THE UNIVERSITY OF GLASGOW

DEPARTMENT OF AERONAUTICS AND FLUID MECHANICS

THE NATURE OF SOLUTIONS OF A SECOND ORDER DIFFERENTIAL EQUATION WITH NON-LINEAR DAMPING AND STIFFNESS

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A.W. Babister, M.A., Ph.D.

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SUMMARY

The general nature of solution of the autonomous equation

$$\ddot{x} + (b + lx + m\dot{x})\,\dot{x} + c_1 x + c_2 x^2 = 0$$

is considered (for $c_1 > 0$). It is shown that limit cycles cannot occur for this system if b, $(-lc_2/c_1)$ and $(-lm/c_1)$ all have the same sign. Trajectories in the $(x,\dot{x})$ phase plane are given for various combinations of sign of b, l, m and c_2. It is shown that self-sustaining oscillations occur if $b = 0$ and $c_1 m + c_2 = 0$.

Limit cycles can occur for numerically small (non-zero) values of $b/\sqrt{c_1}$, for a certain range of values of mb/l. Examples of these cycles are shown for both $c_2 = 0$ and $c_2 > 0$.

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1. Introduction

This report is one of a series on dynamical systems with non-linear characteristics (Babister, 1973 and 1975). In the earlier reports we considered the nature of solutions of the autonomous equations

$$\ddot{x} + (b_0+b_1 x)\,\dot{x} + c_1 x + c_2 x^2 = 0 \qquad (1)$$

and

$$\ddot{x} + b_0\dot{x} + \mu\dot{x}^2 + c_1 x + c_2 x^2 = 0. \qquad (2)$$

These are particular cases of the generalized equation for relaxation oscillations

$$\ddot{x} + f\,(x,\dot{x})\,\dot{x} + g(x) = 0, \qquad (3)$$

or of the equivalent system

$$\left.\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -\,f(x,y)\,y - g(x), \end{aligned}\right\} \qquad (4)$$

considered by Levinson and Smith (1942). If both f and g are polynomials of the second degree (at most), the system (4) is said to be quadratic.

The general topological theory of two-dimensional systems of the type

$$\left.\begin{aligned} \dot{x} &= \alpha_{10}x + \alpha_{01}y + \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 \\ \dot{y} &= \beta_{10}x + \beta_{01}y + \beta_{20}x^2 + \beta_{11}xy + \beta_{02}y^2 \end{aligned}\right\} \qquad (5)$$

has been discussed in a number of papers (Bautin, 1954, Petrovskii, 1958, Yan-Qian, 1963-4 and Coppel, 1966). Such equations occur in a variety of physical problems, notably in the Emden-Fowler equation of astrophysics and in the analysis of an oscillator with quadratic terms. However, due to the large number of coefficients in (5), the characteristics of many types of such systems still remain to be investigated. As pointed out by Yan-Qian (1963), it is still not possible to state whether a given system (5), with explicit numerical coefficients, has limit cycles.

In this paper we shall consider solutions of the autonomous equation

$$\ddot{x} + (b + lx + m\dot{x})\dot{x} + c_1 x + c_2 x^2 = 0, \qquad (6)$$

or of the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(b + lx + my)y - c_1 x - c_2 x^2 , \end{aligned}\right\} \qquad (7)$$

where b, c_1, c_2, l, and m are real coefficients. We shall consider both the global structure of the integral curves and the limit cycles of (7), our aim being to get quantitative data on these matters. We note that if m = 0, (6) reduces to (1); if l = 0, (6) reduces to (2).

2. The general nature of the solutions

Consider first the general nature of the solutions of (7). The system has a singular point at the origin O, this point being a focus (or node) if $c_1 > 0$ and a saddle point (or higher order singular point) if $c_1 < 0$. The focus at O will be stable or unstable according as b > or < 0. As shown by Loud (1964), the origin becomes a centre if both b = 0 and $l(c_1 m + c_2) = 0$ with $c_1 > 0$. This is considered further in § 4 and § 5. The point $(-c_1/c_2, 0)$ is another singular point for the system. On putting $x = z - c_1/c_2$, we see that (6) becomes

$$\ddot{z} + ([b - lc_1/c_2] + lz + m\dot{z})\dot{z} - c_1 z + c_2 z^2 = 0 \qquad (8)$$

which is an equation of the same form as (6). More generally, any equation of the form

$$\ddot{z} + (b + lz + m\dot{z})\dot{z} + c_0 + c_1 z + c_2 z^2 = 0, \qquad (9)$$

in which c_1 and c_2 are not both zero, can be put in the form of (6) with real coefficients on letting $x = z - \gamma$, where γ is a real constant, provided that the equation

$$c_0 + c_1\gamma + c_2\gamma^2 = 0$$

has a real root, that is, provided $c_1^2 \geqslant 4c_0c_2$.

In (6) put $x = \alpha X$, $t = \beta T$, where α and β are constants. Then

$$\frac{d^2X}{dT^2} + \left(\beta b + \alpha\beta l X + \alpha m \frac{dX}{dT}\right)\frac{dX}{dT} + \beta^2 c_1 X + \alpha\beta^2 c_2 X^2 = 0. \tag{10}$$

Thus, if (6) has the solution $x = \phi(t)$, with $x = x_0$, $y = y_0$ at $t = 0$, (10) has the solution $X = \alpha^{-1}\ \phi(\beta T)$,

with $X = x_0/\alpha$ and $dX/dT = \beta y_0/\alpha$ at $T = 0$. In particular we note that a variation in the value of α merely effects the non-linear terms in (6), and that, if β is replaced by $-\beta$, t is changed to -t. Thus, if (6) has a periodic solution, (10) with any non-zero α and β will also have a periodic solution. If $\alpha = -1$ and $\beta = 1$, the coefficients l, m and c_2 in (6) change sign, and the variation of X with T is identical with that of (-x) with t. Again, if $\alpha = 1$ and $\beta = -1$, the coefficients b and l in (6) become -b and -l, and the variation of X with T is identical with that of x with (-t).

Scaling factors were used in the numerical solutions given in this report, many of which were calculated on Glasgow University's analogue computer (GACA). The computer calculations were carried out for the equation

$$0.1\frac{d^2X}{dT^2} + \left(0.1b + 0.2lX + 0.2m\frac{dX}{dT}\right)\frac{dX}{dT} + 0.1c_1 X + 0.2c_2 X^2 = 0 \tag{11}$$

with $c_1 = 1$ and l, m and c_2 each having the value 1, 0, -1. As stated above, the origin is then a focus (or node). Thus the solutions were performed in real time ($\beta = 1$) with a scaling factor $\alpha = 2$.

