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COMPLEX QUANTUM GROUPS AND A DEFORMATION OF THE BAUM–CONNES ASSEMBLY MAP

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ABSTRACT. We define and study an analogue of the Baum–Connes assembly map for complex semisimple quantum groups, that is, Drinfeld doubles of q-deformations of compact semisimple Lie groups.

Our starting point is the deformation picture of the Baum–Connes assembly map for a complex semisimple Lie group G, which allows one to express the K-theory of the reduced group C^*-algebra of G in terms of the K-theory of its associated Cartan motion group. The latter can be identified with the semidirect product of the maximal compact subgroup K acting on $\mathfrak{k}^*$ via the coadjoint action.

In the quantum case the role of the Cartan motion group is played by the Drinfeld double of the classical group K, whose associated group C^*-algebra is the crossed product of $C(K)$ with respect to the adjoint action of K. Our quantum assembly map is obtained by varying the deformation parameter in the Drinfeld double construction applied to the standard deformation K_q of K. We prove that the quantum assembly map is an isomorphism, thus providing a description of the K-theory of complex quantum groups in terms of classical topology.

Moreover, we show that there is a continuous field of C^*-algebras which encodes both the quantum and classical assembly maps as well as a natural deformation between them. It follows in particular that the quantum assembly map contains the classical Baum–Connes assembly map as a direct summand.

1. Introduction

Let G be a second countable locally compact group. The Baum–Connes conjecture asserts that the assembly map

$$\mu : K_*^{\mathsf{top}}(G) \to K_*(C^*_{\mathsf{r}}(G))$$

is an isomorphism [2], [3]. Here $C^*_{\mathsf{r}}(G)$ denotes the reduced group C^*-algebra of G. The conjecture has been established for large classes of groups: in particular, it holds for Lie groups [18], [38], [29], [22], and in fact for arbitrary almost connected groups [9]. This part of the Baum–Connes conjecture is also known as the Connes–Kasparov conjecture.

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In the case that G is a Lie group with finitely many components, there exists an alternative description of the Baum–Connes assembly map [11], [3]. Namely, if $K \subset G$ is a maximal compact subgroup and $\mathfrak{k} \subset \mathfrak{g}$ the corresponding inclusion of Lie algebras, there exists a continuous bundle of groups over $[0,1]$ deforming G into its associated Cartan motion group $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$. Moreover, this bundle is trivial outside 0, and the resulting induced map

$$K_*(C^*(G_0)) \to K_*(C^*_\mathsf{r}(G))$$

in K-theory identifies naturally with the Baum–Connes assembly map. We refer to [25] for a detailed discussion. The deformation picture of the Baum–Connes map can be viewed as a K-theoretic version of the Mackey analogy [24].

In the special case of complex semisimple groups, Higson obtained a proof of the Connes–Kasparov conjecture in the deformation picture by analyzing the structure of the C^*-algebras $C^*(G_0)$ and $C^*_\mathsf{r}(G)$, using the representation theory of complex semisimple groups and the Mackey machine [16].

In the present paper we shall study the K-theory of complex semisimple quantum groups from a similar perspective. These quantum groups are obtained by applying the Drinfeld double construction to q-deformations of compact semisimple Lie groups [30]. We shall determine the K-theory of the associated reduced group C^*-algebras in terms of the K-theory of a quantum analogue of the Cartan motion group. As in [16], we start from an explicit description of the reduced group C^*-algebras.

Our definition of the quantum assembly map is very natural from the point of view of deformation quantization. To explain this, recall that the maximal compact subgroup K of a complex semisimple group G admits an essentially unique standard deformation. The classical double of the corresponding Poisson–Lie group [10] naturally identifies with the group G. Moreover, if we instead consider the trivial Poisson bracket, then the corresponding classical double is precisely the Cartan motion group G_0. The Baum–Connes deformation may thus be viewed as rescaling the standard Poisson structure on K to zero and considering the corresponding classical doubles. In the quantum case we obtain our deformation in the same way by rescaling the deformation parameter q to 1 and applying the Drinfeld double construction. Moreover, we show that the various deformations appearing in this picture are all compatible in a natural sense.

As already indicated above, our approach in the present paper relies on the representation theory of complex quantum groups [35], [36]. It would be nice to find alternative proofs without having to invoke these results. Such an independent approach might shed some new light on the Baum–Connes isomorphism for classical complex semisimple groups as well.

Let us now explain how the paper is organized. In section 2 we collect some preliminaries and fix our notation. Section 3 contains a review of the deformation picture of the Baum–Connes assembly map for almost connected Lie groups, and more specifically complex semisimple groups. The left-hand side of the assembly map is identified with the K-theory of the Cartan motion group $K \ltimes \mathfrak{k}^*$ associated with the complex group G; here K is the maximal compact subgroup of K acting on the dual $\mathfrak{k}^*$ of its Lie algebra $\mathfrak{k}$ by the coadjoint action. The main aim of section 4 is to explain how the Cartan motion group can be quantized in such a way that one obtains a continuous field of C^*-algebras. Section 5 deals with a corresponding quantization on the level of the complex group itself. Again, we obtain a continuous field of C^*-algebras. In section 6 we prove our first main result; more precisely, we define a quantum analogue of the Baum–Connes assembly map from the K-theory of the quantum Cartan motion group with values in the K-theory of the reduced group C^*-algebra of the corresponding complex quantum group and show that this map is an isomorphism. Finally, in section 7 we bring together all the deformations studied in the previous sections. We introduce the concept of a deformation square, by which we mean a specific type of continuous field over the unit square. For each complex semisimple Lie group we construct a deformation square, which encodes both the quantum and classical assembly maps as well as a deformation between them.

Let us conclude with some remarks on notation. The algebra of adjointable operators on a Hilbert space or Hilbert module $\mathcal{E}$ is denoted by $\mathbb{L}(\mathcal{E})$, and we write $\mathbb{K}(\mathcal{E})$ for the algebra of compact operators. Depending on the context, the symbol $\otimes$ denotes the algebraic tensor product over the complex numbers, the tensor product of Hilbert spaces or the minimal tensor product of C^*-algebras.

2. Preliminaries

In this section we review some background material on continuous fields of C^*-algebras and on quantum groups.

2.1. **Continuous fields of C^*-algebras and K-theory.**

In this subsection we recall some basic definitions and facts on continuous fields of C^*-algebras and their K-theory; see [5], [27], [39].

If A is a C^*-algebra, we write $ZM(A)$ for its central multiplier algebra, that is, the center of the multiplier algebra of A.

Definition 2.1. Let X be a locally compact space. A $C_0(X)$-algebra is a C^*-algebra A together with a nondegenerate *-homomorphism $C_0(X) \to ZM(A)$.

We will usually omit the homomorphism $C_0(X) \to ZM(A)$ from our notation and simply write fa for the action of $f \in C_0(X)$ on $a \in A$. A morphism of $C_0(X)$-algebras is a *-homomorphism $\phi : A \to B$ such that $\phi(fa) = f\phi(a)$ for all $a \in A$ and $f \in C_0(X)$. In particular, two $C_0(X)$-algebras are isomorphic if there exists a *-isomorphism between them which is compatible with the $C_0(X)$-module structures.

Assume that A is a $C_0(X)$-algebra, and let $x \in X$. The fiber of A at x is the C^*-algebra

$$A_x = A/I_xA,$$

where $I_x \subset C_0(X)$ is the ideal of all functions vanishing at x. We note that I_xA is a closed two-sided ideal of A due to Cohen's factorization theorem. If $a \in A$, we write $a_x \in A_x$ for the image under the canonical projection homomorphism.

Definition 2.2. Let X be a locally compact space. A continuous field of C^*-algebras over X is a $C_0(X)$-algebra A such that for each $a \in A$ the map $x \mapsto \|a_x\|$ is continuous.

A basic example of a continuous field over X is $A = C_0(X, D)$, where D is an arbitrary C^*-algebra and the action of $C_0(X)$ is by pointwise multiplication. Continuous fields which are isomorphic to fields of this form are called *trivial*.

Let A be a $C_0(X)$-algebra, and assume that $U \subset X$ is a closed subset. In the same way as in the construction of the fiber algebra A_x, one constructs a C^*-algebra A_U by setting

$$A_U = A/I_U A,$$

where $I_U \subset C_0(X)$ is the ideal of all functions vanishing on U. Note that we have $(A_U)_V \cong A_V$ if $V \subset U \subset X$ are closed subsets. In particular, A_U is a $C_0(U)$-algebra with the same fibers as A at the points of U. If A is a continuous field over X, then A_U is a continuous field over U. We call A_U the restriction of A to U.

Now let A be a $C_0(X)$-algebra, and assume that $U \subset X$ is an open subset. We let A_U be the kernel of the canonical projection $A \to A_{X \setminus U}$. Equivalently, we may describe A_U as

$$A_U = C_0(U)A$$

in this case. In the same way as for closed subsets, we call A_U the restriction of A to U. If A is a continuous field over X, then A_U is a continuous field over U, with the same fibers as A at the points of U.

A continuous field of C^*-algebras A over a locally compact space X is called trivial away from $x \in X$ if the restricted field $A_{X \setminus \{x\}}$ is trivial.

Let us now review a basic fact from the K-theory of continuous fields which will play a central role in our considerations. Assume that A is a continuous field of C^*-algebras over the unit interval $X = [0,1]$. If A is trivial away from $x = 0$, then we obtain a canonical induced homomorphism between the K-groups of the fibers A_0 and A_1. Indeed, restriction to 0 yields a short exact sequence

$$0 \longrightarrow A_{(0,1]} \longrightarrow A \longrightarrow A_0 \longrightarrow 0$$

of C^*-algebras, and triviality of the field away from 0 means that the kernel in this sequence is isomorphic to $C_0((0,1], A_1)$. In particular, the K-theory of $A_{(0,1]}$ vanishes, and therefore the six-term exact sequence yields an isomorphism $K_*(A) \to K_*(A_0)$. Combining this with the projection to the fiber at 1, the resulting diagram

$$K_*(A_0) \xleftarrow{\cong} K_*(A) \longrightarrow K_*(A_1)$$

yields a homomorphism $i : K_*(A_0) \to K_*(A_1)$. This construction will be used repeatedly in the sequel.

If the continuous field A is unital, it is easy to describe the map i directly on the level of projections and unitaries. Since this will become important later on, let us briefly review this. Assume that $p_0 \in M_n(A_0)$ is a projection. We can lift p_0 to a positive element q in $M_n(A)$. Since $p_0 - p_0^2 = 0$, we see that $\|q(\tau) - q(\tau)^2\| < \frac{1}{4}$ for τ sufficiently small. In particular, since $\frac{1}{2}$ is not in the spectrum of $q(\tau)$ for such τ, functional calculus allows us to find a lift $p \in M_n(A)$ of p_0 such that all elements $p(\tau) \in M_n(A_\tau)$ are projections for τ small. Since the field is trivial outside 0, we can in fact arrange that $p(\tau)$ is a projection for all $\tau \in [0,1]$. The image of $[p_0] \in K_0(A_0)$ under the map i is then the class $[p(1)] \in K_0(A_1)$. In particular, on the level of K-theory the previous construction is independent of the choice of the lift p.

In a similar way, assume that $u_0 \in M_n(A_0)$ is unitary. Then we can lift u_0 to an element u in $M_n(A)$, and for τ small the resulting elements $u(\tau)$ will satisfy $\|u(\tau)u(\tau)^* - 1\| < 1$ and $\|u(\tau)^*u(\tau) - 1\| < 1$. It follows that $u(\tau)$ is invertible for t small, and by triviality of the field outside 0 we can assume that $u(\tau)$ is invertible for all $\tau \in [0,1]$. Using polar decomposition, we may as well arrange for

$u(\tau)$ to be unitary. In any case, the image of $[u_0] \in K_1(A_0)$ under i is the class $[u(1)] \in K_1(A_1)$. Again, upon passing to K-theory, this does not depend on the choice of the lifting section u.

2.2. **Quantum groups.** In this subsection we give a quick introduction to the theory of quantum groups with a particular focus on complex semisimple quantum groups. We refer to [10], [19], [35] for more details.

Let G be a simply connected complex semisimple Lie group, and let $\mathfrak{g}$ be its Lie algebra. We will fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a set $\alpha_1, \dots, \alpha_N$ of simple roots. Moreover, we let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of the maximal compact subgroup $K \subset G$ with maximal torus T such that $\mathfrak{t} \subset \mathfrak{h}$, where $\mathfrak{t}$ is the Lie algebra of T. Let $(\cdot, \cdot)$ be the bilinear form on $\mathfrak{h}^*$ obtained by rescaling the Killing form such that $(\alpha, \alpha) = 2$ for the shortest root of $\mathfrak{g}$, and set $d_i = (\alpha_i, \alpha_i)/2$. Moreover, we let $\mathbf{Q} \subset \mathbf{P} \subset \mathfrak{h}^*$ be the root and weight lattices of $\mathfrak{g}$, respectively. The set $\mathbf{P}^+ \subset \mathbf{P}$ of dominant integral weights is the set of all nonnegative integer combinations of the fundamental weights $\varpi_1, \dots, \varpi_N$, with the latter being defined by stipulating $(\varpi_i, \alpha_j) = \delta_{ij} d_j$. Equivalently, the fundamental weights are the dual basis to the basis formed by the coroots $\alpha_1^\vee, \dots, \alpha_N^\vee$, where $\alpha_j^\vee = d_j^{-1} \alpha_j$.

