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# Lorentz-transformation and Galileo-transformation windows

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## ABSTRACT

We define Lorentz-transformation windows as windows that change the direction of transmitted light rays like a Lorentz transformation. Similarly, Galileo-transformation windows change the direction of transmitted light rays like a Galileo transformation. This light-ray-direction change distorts the scene seen through such a window in the same way in which the scene would be distorted in a photo taken with a camera moving through the scene. Lorentz-transformation windows can also undo the distortion of the scene when moving at relativistic velocity relative to it. For small angles between the direction of the light rays and the direction of the velocity, Galileo-transformation windows can be realised with relatively simple telescope windows, which consist of arrays of identical micro-telescopes.

**Keywords:** micro-optics, relativistic aberration, generalised refraction

## 1. INTRODUCTION

Even before Einstein's 1905 paper on Special Relativity,[1] physicists discussed the deformation of solid bodies moving at relativistic speeds. A length contraction, nowadays called *Lorentz-FitzGerald contraction* or *Lorentz contraction*, was suggested by FitzGerald[2] as an explanation for the Michelson-Morley experiment.[3] Einstein, in his 1905 paper,[1] discusses the deformation, due to Lorentz contraction, of a sphere into an ellipsoid.

Penrose[4] and Terrell[5] were the first to realise that the apparent distortion of objects moving relative to a camera/observer is more complicated. We discuss here the case of a photo taken with a pinhole camera. The distortion then depends on the precise time when the light rays that contribute to different parts of a photo left the object. This, in turn, depends on the *camera model*, which specifies the position of the shutter surface, through which the light rays that subsequently reach the film/detector chip are assumed to travel simultaneously. Depending on where the shutter is located, different light rays leave the surface of moving objects at different times. Once this subtlety is taken into account, objects no longer appear contracted, but instead rotated. This effect is called *Penrose-Terrell rotation*.

Researchers started to visualise relativistic distortion effects, also known as *relativistic aberration*, using raytracing only much later.[7] There are now several ray tracers capable of such simulations, the most notable perhaps being the free *Real Time Relativity*,[8] which not only simulates static views, but real-time, interactive, simulation of relativistic movement through different scenes (several of which scattered with clocks to allow investigation of time-related phenomena). We also adapted our own raytracer, TIM,[9] to be able to simulate relativistic camera movement with different camera models, resulting in TIM's graduation.[6] Dr TIM, as TIM is now known, including source code, is freely available. Fig. 1 shows an example of the relativistic distortion of a scene, calculated with Dr TIM.

One of the themes in our research is ray optics not constrained by the laws of its underlying theory, wave optics. At first, this seems a bad idea, as ray optics, of course, *is* constrained by wave optics. However, it is possible to make compromises such that, at the expense of limited field of view and diffraction effects, it is possible to create light-ray fields that *look* like no corresponding wave-optical phase fronts can exist.[10] Such light-ray fields can be created using micro-structured windows that re-direct, individually, tiny pieces of the beams, and which introduce discontinuities between the pieces.[11] We have used this idea to design and investigate windows that change the pieces of the beam by passing them through tiny Dove prisms,[12,13] which leads to mirroring of the ray direction, and we have used lenticular arrays (as used in 3D postcards) to build windows with this same functionality.[14] We have also designed and investigated[15–17] and built[18] arrays of micro-telescopes, which we