3. General theorems on limit cycles

For the system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y) ,$$

by Dulac's extension of Bendixson's theorem, we find that, if B(x,y) is a continuous function such that, in a single connected region of the (x,y) plane,

$$\Delta = -\left[\partial(BP)/\partial x + \partial(BQ)/\partial y\right] \tag{12}$$

is of constant sign, then no closed contours exist in that region for the given system.

We apply the above theorem to the system (7). Firstly take

$$B = \exp\left[f(x) + g(y)\right] \tag{13}$$

where $f(x) = px + \frac{1}{2}qx^2$

and $g(y) = ry.$

Then

$$\Delta = B\left[b + (1+c_1 r)x + c_2 rx^2 + (2m-p+br)y + (lr-q)xy + mry^2\right]$$

Put $1+c_1 r = 0$, $2m-p+br = 0$ and $lr-q = 0$.

$$\therefore \quad r = -1/c_1, \quad p = 2m - bl/c_1 \text{ and } q = -l^2/c_1.$$

$$\therefore \quad \Delta = B\left(b - \frac{lc_2}{c_1}x^2 - \frac{lm}{c_1}y^2\right). \tag{14}$$

Now the r.h.s. of (14) will be of constant sign over the whole (x,y) plane provided that b, $(-lc_2/c_1)$ and $(-lm/c_1)$ have the same sign. Thus limit cycles cannot occur for the system (7) if b, $(-lc_2/c_1)$ and $(-lm/c_1)$ all have the same sign.

Again, from (12), with $B = \exp(2mx)$,

$$\Delta = B\,(b + lx) \tag{15}$$

The r.h.s. of (15) will be of constant sign over the whole (x,y) plane if l is zero; no limit cycles can then occur. This case was dealt with in Babister (1975). More generally, from (15) we see that, if $l \neq 0$, any limit cycles of (7) must intersect the line $x = -b/l$.

From (14), if $b = 0$ and $l \neq 0$, limit cycles cannot occur for the system (7) if c_2 and m have the same sign. If both b and c_2 are zero, (7) has no periodic solution (if $lm \neq 0$). By taking $B = (y+\alpha x+\beta)^{\gamma}\exp\delta x$, where α, β, γ and δ can be suitably determined, Yan-Qian (1963) showed that, for $c_1 m + c_2 \neq 0$ with $l \neq 0$, the system (7) with $b = 0$ has no periodic solution. This is considered further in § 4.1.

4. Phase plane trajectories for $c_2 = 0$

Trajectories in the $(x,\dot{x})$ phase plane were found by analogue computer for

$$\ddot{x} + (b + lx + m\dot{x})\dot{x} + c_1 x = 0 \tag{16}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -(b + lx + my)y - c_1 x. \end{aligned}\right\} \tag{17}$$

For this system, as shown in § 3, no limit cycles occur if b and $(-lm/c_1)$ have the same sign. Now the origin will be a saddle point if $c_1 < 0$. Therefore in the subsequent discussion we shall take $c_1 > 0$.

4.1. Systems with $b = 0$, $c_2 = 0$.

As shown by Babister (1973 and 1975), if $b = 0$ the origin becomes a centre if, in addition, either l or m vanishes; self-sustaining oscillations can then occur (for sufficiently small amplitudes in x). As shown in § 3, if $lm \neq 0$, the system (17), with $b = 0$, has no periodic solutions. Figure 1 shows the trajectories for $l = -1$, $m = 1$, $c_1 = 1$; these were found by using the analogue computer, there being no exact first integral. The curves converge very slowly to the origin, the coefficient of the damping term $(lx+m\dot{x})$ being greatest in the second quadrant for $l < 0$ and $m > 0$. Putting

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}c_1 x^2 ,$$

we see that (16) becomes (with $b = 0$),

$$dE/dt = -\dot{x}^2 (lx + m\dot{x}).$$

Thus the total energy E of the system will decrease if $lx+m\dot{x} > 0$. In the neighbourhood of O, the effect of the non-linear terms in (17), and hence the damping, is very small.

Trajectories for $l = \pm 1$, $m = \pm 1$, $c_1 = 1$ can be obtained from figure 1 by applying scaling factors $\alpha = -1$ and $\beta = -1$. It can readily be shown that the trajectories converge to the origin if $l/m < 0$, and diverge (to infinity) if $l/m > 0$. More generally, we find the equations for the trajectories for the system (17), with $b = 0$, can be put in the form

$$\frac{my}{\sqrt{c_1}} = f_1\left(\frac{lx}{\sqrt{c_1}} , \frac{l}{m\sqrt{c_1}}\right) .$$

4.2 Systems with $b \neq 0$, $c_2 = 0$.

If $b > 0$, the system (17) has positive damping in the neighbourhood of the origin, which is then a stable focus (or stable node). Figure 2 shows the trajectories for $b=1$, $l=1$, $m=1$, $c_1=1$, and figure 3 shows those for $b=1$, $l=-1$, $m=1$, $c_1=1$. In both of these figures, with $lx > 0$ there is an appreciable increase in damping. Trajectories for $m=-1$ can be determined by applying a scaling factor $\alpha = -1$ to these curves. The increased damping with $b > 0$ is readily seen by comparing figures 1 and 3. In figure 2, trajectories only converge on the origin if $y > -1$, the line $y = -1$ being a separatrix for the given values of the parameters. More generally, we see that, if $bl = mc_1$, the system (16) has a particular integral

$$\dot{x} = - c_1/l.$$

Trajectories for $b = -1$, $l = \pm 1$, $m = \pm 1$, $c_1 = 1$ can be obtained from figures 2 and 3 by applying scaling factors $\alpha = \pm 1$, $\beta = -1$. For these values of b, l, m and c_1, all the trajectories diverge to infinity as $t \to + \infty$.

Limit cycles can occur for small values of $b/\sqrt{c_1}$, as shown in figure 4. For $b < 0$, the origin is an unstable focus; trajectories which start in the neighbourhood of the origin spiral out to the limit cycle, which is stable. Figure 4 shows how the size of the limit cycle increases as the numerical value of b increases; thus only part of the limit cycle is shown for $b = - 0.3$. We note that all the limit cycles in this figure lie below the line $y = 1$. Now, for the system (17), the slope of trajectories on the line $y = - c_1/l$ is given by

$$dy/dx = - b + mc_1/l$$

which is a constant. Thus (if $b \neq mc_1/l$), as shown in figures 1 and 3, no trajectory can cross this line more than once; in particular, as shown in figure 4, no limit cycle can intersect this line. If $b=mc_1/l$, as shown above, the line $y = -c_1/l$ is a separatrix.

