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Windows into non-Euclidean spaces

Stephen Oxburgh, Chris D. White, Georgios Antoniou, Lena Mertens, Christopher Mullen, Jennifer Ramsay, Duncan McCall, and Johannes Courtial

School of Physics & Astronomy, University of Glasgow, Glasgow G12 8QQ, United Kingdom

ABSTRACT

Two microlens arrays that are separated by the sum of their focal lengths form arrays of micro-telescopes. Parallel light rays that pass through corresponding lenses remain parallel, but the direction of the transmitted light rays is different. This remains true if corresponding lenses do not share an optical axis (i.e. if the two microlens arrays are shifted with respect to each other). The arrays described above are examples of generalised confocal lenslet arrays, and the light-ray-direction change in these devices is well understood [Oxburgh *et al.*, Opt. Commun. **313**, 119 (2014)]. Here we show that such micro-telescope arrays change light-ray direction like the interface between spaces with different metrics. To physicists, the concept of metrics is perhaps most familiar from General Relativity (where it is applied to spacetime, not only space, like it is here) and Transformation Optics [Pendry *et al.*, Science **312**, 1780 (2006)], where different materials are treated like spaces with different optical metrics. We illustrate the similarities between micro-telescope arrays and metric interfaces with raytracing simulations. Our results suggest the possibility of realising transformation-optics devices with micro-telescope arrays, which we investigate elsewhere.

Keywords: micro-optics, generalised refraction, transformation optics

1. INTRODUCTION

One of the themes in our research is ray optics not constricted by the demands of its underlying theory, wave optics. Needless to say, it is not possible to realise ray optics that contradicts wave optics, but it is possible to build optical components that *look like* (but do not actually) turning standard incident light-ray fields into light-ray fields for which no corresponding wave front, i.e. a surface that is perpendicular to the rays at each point, can exist.[1] These components achieve this by criss-crossing the wave front with discontinuities, which develop on propagation into optical vortices;[2] they should open up new possibilities for optical design.

One of the components that can turn standard light-ray fields into apparently wave-optically forbidden ones consists of arrays of micro-telescopes, formed from confocal arrays of micro lenses.[3,4] We call such windows telescope windows; more prosaically, they have also been called generalised confocal lenslet arrays.[4] The generalised law of refraction that describes the light-ray-direction change upon transmission[5] gives telescope windows interesting imaging properties: planar telescope windows can perform imaging that is almost, but not quite, stigmatic (ray-optically perfect).[3,6] The generalised law of refraction is just right for stigmatic imaging, but the imaging is merely *almost* stigmatic as telescope window offset light rays on transmission, by an amount that is small, indeed in principle so small that it can be invisible. Simple telescope windows have been realised experimentally.[7]

Here we describe work aimed at using telescope windows to realise transformation-optics devices. Transformation optics is the idea of taking a part of space in which the light-ray trajectories are well understood (usually empty space, in which light rays travel in straight lines, like in Refs,[8,9] but refractive-index distributions such as those in the Maxwell fisheye, in which all light rays travel in circles, are also a possibility[10]), and distorting the inside of this space and the light-ray trajectories with it. Distorting the light-ray trajectories in this way can be achieved with metamaterial structures.[11] Because of the close relationship with curved spaces, which can be described mathematically in terms of metrics, transformation optics is usually formulated in this way, too. With the aim of realising transformation-optics devices using telescope windows, we compare here the light-ray-direction change performed by telescope windows with that occurring at the interface between spaces with different metrics. We find that a subset of telescope windows can refract like such a metric interface. We illustrate our findings with ray-tracing simulations.

Further author information: JC's e-mail address is johannes.courtial@glasgow.ac.uk

