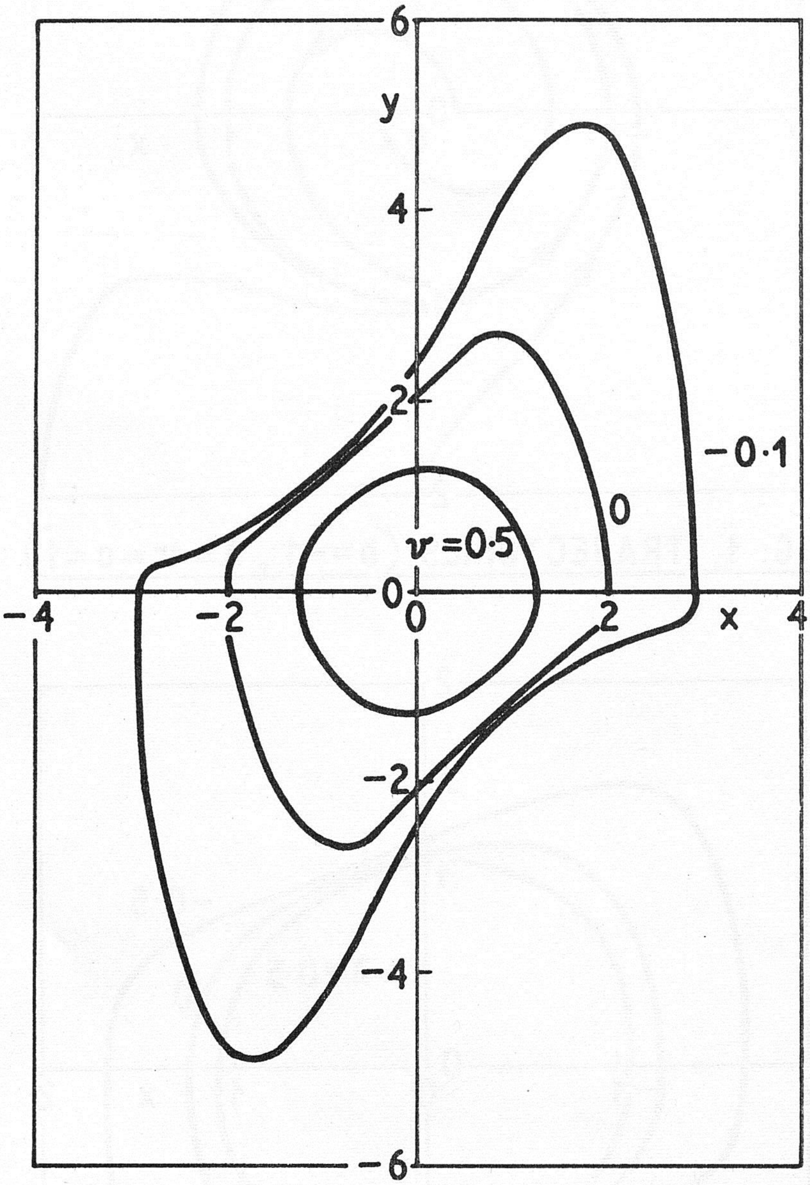

FIG. 3. LIMIT CYCLES FOR VARIOUS VALUES OF v $(b=-1,\ \lambda=c=1)$

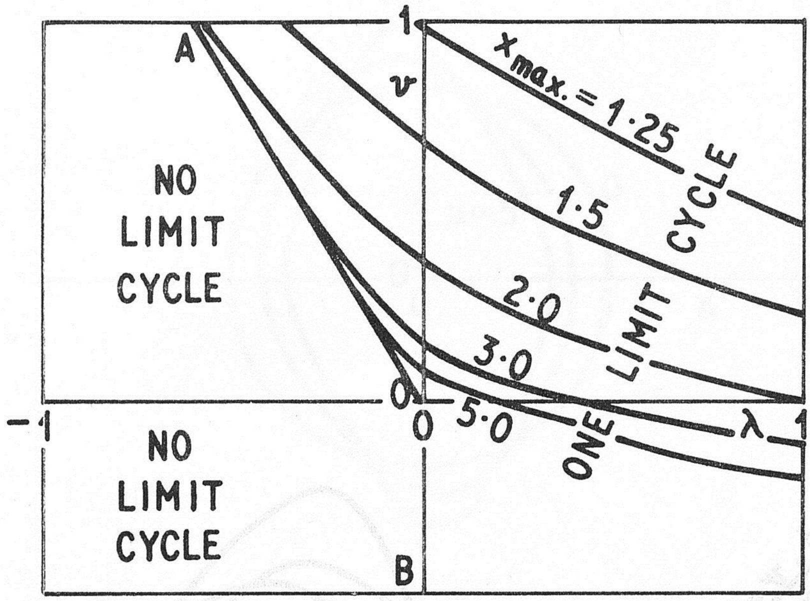

FIG. 4. MAXIMUM VALUES OF x ON LIMIT CYCLES (b = 1, c = 1)