More generally we find the equations for the trajectories for the system (17), with $b \neq 0$, can be put in the form

$$\frac{my}{\sqrt{c_1}} = f_2\left(\frac{lx}{\sqrt{c_1}}, \frac{mb}{l}, \frac{b}{\sqrt{c_1}}\right) .$$

Thus, if l and m are both increased in the ratio $\alpha : 1$, the x and y coordinates of the trajectories will be decreased in the same ratio. For this system, the occurrence of limit cycles will therefore depend on the values of the parameters mb/l and $b/\sqrt{c_1}$; this is illustrated in figure 5, which shows the results of a number of runs on the analogue computer for $b/\sqrt{c_1} < 0$, for $mb/l > 0$. If $mb/l < 0$ and $c_1 > 0$, b and $(-lm/c_1)$ will have the same sign (as in figure 3) and, as stated above, no limit cycles can occur.

Stable limit cycles occurred for $-0.3 < b/\sqrt{c_1} < 0$ for a range of values of mb/l; by applying a scaling factor $\beta = -1$ (as indicated in § 2) it is readily seen that unstable limit cycles will occur for $0 < b/\sqrt{c_1} < 0.3$ for the same range of values of mb/l. For given values of m, b and c_1, the relative size of the limit cycle increases as l decreases in magnitude, until for $mb/l > 0.3$ no limit cycle occurs, the trajectories diverging to infinity for $b < 0$ and converging to the origin for $b > 0$ (as in figure 2). Again, for given values of b, l and m (with $mb/l < 0.3$), the limit cycle increases in size (in the x direction) as c_1 decreases in magnitude, eventually ceasing to exist for sufficiently large values of $|b|/\sqrt{c_1}$ (as indicated in figure 5).

The absence of limit cycles for these values of $|b|/\sqrt{c_1}$ was shown by Yan-Qian and others (1964), It can be demonstrated by using Dulac's extension of Bendixson's theorm (§ 3) for the system (17) with $B=1/(ly+c_1)$. Then

$$\Delta = -\left[\partial(BP)/\partial x + \partial(BQ)/\partial y\right]$$

$$= \frac{lmy^2 + 2mc_1 y + bc_1}{(ly + c_1)^2}$$

Now Δ will have the same sign over the whole of the (x,y) plane if

$lmbc_1 - m^2c_1^2 > 0$,

that is, if $lb/mc_1 > 1$. (18)

We note that B changes sign as the line $y = -c_1/l$ is crossed. Therefore, if (18) is satisfied, any limit cycle of the system must cross the line $y = -c_1/l$. But, as shown above, no limit cycle can intersect this line. Therefore the system (17) has no limit cycle if $(lb/mc_1) > 1$; that is, if $b^2/c_1 > mb/l$. (19)

The limiting condition $b^2/c_1 = mb/l$ is shown in fig. 5.

5. Phase plane trajectories for $c_2 > 0$

Trajectories in the $(x,\dot{x})$ phase plan were found by analogue computer for

$$\ddot{x} + (b + lx + m\dot{x})\dot{x} + c_1 x + c_2 x^2 = 0 \quad (20)$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -(b + lx + my)y - c_1 x - c_2 x^2 \end{aligned}\right\} \quad (21)$$

for c_1 and c_2 both positive. The system then has one singular point at the origin O and another at the point $(-c_1/c_2, 0)$, the latter being a saddle point.

For this system, as shown in § 3, no limit cycles occur if b, $(-lm/c_1)$ and $(-lc_2/c_1)$ all have the same sign. Thus, with c_1 and c_2 both positive, no limit cycles occur if b and (-l) have the same sign and $m > 0$.

5.1 Systems with $b = 0$, $c_2 > 0$

As shown by Loud (1964), if $b = 0$, the origin becomes a centre if, in addition, $l(c_1 m + c_2) = 0$. The case $l = 0$ was considered by Babister (1975); self-sustaining oscillations can then occur (for sufficiently small amplitudes in x). If $b = 0$ and $c_2 = -c_1 m$, (20) has the particular integral

$$\dot{x} = \lambda(mx - 1) \tag{22}$$

with $m^2\lambda^2 + l\lambda - c_1 = 0.$ (23)

With $c_1 > 0$, eq. (22) corresponds to a pair of straight lines through the point $(-c_1/c_2, 0)$ in the phase plane. The general first integral of (20), with $c_2 = -c_1 m$, can be shown to be

$$(\lambda_2 m^2 + l)\log\left[\dot{x} - \lambda_1(mx - 1)\right] - (\lambda_1 m^2 + l)\log\left[\dot{x} - \lambda_2(mx-1)\right] + (\lambda_2 - \lambda_1)m^3 x = A \tag{24}$$

where λ_1 and λ_2 are the roots of (23) and A is a constant. If $l = 0$, from (23), $\lambda_1 = -\lambda_2 = \sqrt{c_1}/m$, and eq. (24) reduces to

$$\log\left[\dot{x} - \lambda_1(mx-1)\right] + \log\left[\dot{x} + \lambda_1(mx-1)\right] + 2mx = -A/\lambda_1 m^2 ,$$

that is, $$\dot{x}^2 = (c_1/m^2)(mx-1)^2 + C e^{-2mx}, \tag{25}$$

where C is a constant, in agreement with Babister (1975),

Figure 6 shows the trajectories for $l = -1$, $m = -1$, $c_1 = 1$, $c_2 = 1$. The general configuration is very similar to that for $l = 0$, given in Babister (1975). The lines $y = -\lambda(x+1)$ are the separatrices of the system, where $\lambda^2 - \lambda - 1 = 0$, that is, $\lambda = 0.5 \pm \sqrt{1.25}$. Thus the separatrices are the perpendicular lines $y = 0.62(x+1)$ and $y = -1.62(x+1)$.

We see that the separatrices divide the phase plane into four regions, all the trajectories going to infinity unless they are within the region to the right of PA and PB. In that region there are cyclic trajectories enclosing the origin, which is a centre.

