The NC *property for Banach algebras, and C^*-equivalence*

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Abstract

Consider a unital Banach algebra $\mathcal{A}$ having the *NC* property: that

$$1 - \mathcal{A}_1^+ \subseteq \mathcal{A}_1^+.$$

Then

$$\left|h\right|_\sigma = \|h\| \qquad (h \in \mathcal{A}^+).$$

Any unital hermitian Banach $*$-algebra with this property is therefore C^*-equivalent.

Introduction

All Banach algebras considered here will be unital, with $\|1\| = 1$.

As usual, given an element a of a unital Banach algebra $\mathcal{A}$ write $\sigma(a)$ for its spectrum and $\left|a\right|_\sigma$ for its spectral radius.

Write $\mathcal{A}^r$ for the set $\{h \in \mathcal{A} : \sigma(h) \subseteq \mathbb{R}\}$, the set of elements in $\mathcal{A}$ with real spectrum; write $\mathcal{A}^+$ for the set $\{h \in \mathcal{A} : \sigma(h) \subseteq \mathbb{R}^+\}$ (the *positive* elements of $\mathcal{A}$); and $\mathcal{A}_1^+$ for the set $\{h \in \mathcal{A}^+ : \|h\| \leq 1\}$. We write $h \gg 0$ to indicate that $\sigma(h) \subseteq \mathbb{R}^+ \setminus \{0\}$ — equivalently, that h is positive and invertible.

The *NC* property introduced in [4] is that

$$1 - \mathcal{A}_1^+ \subseteq \mathcal{A}_1^+,$$

or, alternatively put,

$$\text{if } \sigma(h) \subseteq \mathbb{R}^+ \;\&\; \|h\| \leq 1 \quad \text{then } \|1 - h\| \leq 1.$$

As we shall show, given *NC* one can produce positive square roots of positive elements, with control of their norm, and prove that the norm and spectral radius agree on positive elements. (This was claimed in [4, §11] but the argument there presented was both overclumsy and undercorrect.) The essential tool is the family of real polynomial functions π_n defined recursively by:

$$\begin{aligned} \pi_0(t) &= 0, \\ 2\pi_{n+1}(t) &= t + 2\pi_n(t) - \pi_n(t)^2. \end{aligned}$$

Recall that on $[0, 1]$ these polynomials increase monotonically and uniformly to the square root function.

The *NC* Square Root Lemma

Let $\mathcal{A}$ be a unital Banach algebra. Given an element $a \in \mathcal{A}$ we define

$$a_n = \pi_n(a) \qquad (n = 0, 1, \dots)$$

and check that then, for $n = 0, 1, \dots,$

$$\begin{aligned} 2(1 - a_{n+1}) &= 1 - a + (1 - a_n)^2 \\ 2(a_{n+2} - a_{n+1}) &= (a_{n+1} - a_n)(2 - a_{n+1} - a_n). \end{aligned}$$

Lemma 1. *Consider a unital Banach algebra $\mathcal{A}$ satisfying* NC *and an element $h \in \mathcal{A}^+$ for which $h \gg 0$ and $\|h\| < 1$.*
Define $h_n = \pi_n(h)$ for $n = 0, 1, \dots$. Then (h_n) is a Cauchy sequence and its limit h_∞ lies in $\mathcal{A}_1$. Moreover, $h_\infty^2 = h$.

Proof. Choose an $\varepsilon > 0$ such that

$$2\varepsilon \leq h \qquad \& \qquad \|h\| + 2\varepsilon < 1$$

and put $\lambda = 1 - \varepsilon$. Note that $2\varepsilon < 1 < 2\lambda$.
Now $\|h - \varepsilon\| \leq \|h\| + \varepsilon \leq 1 - \varepsilon = \lambda$, and $1 - h = \lambda\left[1 - \dfrac{h - \varepsilon}{\lambda}\right]$; so

$$\|1 - h\| \leq \lambda.$$

Also, $\|h - 2\varepsilon\| \leq \|h\| + 2\varepsilon < 1$, and $1 - h_1 = 1 - \dfrac{h}{2} = \lambda\left[1 - \dfrac{h - 2\varepsilon}{2\lambda}\right]$; so

$$\|1 - h_1\| \leq \lambda.$$

From this, the induction steps being

$$\begin{aligned} 2\|1 - h_n\| &\leq \|1 - h\| + \left\|(1 - h_n)^2\right\| \\ &\leq \lambda + \lambda^2 \leq 2\lambda \end{aligned}$$

and

$$2\|h_{n+2} - h_{n+1}\| \leq \|h_{n+1} - h_n\|\left(\|1 - h_{n+1}\| + \|1 - h_n\|\right) \leq 2\lambda\|h_{n+1} - h_n\|,$$

we deduce that

$$\begin{aligned} \|1 - h_n\| &\leq \lambda \qquad (\text{whence } \|h_n\| \leq 1) \qquad \text{and} \\ \|h_{n+1} - h_n\| &\leq \lambda^{n+1} \end{aligned}$$

for $n = 1, 2, \dots$. Thus (h_n) is a Cauchy sequence in $\mathcal{A}_1$ and its limit h_∞, also in $\mathcal{A}_1$, must satisfy the relation

$$2h_\infty = h + 2h_\infty - h_\infty^2,$$

that is

$$h_\infty^2 = h.$$

□

The *NC* Theorem

Theorem 2. *If $\mathcal{A}$ is a unital Banach algebra satisfying* NC *then*

$$\big|h\big|_\sigma = \|h\| \qquad (h \in \mathcal{A}^+).$$

Consequently positive square roots of positives exist and are uniquely determined. Further,

$$\|h\| \le 2\big|h\big|_\sigma \qquad (h \in \mathcal{A}^r).$$

Proof. First suppose that $h \gg 0$ and that $\|h^2\| < 1$. Put $k = h^2$ and apply the Lemma to obtain a k_∞, $= \lim \pi_n(k)$, such that $k_\infty^2 = k = h^2$. By construction, k_∞ commutes with h.