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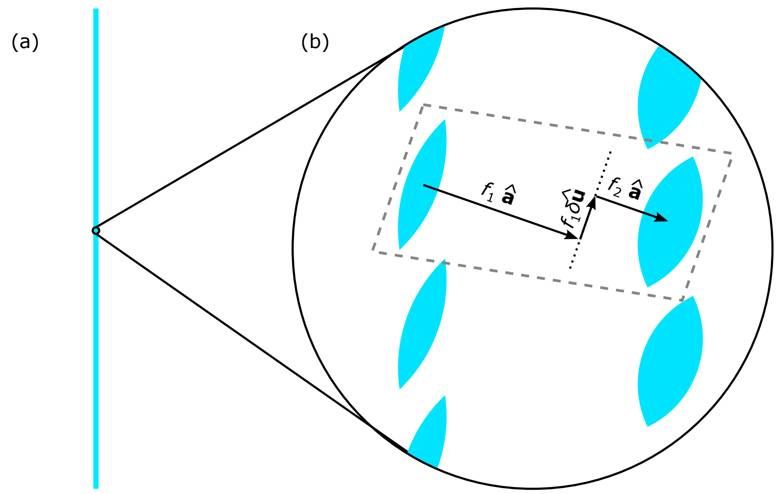

Figure 1. Structure of telescope windows. (a) Side view of a telescope window. (b) Close-up view, showing the detailed structure of the window. One of the telescopelets in the window is enclosed in a dashed box; the dotted line shows the focal plane shared by the telescopelet's two lenslets. The window is characterised in terms of the direction of the optical axis of the telescopelets, $\hat{\mathbf{a}}$, the ratio $\eta = -f_2/f_1$ of the focal lengths of the lenslets in each telescopelet, and the offsets $f_1\delta_u$ and $f_1\delta_v$ between the optical axes of the two lenslets forming a telescopelet relative to each other in two directions perpendicular to the optical axis. A further generalisation (not shown) involves making the lenslets elliptical, resulting in different focal-length ratios, η_u and η_v, in the u and v directions.

2. THE METRIC TENSOR AND ITS ROLE IN TRANSFORMATION OPTICS

In three-dimensional Euclidean space, the length of an infinitesimal line element is given by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{1}$$

This clearly is simply Pythagoras's theorem. In three-dimensional Riemannian spaces the length of such a line element becomes more general and is given by

$$\begin{aligned}\mathrm{d}s^2 =& g_{11}\mathrm{d}x^2 + g_{22}\mathrm{d}y^2 + g_{33}\mathrm{d}z^2 \\ &+ 2g_{12}\mathrm{d}x\mathrm{d}y + 2g_{13}\mathrm{d}x\mathrm{d}z + 2g_{23}\mathrm{d}y\mathrm{d}z,\end{aligned} \tag{2}$$

where the numbers g_{ij} are the elements of the (symmetric) metric tensor, g. As can be seen, in a Euclidean space, $g_{ij} = \delta_{ij}$ for all allowed values (i.e. 1 to 3 for three-dimensional spaces) of i and j, where δ is the Kronecker delta. A Riemannian space where this is not the case is therefore non-Euclidean.

TO involves creating a spatially varying metric tensor, which can be interpreted as defining an effective local refractive index which may or may not be homogenous and/or isotropic. Hence in TO the metric tensor allows us to measure the distance in optical space i.e. the optical path length. For example in a conventional refractive index material of index n the optical path length is $\mathrm{d}s^2 = n^2\mathrm{d}x^2 + n^2\mathrm{d}y^2 + n^2\mathrm{d}z^2$. According to Fermat's principle, light rays travel along paths for which the optical path length is stationary, and so light rays in general change direction when the metric changes along their trajectory. Hence light should refract when it passes through a plane with different metrics on either side. We aim to show that the same light-ray-direction change can be achieved with telescope windows.

3. FERMAT'S PRINCIPLE AT THE INTERFACE BETWEEN DIFFERENT RIEMANNIAN SPACES

Fermat's principle states that light rays that travel between two points, **A** and **B**, follow the path of stationary optical path length. We consider the situation where two homogeneous spaces with different metrics meet at a planar interface, and where **A** and **B** are positioned on either side of this metric interface, each a distance 1 from the interface. We calculate the light-ray-direction change at the interface.

We start by choosing a coordinate system such that the interface is in the $z = 0$ plane and that the path of stationary optical path length passes through the origin. We let the space on one side of the interface (when $z < 0$) be characterised by the metric tensor g and the space on the other side of the interface (when $z > 0$) by the metric tensor h. We call the vector from **A** to the origin $\mathbf{d} = (d_x, d_y, 1)$, and notice that this vector points in the direction of the incident light ray. Similarly, we call the vector from the origin to **B** $\mathbf{e} = (e_x, e_y, 1)$; this vector points in the direction of the outgoing light ray.