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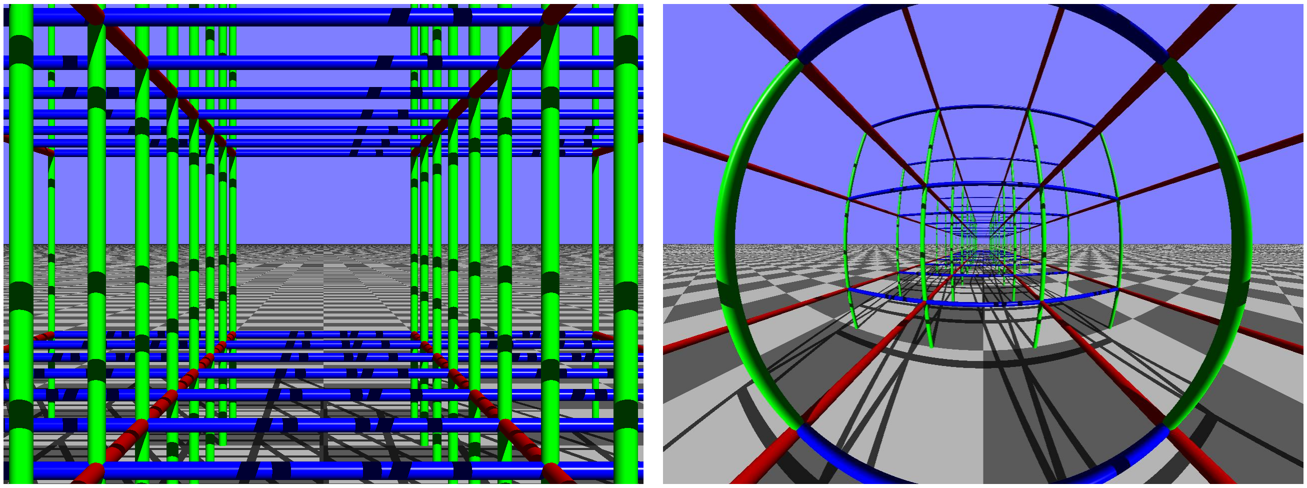


Figure 1. Example of relativistic distortion. The left image shows a simulated photo of a lattice taken with a camera at rest. In the right image, the virtual camera is moving with speed $0.99c$ into the image. The simulated field of view is $20°$ in the horizontal direction. The images were calculated with our raytracer Dr TIM.[6]

call *telescope windows*, which change the direction of transmitted light rays according to a law of refraction that enables stigmatic imaging by planar, homogeneous, interfaces.[19] Because of the particular light-ray-direction changes in these components, they can turn wave-optically allowed light-ray fields into apparently forbidden ones.[20]

Here we investigate whether or not planar telescope windows can simulate relativistic aberration, i.e. distort the view through them such that it looks, to a good approximation, like a photo taken with a camera moving at relativistic speed.

## 2. LIGHT-RAY DIRECTION IN DIFFERENT INERTIAL FRAMES

Special relativity relates events, described by spatial and time coordinates, in different inertial frames. In this paper, the scene is at rest in one of these frames, the observer in the other. We have placed our coordinate systems such that, at the relevant moment, the origins of both frames coincide. We describe events in one frame, the "unprimed frame", in terms of unprimed coordinates, those in the other, "primed", frame by primed coordinates. In the primed frame, the unprimed frame is moving with velocity

$$\boldsymbol{v} = \boldsymbol{\beta} c. \tag{1}$$

We now consider two events, namely the same light ray passing through two different points. As light rays travel in straight lines in inertial frames, the difference in the spatial parts of these events is a vector parallel to the direction of the light ray. By transforming the events between the frames, the light-ray direction can be calculated in different frames.

We first describe the events in the unprimed frame. As our events, we choose the moments the light ray passes through position $\boldsymbol{x}_1$ and

$$\boldsymbol{x}_2 = \boldsymbol{x}_1 + \alpha \hat{\boldsymbol{d}}, \tag{2}$$

where $\hat{\boldsymbol{d}}$ is the normalised light-ray direction and $\alpha$ is an arbitrary real number. We call the times at which these events happen $t_1$ and

$$t_2 = t_1 + \alpha / c, \tag{3}$$

where $c$ is the speed of light.