The C^*-algebra $C(K)$ of continuous functions on the maximal compact subgroup K can be obtained as a completion of the algebra of matrix coefficients of all finite dimensional representations of $\mathfrak{g}$. In the same way one constructs a C^*-algebra $C(K_q)$ for $q \in (0,1)$ as the completion of the algebra of matrix coefficients of all finite dimensional integrable representations of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated with $\mathfrak{g}$. Let us briefly review these constructions.

We recall that $U_q(\mathfrak{g})$ is generated by elements K_μ^q for $\mu \in \mathbf{P}$ and E_i^q, F_j^q for $1 \leq i, j \leq N$ satisfying a deformed version of the Serre presentation for $U(\mathfrak{g})$. More precisely, we will work with the Hopf algebra $U_q(\mathfrak{g})$ as defined in [35]. We let $U_1(\mathfrak{g}) = U(\mathfrak{g})$ be the classical universal enveloping algebra of $\mathfrak{g}$ and write H_i^1, E_i^1, F_i^1 for the Serre generators of $U(\mathfrak{g})$.

In analogy with the classical case one has the notion of a weight module for $U_q(\mathfrak{g})$. Every finite dimensional weight module is completely reducible, and the irreducible finite dimensional weight modules of $U_q(\mathfrak{g})$ are parameterized by their highest weights in $\mathbf{P}^+$ as in the classical theory. We will write $V(\mu)^q$ for the module associated with $\mu \in \mathbf{P}^+$ and $\pi_\mu^q : U_q(\mathfrak{g}) \to \operatorname{End}(V(\mu)^q)$ for the corresponding representation. The direct sum of the maps π_μ^q induces an embedding $\pi^q : U_q(\mathfrak{g}) \to \prod_{\mu \in \mathbf{P}^+} \operatorname{End}(V(\mu)^q)$. For $q \in (0,1)$ we may use this to define H_j^q to be the unique element in $\prod_{\mu \in \mathbf{P}^+} \operatorname{End}(V(\mu)^q)$ such that $K_j^q = q^{d_j H_j^q}$, where $K_j^q = K_{\alpha_j}^q$.

If we fix a *-structure on $U_q(\mathfrak{g})$ as in [35], then the representations $V(\mu)^q$ for $\mu \in \mathbf{P}^+$ are unitarizable. In fact, one can identify the underlying Hilbert spaces $V(\mu)^q$ with the Hilbert space $V(\mu)^1 = V(\mu)$ of the corresponding representation of $U(\mathfrak{g})$, in such a way that the operators H_i^q, E_i^q, F_i^q define continuous families in $\mathbb{K}(V(\mu))$ for all $\mu \in \mathbf{P}^+$; compare [26].

For $q \in (0,1)$ we define $\mathfrak{C}^\infty(K_q)$ to be the space of all matrix coefficients of finite dimensional weight modules over $U_q(\mathfrak{g})$. This space becomes a Hopf *-algebra with multiplication, comultiplication, counit, and antipode defined in such a way that the canonical evaluation pairing $U_q(\mathfrak{g}) \times \mathfrak{C}^\infty(K_q) \to \mathbb{C}$ satisfies

$$(XY, f) = (X, f_{(1)})(Y, f_{(2)}), \qquad (X, fg) = (X_{(2)}, f)(X_{(1)}, g)$$

and

$$(\hat{S}(X), f) = (X, S^{-1}(f)), \qquad (\hat{S}^{-1}(X), f) = (X, S(f)),$$

$$(X, f^*) = \overline{(\hat{S}^{-1}(X)^*, f)}$$

for $X, Y \in U_q(\mathfrak{g})$ and $f, g \in \mathfrak{C}^\infty(K_q)$. Here we use the Sweedler notation $\hat{\Delta}(X) = X_{(1)} \otimes X_{(2)}$ for the coproduct of $X \in U_q(\mathfrak{g})$ and write $\Delta(f) = f_{(1)} \otimes f_{(2)}$ for the coproduct of $f \in \mathfrak{C}^\infty(K_q)$. Similarly, we denote by $\hat{S}, \hat{\epsilon}$ the antipode and counit of $U_q(\mathfrak{g})$ and write S, ϵ for the corresponding maps for $\mathfrak{C}^\infty(K_q)$, respectively.

By definition, the Hopf *-algebra $\mathfrak{C}^\infty(K_q)$ is the algebra of polynomial functions on the compact quantum group K_q. We note that $\mathfrak{C}^\infty(K_q)$ has a linear basis of matrix coefficients $u^\mu_{ij} = \langle e^\mu_i | \bullet | e^\mu_j \rangle$, where $\mu \in \mathbf{P}^+ = \mathsf{Irr}(K_q)$ is the set of equivalence classes of irreducible representations of K_q and $e^\mu_1, \dots, e^\mu_n$ is an orthonormal basis of $V(\mu)$. We will always consider matrix coefficients with respect to a basis of weight vectors. In terms of matrix elements the normalized Haar functional $\phi : \mathfrak{C}^\infty(K_q) \to \mathbb{C}$ is given by

$$\phi(u^\mu_{ij}) = \begin{cases} 1 & \text{if } \mu = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since K_q is a compact quantum group, the algebra $\mathfrak{C}^\infty(K_q)$ is an algebraic quantum group in the sense of Van Daele [34]. We shall write $\mathfrak{D}(K_q)$ for its dual in the sense of algebraic quantum groups. Explicitly, the dual is is given by the algebraic direct sum

$$\mathfrak{D}(K_q) = \text{alg-}\bigoplus_{\mu \in \mathbf{P}^+} \mathbb{K}(V(\mu)),$$

with the *-structure arising from the C^*-algebras $\mathbb{K}(V(\mu))$. We denote by p_η the central projection in $\mathfrak{D}(K_q)$ corresponding to the matrix block $\mathbb{K}(V(\eta))$ for $\eta \in \mathbf{P}^+$.

The left and right Haar functionals for $\mathfrak{D}(K_q)$ are given by

$$\hat{\phi}(x) = \sum_{\mu \in \mathbf{P}^+} \dim_q(V(\mu)) \operatorname{tr}(K_{2\rho} p_\mu x), \qquad \hat{\psi}(x) = \sum_{\mu \in \mathbf{P}^+} \dim_q(V(\mu)) \operatorname{tr}(K_{-2\rho} p_\mu x),$$

respectively. Here $\dim_q(V(\mu))$ denotes the quantum dimension of $V(\mu)$, and $K_{2\rho} \in U_q(\mathfrak{g})$ is the generator associated with 2ρ, where ρ is the half-sum of all positive roots.

Given the basis of matrix coefficients u^μ_{ij} in $\mathfrak{C}^\infty(K_q)$, we obtain a linear basis of matrix units ω^μ_{ij} of $\mathfrak{D}(K_q)$ satisfying

$$(\omega^\mu_{ij}, u^\nu_{kl}) = \delta_{\mu\nu} \delta_{ik} \delta_{jl}.$$

Both $\mathfrak{C}^\infty(K_q)$ and $\mathfrak{D}(K_q)$ admit universal C^*-completions, which we will denote by $C(K_q)$ and $C^*(K_q)$, respectively.

The fundamental multiplicative unitary W is the algebraic multiplier of $\mathfrak{C}^\infty(K_q) \otimes \mathfrak{D}(K_q)$ given by

$$W = \sum_{\mu \in \mathbf{P}^+} \sum_{i,j=1}^{\dim(V(\mu))} u^\mu_{ij} \otimes \omega^\mu_{ij}.$$

We may also view W as a unitary operator on the tensor product $L^2(K_q) \otimes L^2(K_q)$, where $L^2(K_q)$ is the GNS-Hilbert space of the Haar integral ϕ.

Let us now define the Drinfeld double $G_q = K_q \bowtie \hat{K}_q$. By definition, this is the algebraic quantum group given by the *-algebra

$$\mathfrak{C}_c^\infty(G_q) = \mathfrak{C}^\infty(K_q) \otimes \mathfrak{D}(K_q),$$

with comultiplication

$$\Delta_{G_q} = (\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathsf{ad}(W) \otimes \mathrm{id})(\Delta \otimes \hat{\Delta}),$$

counit

$$\epsilon_{G_q} = \epsilon \otimes \hat{\epsilon},$$

and antipode

$$S_{G_q}(f \otimes x) = W^{-1}(S(f) \otimes \hat{S}(x))W = (S \otimes \hat{S})(W(f \otimes x)W^{-1}).$$

Here W denotes the multiplicative unitary from above. A positive left and right invariant Haar integral for $\mathfrak{C}_c^\infty(G_q)$ is given by

$$\phi_{G_q}(f \otimes x) = \phi(f)\hat{\psi}(x),$$

as seen in [30], [35].

By taking the dual of $\mathfrak{C}_c^\infty(G_q)$ in the sense of algebraic quantum groups, we obtain the convolution algebra $\mathfrak{D}(G_q) = \mathfrak{D}(K_q) \bowtie \mathfrak{C}^\infty(K_q)$, which has $\mathfrak{D}(K_q) \otimes \mathfrak{C}^\infty(K_q)$ as underlying vector space, equipped with the tensor product comultiplication and the multiplication

$$(x \bowtie f)(y \bowtie g) = x(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}(\hat{S}(y_{(3)}), f_{(3)})g.$$

The *-structure of $\mathfrak{D}(G_q)$ is defined in such a way that the natural inclusion homomorphisms $\mathfrak{D}(K_q) \to \mathcal{M}(\mathfrak{D}(G_q))$ and $\mathfrak{C}^\infty(K_q) \to \mathcal{M}(\mathfrak{D}(G_q))$ are *-homomorphisms. Here $\mathcal{M}(\mathfrak{D}(G_q))$ denotes the algebraic multiplier algebra of $\mathfrak{D}(G_q)$.

Both algebras $\mathfrak{C}_c^\infty(G_q)$ and $\mathfrak{D}(G_q)$ admit universal C^*-completions, which will be denoted by $C_0(G_q)$ and $C^*_{\mathsf{f}}(G_q)$, respectively.

A unitary representation of G_q on a Hilbert space $\mathcal{H}$ is defined to be a nondegenerate *-homomorphism $\pi : C^*_{\mathsf{f}}(G_q) \to \mathbb{L}(\mathcal{H})$. A basic example is the left regular representation of G_q. It is obtained from a canonical *-homomorphism $C^*_{\mathsf{f}}(G) \to \mathbb{L}(L^2(G_q))$, where $L^2(G_q)$ is the GNS-construction of the left Haar weight of G_q. By definition, the reduced group C^*-algebra $C^*_{\mathsf{r}}(G_q)$ of G_q is the image of $C^*_{\mathsf{f}}(G_q)$ under the left regular representation.

3. The classical Baum–Connes deformation

In this section we review the definition of the Baum–Connes assembly map in the deformation picture. For background and more information we refer to [11, section II.10], [3, section 4], [25].

Let us first recall a geometric construction called *deformation to the normal cone* [11], [16], [13], which is a variant of a classical concept in algebraic geometry. We start with a smooth manifold M and a closed submanifold $Z \subset M$, and we write NZ for the normal bundle of this inclusion. The associated deformation to the normal cone is defined as

$$D_{Z \subset M} = \{0\} \times NZ \sqcup \mathbb{R}^\times \times M,$$

which becomes a smooth manifold in the following way. The topology and smooth structure are the obvious ones on $\{0\} \times NZ$ and $\mathbb{R}^\times \times M$, and these subsets are glued together using an exponential map for the normal bundle. More precisely, assume that $\exp : V \to W$ is a diffeomorphism such that $\exp(z, 0) = z$ and $d_{(z,0)} \exp = \mathrm{id}$ for all $z \in Z$, where $V \subset NZ$ is an open neighborhood of the zero section and $W \subset M$ is an open set containing Z. Then the map θ given by

$$\theta(\tau, z, X) = \begin{cases} (0, z, X) & \tau = 0, \\ (\tau, \exp(z, \tau X)) & \tau \neq 0 \end{cases}$$

is a diffeomorphism from a suitable neighborhood of $\mathbb{R} \times Z \subset \mathbb{R} \times NZ$ onto $\{0\} \times NZ \sqcup \mathbb{R}^\times \times W \subset D_{Z\subset M}$.

In the sequel we will always restrict the parameter space in this construction from $\mathbb{R}$ to $[0, 1]$, as this is better suited to the questions we are interested in. That is, we will consider $\{0\} \times NZ \sqcup (0, 1] \times M \subset D_{Z\subset M}$ and, by slight abuse of language, refer to this again as deformation to the normal cone.