Figure 7 shows the trajectories for $l = -1$, $m = 1$, $c_1 = 1$, $c_2 = 1$ (there is no general first integral in this case). Trajectories which come sufficiently close to O spiral in very slowly to the origin (which is a stable focus); all other trajectories go off to infinity, apart from the two which converge on to the saddle point $(-1,0)$. There are no cyclic trajectories. The destabilising effect of the saddle point (i.e, the effect of the non-linear stiffness term $c_2 x^2$) can be seen by comparing figures 1 and 7.

Trajectories for $l = 1$, $m = \pm 1$, $c_1 = 1$, $c_2 = 1$ can be obtained from figures 6 and 7 by applying the scaling factor $\beta = -1$. Cyclic trajectories occur for $m = -1$ (as above). All trajectories diverge to infinity if l and m are both positive (apart from certain separatrices). More generally we find the equations for the trajectories for the system (21) with $b = 0$ can be put in the form

$$\frac{my}{\sqrt{c_1}} = f_3\left(\frac{lx}{\sqrt{c_1}}, \frac{l}{m\sqrt{c_1}}, \frac{c_2}{mc_1}\right) .$$

5.2 Systems with $b \neq 0$, $c_2 > 0$

Figures 8, 9, 10 and 11 show the trajectories for $b = 1$, $l = \pm 1$, $m = \pm 1$, $c_1 = 1$, $c_2 = 1$. The point $(-1,0)$ is then a saddle point. If $b > 0$, the origin is a stable focus (or stable node), and trajectories which come sufficiently close to O spiral in to the origin. The nature of the trajectories near the two singularities $(0,0)$ and $(-1,0)$ is determined by the linear terms in eq. (21). There are no limit cycles for these values of the parameters. Trajectories for $b = -1$, $l = \pm 1$, $m = \pm 1$, $c_1 = 1$, $c_2 = 1$ can be obtained from figures 8 - 11 by applying a scaling factor $\beta = -1$. For these values of the parameters, all the trajectories go off to infinity, apart from the two separatrices which converge on the saddle point $(-1,0)$. We find the general equation for the trajectories for the system (21) can be put in the form

$$\frac{my}{c_1} = f_4\left(\frac{lx}{\sqrt{c_1}}, \frac{1}{m\sqrt{c_1}}, \frac{b}{\sqrt{c_1}}, \frac{c_2}{mc_1}\right)$$

or

$$\frac{c_2 y}{c_1^{3/2}} = f_5\left(\frac{c_2 x}{c_1} \quad \frac{m\sqrt{c_1}}{l} \quad \frac{b}{\sqrt{c_1}}, \frac{c_2}{l\sqrt{c_1}}\right),$$

the second form being more useful in comparing trajectories of this report with those of an earlier report (Babister, 1973) in which m was taken to be zero.

As in §4, we find that limit cycles can occur for numerically small (non-zero) values of $b/\sqrt{c_1}$ (and $bl/c_2 > 0$). With $b = -0.1$, $l = -1$, $c_1 = 1$, $c_2 = 1$, figure 12 shows the limit cycle for $m = 0$ and figure 13 that for $m = 1$, both limit cycles being stable. Limit cycles were also obtained for small negative values of m, but none were obtained for $m < -0.75$ and the above values of the other parameters.

By applying the scaling factor $\beta = -1$, we see that an unstable limit cycle occurs for $b = 0.1$, $l = 1$, $c_1 = 1$, $c_2 = 1$ and $m = 0$ and 1. This is in agreement (for $m = 0$) with the results of Obi, which show that a limit cycle of the system represented by eq. (20) will occur if $0 < bc_2/lc_1 \ll 1/7$. In the limiting condition, the limit cycle will merge with one branch of the separatrix through the point $(-c_1/c_2, 0)$.

For higher values of bc_2/lc_1 (corresponding to an increase in the strength of the focus at O) the trajectories diverge to infinity if $b < 0$, as shown in our earlier report. As $|b|$ decreases to zero, the amplitude of the limit cycle also decreases to zero, there being no limit cycle when $b = 0$ (as shown above).

6. Phase-plane trajectories for $c_2 < 0$

These can be obtained from those discussed in §5 by applying a scaling factor $\alpha = -1$. As shown in §2, this changes the signs of x, y, l, m and c_2. The point (x,y) is transformed into the point $(-x,-y)$. Thus, with $c_1 = 1$ and $c_2 = -1$, there is a singular point (saddle point) at $(1,0)$. As in §5, limit cycles can occur for numerically small (non-zero) values of $b/\sqrt{c_1}$ (and $bl/c_2 > 0$).

References

Babister, A.W. Non-linear differential equations with quadratic stiffness terms. U. Glasgow Dept. Aero. Fluid Mechs. Report 7301 (1973)

Babister, A.W. Non-linear differential equations having both quadratic damping and stiffness. U. Glasgow Dept. Aero. Fluid Mechs. Report 7501 (1975)

Bautin, N.N. On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or centre type. American Math. Soc. Translation (Series 1) 100 (1954)

Coppel, W.A. A survey of quadratic systems. Jour. of Differential Equations, 2 (1966), 293-304

Levinson, N. and Smith, O.K. A general equation for relaxation oscillations. Duke Math. Journal, 9 (1942), 382-403

Loud, W.S. Behaviour of the period of solution of certain plane autonomous systems near centres. Contributions to Differential equations (Interscience, 1964)

Obi, C. Researches on the equation

$$\ddot{x} + (\epsilon_1 + \epsilon_2 x)\dot{x} + x + \epsilon_3 x^2 = 0.$$

Proc. Cambridge Phil. Soc. 50, (1954), 26-32.

Petrovskii, I.G. Landis, E.M. On the number of limit cycles of the equation $dy/dx = P(x,y)/Q(x,y)$, where P and Q are polynomials of the second degree. American Math. Soc. Translation (Series 2) 10 (1958).

Yan-Qian, Y. et al. A qualitative study of the integral curves of the differential equation

$$\frac{dy}{dx} = \frac{q_{00}+q_{10}x+q_{01}y+q_{20}x^2+q_{11}xy+q_{02}y^2}{p_{00}+p_{10}x+p_{01}y+p_{20}x^2+p_{11}xy+p_{02}y^2}.$$

Chinese Mathematics, 3 (1963), 1-18 and 62-70, and 5 (1964), 129-138.