The coordinates of these events in the primed frame are related to those in the unprimed frame through the Lorentz transformation[7]

$$\boldsymbol{x}_i' = \boldsymbol{x}_i + (\gamma - 1)\frac{(\boldsymbol{\beta} \cdot \boldsymbol{x}_i)\boldsymbol{\beta}}{\beta^2} + \gamma \boldsymbol{\beta} c t_i, \tag{4}$$

$$c t_i' = \gamma c t_i + \gamma(\boldsymbol{\beta} \cdot \boldsymbol{x}_i), \tag{5}$$

where $i = 1, 2$ and $\gamma = 1/\sqrt{1 - \beta^2}$. What matters to us is the spatial parts of the events, which are

$$\boldsymbol{x}_1' = \boldsymbol{x}_1 + (\gamma - 1)\frac{(\boldsymbol{\beta} \cdot \boldsymbol{x}_1)\boldsymbol{\beta}}{\beta^2} + \gamma \boldsymbol{\beta} c t_1, \tag{6}$$

$$\boldsymbol{x}_2' = \boldsymbol{x}_2 + (\gamma - 1)\frac{(\boldsymbol{\beta} \cdot \boldsymbol{x}_2)\boldsymbol{\beta}}{\beta^2} + \gamma \boldsymbol{\beta} c t_2. \tag{7}$$

After substitution of the coordinates of the second event in the unprimed frame, given by Eqns (2) and (3), the spatial part of the second event can be written in the form

$$\boldsymbol{x}_2' = \boldsymbol{x}_1' + \alpha \boldsymbol{d}', \tag{8}$$

where

$$\boldsymbol{d}' = \hat{\boldsymbol{d}} + (\gamma - 1)(\hat{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{d}})\hat{\boldsymbol{\beta}} + \gamma \boldsymbol{\beta} \tag{9}$$

and $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}/\beta$ is a unit vector in the direction of $\boldsymbol{\beta}$. The vector $\boldsymbol{d}'$ points in the direction of the straight line between the two events in the primed frame, i.e. the light-ray direction in the primed frame. Eqn (9) expresses this light-ray direction in the primed frame in terms of that in the unprimed frame and various quantities that describe the relative velocity between the two frames. Alternatively, it can be written in terms of the angle $\theta$ between $\hat{\boldsymbol{d}}$ and $\boldsymbol{\beta}$:

$$\boldsymbol{d}' = \hat{\boldsymbol{d}} + (\gamma - 1)\cos\theta \hat{\boldsymbol{\beta}} + \gamma \boldsymbol{\beta}. \tag{10}$$

## 3. LORENTZ-TRANSFORMATION WINDOWS

We are now ready to define *Lorentz-transformation windows* as windows that change the direction of transmitted light rays according to the equation describing the direction change upon changing frames, Eqn (9). In other words, we define Eqn (9) as a generalised law of refraction.

Fig. 2 shows ray-tracing simulations of an idealised Lorentz-transformation window that corresponds to $\boldsymbol{\beta} = (0, 0, -0.99)$, defined in a coordinate system associated with the window. The window is idealised in that it exactly refracts according to Eqn (9), without any offset, absorption or reflections. If the window is very close to the camera, the distortion of the view is the same as that of a photo taken with a camera moving into the image with a speed $0.99c$, i.e. with $\boldsymbol{\beta} = (0, 0, 0.99)$ (see Fig. 1).