Now let G be a complex semisimple Lie group, and let $G = KAN$ be the Iwasawa decomposition of G. We consider

$$\mathcal{G}_G = \{0\} \times K \times \mathfrak{k}^* \sqcup (0, 1] \times G,$$

the deformation to the normal cone of the inclusion $K \subset G$. Note here that the normal bundle NK is the trivial bundle $K \times \mathfrak{an}$, and that $\mathfrak{an} \cong \mathfrak{k}^*$ naturally. If we write $G_0 = K \ltimes \mathfrak{k}^*$ for the semidirect product of K acting on the abelian Lie group $\mathfrak{k}^*$ via the coadjoint action, then the fiberwise group structure on $\mathcal{G}_G$ turns the latter into a Lie groupoid over the base space $[0, 1]$.

The corresponding reduced groupoid C^*-algebra $C^*_{\mathsf{r}}(\mathcal{G}_G)$ is a continuous field of C^*-algebras over $[0, 1]$ with fibers $C^*(G_0)$ at 0, and $C^*_{\mathsf{r}}(G)$ otherwise. Moreover, the bundle $\mathcal{G}_G$ is trivial outside 0 and therefore induces a map $K_*(C^*(G_0)) \to K_*(C^*_{\mathsf{r}}(G))$ in K-theory. Note that G_0 is amenable and that we have $C^*(G_0) \cong K \ltimes C_0(\mathfrak{k})$ by Fourier transform.

The following result describes the deformation picture of the Baum–Connes assembly map; see [11, section II.10].

Theorem 3.1. *Let G be a complex semisimple Lie group. There exists an isomorphism $K^{\mathsf{top}}_*(G) \cong K_*(K \ltimes C_0(\mathfrak{k}))$ such that the diagram*

$$\begin{tikzcd} & K_*(K \ltimes C_0(\mathfrak{k})) \arrow[dr] & \\ K^{\mathsf{top}}_*(G) \arrow[ur, "\cong"'] \arrow[rr, "\mu"'] & & K_*(C^*_{\mathsf{r}}(G)) \end{tikzcd}$$

is commutative, where $\mu : K^{\mathsf{top}}_*(G) \to K_*(C^*_{\mathsf{r}}(G))$ is the Baum–Connes assembly map and the right downward arrow is the map arising from the deformation groupoid.

This result allows us to reinterpret the Baum–Connes assembly map in terms of the groupoid $\mathcal{G}_G$. For a detailed account of its proof in the setting of arbitrary almost connected groups we refer to [25].

4. Quantization of the Cartan motion group

In this section we explain how the classical Cartan motion group of G can be deformed into a quantum group. The corresponding continuous field of C^*-algebras is a variant of the deformation to the normal cone construction.

4.1. **Representations of crossed products by compact groups.** Let us review some general facts regarding crossed product C^*-algebras by actions of compact groups. For more information see [33], [39].

Let K be a compact group, and let $H \subset K$ be a closed subgroup. If V is a unitary representation of H, then the induced representation is the Hilbert space

$$\mathrm{ind}_H^K(V) = L^2(K, V)^H$$

equipped with the left regular representation of K, where H acts by $(r \cdot \xi)(t) = r \cdot \xi(tr)$ on $\xi \in L^2(K, V)$. Together with the action of $C(K/H)$ on $\mathrm{ind}_H^K(V)$ by pointwise multiplication, we obtain a covariant representation on $\mathrm{ind}_H^K(V)$. Hence $\mathrm{ind}_H^K(V)$ is naturally a representation of $K \ltimes C(K/H)$. In fact, the passage from V to $\mathrm{ind}_H^K(V)$ corresponds to Rieffel induction under the Morita equivalence between $C^*(H)$ and $K \ltimes C(K/H)$.

Now let X be a locally compact space equipped with a continuous action of K. It is well known that the irreducible representations of the crossed product $K \ltimes C_0(X)$ can be described as follows.

Proposition 4.1. *Let K be a compact group acting on a locally compact space X. Then all irreducible *-representations of $K \ltimes C_0(X)$ are of the form*

$$\mathrm{ind}_{K_x}^K(V),$$

where $x \in X$ and where V is an irreducible unitary representation of the stabilizer group K_x of x. Moreover, two representations $\mathrm{ind}_{K_x}^K(V)$ and $\mathrm{ind}_{K_y}^K(W)$ of this form are equivalent if and only if $x = t \cdot y$ for some $t \in K$ and W corresponds to V under the identification of K_x and K_y induced by conjugation with t.

Proof. Note that $\mathrm{ind}_{K_x}^K(V)$ is a representation of $K \ltimes C_0(X)$ by first projecting $K \ltimes C_0(X) \to K \ltimes C(K \cdot x) \cong K \ltimes C(K/K_x)$ and then considering the Morita equivalence between $K \ltimes C(K/K_x)$ and $C^*(K_x)$.

Since Morita equivalence preserves irreducibility, it follows that inducing irreducible representations of stabilizers produces irreducible representations of the crossed product. In fact, all irreducible representations are obtained in this way; see, for instance, [39, Proposition 8.7].

If $y = t \cdot x$ for some $t \in K$ and W corresponds to V under $\mathsf{ad}_t : K_x \to K_y$, $\mathsf{ad}_t(r) = trt^{-1}$, the corresponding isomorphism of stabilizer groups, then we obtain a unitary intertwiner $T : \mathrm{ind}_{K_x}^K(V) \to \mathrm{ind}_{K_y}^K(W)$ by setting $T(\xi)(x) = \xi(xt)$.

Now assume that $\mathrm{ind}_{K_x}^K(V)$ and $\mathrm{ind}_{K_y}^K(W)$ are equivalent. If x and y are on the same K-orbit, we can identify the stabilizers K_x and K_y via conjugation, so we may assume that $x = y$. In this case we obtain $V \cong W$ via the Morita equivalence between $K \ltimes C(K/K_x)$ and $C^*(K_x)$.

If $x, y \in X$ are not on the same K-orbit, then the primitive ideals of $K \ltimes C_0(X)$ corresponding to induced representations at x and y differ, and in particular there can be no nonzero intertwiner between such representations. □

The following well-known fact regarding the structure of crossed products for actions of compact groups follows from the Imai–Takai biduality theorem [17].

Proposition 4.2. *Let K be a compact group acting on a locally compact space X. Then the crossed product $K \ltimes C_0(X)$ is naturally isomorphic to the C^*-algebra*

$$C_0(X, \mathbb{K}(L^2(K)))^K,$$

where K acts on $C_0(X)$ with the given action and on $\mathbb{K}(L^2(K))$ with the conjugation action with respect to the right regular representation on $L^2(K)$.

We note that under the identification in Proposition 4.2, the (reduced) crossed product $K \ltimes C_0(X)$ is the closed linear span of all elements of the form $(y \otimes 1)\gamma(f)$ inside $\mathbb{K}(L^2(K)) \otimes C_0(X)$, where $y \in C^*(K)$ and $f \in C_0(X)$. Here $\gamma : C_0(X) \to C(K) \otimes C_0(X)$ is given by $\gamma(f)(t, x) = f(t \cdot x)$.

4.2. **The representation theory of $K \ltimes C_0(\mathfrak{k})$.** Using the general method outlined above, we shall now compute the irreducible representations of the Cartan motion group $K \ltimes \mathfrak{k}^*$. For this it will be convenient to identify $C^*(K \ltimes \mathfrak{k}^*)$ with the crossed product $K \ltimes C_0(\mathfrak{k})$. Unless explicitly stated otherwise, we assume throughout that K is a simply connected semisimple compact Lie group.

We note first that each point of $\mathfrak{k}$ is conjugate to a point of $\mathfrak{t}$; see, for instance, [6, chapter IX]. Therefore by Proposition 4.1 all irreducible representations of $K \ltimes C_0(\mathfrak{k})$ are of the form

$$\mathrm{ind}_{K_X}^K(V),$$

where $X \in \mathfrak{t}$ and where V is an irreducible representation of the stabilizer subgroup $K_X \subset K$ of X. More concretely, the induced representation $\mathrm{ind}_{K_X}^K(V)$ corresponds to the covariant pair given by the left regular representation of K on $\mathrm{ind}_{K_X}^K(V)$ and

$$\pi_{(V,X)} : C_0(\mathfrak{k}) \to \mathbb{L}(\mathrm{ind}_{K_X}^K(V)), \qquad \pi_{(V,X)}(f)(\xi)(k) = f(k \cdot X)\xi(k),$$

where we write $k \cdot X$ for the adjoint action of k on X.

If W denotes the Weyl group of K, then the intersections of the orbits of K in $\mathfrak{k}$ with $\mathfrak{t}$ are the orbits of W in $\mathfrak{t}$. For each $X \in \mathfrak{t}$ the stabilizer subgroup K_X is a connected subgroup, and we have $T \subset K_X \subset K$. Since K_X has again T as maximal torus, we obtain $\mathsf{Irr}(K_X) = \mathsf{Irr}(T)/W_X$. Here W_X is the Weyl group of K_X, and we note that this group identifies with the stabilizer of X in W. Putting all of these facts together, we see that the spectrum of $K \ltimes C_0(\mathfrak{k})$, as a set, is

$$\bigsqcup_{X \in \mathfrak{t}/W} \mathbf{P}/W_X \cong (\mathbf{P} \times \mathfrak{t})/W.$$

Here we have identified $\mathsf{Irr}(T)$ with the weight lattice $\mathbf{P}$.

In order to describe the algebra $K \ltimes C_0(\mathfrak{k})$ more concretely, we use Proposition 4.2 to write

$$K \ltimes C_0(\mathfrak{k}) \cong C_0(\mathfrak{k}, \mathbb{K}(L^2(K)))^K.$$

Let $\mathbb{K}(L^2(K))^{K_X}$ be the fixed point subalgebra with respect to the right regular action of K_X on $L^2(K)$. We note that if $f \in C_0(\mathfrak{k}, \mathbb{K}(L^2(K)))^K$, then $f(X) \in \mathbb{K}(L^2(K))^{K_X}$. Using the Peter–Weyl theorem, one obtains

$$\mathbb{K}(L^2(K))^{K_X} \cong \bigoplus_{V \in \mathsf{Irr}(K_X)} \mathbb{K}(L^2(K, V)^{K_X})$$

by inducing the left regular representation of K_X to K and then applying Schur's lemma. In particular, if $f \in C_0(\mathfrak{k}, \mathbb{K}(L^2(K)))^K$ and if V is an irreducible representation of K_X for some $X \in \mathfrak{t}$, then the projection of $f(X)$ to the direct summand $\mathbb{K}(L^2(K, V)^{K_X})$ in $\mathbb{K}(L^2(K))^{K_X}$ corresponds to $\mathrm{ind}_{K_X}^K(V)$.

Consider the map

$$\pi : C_0(\mathfrak{k}, \mathbb{K}(L^2(K)))^K \to C_0(\mathfrak{t}, \mathbb{K}(L^2(K)))$$

obtained by restriction to $\mathfrak{t} \subset \mathfrak{k}$. This map is injective since every orbit of the adjoint action of K on $\mathfrak{k}$ meets $\mathfrak{t}$.

Our aim is to describe the image of π. Given $\mu \in \mathbf{P}$, let us write $V(\mu)$ for the corresponding representation of T and define

$$L^2(\mathcal{E}_\mu) = \mathrm{ind}_T^K(V(\mu)) = L^2(K, V(\mu))^T.$$

The fixed point algebra of $\mathbb{K}(L^2(K))$ with respect to the action of T is

$$\mathbb{K}(L^2(K))^T = \bigoplus_{\mu \in \mathbf{P}} \mathbb{K}(L^2(\mathcal{E}_\mu)).$$

Since $T \subset K_X$, we have $f(X) \in \mathbb{K}(L^2(K))^T$ for all $f \in C_0(\mathfrak{k}, \mathbb{K}(L^2(K)))^K$ and $X \in \mathfrak{t}$. In particular, we see that π takes values in $C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H}))$, where $\mathcal{H} = (\mathcal{H}_{\mu,X})$ denotes the locally constant bundle of Hilbert spaces over $\mathbf{P} \times \mathfrak{t}$ with fibers $\mathcal{H}_{\mu,X} = L^2(\mathcal{E}_\mu)$. Here $C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H}))$ is the C^*-algebra of sections of the $C_0(\mathbf{P} \times \mathfrak{t})$-algebra of compact operators on the Hilbert $C_0(\mathbf{P} \times \mathfrak{t})$-module corresponding to $\mathcal{H}$.

The Weyl group W acts naturally on $C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H})) = C_0(\mathfrak{t}, \bigoplus_{\mu \in \mathbf{P}} \mathbb{K}(L^2(\mathcal{E}_\mu)))$. Indeed, if $w \in W$, there is a unitary isomorphism $U_w : L^2(\mathcal{E}_\mu) \to L^2(\mathcal{E}_{w\mu})$ defined by $U_w(\xi)(k) = \xi(kw)$. Note that here we are viewing $w \in K$, and strictly speaking we are picking a representative for an element of W. However, the induced isomorphism $\mathbb{K}(L^2(\mathcal{E}_\mu)) \cong \mathbb{K}(L^2(\mathcal{E}_{w\mu}))$ is independent of this choice. The image of π is invariant under the action of the Weyl group, by the invariance under the action of K.