For a light ray that passes the interface at a point $\mathbf{P} = (x, y, 0)$, the optical path length from **A** to **B** is the sum of the optical path length between **A** and **P**, $\pm\sqrt{(\mathbf{d}+\mathbf{P})^T g(\mathbf{d}+\mathbf{P})}$, and the optical path length between **P** and **B**, $\pm\sqrt{(\mathbf{e}-\mathbf{P})^T h(\mathbf{e}-\mathbf{P})}$. In both cases, the '+' sign has to be chosen if the optical path length is positive on the relevant side of the interface, otherwise the '$-$' sign has to be chosen. Fermat's principle requires that the total optical path length be stationary, i.e. the derivatives of the total optical path length w.r.t. x and y must be zero. This then means that

$$\begin{aligned} \left.\frac{\partial\sqrt{(\mathbf{d}+\mathbf{P})^T g(\mathbf{d}+\mathbf{P})}}{\partial x}\right|_{\mathbf{P}=(0,0,0)} &= \pm \left.\frac{\partial\sqrt{(\mathbf{e}-\mathbf{P})^T h(\mathbf{e}-\mathbf{P})}}{\partial x}\right|_{\mathbf{P}=(0,0,0)}, \\ \left.\frac{\partial\sqrt{(\mathbf{d}+\mathbf{P})^T g(\mathbf{d}+\mathbf{P})}}{\partial y}\right|_{\mathbf{P}=(0,0,0)} &= \pm \left.\frac{\partial\sqrt{(\mathbf{e}-\mathbf{P})^T h(\mathbf{e}-\mathbf{P})}}{\partial y}\right|_{\mathbf{P}=(0,0,0)}, \end{aligned} \tag{3}$$

where either the '+' sign or the '$-$' sign applies in both equations.

By using index notation and the Einstein summation convention it is possible to re-write the terms inside the square root of Eqs.(3) as

$$(\mathbf{c} \pm \mathbf{P})^T f(\mathbf{c} \pm \mathbf{P}) = (c_k \pm x_k) f_{jk} (c_j \pm x_j) \tag{4}$$

(where $\mathbf{c} = \mathbf{d}$, $f = g$, and the + sign is chosen, or where $\mathbf{c} = \mathbf{e}$, $f = h$, and the $-$ sign is chosen). The derivative with respect to x_i then becomes

$$\frac{\partial\sqrt{(\mathbf{c}\pm\mathbf{P})^T f(\mathbf{c}\pm\mathbf{P})}}{\partial x_i} \tag{5}$$

$$= a\frac{\partial}{\partial x_i}[(c_k \pm x_k) f_{jk} (c_j \pm x_j)] \tag{6}$$

$$= a\frac{\partial}{\partial x_i}[c_k f_{jk} c_j \pm c_k f_{jk} x_j \pm x_k f_{jk} c_j + x_k f_{jk} x_j] \tag{7}$$

$$= a[\pm c_k f_{jk} \delta_{ij} \pm \delta_{ik} f_{jk} c_j + \delta_{ik} f_{jk} x_j + x_k f_{jk} \delta_{ij}], \tag{8}$$

where

$$a = \frac{1}{2\sqrt{(\mathbf{c}\pm\mathbf{P})^T f(\mathbf{c}\pm\mathbf{P})}}. \tag{9}$$

As stated earlier, the light ray incident on the interface strikes the origin, hence in order to find the law of refraction at this point we must evaluate Fermat's principle at the point $\mathbf{P} = 0$ (i.e. $x_1 = x_2 = x_3 = 0$). There, these derivatives become

$$\left.\frac{\partial\sqrt{(\mathbf{c}\pm\mathbf{P})^T f(\mathbf{c}\pm\mathbf{P})}}{\partial x_i}\right|_{\mathbf{P}=0} = \pm\frac{c_k f_{jk} \delta_{ij} + \delta_{ik} f_{jk} c_j}{2\sqrt{\mathbf{c}^T f \mathbf{c}}} \tag{10}$$

$$= \pm\frac{f_{ik} c_k}{\sqrt{\mathbf{c}^T f \mathbf{c}}} \tag{11}$$

(where, in the last line, we have used the fact that $f_{ij} = f_{ji}$ (f is symmetric) and we have re-named the dummy variable j to k in the right-hand term). In vector notation this is then written as

$$\left.\frac{\partial\sqrt{(\mathbf{c}\pm\mathbf{P})^T f(\mathbf{c}\pm\mathbf{P})}}{\partial x_i}\right|_{\mathbf{P}=0} = \pm\frac{(f\mathbf{c})_i}{\sqrt{\mathbf{c}^T f\mathbf{c}}}. \tag{12}$$