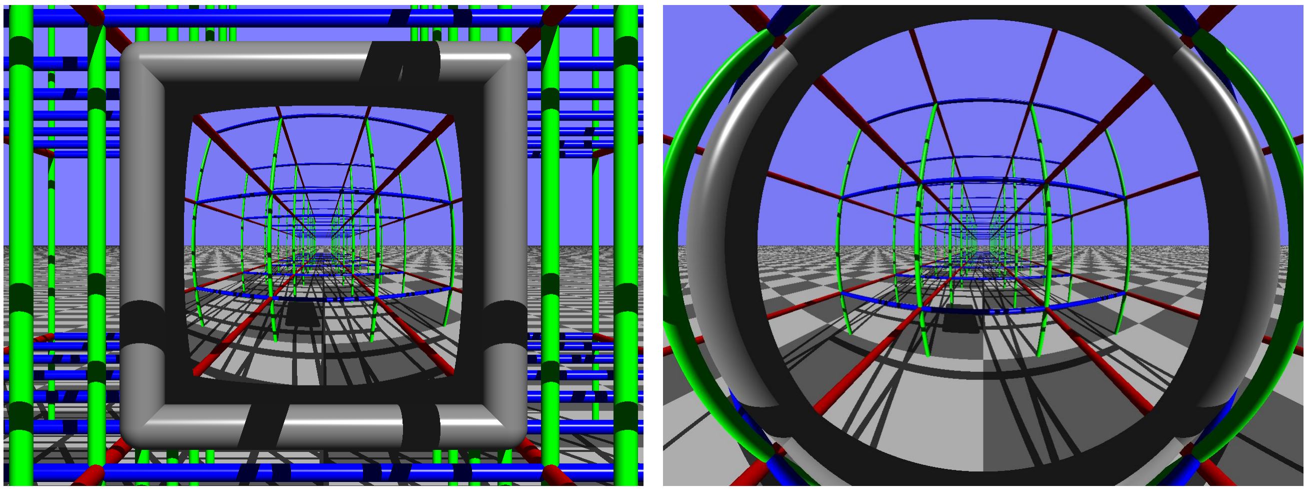
It is worth noting that ray-tracing simulations (not shown) and numerical calculations show that a Lorentz-transformation window immediately in front of a moving camera can undo the distortion of the image.

## 4. TELESCOPE WINDOWS AS LORENTZ-TRANSFORMATION WINDOWS?

Telescope windows, more prosaically known as generalised confocal lenslet arrays,[16] consist of arrays of pairs of lenslets forming telescopelets (Fig. 3). Provided a light ray enters and exits lenses that belong to the same telescopelet, it gets refracted according to the generalised law of refraction[17]

$$\boldsymbol{d}' = \frac{\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{u}}/(\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{a}}) - \delta_u}{\eta_u}\hat{\boldsymbol{u}} + \frac{\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{v}}/(\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{a}}) - \delta_v}{\eta_v}\hat{\boldsymbol{v}} + \hat{\boldsymbol{a}}, \tag{11}$$

where $\hat{\boldsymbol{a}}$ is a unit vector in the direction of the telescopelets' optical axes, $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ are two unit vectors perpendicular to $\hat{\boldsymbol{a}}$ and to each other, $\eta_u$ and $\eta_v$ are the focal-length ratios of the telescope lenslets in different projections



$z$=2 $z$=0.1

Figure 2. View through a Lorentz-transformation window with $\boldsymbol{\beta} = (0, 0, -0.99)$. The window is positioned at distance $z$ (in units of the floor-tile side length) in front of the camera. In the image for $z = 0.1$, the grey window frame is distorted into a torus-like shape. This is not the window frame seen directly (like in the $z = 2$ frame), but only after the relevant light rays have been redirected by the window. The images were calculated with Dr TIM.[6]