For $X \in \mathfrak{t}$ and $f \in C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H}))$ denote by $f_X \in \mathbb{K}(L^2(K))$ the element obtained by evaluating f in the second variable and viewing the resulting section of the bundle $\mathbb{K}(\mathcal{H})$ over $\mathbf{P}$ as an element of

$$\bigoplus_{\mu \in \mathbf{P}} \mathbb{K}(L^2(\mathcal{E}_\mu)) \subset \mathbb{K}(L^2(K)).$$

With this notation in place we can describe the image of π as follows.

Theorem 4.3. *The C^*-algebra $K \ltimes C_0(\mathfrak{k})$ is isomorphic to the subalgebra*

$$A_0^L = \{f \in C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H}))^W \mid f_X \in \mathbb{K}(L^2(K))^{K_X} \text{ for all } X \in \mathfrak{t}\}$$

of $C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H}))^W$.

Proof. We have already observed that the map π defined above is injective and takes values in A_0^L. For surjectivity, we start by understanding the form of the irreducible representations of A_0^L.

Let $\theta : A_0^L \to \mathbb{L}(\mathcal{V})$ be an irreducible representation of A_0^L on a Hilbert space $\mathcal{V}$. By standard C^*-algebra theory there exists an irreducible representation $\Theta : C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H})) \to \mathbb{L}(\mathcal{K})$ on some Hilbert space $\mathcal{K}$ and an A_0^L-invariant subspace $\mathcal{L} \subset \mathcal{K}$ such that the restriction of Θ to A_0^L, acting on $\mathcal{L}$, is equivalent to θ. In particular, we can assume $\mathcal{V} \subset \mathcal{K}$ such that the action of A_0^L on $\mathcal{V}$ is given by Θ. Now note that $C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H})) = C_0(\mathfrak{t}, \bigoplus_{\mu \in \mathbf{P}} \mathbb{K}(L^2(\mathcal{E}_\mu)))$. Then, up to isomorphism, Θ must be given by point evaluation at an element $X \in \mathfrak{t}$, followed by projection to a direct summand of $\oplus_{\mu \in \mathbf{P}} \mathbb{K}(L^2(\mathcal{E}_\mu))$.

In particular, each irreducible representation of A_0^L factors through point evaluation at some element of $\mathfrak{t}$. If $f \in A_0^L$, then $f_X \in \mathbb{K}(L^2(K))^{K_X}$, which we saw has a direct sum decomposition

$$\mathbb{K}(L^2(K))^{K_X} \cong \bigoplus_{V \in \mathsf{Irr}(K_X)} \mathbb{K}(L^2(K, V)^{K_X}).$$

We conclude that any irreducible representation of A_0^L is given by point evaluation at an element $X \in \mathfrak{t}$, followed by projection to one of these direct summands of $\mathbb{K}(L^2(K))^{K_X}$. Notice that restricting an irreducible representation of A_0^L to the image of π gives an irreducible representation of the latter, by our concrete description of the irreducible representations of $C_0(\mathfrak{k}, \mathbb{K}(L^2(K)))^K$. By construction of the algebra A_0^L and the structure of the spectrum of $K \ltimes C_0(\mathfrak{k})$ explained further above, any two irreducible representations of A_0^L which become equivalent when viewed as representations of $\mathrm{im}(\pi)$ are already equivalent as representations of A_0^L.

The result then follows from Dixmier's Stone–Weierstrass theorem; see [14, section 11.1]. □

In order to illustrate Theorem 4.3, let us consider the special case $K = \mathrm{SU}(2)$. In this case the stabilizer group K_X is equal to T if $X \in \mathfrak{t} \cong \mathbb{R}$ is nonzero, and $K_X = K$ if $X = 0$. It follows that the algebra described in Theorem 4.3 identifies with

$$A_0^L = \{f \in C_0(\mathbb{Z} \times \mathbb{R}, \mathbb{K}(\mathcal{H}))^{\mathbb{Z}_2} \mid f_0 \in \bigoplus_{\nu \in \mathbf{P}^+} \mathbb{K}(V(\nu)) \subset \mathbb{K}(L^2(K))\},$$

where $\mathbb{K}(V(\nu))$ is represented via the left regular representation of K.

4.3. **The representation theory of $K \ltimes C(K)$.** Let us next compute the irreducible representations of the C^*-algebra $K \ltimes C(K)$ of the quantum Cartan motion group. The arguments are parallel to the ones in the previous subsection. We remark that $K \ltimes C(K)$ is the group C^*-algebra of the Drinfeld double of the classical group K.

The irreducible representations of $K \ltimes C(K)$ are labeled by $(\mathbf{P} \times T)/W$. Indeed, each point of K is conjugate to a point of T, so Proposition 4.1 tells us that all irreducible representations of $K \ltimes C(K)$ are of the form

$$\mathrm{ind}_{K_t}^K(V),$$

where $t \in T$ and V is an irreducible representation of the centralizer subgroup $K_t \subset K$ of t in K. The intersections of the conjugacy classes in K with T are the orbits of W on T. For each $t \in T$, the centralizer subgroup K_t is a connected subgroup, and we have $T \subset K_t \subset K$. Therefore $\mathsf{Irr}(K_t) = \mathsf{Irr}(T)/W_t$, where W_t is the Weyl group of K_t, which identifies with the stabilizer of t in W. We conclude that the spectrum of $K \ltimes C(K)$, as a set, is

$$\bigsqcup_{t \in T/W} \mathbf{P}/W_t \cong (\mathbf{P} \times T)/W.$$

In the same way as in the analysis of the classical Cartan motion group, let us consider the isomorphism

$$K \ltimes C(K) \cong C(K, \mathbb{K}(L^2(K)))^K$$

obtained from Proposition 4.2. If $f \in C(K, \mathbb{K}(L^2(K)))^K$, then $f(t)$ is contained in $\mathbb{K}(L^2(K))^{K_t}$, and each irreducible representation of $C(K, \mathbb{K}(L^2(K)))^K$ can be described as point evaluation at some $t \in T$ followed by projection onto a direct summand of

$$\mathbb{K}(L^2(K))^{K_t} \cong \bigoplus_{V \in \mathsf{Irr}(K_t)} \mathbb{K}(L^2(K, V)^{K_t}).$$

The map

$$\pi : C(K, \mathbb{K}(L^2(K)))^K \to C(T, \mathbb{K}(L^2(K)))$$

obtained by restricting operator-valued functions to $T \subset K$ is injective, and the image of π is contained in $C_0(\mathbf{P}\times T, \mathbb{K}(\mathcal{H}))^W$, where $\mathcal{H} = (\mathcal{H}_{\mu,t})$ denotes the locally constant bundle of Hilbert spaces over $\mathbf{P} \times T$ with fibers $\mathcal{H}_{\mu,t} = L^2(\mathcal{E}_\mu)$.

For $t \in T$ and $f \in C_0(\mathbf{P} \times T, \mathbb{K}(\mathcal{H}))^W$ denote by $f_t \in \mathbb{K}(L^2(K))$ the element obtained by evaluating f in the second variable and viewing the resulting section of the bundle $\mathbb{K}(\mathcal{H})$ over $\mathbf{P}$ as an element of

$$\bigoplus_{\mu\in\mathbf{P}} \mathbb{K}(L^2(\mathcal{E}_\mu)) \subset \mathbb{K}(L^2(K)).$$

We then have the following analogue of Theorem 4.3, with essentially the same proof.

Theorem 4.4. *The C^*-algebra $K \ltimes C(K)$ is isomorphic to the subalgebra*

$$A_1^L = \{f \in C_0(\mathbf{P} \times T, \mathbb{K}(\mathcal{H}))^W \mid f_t \in \mathbb{K}(L^2(K))^{K_t} \text{ for all } t \in T\}$$

of $C_0(\mathbf{P} \times T, \mathbb{K}(\mathcal{H}))^W$.

Let us again consider explicitly the case $K = \mathrm{SU}(2)$. In this case the centralizer group K_t is equal to K if $t = \pm 1$, and $K_t = T \cong S^1$ otherwise.

For $t \in T$ let us write

$$V_{\mu,t} = \mathrm{ind}_{K_t}^K(V(\mu))$$

for the induced representation of the irreducible representation $V(\mu)$ of K_t associated with the weight $\mu \in \mathbf{P}$. Here we tacitly declare $V(\mu) = V(-\mu)$ if $\mu \in \mathbf{P}^+$ and $t = \pm 1$. Two representations $V_{\mu,t}, V_{\mu',t'}$ are equivalent if and only if $(\mu', t') = (\pm\mu, t^{\pm 1})$.

Let us write down the representations $V_{\mu,t}$ explicitly. If $t \neq \pm 1$, then $V_{\mu,t}$ is the space $L^2(\mathcal{E}_\mu)$ with the action of $C(K)$ given by

$$(f \cdot \xi)(k) = f(ktk^{-1})\xi(k),$$

and the action of K given by

$$(s \cdot \xi)(k) = \xi(s^{-1}k).$$

If $t = \pm 1$, we have $V_{\mu,t} = V(\mu)$ and the action of $C(K)$ reduces to

$$f \cdot \xi = f(t)\xi.$$

The algebra described in Theorem 4.4 is

$$A_1^L = \{f \in C_0(\mathbb{Z} \times S^1, \mathbb{K}(\mathcal{H}))^{\mathbb{Z}_2} \mid f_{\pm 1} \in \bigoplus_{\nu\in\mathbf{P}^+} \mathbb{K}(V(\nu)) \subset \mathbb{K}(L^2(K))\},$$

where $\mathbb{K}(V(\nu))$ is again represented via the left regular representation of K.

4.4. **The quantization field for the Cartan motion group.** In this section we discuss the quantization of the classical Cartan motion group, represented by the C^*-algebra $K \ltimes C_0(\mathfrak{k})$, towards the quantum Cartan motion group, which is represented by the C^*-algebra $K \ltimes C(K)$.

Let us consider the deformation to the normal cone construction outlined in section 3 for the manifold K and its one-point submanifold consisting of the identity element of K. We obtain a Lie groupoid $\mathcal{G}_K$ over the base space $[0, 1]$, also known as the corresponding adiabatic groupoid [13], with object space

$$\mathcal{G}_K = \{0\} \times \mathfrak{k} \sqcup (0, 1] \times K,$$

equipped with the topology and smooth structure as defined in section 3. The groupoid structure is given by the fiberwise group operations. Since the canonical projection map $\mathcal{G}_K \to [0,1]$ is continuous and open, the algebra $C_0(\mathcal{G}_K)$ is naturally a continuous field of C^*-algebras over $[0,1]$, with fibers

$$C_0(\mathcal{G}_K)_\tau = \begin{cases} C_0(\mathfrak{k}) & \text{if } \tau = 0, \\ C(K) & \text{otherwise.} \end{cases}$$

For our purposes it is convenient to describe this construction directly on the level of functions. Given any $f_0 \in C_c(\mathfrak{k})$, we obtain a continuous function f on $\mathcal{G}_K$ by considering the family $(f_\tau)_{\tau\in[0,1]}$ of functions defined as follows. We set $f_\tau(\exp(X)) = f(\tau^{-1}X)$ for $\tau > 0$ small and X in a neighborhood of 0 on which the exponential map is a diffeomorphism. Extending by zero and cutting off in the τ-direction, we obtain continuous functions $f_\tau \in C(K)$ for all $\tau > 0$ in this way. Evaluating the family $(f_\tau)_{\tau\in[0,1]}$ in the fibers of $\mathcal{G}_K$ then yields $f \in C_0(\mathcal{G}_K)$. We also consider the continuous functions on $\mathcal{G}_K$ obtained from all families $(f_\tau)_{\tau\in[0,1]}$ such that $f_\tau = 0$ for τ small and $(\tau, x) \mapsto f_\tau(x)$ is continuous on $(0,1] \times K$. By definition of the topology of $\mathcal{G}_K$, the collection of all functions described above yields a *-algebra which is dense in the C^*-algebra $C_0(\mathcal{G}_K)$ of functions on $\mathcal{G}_K$.

Using the fact that continuous fields are preserved by taking crossed products with respect to fiberwise actions of amenable groups (see [28, Theorem 2.1]), we obtain the following result.

Proposition 4.5. *Let K be a connected compact Lie group, and let $\mathcal{G}_K$ be the associated adiabatic groupoid as above. The crossed product $L = K \ltimes C_0(\mathcal{G}_K)$ with respect to the fiberwise adjoint action is a continuous field of C^*-algebras with fibers $L_0 = K \ltimes C_0(\mathfrak{k})$ and $L_\tau = K \ltimes C(K)$ for $\tau > 0$.*

We call the continuous field constructed in Proposition 4.5 the *quantization field for the Cartan motion group.*

Using Theorems 4.3 and 4.4, let us define another continuous field of C^*-algebras A^L over $[0,1]$ whose fibers are isomorphic to the fibers of the quantization field L. More specifically, we stipulate that the field A^L is trivial away from 0, and if $f_0 \in K \ltimes C_0(\mathfrak{k}) \subset C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H}))^W$ has compact support as a function on $\mathbf{P} \times \mathfrak{t}$, then a continuous section $(f_\tau)_{\tau\in[0,1]}$ of A^L is obtained by setting

$$f_\tau(\mu, \exp(X)) = f(\mu, \tau^{-1}X)$$

for $\tau > 0$ small and X in a neighborhood of 0 on which exp is a diffeomorphism, extended by 0 to all of T. Note here that f_τ for $\tau > 0$ defines an element of $K \ltimes C(K) \subset C_0(\mathbf{P} \times T, \mathbb{K}(\mathcal{H}))^W$. The fiber at $\tau = 0$ of the field A^L identifies with A^L_0 as in Theorem 4.3, and the fibers of A^L at points $\tau > 0$ can be identified with A^L_1 as in Theorem 4.4.