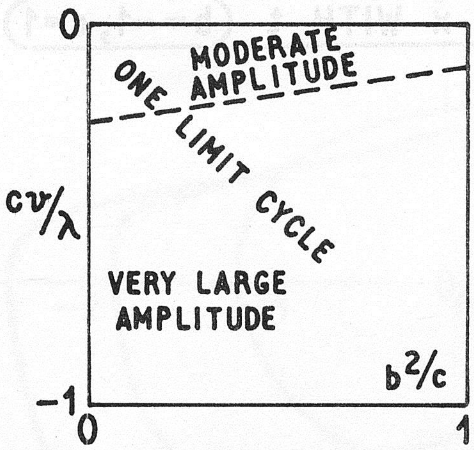

FIG. 5.
OCCURRENCE OF
LIMIT CYCLES ($\lambda > 0$)

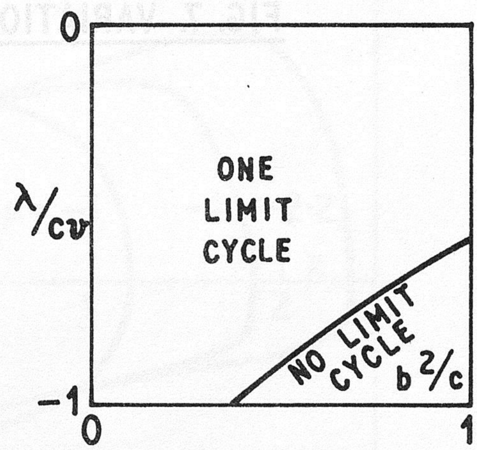

FIG. 6.
OCCURRENCE OF
LIMIT CYCLES ($v > 0$)