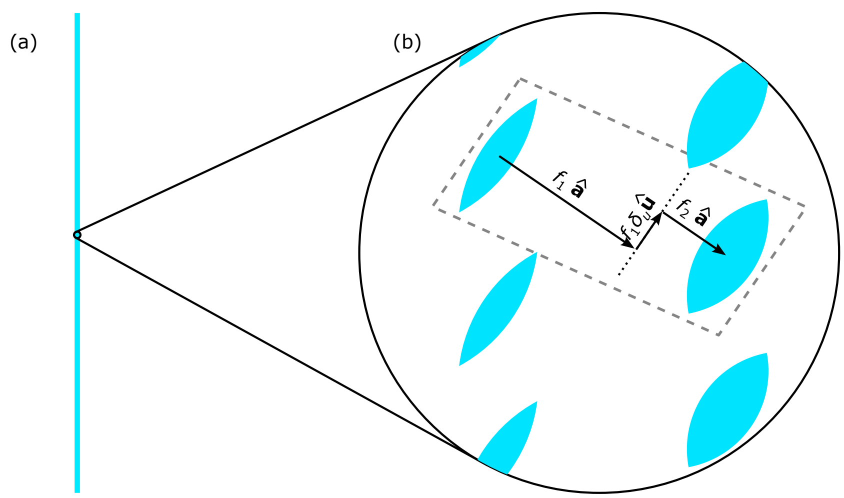



Figure 3. Structure of telescope windows. Side view of a telescope window (vertical line) (a) and close-up view of the window's structure (b). The window consists of an array of parallel telescopelets (one is outlined with a dashed rectangle; the dotted line indicates the focal plane shared by the telescopelet's two lenses). It is characterised by the unit vector $\hat{\boldsymbol{a}}$ in the direction of the telescopelets' optical axes, the ratio of the focal lengths of the two lenses forming each telescopelet, $\eta = -f_2/f_1$, and $\delta_u$ and $\delta_v$, which describe the offsets between the optical axes of the two lenslets forming each telescopelet ($f_1\delta_u$ and $f_1\delta_v$ are the offsets in two directions, $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$, which are perpendicular to $\hat{\boldsymbol{a}}$ and each other; in the sketch shown, $\hat{\boldsymbol{u}}$ is in the paper plane, $\hat{\boldsymbol{v}}$ is perpendicular to it).

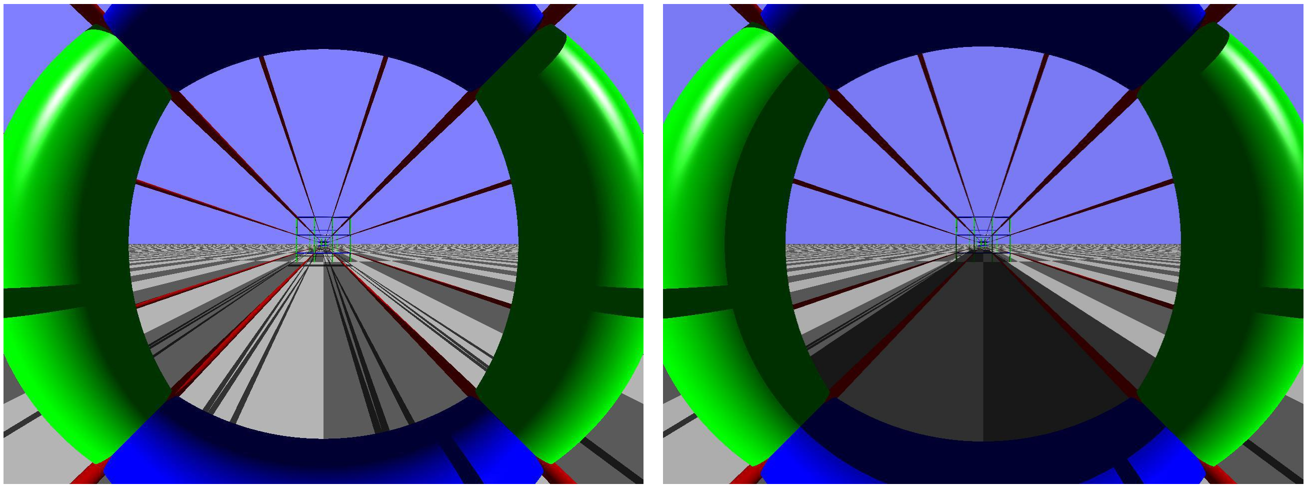


Figure 4. Example of the distortion of the view due to the Galileo transformation (left), and view through a corresponding Galileo-transformation window. The images, calculated with Dr TIM,[6] show a simulated photo of the lattice shown in Fig. 1. In the left image, the virtual camera is again moving with speed $0.99\,c$ into the image, but this time events that defined the light-ray trajectories were calculated not according to the Lorentz transformation (like in section 2), but according to the Galileo transformation. In the right image, the camera is stationary but a Galileo-transformation window with $\boldsymbol{\beta} = (0, 0, -0.99)$ is positioned a distance $z = 0.001$ floor-tile lengths in front of the camera.