Consider any character φ on the unital commutative Banach subalgebra of $\mathcal{A}$ generated by h. We have $\varphi(k_\infty)^2 = \varphi(h)^2$; and, since both k_∞ and h are positive, we have $\varphi(k_\infty) = \varphi(h)$. Thus $\min \sigma(k_\infty + h) = 2 \min \sigma(h) > 0$, which shows that $k_\infty + h$ is invertible.

Now $(k_\infty - h)(k_\infty + h) = k_\infty^2 - h^2 = 0$, whence $k_\infty = h$. That is, $h = \lim \pi_n(h^2)$ and therefore

$$\|h\| \le 1, \quad \text{so long as } h \gg 0 \text{ and } \big\|h^2\big\| < 1.$$

Suppose now that $h \ge 0$ and $\|h^2\| < 1$. Then $h + \varepsilon \gg 0$ and $\|(h+\varepsilon)^2\| < 1$ for ε positive and small enough. Thus $\|h + \varepsilon\| \le 1$ for ε small enough: so $\|h\| \le 1$. The usual scaling argument shows that $\|h\|^2 \le \|h^2\|$ and therefore

$$\big|h\big|_\sigma = \|h\|$$

for all $h \in \mathcal{A}^+$.

Since the norm and spectral norm agree on positives we have

$$k = \lim \pi_n(k^2) \qquad (k \in \mathcal{A}_1^+).$$

Thus, if k and l are positive square roots of the same element (which, without loss of generality, we may assume to be in $\mathcal{A}_1^+$) then $k = \lim \pi_n(k^2) = \lim \pi_n(l^2) = l$. That is, positives have unique positive square roots.

Given $h \in \mathcal{A}$ with $\sigma(h) \subseteq \mathbb{R}$ we can define its absolute value $|h| = (h^2)^{\frac{1}{2}}$ and then define h^+ and h^- by $2h^\pm = |h| \pm h$. This provides the decomposition $h = h^+ - h^-$ where $h^\pm \ge 0$, $h^+h^- = 0$ and $\|h^\pm\| \le \big|h\big|_\sigma \le \|h\|$. Hence $\|h\| \le \|h^+\| + \|h^-\| \le 2\big|h\big|_\sigma$. □

Hermitian Banach *-algebras with *NC*

When $\mathcal{A}$ is a *-algebra write $\mathcal{A}^h = \{h \in \mathcal{A} : h = h^*\}$.

A unital Banach *-algebra $\mathcal{A}$ is *hermitian* if $\sigma(h) \subseteq \mathbb{R}$ for all $h \in \mathcal{A}^h$: that is, if $\mathcal{A}^h \subseteq \mathcal{A}^r$. Such an algebra is *symmetric*: $\sigma(a^*a) \subseteq \mathbb{R}^+$ for all $a \in \mathcal{A}$ (the *Shirali-Ford Theorem*).

In his seminal paper [2, §5] Ptak showed that on a hermitian Banach *-algebra the *Ptak seminorm*, the function $a \mapsto \big|a\big|_\Sigma = \big|a^*a\big|_\sigma^{\frac{1}{2}}$, is a C^*-seminorm. Moreover, he showed that a hermitian Banach *-algebra $\mathcal{A}$ is C^*-equivalent if there exists a $\beta > 0$ such that $\|h\| \le \beta\big|h\big|_\sigma$ for all $h \in \mathcal{A}^h$. Hence

Corollary 3. *If $\mathcal{A}$ is a unital hermitian Banach $*$-algebra with the* NC *property then $\mathcal{A}$ is C^*-equivalent.* $\square$

Remark. We have $\|a\| \leq 4\big|a\big|_\Sigma$ for all $a \in \mathcal{A}$. When the involution is *isometric*, as was hypothesized in [4], we also have $\big|a\big|_\Sigma \leq \|a\|$ for all a.

Remark. One cannot improve this: *NC* does not imply that the original norm is C^*. As remarked by B.A. Barnes, reported by R.B. Burckel in his review of [1], if we take $\mathcal{B} = (\mathbb{R}[0,1], \|\cdot\|_\infty)$ and construct a Banach algebra norm $\|\cdot\|_c$ on $\mathcal{A} = \mathbb{C}[0,1]$ as the complexification of $\mathcal{B}$ following [3, Theorem 1.3.2] then, taking $h, k \in \mathcal{B}$, $hk = 0$, $\|h\|_\infty = \|k\|_\infty > 0$, we find that $\|h + ik\|_c \geq \sqrt{2}\,\|h + ik\|_\infty$. So the norm $\|\cdot\|_c$ is a C^*-norm on $\mathcal{A}^+$ but is not a C^*-norm on $\mathcal{A}$.

References

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