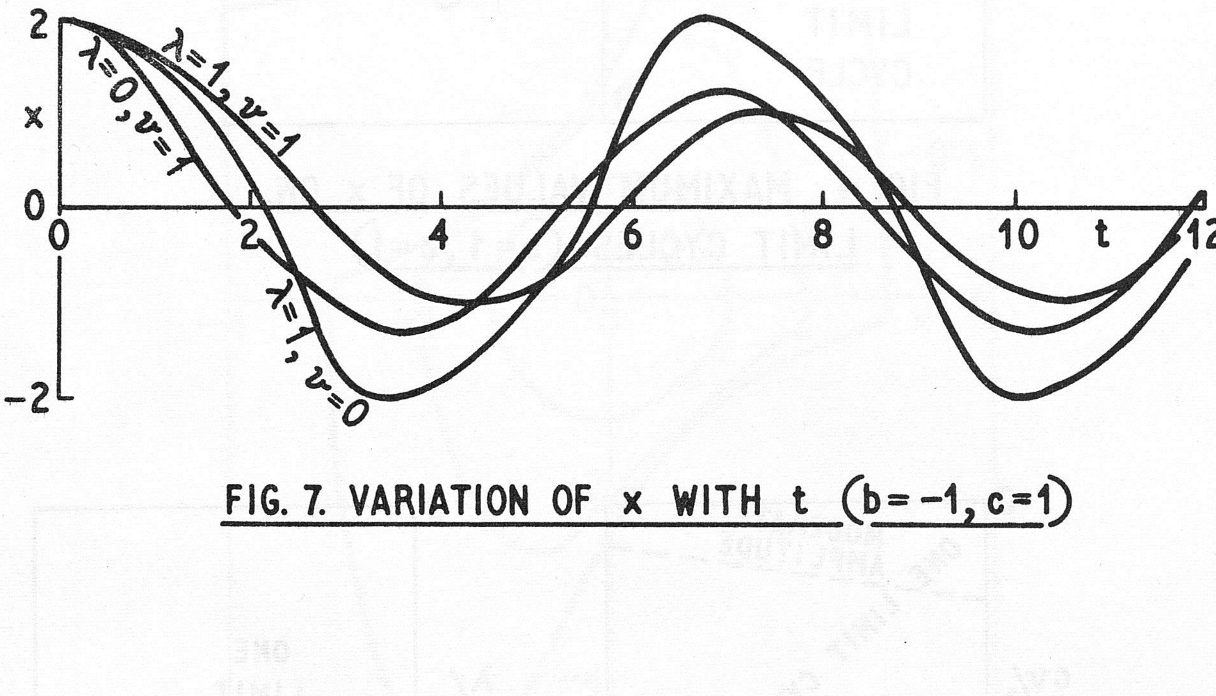

FIG. 7. VARIATION OF x WITH t (b = −1, c = 1)

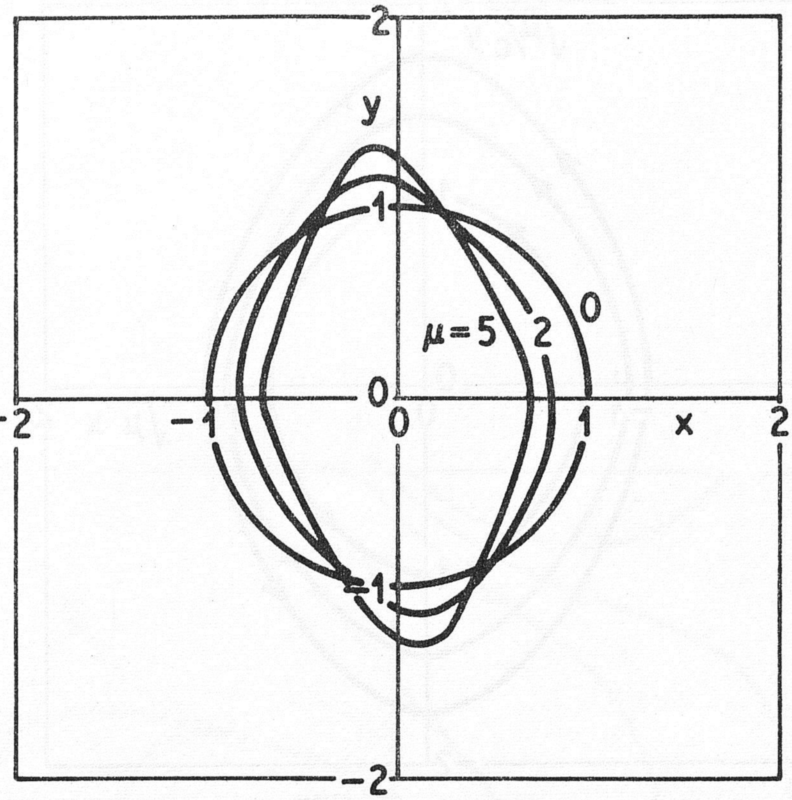

FIG. 8. LIMIT CYCLES FOR $\mu \geq 0$

($b=-1$, $\lambda=\nu=c=1$)