(this applies only if the lenslets forming the telescopelets are elliptical, a case we do not consider here, and so we will not explain it in detail), and $\delta_u$ and $\delta_v$ are the offsets between the optical axes of the two lenslets forming each telescopelets in the $u$ and $v$ directions, divided by $f_1$.[16]

The telescope windows relevant for our current purposes are those with $\eta_u = \eta_v = \eta$. Multiplying the RHS of Eqn (11) by $\eta(\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{a}})$ gives

$$\boldsymbol{d}' \propto \left[\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{u}} - \delta_u(\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{a}})\right] \hat{\boldsymbol{u}} + \left[\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{v}} - \delta_v(\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{a}})\right] \hat{\boldsymbol{v}} + (1 + \eta - 1)(\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{a}})\hat{\boldsymbol{a}} \tag{12}$$

$$= \hat{\boldsymbol{d}} + (\eta - 1)\cos\Theta\hat{\boldsymbol{a}} - \cos\Theta\boldsymbol{\delta}, \tag{13}$$

where $\cos\Theta = \hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{a}}$, i.e. $\Theta$ is the angle between $\hat{\boldsymbol{d}}$ and $\hat{\boldsymbol{a}}$, and $\boldsymbol{\delta} = \delta_u\hat{\boldsymbol{u}} + \delta_v\hat{\boldsymbol{v}}$.

Comparison of this form of the generalised law of refraction for the relevant telescope windows, Eqn (13), and the generalised law of refraction for Lorentz-transformation windows, Eqn (10), reveals that the first two terms on the RHS of both equations are identical, whereby $\hat{\boldsymbol{a}}$ plays the role of $\hat{\boldsymbol{\beta}}$, and where $\eta$ plays the role of $\gamma$. This is the good news.

The bad news is that the third terms differ. The third term in Eqn (10) points in the direction of $\boldsymbol{\beta}$, whereas the third term in Eqn (13) points in the direction of $\boldsymbol{\delta}$, which is perpendicular to $\hat{\boldsymbol{a}}$ and therefore also to $\boldsymbol{\beta}$, as $\hat{\boldsymbol{a}}$ has to be chosen to equal $\hat{\boldsymbol{\beta}}$ for the second terms to agree. The last hope, then, is to make both third terms equal to zero. The third term in Eqn (13) becomes zero for the choice $\boldsymbol{\delta} = 0$. The third term in Eqn (10) is of magnitude $\gamma\beta = \beta/\sqrt{1-\beta^2}$, which goes to zero for $\beta \ll 1$. This is a problem, as this limit corresponds to non-relativistic speeds, which are arguably not so interesting. What is worse, however, is that, for small values of $\beta$, the ratio of the magnitudes of the wrong, third, term and the correct, second, term is $\gamma\beta/(\gamma - 1) \approx 2/\beta$, which diverges in the limit $\beta \to 0$, i.e. the correct change in light-ray direction is eclipsed by a wrong change. This means that telescope windows are not Lorentz-transformation windows.

## 5. GALILEO-TRANSFORMATION WINDOWS

In addition to Lorentz-transformation windows, which mimic some of the visual effects of Special Relativity, we can consider Galileo-transformation windows, which visualise some of the visual effects of non-relativistic motion.