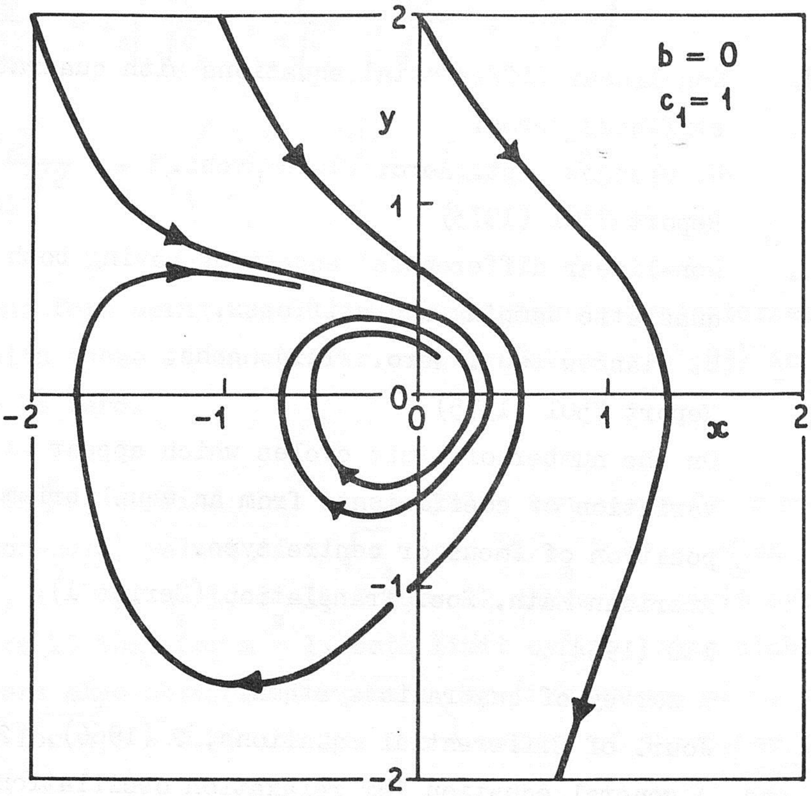

Fig. 1. TRAJECTORIES $\ell = -1$, $m = 1$, $c_2 = 0$

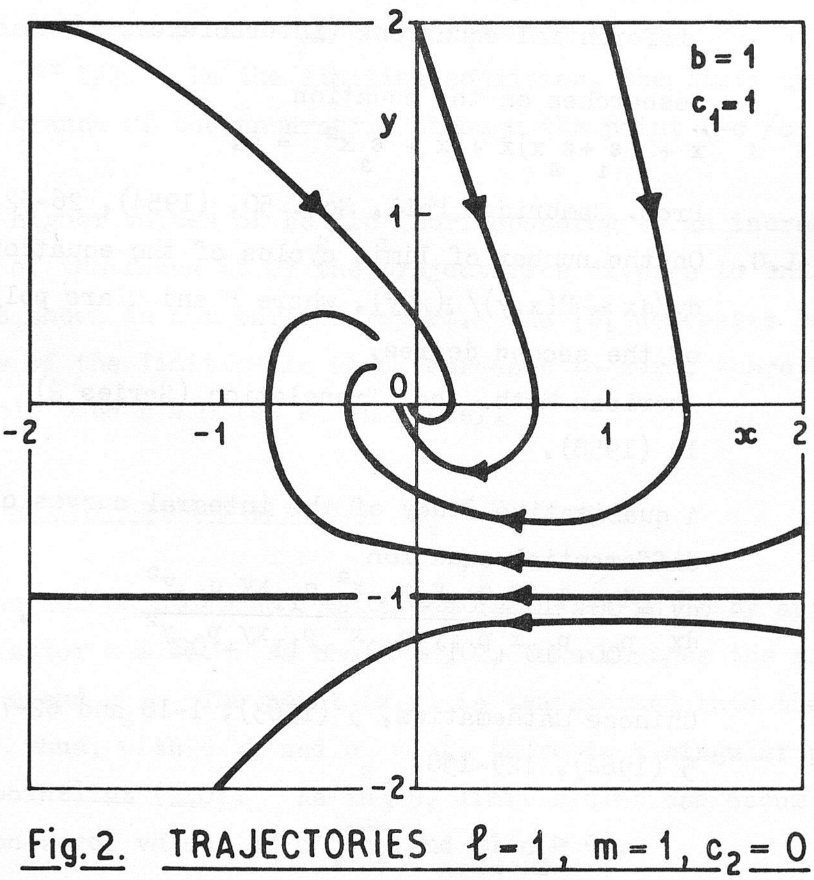

Fig. 2. TRAJECTORIES $\ell = 1$, $m = 1$, $c_2 = 0$

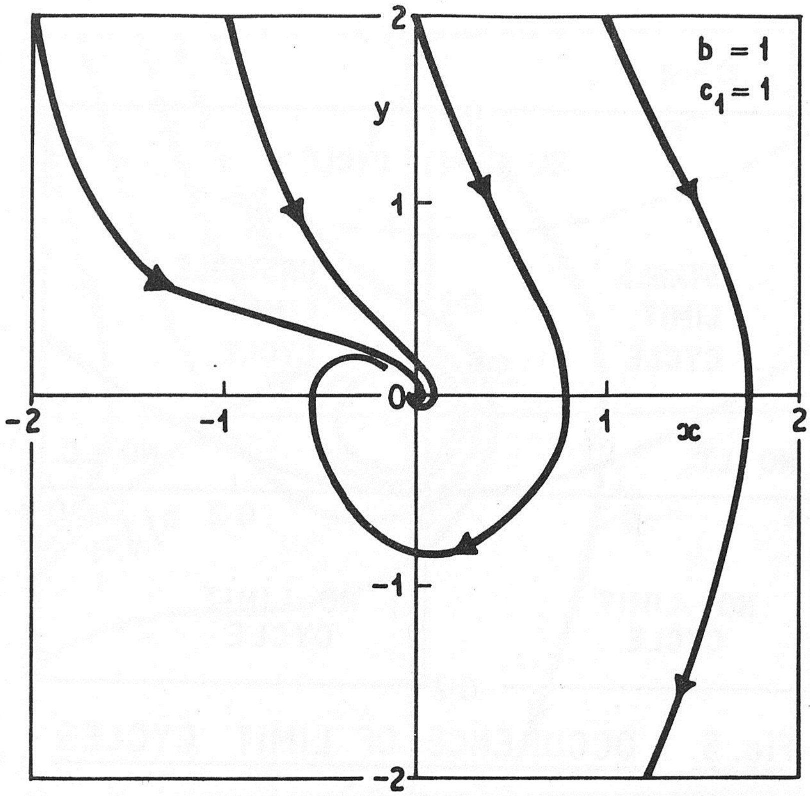