Recall that this expression must be zero. By writing in terms of the ingoing and outgoing light ray direction we therefore have

$$\frac{(g\mathbf{d})_i}{\sqrt{\mathbf{d}^T g\mathbf{d}}} = \mp\frac{(h\mathbf{e})_i}{\sqrt{\mathbf{e}^T h\mathbf{e}}}, \tag{13}$$

which has to be satisfied for $i = 1, 2$. Projecting both sides into the plane of the interface gives the full three-dimensional vector equation

$$\frac{g\mathbf{d} - (g\mathbf{d}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}}{\sqrt{\mathbf{d}^T g\mathbf{d}}} = \mp\frac{h\mathbf{e} - (h\mathbf{e}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}}{\sqrt{\mathbf{e}^T h\mathbf{e}}}, \tag{14}$$

where $\hat{\mathbf{n}}$ is the normalised surface normal.

4. TELESCOPE WINDOWS AS METRIC INTERFACES

We now move on to show how the light-ray-direction change due to telescope windows can be the same as that across a metric interface.

In the simplest telescope windows,[3] the two lenslets forming a telescopelet are spherical and they share the same optical axis, which is perpendicular to the window plane. These windows can be generalised without losing their ability to change the direction of transmitted light rays according to a single generalised law of refraction that is independent of the position where a light ray hits the window.[4] The type of generalisations that can be made fall in to two categories: orientation and lens type. The two things we can change as regards orientation are:

1. The optical axes of the two lenses. The optical axes of both lenses have to be parallel, however they can be offset relative to one another in both directions that are orthogonal to the optical axes.
2. The entire telescopelet, generalised as described above, can be rotated. The sheet itself remains in the same plane, but the individual telescoplets within the sheet are rotated.

We can also make the lenslets elliptical, such that they are equivalent to a combination of two cylindrical lenses with crossed axes. This generalisation must be applied to both lenslets if it is to work, as the sum of the focal lengths for each transverse direction has to equal the distance between the two lenses as the lenses have to remain confocal.

The orientations of the axes of the cylindrical lenses are $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$; the focal lengths of the two cylindrical lenses whose cylinder axis is in the $\mathbf{v}$ direction, and which therefore focus in the (u, a) projection, are $f_{1,u}$ and $f_{2,u} = -\eta_u f_{1,u}$; those of the two cylindrical lenses whose cylinder axis is in the $\mathbf{u}$ direction are $f_{1,v}$ and $f_{2,v} = -\eta_v f_{1,v}$. The offset between the optical axes of the two lenses is $f_{1,u}\delta_u\hat{\mathbf{u}} + f_{1,v}\delta_v\hat{\mathbf{v}}$; the direction of the optical axes is $\hat{\mathbf{a}}$. Fig. 1 indicates a few of these parameters.

The law of refraction for completely general telescope windows, formulated for a general coordinate system (u, v, a),[5] is given by

$$e_u = \frac{d_u/d_a - \delta_u}{\eta_u}, \quad e_v = \frac{d_v/d_a - \delta_v}{\eta_v}, \quad e_a = 1, \tag{15}$$

where the the vector $\mathbf{d} = d_u\hat{\mathbf{u}} + d_v\hat{\mathbf{v}} + \hat{\mathbf{a}}$ points in the direction of the incident light-ray direction and the vector $\mathbf{e} = e_u\hat{\mathbf{u}} + e_v\hat{\mathbf{v}} + e_a\hat{\mathbf{a}}$ describes the outgoing light-ray direction.