Proposition 4.6. *The continuous field A^L described above is isomorphic to the quantization field of the Cartan motion group.*

Proof. Let f be an element of the convolution algebra $C(K, C_0(\mathcal{G}_K)) \subset K \ltimes C_0(\mathcal{G}_K)$ of the form $f = g \otimes h$, where $g \in C(K)$ and $h \in C_0(\mathcal{G}_K)$ is the generating section associated with a compactly supported function h_0 in $C_c(\mathfrak{k})$ as in the description of $C_0(\mathcal{G}_K)$ explained further above. Since linear combinations of elements of this type form a dense subspace of L, it suffices to show that f defines a continuous section of A^L.

Under the isomorphism provided by Theorem 4.3, we have

$$((g \otimes h_0)(\mu, X))(\xi)(k) = \int_K g(s)h(s^{-1}k \cdot X)\xi(s^{-1}k)ds$$

for $\mu \in \mathbf{P}, X \in \mathfrak{t}, \xi \in L^2(\mathcal{E}_\mu)$, and $k \in K$. Similarly, under the isomorphism provided by Theorem 4.4, we have

$$((g \otimes h_\tau)(\mu, \exp(X)))(\xi)(k) = \int_K g(s)h(\tau^{-1}s^{-1}k \cdot X)\xi(s^{-1}k)ds$$

for $\tau > 0$ small. From these formulas we can see that under these isomorphisms on each fiber, the element f is mapped to a continuous section of A^L, as required. $\square$

5. Quantization of complex semisimple Lie groups

In this section we describe a deformation of the reduced group C^*-algebra of a complex semisimple Lie group G. The resulting continuous field of C^*-algebras is another instance of the deformation to the normal cone construction.

5.1. **The tempered representations of classical complex groups.** In this subsection we review the structure of the space of irreducible tempered representations of complex semisimple groups. Throughout we fix a simply connected complex semisimple Lie group G with maximal compact subgroup K, and a compatible choice of Borel subgroup $B \subset G$ and maximal torus $T \subset K$.

Let $\mu \in \mathbf{P}$. Then the space of smooth sections $\Gamma(\mathcal{E}_\mu) \subset L^2(\mathcal{E}_\mu)$ of the induced vector bundle $\mathcal{E}_\mu = K \times_T \mathbb{C}_\mu$ over $G/B = K/T$ corresponding to μ is the subspace of $C^\infty(K)$ of weight μ with respect to the right translation action of T given by

$$(t \rightharpoonup \xi)(x) = \xi(xt).$$

Equivalently, we have

$$\Gamma(\mathcal{E}_\mu) = \{\xi \in C^\infty(K) \mid (\mathrm{id} \otimes \pi_T)\Delta(\xi) = \xi \otimes z^\mu\},$$

where $\pi_T : C^\infty(K) \to C^\infty(T)$ is the projection homomorphism and $z^\mu \in C^\infty(T)$ is the unitary corresponding to the weight μ.

Given $\lambda \in \mathfrak{t}^*$, we can identify

$$\Gamma(\mathcal{E}_\mu) = \{\xi \in C^\infty(G) \mid (\mathrm{id} \otimes \pi_B)\Delta(\xi) = \xi \otimes z^\mu \otimes \chi_{\lambda+2\rho}\} \subset C^\infty(G),$$

where we view $G = KAN = K \times \exp(\mathfrak{a}) \times \exp(\mathfrak{n})$ and write $\chi_{\lambda+2\rho}$ for the character on AN given by

$$\chi_{\lambda+2\rho}(\exp(a)\exp(n)) = e^{(\lambda+2\rho, a)};$$

recall that ρ denotes the half-sum of all positive roots. The left regular action of G on $\Gamma(\mathcal{E}_\mu) = \Gamma(\mathcal{E}_{\mu,\lambda})$ defines a representation of G which is called the *principal series representation* with parameter $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{t}^*$. Since $\lambda \in \mathfrak{t}^*$, the representation $\Gamma(\mathcal{E}_{\mu,\lambda}) \subset C^\infty(K)$ is unitary for the standard scalar product on $C^\infty(K)$ induced from the Haar measure on K. In particular, we obtain a corresponding nondegenerate *-representation $\pi_{\mu,\lambda} : C^*_\mathfrak{f}(G) \to \mathbb{L}(\mathcal{H}_{\mu,\lambda})$, where $\mathcal{H}_{\mu,\lambda} \subset L^2(K)$ is the Hilbert space completion of $\Gamma(\mathcal{E}_{\mu,\lambda})$.

The following result is due to Želobenko [40] and Wallach [37].

Theorem 5.1. *For all $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{t}^*$ the unitary principal series representation $\mathcal{H}_{\mu,\lambda}$ is an irreducible unitary representation of G.*

The Weyl group W acts on the parameter space $\mathbf{P} \times \mathfrak{t}^*$ by

$$w(\mu, \lambda) = (w\mu, w\lambda).$$

Using this action, one can describe the isomorphisms between unitary principal series representations as follows; see [21].

Theorem 5.2. *Let $(\mu, \lambda), (\mu', \lambda') \in \mathbf{P} \times \mathfrak{t}^*$. Then $\mathcal{H}_{\mu,\lambda}$ and $\mathcal{H}_{\mu',\lambda'}$ are equivalent representations of G if and only if $(\mu, \lambda), (\mu', \lambda')$ are on the same Weyl group orbit, that is, if and only if*

$$(\mu', \lambda') = (w\mu, w\lambda)$$

for some $w \in W$.

Combining these results with Harish-Chandra's Plancherel theorem [15], one obtains the following description of the reduced group C^*-algebra of G.

Theorem 5.3. *Let G be a complex semisimple Lie group, and let $\mathcal{H} = (\mathcal{H}_{\mu,\lambda})_{\mu,\lambda}$ be the Hilbert space bundle of unitary principal series representations of G over $\mathbf{P} \times \mathfrak{t}^*$. Then one obtains an isomorphism*

$$C^*_\mathsf{r}(G) \cong C_0(\mathbf{P} \times \mathfrak{t}^*, \mathbb{K}(\mathcal{H}))^W$$

induced by the canonical *-homomorphism $\pi : C^*_\mathsf{f}(G) \to C_0(\mathbf{P} \times \mathfrak{t}^*, \mathbb{K}(\mathcal{H}))$.

5.2. **The tempered representations of complex quantum groups.** In this subsection we review some aspects of the representation theory of complex quantum groups. For more details and background we refer to [35].

Throughout we fix $q = e^h$ such that $0 < q < 1$ and set

$$\mathfrak{t}^*_q = \mathfrak{t}^*/i\hbar^{-1}\mathbf{Q}^\vee,$$

where $\hbar = \frac{h}{2\pi}$ and $\mathbf{Q}^\vee$ is the coroot lattice of $\mathfrak{g}$.

Let $\mu \in \mathbf{P}$. Then we define the space of sections $\Gamma(\mathcal{E}_\mu) \subset \mathfrak{C}^\infty(K_q)$ of the induced vector bundle $\mathcal{E}_\mu$ corresponding to μ to be the subspace of $\mathfrak{C}^\infty(K_q)$ of weight μ with respect to the $\mathfrak{D}(K_q)$-module structure

$$x \rightharpoonup \xi = \xi_{(1)}(x, \xi_{(2)}).$$

Equivalently, we have

$$\Gamma(\mathcal{E}_\mu) = \{\xi \in \mathfrak{C}^\infty(K_q) \mid (\mathrm{id} \otimes \pi_T)\Delta(\xi) = \xi \otimes z^\mu\},$$

where $\pi_T : \mathfrak{C}^\infty(K_q) \to \mathfrak{C}^\infty(T)$ is the canonical projection homomorphism and $z^\mu \in \mathfrak{C}^\infty(T) = \mathbb{C}[\mathbf{P}]$ is the generator corresponding to the weight μ.

For $\lambda \in \mathfrak{t}^*_q$ we define the twisted left adjoint representation of $\mathfrak{C}^\infty(K_q)$ on $\Gamma(\mathcal{E}_\mu)$ by

$$f \cdot \xi = f_{(1)}\, \xi\, S(f_{(3)})(K_{2\rho+\lambda}, f_{(2)}).$$

Together with the left coaction $\Gamma(\mathcal{E}_\mu) \to \mathfrak{C}^\infty(K_q) \otimes \Gamma(\mathcal{E}_\mu)$ given by comultiplication, this turns $\Gamma(\mathcal{E}_\mu)$ into a Yetter–Drinfeld module.

It is convenient to switch from the left coaction on $\Gamma(\mathcal{E}_\mu)$ to the left $\mathfrak{D}(K_q)$-module structure given by

$$x \cdot \xi = (\hat{S}(x), \xi_{(1)})\xi_{(2)}$$

for $x \in \mathfrak{D}(K_q)$. The space $\Gamma(\mathcal{E}_\mu) = \Gamma(\mathcal{E}_{\mu,\lambda})$ with the actions as above is called the *principal series Yetter–Drinfeld module* with parameter $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{t}^*_q$. Since $\lambda \in \mathfrak{t}^*_q$, the Yetter–Drinfeld module $\Gamma(\mathcal{E}_{\mu,\lambda}) \subset \mathfrak{C}^\infty(K_q)$ is unitary for the scalar product induced from the Haar state on $\mathfrak{C}^\infty(K_q)$. In particular, we obtain a corresponding nondegenerate *-representation $\pi_{\mu,\lambda} : C^*_{\mathsf{f}}(G_q) \to \mathbb{L}(\mathcal{H}_{\mu,\lambda})$, where $\mathcal{H}_{\mu,\lambda} \subset L^2(K_q)$ is the Hilbert space completion of $\Gamma(\mathcal{E}_{\mu,\lambda})$.

The unitary representations of G_q on $\mathcal{H}_{\mu,\lambda}$ for $(\mu,\lambda) \in \mathbf{P} \times \mathfrak{t}^*_q$ as above are called unitary principal series representations.

For a proof of the following result we refer to [35].

Theorem 5.4. *For all $(\mu,\lambda) \in \mathbf{P} \times \mathfrak{t}^*_q$ the unitary principal series representation $\mathcal{H}_{\mu,\lambda}$ is an irreducible unitary representation of G_q.*

As in the classical situation, there are nontrivial intertwiners between unitary principal series representations of G_q. The classical Weyl group W acts on the parameter space $\mathbf{P} \times \mathfrak{t}^*_q$ by

$$w(\mu,\lambda) = (w\mu, w\lambda).$$

The following result describes the isomorphisms between unitary principal series representations in the quantum case [35].

Theorem 5.5. *Let $(\mu,\lambda), (\mu',\lambda') \in \mathbf{P} \times \mathfrak{t}^*_q$. Then $\mathcal{H}_{\mu,\lambda}$ and $\mathcal{H}_{\mu',\lambda'}$ are equivalent representations of G_q if and only if $(\mu,\lambda), (\mu',\lambda')$ are on the same Weyl group orbit, that is, if and only if*

$$(\mu',\lambda') = (w\mu, w\lambda)$$

for some $w \in W$.

Let $\mathcal{H} = (\mathcal{H}_{\mu,\lambda})_{\mu,\lambda}$ be the Hilbert space bundle of unitary principal series representations of G_q over $\mathbf{P} \times \mathfrak{t}^*_q$. Then we obtain a *-homomorphism $\pi : C^*_{\mathsf{f}}(G) \to C_0(\mathbf{P} \times \mathfrak{t}^*_q, \mathbb{K}(\mathcal{H}))$ by setting $\pi(x)(\mu,\lambda) = \pi_{\mu,\lambda}(x)$, and this map is used to describe the structure of the reduced C^*-algebra of G_q as follows [36].

Theorem 5.6. *Let G_q be a complex semisimple quantum group, and let $\mathcal{H} = (\mathcal{H}_{\mu,\lambda})_{\mu,\lambda}$ be the Hilbert space bundle of unitary principal series representations of G_q over $\mathbf{P} \times \mathfrak{t}^*_q$. Then one obtains an isomorphism*

$$C^*_{\mathsf{r}}(G_q) \cong C_0(\mathbf{P} \times \mathfrak{t}^*_q, \mathbb{K}(\mathcal{H}))^W$$

induced by the canonical *-homomorphism $\pi : C^*_{\mathsf{f}}(G_q) \to C_0(\mathbf{P} \times \mathfrak{t}^*_q, \mathbb{K}(\mathcal{H}))$.

In the case $G_q = \mathrm{SL}_q(2,\mathbb{C})$ the statement of Theorem 5.6 follows already from the work of Pusz and Woronowicz [32], [31] and Buffenoir and Roche [8].