Fig. 3. TRAJECTORIES $\ell = -1$, $m = 1$, $c_2 = 0$

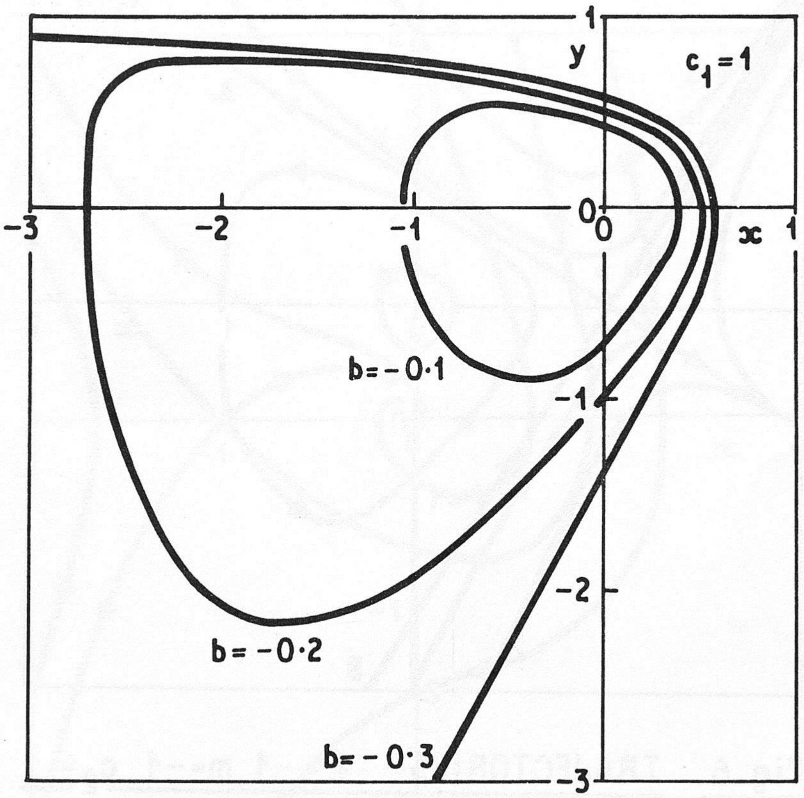

Fig. 4. LIMIT CYCLES $\ell = -1$, $m = 1$, $c_2 = 0$

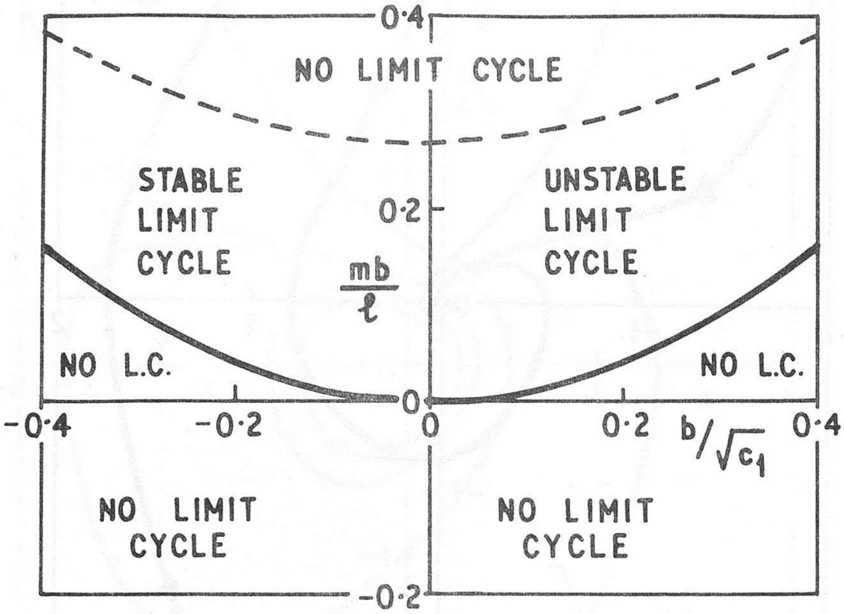

Fig. 5. OCCURENCE OF LIMIT CYCLES ($c_2 = 0$)