For telescope windows in the $z = 0$ plane in which the telescoplets are perpendicular to the window plane (i.e. the optical-axis direction is in the z direction), and with the cylinder axes lying in the x and y directions, the law of refraction, for an incident light-ray vector $\mathbf{d}$, is given by

$$e_x = \frac{d_x - \delta_x}{\eta_x}, \quad e_y = \frac{d_y - \delta_y}{\eta_y}, \quad e_z = 1. \tag{16}$$

These equations can be written in the form

$$\mathbf{e} = \mathrm{M} \cdot \mathbf{d}, \tag{17}$$

where

$$\mathrm{M} = \begin{pmatrix} 1/\eta_x & 0 & -\delta_x/\eta_x \\ 0 & 1/\eta_y & -\delta_y/\eta_y \\ 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

It is easy to calculate the matrix inverse of M, which is

$$\mathrm{M}^{-1} = \begin{pmatrix} \eta_x & 0 & \delta_x \\ 0 & \eta_y & \delta_y \\ 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

We can now perform some mathematical operations to find expressions for the elements of the metric h. Strategically inserting identity matrices of the form $\mathrm{I} = \mathrm{M}\mathrm{M}^{-1} = (\mathrm{M}^T)^{-1}\mathrm{M}^T$ into the RHS of Eqn (14), and using the fact that $\mathbf{d} = \mathrm{M}^{-1}\mathbf{e}$, yields

$$\frac{g\mathbf{d} - (g\mathbf{d} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{\sqrt{\mathbf{d}^T g \mathbf{d}}} = \mp \frac{h\mathrm{M}\mathrm{M}^{-1}\mathbf{e} - (h\mathrm{M}\mathrm{M}^{-1}\mathbf{e} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{\sqrt{\mathbf{e}^T(\mathrm{M}^{-1})^T\mathrm{M}^T h \mathrm{M}\mathrm{M}^{-1}\mathbf{e}}} = \mp \frac{h\mathrm{M}\mathbf{d} - (h\mathrm{M}\mathbf{d} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{\sqrt{\mathbf{d}^T\mathrm{M}^T h\mathrm{M}\mathbf{d}}}. \tag{20}$$

Now consider a matrix N of the form

$$\mathrm{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ N_{13} & N_{23} & N_{33} \end{pmatrix}. \tag{21}$$

Multiplying the RHS of Eqn (20) by N does not alter the transverse components of the vector in Eqn (20). Provided $\eta_x = \eta_y = \eta$, the matrix $\eta\mathrm{M}^T$ as defined in Eqn (18) is precisely of this form, and so we can write the RHS of Eqn (20) as

$$\mp \frac{\eta\mathrm{M}^T h\mathrm{M}\mathbf{d} - (\eta\mathrm{M}^T h\mathrm{M}\mathbf{d} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{\sqrt{\mathbf{d}^T\mathrm{M}^T h\mathrm{M}\mathbf{d}}}. \tag{22}$$

If

$$h' = \eta^2\mathrm{M}^T h\mathrm{M}, \tag{23}$$

this can be written as

$$\mp \frac{h'\mathbf{d} - (h'\mathbf{d} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{\sqrt{\mathbf{d}^T h'\mathbf{d}}}. \tag{24}$$

Eqn (20) then becomes

$$\frac{g\mathbf{d} - (g\mathbf{d} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{\sqrt{\mathbf{d}^T g\mathbf{d}}} = \mp \frac{h'\mathbf{d} - (h'\mathbf{d} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{\sqrt{\mathbf{d}^T h'\mathbf{d}}}. \tag{25}$$