5.3. **The quantization field of a complex group.** Using the structure of the reduced group C^*-algebras $C^*_{\mathsf{r}}(G)$ and $C^*_{\mathsf{r}}(G_q)$ described above, we shall now construct a continuous field of C^*-algebras relating them.

Consider the parameter space $M = \mathbf{P} \times \mathfrak{t}^*_q$ of unitary principal series representations of the quantum group G_q. If we let $Z = \mathbf{P} \times \{0\} \subset M$, then we can view the normal bundle $NZ = \mathbf{P} \times \mathfrak{t}^*$ as the parameter space of unitary principal series representations of the classical group G. Moreover, due to Peter–Weyl theory the underlying Hilbert spaces of classical and quantum unitary principal series representations associated with the same parameter $\mu \in \mathbf{P}$ can be identified. Using the same techniques as in section 4 we obtain a continuous field of C^*-algebras B over $[0,1]$ with fibers $B_0 = C_0(\mathbf{P} \times \mathfrak{t}^*, \mathbb{K}(\mathcal{H}))$ and $B_\tau = C_0(\mathbf{P} \times \mathfrak{t}^*/(i\hbar^{-1}\tau^{-1}\mathbf{Q}^\vee), \mathbb{K}(\mathcal{H}))$ for $\tau > 0$.

The Weyl group action on the bundle $\mathcal{H}$ arising from Theorems 5.2 and 5.5 is compatible with this continuous field structure; see the explicit formulas for intertwining operators obtained in [35, section 5]. Therefore we obtain an action of

W on B. Taking the W-invariant part of B yields a continuous field of C^*-algebras $R = B^W$ over $[0,1]$ with fibers $R_0 = C^*_{\mathsf{r}}(G)$ and $R_\tau = C^*_{\mathsf{r}}(G_{q^\tau})$ for $\tau > 0$. We shall refer to this continuous field as the *quantization field* of G.

For our purposes it will be convenient to reparameterize the spaces of principal series representations of G and G_{q^τ}, respectively. Let $X_*(T) = \ker(\exp : \mathfrak{t} \to T)$ denote the coweight lattice of T. Then we obtain a linear isomorphism $\gamma : \mathfrak{t} \to \mathfrak{t}^*$ determined by $(\gamma(X), \mu) = \mu(X)$ for all $\mu \in \mathbf{P}$. Similarly, let us write $\mathfrak{t}_{q^\tau} = \mathfrak{t}/h^{-1}\tau^{-1}X_*(T)$. Using this notation, we obtain a linear isomorphism $\gamma_{q^\tau} : \mathfrak{t}_{q^\tau} \to \mathfrak{t}^*_{q^\tau}$ determined by the same formula as γ.

These isomorphisms are W-equivariant. According to Theorems 5.3 and 5.6, we can therefore identify the fibers of the quantization field of G with $A^R_0 = C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H}))^W$ and $A^R_\tau = C_0(\mathbf{P} \times T, \mathbb{K}(\mathcal{H}))^W$ for $\tau > 0$, taking into account the canonical rescaling $\mathfrak{t}_{q^\tau} = \mathfrak{t}/h^{-1}\tau^{-1}X_*(T) \cong \mathfrak{t}/X_*(T) \cong T$ for all $\tau > 0$. Let us write A^R for the resulting continuous field.

The quantization field of G can be viewed as a noncommutative deformation to the normal cone construction applied to the tempered dual of G_q.

6. The quantum Baum–Connes assembly map

In this section we define a quantum analogue of the Baum–Connes assembly field and show that it induces an isomorphism in K-theory.

6.1. **The quantum Baum–Connes field.** The key ingredient in our approach is to obtain a continuous field of C^*-algebras by varying the deformation parameter in the construction of the Drinfeld double. More precisely, we shall fix $q = e^h$ and construct a continuous field $Q = C^*_{\mathsf{r}}(\mathbf{G})$ of C^*-algebras over $[0,1]$ with fibers $Q_\sigma = C^*_{\mathsf{r}}(G_{q^\sigma})$.

Let us first recall that the family of C^*-algebras $(C(K_{q^\sigma}))_{\sigma \in [0,1]}$ assembles into a continuous field $C(\mathbf{K})$ of C^*-algebras over $[0,1]$ as follows [26]. If $V(\mu)$ denotes the underlying Hilbert space of the irreducible representation of K of highest weight $\mu \in \mathbf{P}^+$, and $V(\mu)_\sigma$ the corresponding irreducible representation of K_{q^σ}, then we may fix a continuous family of unitary isomorphisms $V(\mu)_\sigma \cong V(\mu)$ which is the identity for $\sigma = 0$. Here continuity means that the unbounded multipliers $E_i^{q^\sigma}, F_i^{q^\sigma}, H_i^{q^\sigma}$ of $C^*(K_{q^\sigma})$ define continuous families of operators in $\mathbb{K}(V(\mu))$ for $\sigma \in [0,1]$ under these isomorphisms.

Then for each $v \in V(\mu)^*, w \in V(\mu)$ the matrix element $\langle v| \bullet |w\rangle$ naturally becomes a continuous section of the field $C(\mathbf{K})$, and this in turn determines the continuous field structure of $C(\mathbf{K})$.

Dually, note that the C^*-algebras $C^*(K_{q^\sigma})$ naturally identify with $C^*(K) = \bigoplus_{\mu \in \mathbf{P}^+} \mathbb{K}(V(\mu))$ under the above continuous family of isomorphisms. In particular, these algebras assemble into a constant continuous field $C^*(\mathbf{K})$ over $[0,1]$ in an obvious way.

Roughly, in order to construct the field $Q = C^*_{\mathsf{r}}(\mathbf{G})$, we need to combine the fields $C(\mathbf{K})$ and $C^*(\mathbf{K})$ as in the Drinfeld double construction.

More precisely, consider the C^*-algebra $E = \prod_{\sigma \in [0,1]} C^*_{\mathsf{r}}(G_{q^\sigma})$ consisting of all uniformly norm-bounded families of elements in $C^*_{\mathsf{r}}(G_{q^\sigma})$. Let $C^*_{\mathsf{r}}(\mathbf{G}) \subset E$ be the C^*-algebra generated by all operators of the form $(\omega^\mu_{ij} \bowtie u^\nu_{kl})f$, where $f \in C[0,1], \mu, \nu \in \mathbf{P}^+$, and where the indices run over all allowed values. Note that $C^*_{\mathsf{r}}(\mathbf{G})$ is indeed contained in E since the norms of ω^μ_{ij} and u^ν_{kl} are uniformly bounded by 1; in the case of ω^μ_{ij} this holds because the elements ω^μ_{ij} for fixed μ form a full matrix algebra, and for u^ν_{kl} it suffices to note that the matrix $u^\nu = (u^\nu_{kl})$ over $C(K_{q^\sigma})$ is unitary for all σ.

By construction, the C^*-algebra $C^*_\mathsf{r}(\mathbf{G})$ is a $C[0,1]$-algebra. In order to describe this algebra further, it is convenient to reparameterize the space of principal series representations of G_{q^σ} as described at the end of section 5. Then according to Theorems 4.4 and 5.6 we obtain a natural inclusion

$$C^*_\mathsf{r}(\mathbf{G}) \subset \prod_{\sigma\in[0,1]} C_0(\mathbf{P}\times T, \mathbb{K}(\mathcal{H}))^W,$$

again taking into account the canonical rescaling $\mathfrak{t}_{q^\sigma} \cong \mathfrak{t}/X_*(T) \cong T$ for all $\sigma > 0$.

Proposition 6.1. *The image of $C^*_\mathsf{r}(\mathbf{G})$ under the above inclusion map is contained in $C_0([0,1]\times\mathbf{P}\times T, \mathbb{K}(\mathcal{H}))^W$.*

Proof. We need to show that the sections $\omega^\alpha_{ij} \bowtie u^\beta_{kl}$ in $C^*_\mathsf{r}(G_{q^\sigma}) \subset C_0(\mathbf{P}\times T, \mathbb{K}(\mathcal{H}))^W$ depend continuously on σ. Since the images of these elements in the natural representations on $\mathcal{H}_{\mu,\lambda}$ are finite rank operators, it is enough to show that they are strongly continuous in σ. For the operators ω^α_{ij} this is obvious because their action is constant across the interval. For the operators u^β_{kl} strong continuity follows by inspecting the explicit formulas for the natural representations and the fact that multiplication of matrix elements in $\mathfrak{C}^\infty(K_{q^\sigma})$ depends continuously on $\sigma\in[0,1]$. □

Theorem 6.2. *The algebra $C^*_\mathsf{r}(\mathbf{G})$ is a continuous field of C^*-algebras over $[0,1]$ with fibers $C^*_\mathsf{r}(\mathbf{G})_\sigma = C^*_\mathsf{r}(G_{q^\sigma})$, trivial away from 0.*

Proof. It follows immediately from Proposition 6.1 that $C^*_\mathsf{r}(\mathbf{G})$ is a continuous field of C^*-algebras over $[0,1]$, and that the restriction of this field to the open interval $(0,1]$ agrees with the trivial field $C_0((0,1]\times\mathbf{P}\times T, \mathbb{K}(\mathcal{H}))^W$. This shows in particular that the fiber $C^*_\mathsf{r}(\mathbf{G})_\sigma$ of $C^*_\mathsf{r}(\mathbf{G})$ for $\sigma>0$ is equal to $C^*_\mathsf{r}(G_{q^\sigma})$.

To compute the fiber at $\sigma = 0$, remark that $C^*_\mathsf{r}(G_1) \cong K \ltimes C(K)$ is a full crossed product. Therefore the natural quotient homomorphism $C^*_\mathsf{r}(\mathbf{G})_0 \to C^*_\mathsf{r}(G_{q^0}) = C^*_\mathsf{r}(G_1)$ splits. Moreover, the image of the splitting homomorphism $C^*_\mathsf{r}(G_1) \to C^*_\mathsf{r}(\mathbf{G})_0$ is dense. In other words, the fiber of $C^*_\mathsf{r}(\mathbf{G})$ at $\sigma = 0$ identifies canonically with $C^*_\mathsf{r}(G_1)$. □

The continuous field obtained in Theorem 6.2 will be called the *quantum assembly field* for the group G.

Let us also define a continuous field of C^*-algebras A^Q over $[0,1]$ by setting

$$A^Q = \{F\in C_0([0,1]\times\mathbf{P}\times T, \mathbb{K}(\mathcal{H}))^W \mid F_0(t)\in\mathbb{K}(L^2(K))^{K_t} \text{ for all } t\in T\}.$$

Here $F_0 \in C_0(\mathbf{P}\times T, \mathbb{K}(\mathcal{H}))^W$ is the evaluation of F at 0 in the first variable. This field is clearly trivial away from 0 and has fibers $A^Q_\sigma \cong C^*_\mathsf{r}(G_{q^\sigma})$ due to Theorems 4.4 and 5.6. From our above considerations we immediately obtain the following result.

Proposition 6.3. *The quantum assembly field is isomorphic to the continuous field A^Q defined above.*

Proof. Theorem 6.2 shows that $C^*_\mathsf{r}(\mathbf{G})$ embeds in A^Q, and it is straightforward to check directly that this embedding is surjective. □

6.2. **The quantum assembly map.** In this subsection we show that the quantum assembly field induces an isomorphism

$$\mu_q : K_*(K \ltimes C(K)) \to K_*(C^*_{\mathsf{r}}(G_q)),$$

following the approach in [16] for the classical Baum–Connes assembly map.

Recall the following lemma from [16]. For the convenience of the reader we include a proof.

Lemma 6.4. *Let B be a C^*-algebra, and let $p \in M(B)$ be a projection. If p acts as a rank one projection in all irreducible representations of B, then pBp is a commutative C^*-algebra which is Morita equivalent to B.*

Proof. Consider the representation $\pi : B \to \bigoplus_{\omega \in PS(B)} \mathbb{L}(\mathcal{H}_\omega)$, where ω ranges over the set $PS(B)$ of pure states of B and $\mathcal{H}_\omega$ is the GNS-representation associated with ω. Then π is injective, and by assumption the image of p in $\mathbb{L}(\mathcal{H}_\omega)$ is a rank one projection for all $\omega \in PS(B)$. It follows that $pBp \cong \pi(pBp) = \pi(p)\pi(B)\pi(p)$ is a commutative C^*-algebra.

We claim that $B = BpB$. Indeed, if $BpB \subset B$ is a proper ideal, then there exists an irreducible representation $\rho : B \to \mathbb{L}(\mathcal{H})$ which vanishes on BpB. This implies that $\rho(p) = 0$, which contradicts our assumption that $\rho(p)$ must be a rank one projection.

It follows that the Hilbert B-module $\mathcal{E} = pB$ with the structure inherited from B viewed as a Hilbert B-module over itself implements a Morita equivalence between B and pBp; see, for instance, [33, example 3.6]. □

In particular, under the assumptions of Lemma 6.4 the spectrum of B agrees with the spectrum of the commutative C^*-algebra pBp.