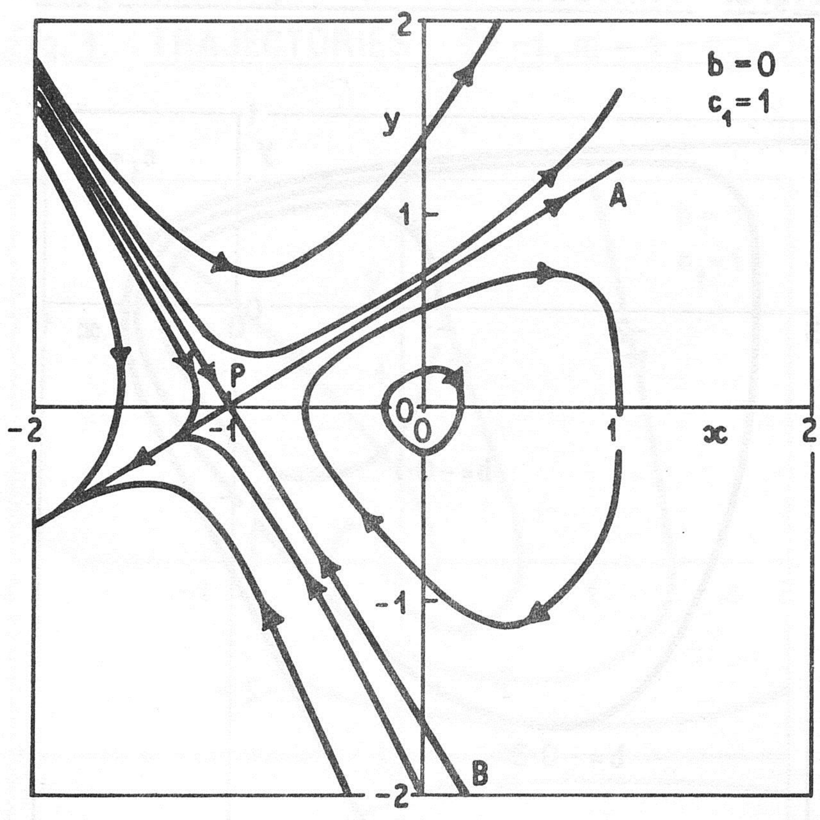

Fig. 6. TRAJECTORIES $\ell = -1, m = -1, c_2 = 1$

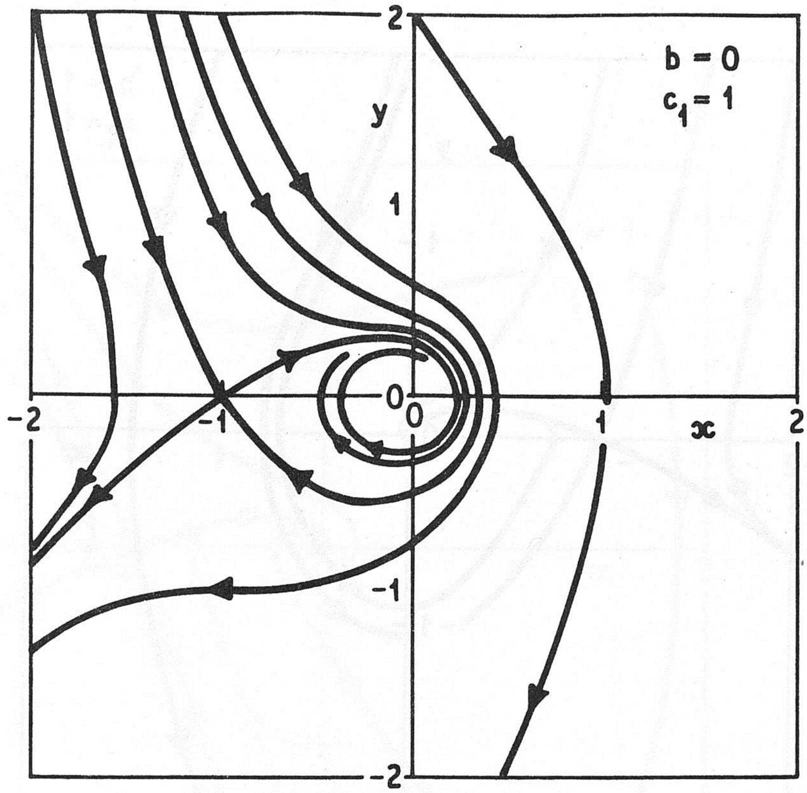

Fig. 7. TRAJECTORIES $\ell = -1$, $m = 1$, $c_2 = 1$

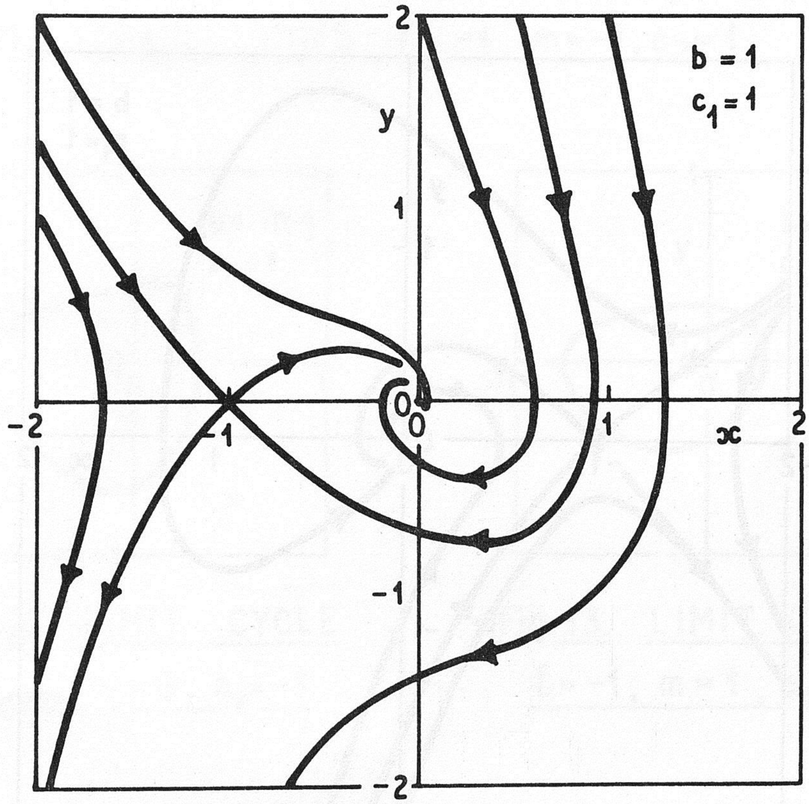

Fig. 8. TRAJECTORIES $\ell = 1$, $m = 1$, $c_2 = 1$

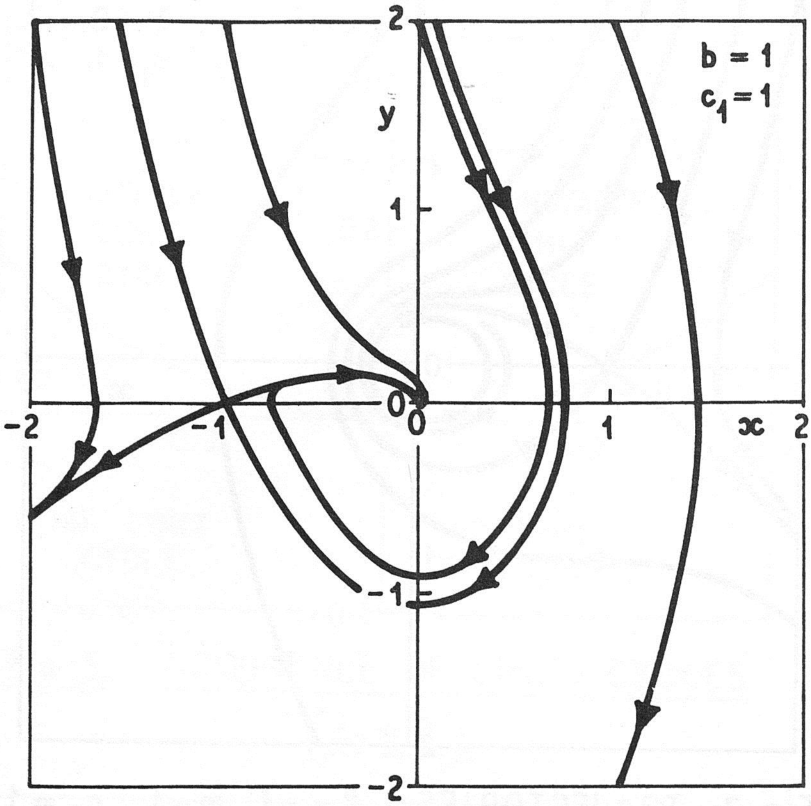

Fig. 9. TRAJECTORIES $\ell = -1, m = 1, c_2 = 1$