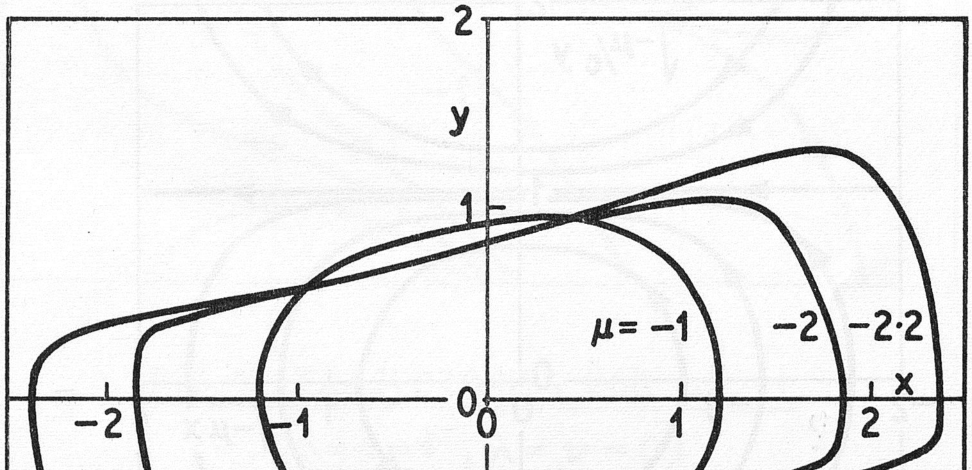

FIG. 9. LIMIT CYCLES FOR $\mu < 0$

($b=-1$, $\lambda=\nu=c=1$)

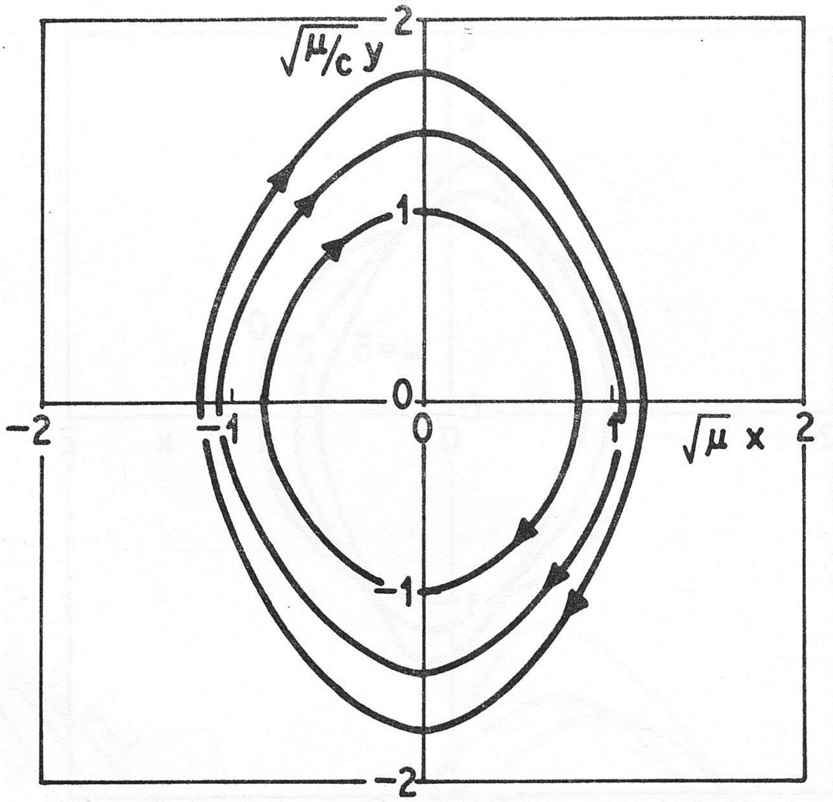

FIG. 10. TRAJECTORIES FOR $\mu > 0$
($b = \lambda = \nu = 0$, $c > 0$)