Because this holds for any vector $\mathbf{d}$, this implies that

$$h' = g, \tag{26}$$

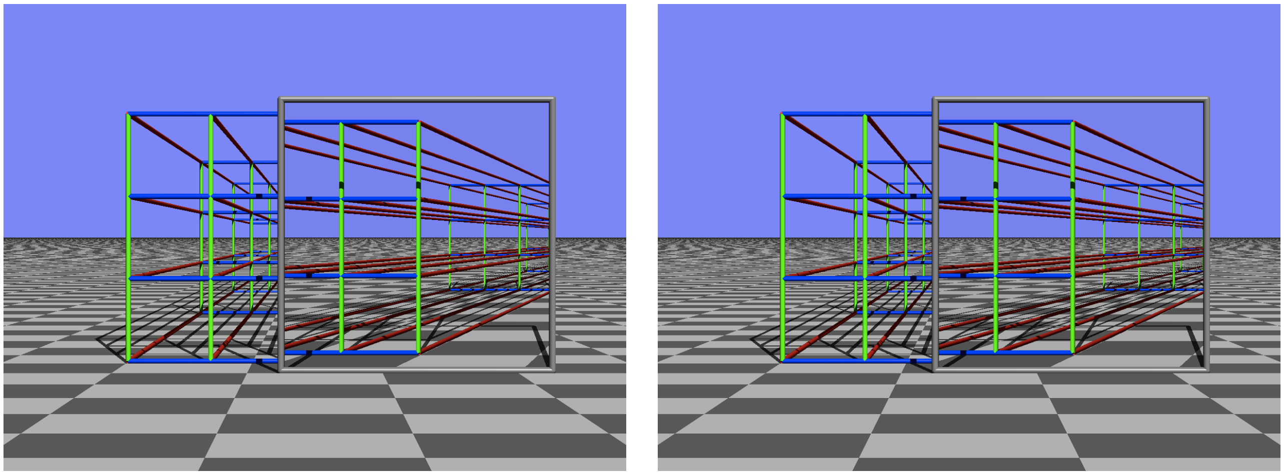

Figure 2. Simulated view through telescope windows (left) and through an equivalent metric interface (right). The parameters of the telescope windows are $\eta = 0.5$, $\delta_x = 0.2$, $\delta_y = 0$. Refraction at the telescope windows is calculated according to Eqns (16); refraction at the metric interface is calculated according to equations derived from Fermat's principle.[12] The images were calculated with the custom ray tracer Dr TIM.[12]

or

$$h = \frac{1}{\eta^2}(\mathrm{M}^T)^{-1} g \mathrm{M}^{-1}. \tag{27}$$

The elements of the metric h are

$$\begin{aligned} h_{11} &= g_{11}, \quad h_{22} = g_{22}, \quad h_{12} = g_{12}, \\ h_{13} &= \frac{g_{13} + g_{12}\delta_y + g_{11}\delta_x}{\eta}, \quad h_{23} = \frac{g_{23} + g_{12}\delta_x + g_{22}\delta_y}{\eta}, \\ h_{33} &= \frac{g_{33} + 2\delta_y g_{23} + \delta_y^2 g_{22} + 2\delta_x g_{13} + \delta_x^2 g_{11} + 2\delta_x\delta_y g_{12}}{\eta^2}. \end{aligned} \tag{28}$$

With the vector $\delta = (\delta_x, \delta_y, 1)^T$, this can be written in the compact form

$$\begin{aligned} h_{\alpha\beta} &= g_{\alpha\beta}, \quad h_{\alpha 3} = \sum_{i=1}^{3} g_{\alpha i}\frac{\delta_i}{\eta}, \quad (\alpha, \beta = 1, 2 \text{ or } x, y), \\ h_{33} &= \sum_{i,j=1}^{3} g_{ij}\frac{\delta_i \delta_j}{\eta^2}. \end{aligned} \tag{29}$$

We can simulate the view through a telescope window and a metric interface using the prescribed metric above. The views through both are shown in Fig. 2 and it is seen that the image produced is the same in both cases as expected.

5. METRIC INTERFACES AND PERFECT IMAGING

The above results can be verified and indeed more easily calculated by utilising the fact that planar telescope windows with $\eta_x = \eta_y = \eta$ and with the telescopes aligned perpendicular to the window plane have the property of imaging, almost stigmatically, all of object space into all of image space.[6] The equation

$$\mathbf{P}' = \mathbf{P} - z \begin{pmatrix} \delta_x \\ \delta_y \\ 1 - \eta \end{pmatrix}, \tag{30}$$

gives a mapping between the object positions, $\mathbf{P}$, and the corresponding image positions, $\mathbf{P}'$.

The Jacobian matrix for this mapping is

$$\Lambda_J = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\delta_x \\ 0 & 1 & -\delta_y \\ 0 & 0 & \eta \end{pmatrix}, \tag{31}$$

where $\mathbf{P} = (x, y, z)^T$ and $\mathbf{P}' = (x', y', z')^T$. The metric h is then related to the metric g according to the equation

$$h = (\Lambda_J^{-1})^T g (\Lambda_J^{-1}). \tag{32}$$

6. CONCLUSIONS

We have found that a subset of telescope windows changes light-ray direction like the interface between spaces with different metrics.

The aim was to realise transformation-optics devices with telescope windows, and — excitingly — we have actually found a way of doing just that. This particular application of the results presented here will be described elsewhere.

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