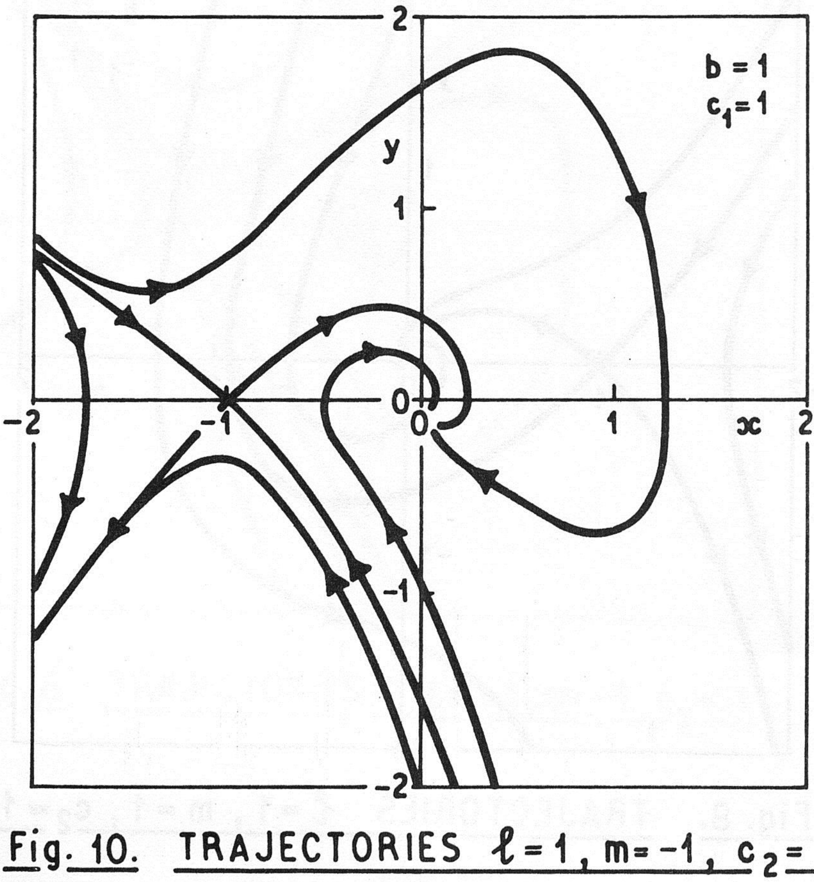

Fig. 10. TRAJECTORIES $\ell = 1, m = -1, c_2 = 1$

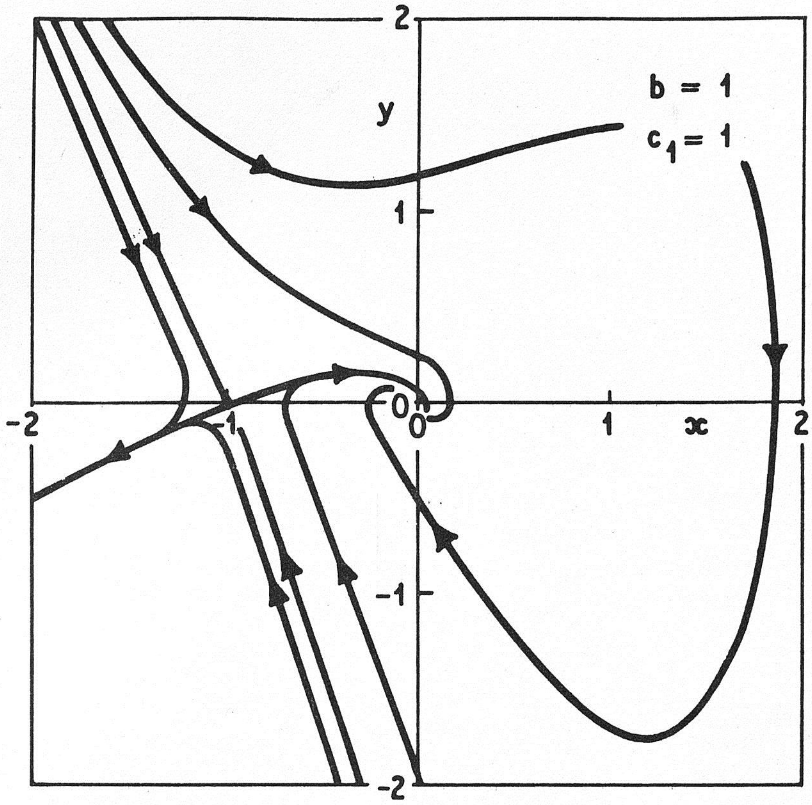

Fig. 11. TRAJECTORIES $\ell = -1$, $m = -1$, $c_2 = 1$

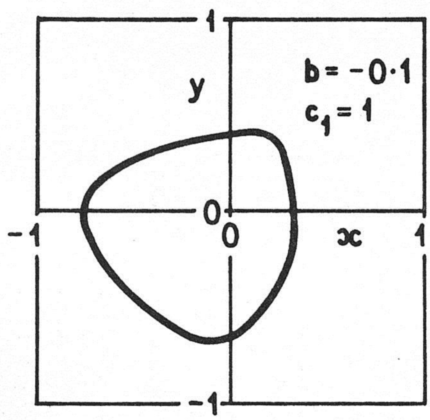

Fig. 12. LIMIT CYCLE
$\ell = -1$, $m = 0$, $c_2 = -1$

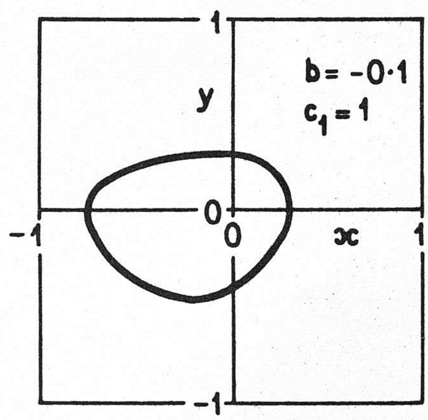

Fig. 13. LIMIT CYCLE
$\ell = -1$, $m = 1$, $c_2 = 1$