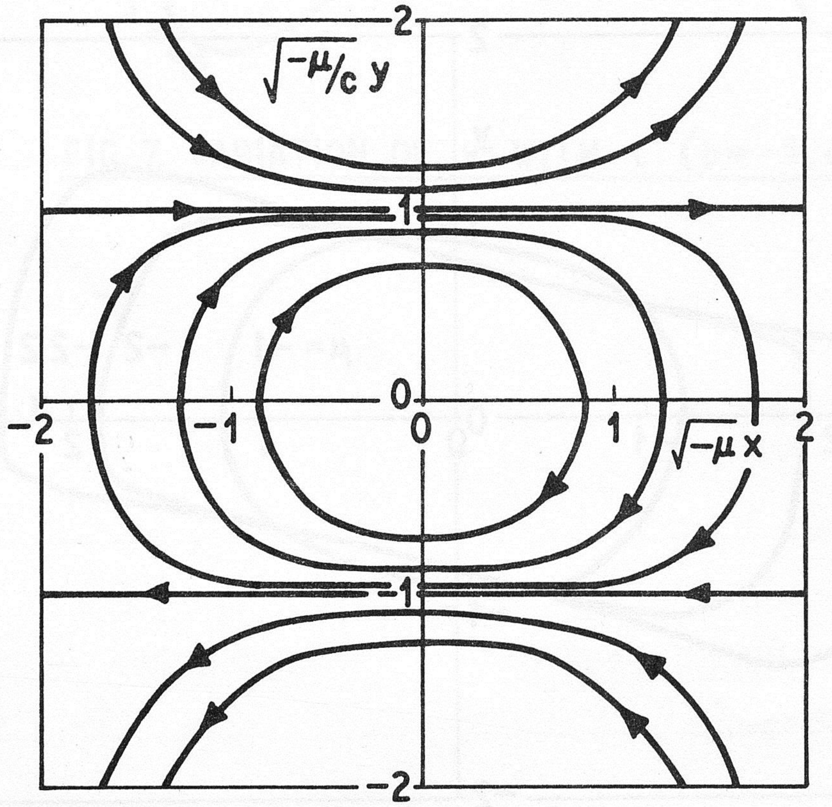

FIG. 11. TRAJECTORIES FOR $\mu < 0$
($b = \lambda = \nu = 0$, $c > 0$)

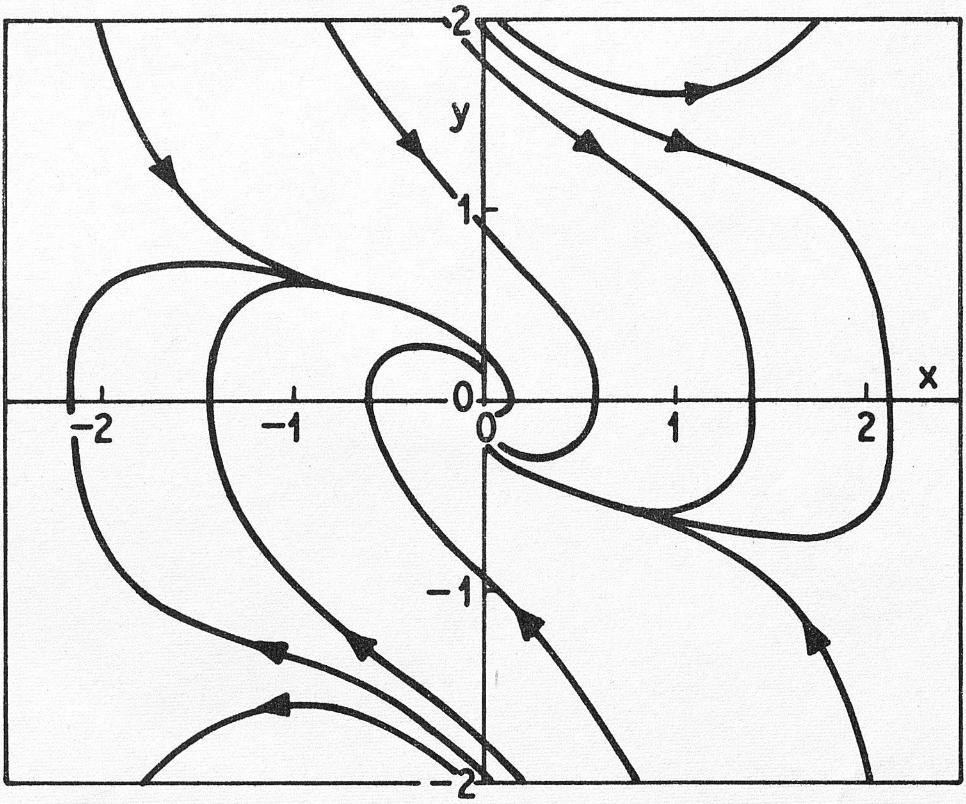

FIG. 12. TRAJECTORIES FOR $\mu = -1$

$(b = c = 1\ ,\ \lambda = \nu = 0)$