Due to Theorem 6.2 the quantum assembly field $C^*_{\mathsf{r}}(\mathbf{G})$ is trivial away from zero. Hence it induces a map in K-theory

$$K_*(K \ltimes C(K)) = K_*(C^*_{\mathsf{r}}(\mathbf{G})_0) \to K_*(C^*_{\mathsf{r}}(\mathbf{G})_1) = K_*(C^*_{\mathsf{r}}(G_q)),$$

as in the classical case. We will call this map the *quantum assembly map* for G.

Theorem 6.5. *Let G_q be a semisimple complex quantum group. The quantum assembly map*

$$\mu_q : K_*(K \ltimes C(K)) \to K_*(C^*_{\mathsf{r}}(G_q))$$

is an isomorphism.

Proof. We follow closely the argument given by Higson in [16] for the classical assembly map.

Fix a labeling $\mathbf{P}^+ = \{\mu_1, \mu_2, \mu_3, \dots\}$ of the set of dominant integral weights such that if $\mu_i \leq \mu_j$ with respect to the natural order on weights, then $i \leq j$.

For each $n \in \mathbb{N}$ we consider the projections $p_n \in C^*_{\mathsf{r}}(\mathbf{G}) = Q$ onto the highest weight vector of the minimal K-type μ_n and set

$$J_n = Qp_nQ.$$

In this way we obtain a filtration $(\mathcal{F}^n(Q))_{n \in \mathbb{N}}$ of Q by ideals where

$$\mathcal{F}^n(Q) = \sum_{1 \leq m \leq n} J_n.$$

Let $Q_1 = \mathcal{F}^1(Q)$ and

$$Q_n = \mathcal{F}^n(Q)/\mathcal{F}^{n-1}(Q)$$

for $n > 1$ be the corresponding subquotients. The projections $p_n \in Q_n$ satisfy the assumptions of Lemma 6.4, and according to Theorems 5.6 and 4.4, combined with Proposition 6.1, we get

$$p_n Q_n p_n \cong C([0,1], C(T)^{W_{\mu_n}}),$$

where $W_{\mu_n} \subset W$ is the subgroup fixing $\mu_n \in \mathbf{P}^+$.

In the same way we obtain a filtration of $D = C^*_{\mathsf{r}}(G_q)$ by considering the ideals

$$\mathcal{F}^n(D) = \sum_{1 \leq m \leq n} I_n,$$

where $I_n = D p_n D$, and we will write D_n for the resulting subquotients. From the description of $p_n Q_n p_n$ above and $p_n D_n p_n \cong C(T)^{W_{\mu_n}}$ it follows that evaluation at 1 yields an isomorphism $K_*(p_n Q_n p_n) \to K_*(p_n D_n p_n)$ in K-theory.

According to Lemma 6.4, in the commutative diagram

$$\begin{tikzcd} K_*(Q_n) \arrow[r] & K_*(D_n) \\ K_*(p_n Q_n p_n) \arrow[u] \arrow[r] & K_*(p_n D_n p_n) \arrow[u] \end{tikzcd}$$

the vertical maps, induced by the canonical inclusions, are isomorphisms. Hence evaluation at 1 yields isomorphisms $K_*(Q_n) \to K_*(D_n)$ for all $n \in \mathbb{N}$. As a consequence, an iterated application of the six-term exact sequence in K-theory shows that evaluation at 1 induces isomorphisms $K_*(\mathcal{F}^n(Q)) \to K_*(\mathcal{F}^n(D))$ for all $n \in \mathbb{N}$.

Let us write $\mathcal{F}^\infty(Q)$ and $\mathcal{F}^\infty(D)$ for the union of the subalgebras $\mathcal{F}^n(Q)$ and $\mathcal{F}^n(D)$, respectively. Then $\mathcal{F}^\infty(Q) \subset Q$ and $\mathcal{F}^\infty(D) \subset D$ are dense, and continuity of K-theory implies that evaluation at $\sigma = 1$ induces an isomorphism

$$K_*(C^*_{\mathsf{r}}(\mathbf{G})) = \varinjlim_{n \in \mathbb{N}} K_*(\mathcal{F}^n(Q)) \cong \varinjlim_{n \in \mathbb{N}} K_*(\mathcal{F}^n(D)) = K^*(C^*_{\mathsf{r}}(G_q)).$$

The map $K_*(Q) = K_*(C^*_{\mathsf{r}}(\mathbf{G})) \to K^*(C^*_{\mathsf{r}}(G_1)) = K_*(Q_0)$ induced by evaluation at $\sigma = 0$ is an isomorphism by the triviality of $C^*_{\mathsf{r}}(\mathbf{G})$ away from 0. Hence the commutative diagram

$$\begin{tikzcd} & K_*(C^*_{\mathsf{r}}(\mathbf{G})) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ K_*(C^*_{\mathsf{r}}(G_1)) \arrow[rr, "\mu_q"'] & & K_*(C^*_{\mathsf{r}}(G_q)) \end{tikzcd}$$

yields the claim. □

7. The deformation square

We shall now assemble the constructions in previous sections and construct the *deformation square* of a complex semisimple Lie group. This continuous field of C^*-algebras encapsulates all the deformations that we have studied so far.

7.1. **Deformation squares and their K-theory.** In this subsection we introduce the abstract concept of deformation squares and discuss their basic K-theoretic properties.

Definition 7.1. Let $X = [0,1] \times [0,1]$ be the unit square, and let A be a continuous field of C^*-algebras over X. We shall say that A is a deformation square if the following conditions hold.

(a) The restriction of A to $(0,1] \times (0,1]$ is a trivial field.
(b) The restriction of A to $\{0\} \times [0,1]$ is trivial away from $(0,0)$.
(c) The restriction of A to $[0,1] \times \{0\}$ is trivial away from $(0,0)$.

If A is a deformation square, then the restriction of A to any boundary edge of $[0,1] \times [0,1]$ yields a continuous field of C^*-algebras which is trivial away from a single point. The following result confirms that the corresponding induced maps in K-theory behave as expected.

Proposition 7.2. *Let A be a deformation square. Then one obtains a commutative diagram*

$$\begin{tikzcd} K_*(A_{0,1}) \arrow[r] & K_*(A_{1,1}) \\ K_*(A_{0,0}) \arrow[r] \arrow[u] & K_*(A_{1,0}) \arrow[u] \end{tikzcd}$$

induced from the restriction of A to the boundary edges of $X = [0,1] \times [0,1]$.

Proof. Using split exactness of K-theory and [5, Proposition 3.2], it suffices to consider the case in which A is a unital deformation square.

Let $p_{0,0} \in M_n(A_{0,0})$ be a projection. In the same way as in our discussion in section 2, we can lift this to a positive element q in $M_n(A)$. Since $p_{0,0}^2 = p_{0,0}$, we know that $q(\sigma,\tau)^2 - q(\sigma,\tau)$ has small norm for (σ,τ) near $(0,0)$. Hence $\frac{1}{2}$ is not in the spectrum of $q(\sigma,\tau)$, and using functional calculus, we obtain an element $p \in A$ lifting $p_{0,0}$ such that $p(\sigma,\tau)$ is a projection provided that $0 \leq \sigma,\tau \leq \epsilon$ for some $\epsilon > 0$. Using triviality of the field along the boundary and the interior, we can actually assume that $p(\sigma,\tau)$ is a projection for all $(\sigma,\tau) \in X$. By the very construction of the induced maps in K-theory, both compositions of the above diagram map $[p_{0,0}]$ to $[p(1,1)] \in K_0(A_{1,1})$. We conclude that the diagram for K_0 is commutative.

In a similar way, one proceeds for K_1. If $u_{0,0} \in M_n(A_{0,0})$ is unitary and $u \in M_n(A)$ a lift of $u_{0,0}$, then since $u(\sigma,\tau)u(\sigma,\tau)^* - 1$ and $u(\sigma,\tau)^*u(\sigma,\tau) - 1$ have small norm near 0, the elements $u(\sigma,\tau)$ will be invertible provided that $0 \leq \sigma,\tau \leq \epsilon$ for some $\epsilon > 0$. Using triviality of the field along the boundary and the interior, we can in fact assume that $u(\sigma,\tau)$ is invertible for all $(\sigma,\tau) \in X$. Again, by construction of the induced maps in K-theory, both compositions in the above diagram map $[u_{0,0}] \in K_1(A_{0,0})$ to $[u(1,1)] \in K_1(A_{1,1})$. $\square$

7.2. **The classical assembly field revisited.** In this subsection we shall give an alternative description of the Baum–Connes assembly field for the classical group G. More precisely, we shall prove that this field agrees with the continuous field

$$A^C = \{F \in C_0([0,1] \times \mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H}))^W \mid F_0 \in A_0^L\}$$

over $[0,1]$, where A_0^L is the C^*-algebra in Theorem 4.3 and $F_0 \in C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H}))^W$ is the evaluation of F at $0 \in [0,1]$. The field A^C is clearly trivial away from 0 and has fibers $A_0^C \cong K \ltimes C_0(\mathfrak{k})$ and $A_\sigma^C \cong C^*_\mathsf{r}(G)$ for $\sigma > 0$ according to Theorems 4.3 and 5.3, respectively.

Recall that the group G admits the Iwasawa decomposition $G = KAN$. That is, any element $g \in G$ can be uniquely decomposed as $g = kb$ for $k \in K$ and $b \in AN$. In particular, for any $b \in AN$ and $k \in K$ there exist unique elements $b \rightharpoonup k \in K$ and $b \leftharpoonup k \in AN$ such that

$$bk = (b \rightharpoonup k)(b \leftharpoonup k).$$

This leads to a left action of AN on K and a right action of K on AN, respectively, which are called the left and right dressing actions. For more information see [23], [10, section 1.5].

Theorem 7.3. *The continuous field A^C defined above is isomorphic to the Baum–Connes assembly field for the group G.*

Proof. We shall verify that a set of generating sections of the assembly field defines elements of the C^*-algebra A^C. For this we need to show that these sections are norm continuous once we identify all fibers with the corresponding subalgebras of $C_0(\mathbf{P} \times \mathfrak{t}, \mathbb{K}(\mathcal{H}))^W$.

Let us first recall that the strong-* topology on bounded subsets of $\mathbb{L}(\mathcal{H})$ coincides with the strict topology. In particular, for a strong-* convergent net $(T_i)_{i\in I}$ of uniformly bounded operators and a compact operator S, the net $(ST_i)_{i\in I}$ is norm convergent.

Now let $(f_\sigma)_{\sigma\in[0,1]}$ be a generating continuous section of the assembly field obtained from a function $f \in C^\infty_c(K \times \mathfrak{an}) \subset C^*(K \ltimes \mathfrak{k}^*)$ which is invariant under convolution with some finite rank projection in $\mathfrak{D}(K)$ from the left and right. According to our above remarks, it suffices to show that the operators $\pi_{\mu,X}(f_\sigma) \in \mathbb{K}(L^2(\mathcal{E}_\mu))$ depend strong-* continuously on $\sigma \in [0,1]$ and $X \in \mathfrak{t}$ for all $\mu \in \mathbf{P}$.

Let us compute $\pi_{\mu,X}(f_\sigma)(\xi)$ for a generating section $(f_\sigma)_{\sigma\in[0,1]}$ associated with $f \in C^\infty_c(K \times \mathfrak{an})$ as above and $\xi \in L^2(\mathcal{E}_\mu)$. That is, we consider

$$f_0(k, a + n) = f(k, a + n)$$

and

$$f_\sigma(k, \exp(a)\exp(n)) = f(k, \sigma^{-1}a + \sigma^{-1}n), \qquad \sigma > 0.$$

Let $\mathcal{S}(\mathfrak{k})$ be the Schwartz space of $\mathfrak{k}$, and let $\mathcal{F} : C^\infty_c(\mathfrak{an}) \to \mathcal{S}(\mathfrak{k})$ be the Fourier transform given by

$$\mathcal{F}(h)(Y) = \int_{\mathfrak{an}} e^{-i\Im(Y,a+n)} h(a + n) da dn.$$

Here $\Im(\cdot,\cdot)$ is the imaginary part of the invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$. We note that this form induces a linear isomorphism between $\mathfrak{an}$ and the real dual space $\mathfrak{k}^* = \mathrm{Hom}(\mathfrak{k}, \mathbb{R})$.