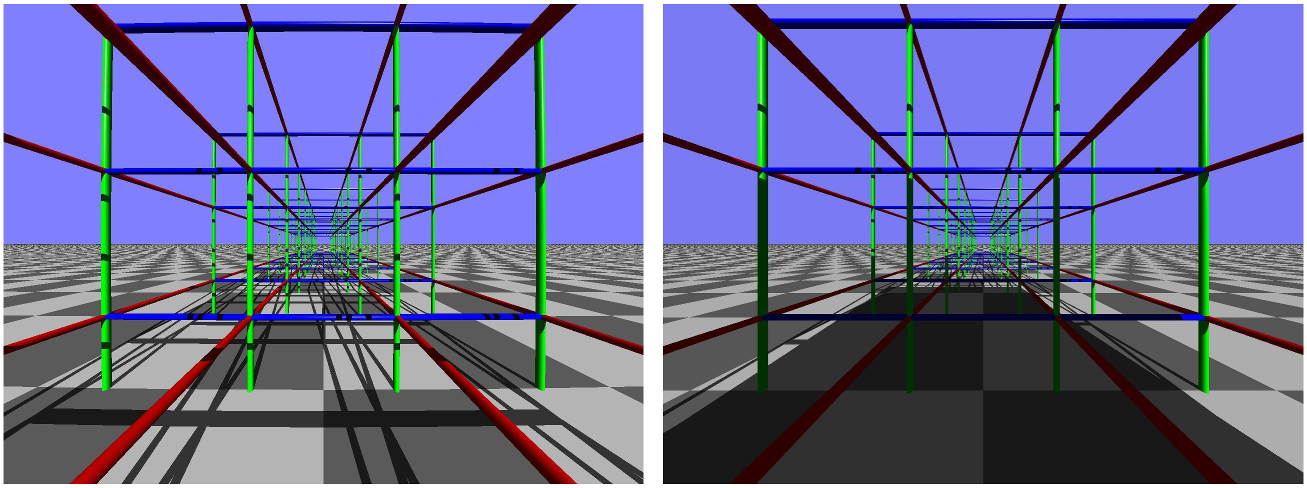


Figure 5. Comparison of the distortion due to the Galileo transformation (left) and the view through the corresponding Galileo-transformation window approximated with a telescope window (right). The left image shows the centre of the left image in Fig. 4; there, the camera's horizontal field of view is 20°, here it is 2.5°. The right image shows the view through a telescope window with $\eta = 1 - 0.99 = 0.01$, positioned a distance $z = 0.01$ floor-tile lengths in front of the camera. Note that the telescopes that form the telescope window are Galilean telescopes, so the connection to Galileo is two-fold. The images were calculated with Dr TIM.[6]

The light-ray-direction change upon a Galilean transformation is

$$\boldsymbol{d}'_G = \hat{\boldsymbol{d}} + \boldsymbol{\beta}, \tag{14}$$

and this is the generalised law of refraction for what we define as *Galileo-transformation windows*. Fig. 4 shows ray-tracing simulations of the view distortion due to the Galileo transformation and due to the view through a corresponding Galileo-transformation window very close to the camera. The two views are virtually identical, differing mainly in the presence of a shadow thrown by the simulated Galileo-transformation window.

For small angles $\theta$, telescope windows can be good approximations to Galileo-transformation windows. This is the case if we choose $\hat{\boldsymbol{a}} = \hat{\boldsymbol{\beta}}$ and $\eta = \beta + 1$, which makes equal (for small angles $\theta$) the second term in the generalised law of refraction for telescope windows, Eqn (13), and in the generalised law of refraction for Galileo-transformation windows, Eqn (14), and if we also choose $\boldsymbol{\delta} = 0$, which makes zero the third term in Eqn (13). Fig. 5 demonstrates that telescope windows can very closely mimic the visual effects of non-relativistic motion over a range of light-ray directions centred around the direction of the velocity of the corresponding motion.

## 6. CONCLUSIONS

We have defined hypothetical Lorentz-transformation and Galileo-transformation windows, and investigated their basic properties. We have found that the relatively simple telescope windows are not Lorentz-transformation windows, but for small angles between the light-ray direction and the direction of the velocity can be good approximations to Galileo-transformation windows.

Perhaps the law of refraction required by Lorentz-transformation windows can be realised with other windows. For example, a combination of two telescope windows might well be able to refract more generally than one telescope window on its own. We also have ideas for pixellated windows that can perform almost arbitrary mappings between the directions of incident and outgoing light rays.

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