Let us also write $\mathcal{F}_{\mathfrak{an}}(f)$ for the Fourier transform of f in the $\mathfrak{an}$-direction and denote by $s \cdot X$ the adjoint action of $s \in K$ on $X \in \mathfrak{t} \subset \mathfrak{k}$. Then we have

$$\begin{aligned}\pi_{\mu,X}(f_0)(\xi)(r) &= \int_K \mathcal{F}_{\mathfrak{an}}(f)(k, k^{-1}r \cdot X)\xi(k^{-1}r) dk \\ &= \int_{K\times\mathfrak{an}} f(k, a + n)e^{-(k^{-1}r\cdot X,a)}\xi(k^{-1}r) dk da dn.\end{aligned}$$

The left Haar measure of $G = KAN$ can be written as $dx = e^{2(\rho,a)} dkdadn$, where dk is the Haar measure of K and da, dn is the Lebesgue measure on $\mathfrak{a} \cong A$ and $\mathfrak{n} \cong N$, respectively; see [20, Proposition 8.43]. Under the identification of the fiber algebra of the assembly field at $\sigma > 0$ with $C^*_\mathsf{r}(G)$, we have to rescale the $\mathfrak{an}$-part of this measure with σ^{-1}. Using our notation $b \rightharpoonup k$ and $b \leftharpoonup k$ for the dressing actions of $b \in AN$ on $k \in K$, and vice versa, we therefore obtain

$$
\begin{aligned}
&\pi_{\mu,X}(f_\sigma)(\xi)(r) \\
&\quad = \int_{K\times\mathfrak{an}} f_\sigma(k, \exp(a)\exp(n))\xi(\exp(-n)\exp(-a)k^{-1}r)e^{(2\rho,a)} dkd(\sigma^{-1}a)d(\sigma^{-1}n) \\
&\quad = \int_{K\times\mathfrak{an}} f(k, \sigma^{-1}a + \sigma^{-1}n)\xi(\exp(-n)\exp(-a)k^{-1}r)e^{(2\rho,a)} dkd(\sigma^{-1}a)d(\sigma^{-1}n) \\
&\quad = \int_{K\times\mathfrak{an}} f(k, \sigma^{-1}a + \sigma^{-1}n)\xi(\exp(-n)\exp(-a) \rightharpoonup (k^{-1}r)) \\
&\qquad\quad \times \chi_{\sigma^{-1}X+2\rho}(\exp(-n)\exp(-a) \leftharpoonup k^{-1}r)e^{(2\rho,a)} dkd(\sigma^{-1}a)d(\sigma^{-1}n) \\
&\quad = \int_{K\times\mathfrak{an}} f(k, a+n)\xi(\exp(-\sigma n)\exp(-\sigma a) \rightharpoonup (k^{-1}r)) \\
&\qquad\quad \times \chi_{\sigma^{-1}X+2\rho}(\exp(-\sigma a)\exp(-\sigma n) \leftharpoonup k^{-1}r)e^{(2\rho,\sigma a)} dkdadn
\end{aligned}
$$

for $\sigma > 0$. Here $\chi_{\sigma^{-1}X+2\rho}$ denotes the character of AN entering the definition of principal series representations as discussed in section 5.

Let us compare the integrand of the last integral with the integrand in our formula for $\pi_{\mu,X}(f_0)(\xi)$ above, splitting up the contribution coming from the character $\chi_{\sigma^{-1}X+2\rho}$ as

$$
\chi_{\sigma^{-1}X}(\exp(-\sigma a)\exp(-\sigma n) \leftharpoonup k^{-1}r)\chi_{2\rho}(\exp(-\sigma a)\exp(-\sigma n) \leftharpoonup k^{-1}r).
$$

Assume that f has support in $K \times D$, where $D \subset \mathfrak{an}$ is compact. The function

$$
D \ni a + n \mapsto \chi_{2\rho}(\exp(-\sigma a)\exp(-\sigma n) \leftharpoonup k^{-1}r)e^{(2\rho,\sigma a)}
$$

depends continuously on $r, k \in K$ and converges uniformly to 1 as $\sigma \to 0$. Moreover, if $C \subset \mathfrak{t}$ is compact, we have

$$
\begin{aligned}
&\lim_{\sigma\to 0} \chi_{\sigma^{-1}X}(\exp(-\sigma a)\exp(-\sigma n) \leftharpoonup k^{-1}r) \\
&\qquad = \chi_X\left(\frac{d}{d\sigma}(\exp(-\sigma a)\exp(-\sigma n) \leftharpoonup k^{-1}r)_{\sigma=0}\right) \\
&\qquad = e^{-(X, a\cdot(k^{-1}r))} = e^{-(k^{-1}r\cdot X, a)},
\end{aligned}
$$

uniformly in $X \in C, r, k \in K$, and $a + n \in D$. Here we use the fact that the linearization of the right dressing action of K at the identity of AN identifies with the right coadjoint action.

Consider now $\xi \in \Gamma(\mathcal{E}_\mu) \subset L^2(\mathcal{E}_\mu) \subset L^2(K)$. Then ξ is a continuous function on K, and since the dressing action is continuous, it follows that for $\sigma \to 0$ the function $\xi(\exp(-\sigma n)\exp(-\sigma a) \rightharpoonup (k^{-1}r))$ converges uniformly to $\xi(k^{-1}r)$ as a function of $r, k \in K$ and $a + n \in D$.

In summary, we see that $\pi_{\mu,X}(f_\sigma)(\xi) \to \pi_{\mu,X}(f_0)(\xi)$ in L^2-norm for all $\xi \in \Gamma(\mathcal{E}_\mu)$, uniformly for $X \in C$. Since $\Gamma(\mathcal{E}_\mu) \subset L^2(\mathcal{E}_\mu)$ is dense, we conclude that $[0,1] \times \mathfrak{t} \ni (\sigma, X) \mapsto \pi_{\mu,X}(f_\sigma) \in \mathbb{K}(L^2(\mathcal{E}_\mu))$ is strongly continuous.

In the same way one checks that $[0,1] \times \mathfrak{t} \ni (\sigma, X) \mapsto \pi_{\mu,X}(f_\sigma)^* \in \mathbb{K}(L^2(\mathcal{E}_\mu))$ is strongly continuous; in fact, this follows from our above considerations because $\pi_{\mu,X}(f_\sigma)^* = \pi_{\mu,X}((f^*)_\sigma)$ and f^* satisfies the same conditions as f.

Since the operators $\pi_{\mu,X}(f_\sigma)$ are uniformly bounded, we conclude that $[0,1] \times \mathfrak{t} \ni (\sigma, X) \mapsto \pi_{\mu,X}(f_\sigma) \in \mathbb{K}(L^2(\mathcal{E}_\mu))$ is norm continuous. We thus obtain a canonical $C[0,1]$-linear *-homomorphism from the assembly field into A^C, and it is easy to check directly that this map is an isomorphism. □

7.3. **The deformation square of a complex semisimple Lie group.** Let G be a complex semisimple Lie group. In this subsection we shall construct a deformation square over $[0,1] \times [0,1]$ such that

$$\begin{aligned}
&A_{0,0} \cong K \ltimes C_0(\mathfrak{k}), && \\
&A_{\sigma,0} \cong C^*_\mathsf{r}(G), && \sigma > 0, \\
&A_{0,\tau} \cong K \ltimes C(K), && \tau > 0, \\
&A_{\sigma,\tau} \cong C^*_\mathsf{r}(G_{q^{\sigma\tau}}) && \text{otherwise.}
\end{aligned}$$

More precisely, let $R \cong A^R$ be the quantization field for G, and let $L \cong A^L \subset A^R \cong R$ be the quantization field of the Cartan motion group. Setting

$$A = \{F \in C([0,1], A^R) \mid F_0 \in A^L\}$$

yields a continuous field of C^*-algebras as desired. Here $F_0 \in A^R$ denotes the evaluation of F at 0.

Theorem 7.4. *The above construction yields a deformation square for any complex semisimple Lie group G.*

Proof. Due to Theorem 7.3 and Proposition 6.3, we see that the sections $a = (a_{\sigma,\tau})$ of this field satisfy the following:

(a) For each fixed $\tau \in [0,1]$ the section $\sigma \mapsto a_{\sigma,\tau}$ is a continuous section of the (quantum) assembly field.
(b) For each fixed $\sigma \in [0,1]$ the section $\tau \mapsto a_{\sigma,\tau}$ is a continuous section of the quantization field.
(c) On $(0,1] \times (0,1]$ the section a can be identified with an element of the trivial continuous field with fiber $C^*_\mathsf{r}(G_q)$.

This yields the claim. □

It is natural to expect that the deformation square in Theorem 7.4 is the total algebra of a locally compact quantum groupoid obtained from a continuous bundle of locally compact quantum groups, in the same way as the deformation picture of the classical Baum–Connes assembly map is associated with a bundle of locally compact groups. We shall not attempt to make this precise; for recent work relevant to this problem, we refer the reader to [12].

7.4. **Classical and quantum Baum–Connes.** We are now ready to assemble and summarize the results obtained above.

The deformation picture of the classical Baum–Connes assembly map provides an isomorphism between the K-theory of the Cartan motion group $K \ltimes \mathfrak{k}^*$ and the K-theory of the reduced group C^*-algebra $C^*_\mathsf{r}(G)$. According to Theorem 6.5, there is an analogous isomorphism between the K-theory of the quantum Cartan motion group and the K-theory of the reduced group C^*-algebra $C^*_\mathsf{r}(G_q)$.

The K-groups of the quantum Cartan motion group have been computed by Brylinski and Zhang [7], and for the classical Cartan motion group the K-groups are well known [3]; see also [1] for a quite general treatment. Let us summarize the results as follows.

Theorem 7.5. *Let K be a simply connected compact Lie group of rank N. Then we have*

$$K_*(K \ltimes C_0(\mathfrak{k})) = \begin{cases} R(K) & * = \dim(K), \\ 0 & \textit{otherwise}, \end{cases}$$

where K acts on $\mathfrak{k}$ via the adjoint action. Similarly,

$$K_*(K \ltimes C(K)) = \Lambda^* R(K)^N$$

for the crossed product with respect to the adjoint action of K on itself.

Here $R(K)$ is the representation ring of K and $\Lambda^* R(K)^N$ denotes the exterior algebra of the free $R(K)$-module $R(K)^N$ over $R(K)$. In particular, the rank of $K_*(K \ltimes C(K))$ as an $R(K)$-module is 2^N.

Using the deformation square of G obtained in Theorem 7.4, we see that the classical and quantum assembly maps are related as follows.

Theorem 7.6. *Let G be a simply connected complex semisimple Lie group with maximal compact subgroup K, and let $q \in (0,1)$. Then we obtain a commutative diagram*

$$\begin{tikzcd} K_*(K \ltimes C(K)) \arrow[r, "\mu_q"] & K_*(C^*_\mathsf{r}(G_q)) \\ K_*(K \ltimes C_0(\mathfrak{k})) \arrow[u] \arrow[r, "\mu"] & K_*(C^*_\mathsf{r}(G)) \arrow[u] \end{tikzcd}$$

in K-theory. Both horizontal maps are isomorphisms, and both vertical maps are split injective.

Proof. It follows from Theorem 7.4 and Proposition 7.2 that we obtain a commutative diagram as stated. The lower horizontal map is an isomorphism because the Baum–Connes conjecture holds for G, and the upper horizontal map is an isomorphism by Theorem 6.5.

It remains to verify the claim regarding the vertical maps. For this it suffices to consider the map $K_*(K \ltimes C_0(\mathfrak{k})) \cong K^K_*(C_0(\mathfrak{k})) \to K^K_*(C(K)) \cong K_*(K \ltimes C(K))$ on the left-hand side. The generator of the free $R(K)$-module $K^K_*(C_0(\mathfrak{k})) \cong K^K_{*+n}(C_\tau(\mathfrak{k}))$ is given by the Bott element, here $C_\tau(\mathfrak{k})$ denotes the Clifford algebra bundle over $\mathfrak{k}$ and $n = \dim(\mathfrak{k})$. The map $K^K_*(C_0(\mathfrak{k})) \to K^K_*(C(K))$ under consideration can be identified with ι_*, where $\iota : C_0(\mathfrak{k}) \cong C_0(U) \to C(K)$ is the *-homomorphism corresponding to the inclusion $U \subset K$ of an open neighborhood U of $e \in K$ which is K-equivariantly diffeomorphic to $\mathfrak{k}$.

Let $D_K \in KK^K_0(C_\tau(K), \mathbb{C})$ be the Dirac element for K; see [18], [4]. By slight abuse of notation we shall write again ι for the *-homomorphism $C_\tau(\mathfrak{k}) \to C_\tau(K)$ corresponding to the inclusion $\mathfrak{k} \cong U \subset K$. Then $\iota^*(D_K) \in KK^K_0(C_\tau(\mathfrak{k}), \mathbb{C})$ identifies with the Dirac element $D_\mathfrak{k} \in KK^K_0(C_\tau(\mathfrak{k}), \mathbb{C})$ for $\mathfrak{k}$. It follows that $R(K) \cong K^K_*(C_0(\mathfrak{k})) \to K^K_*(C(K))$ is split injective, with splitting implemented by D_K. $\square$

Let us conclude by taking a look at the special case $G = \mathrm{SL}(2,\mathbb{C})$. In this case we have $K = \mathrm{SU}(2)$, and Theorem 7.5 reduces to

$$K_*(K \ltimes C_0(\mathfrak{k})) = \begin{cases} 0 & * \text{ even}, \\ R(K) & * \text{ odd}, \end{cases}$$

$$K_*(K \ltimes C(K)) = \begin{cases} R(K) & * \text{ even}, \\ R(K) & * \text{ odd}. \end{cases}$$

Moreover, the natural map between these groups obtained from Theorem 7.6 corresponds to the identity in degree 1. Using the explicit descriptions given in Theorems 5.3 and 5.6, respectively, one can of course also calculate the K-groups of $C^*_\mathsf{r}(G)$ and $C^*_\mathsf{r}(G_q)$ directly.

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