Ultrahermitian Projections on Banach Spaces

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Introduction

I introduce a new class of projections on Banach spaces and analyse some of their properties. I also present improvements on known results for hermitian projections (in Theorem 2.6) and on their order properties (see Theorem 3.3). This leads to results on the compression and patching of hermitian operators: see Theorem 3.1 & Theorem 4.6 and its corollaries.

1 Background & Terminology

I follow the standard notation and sketch only a few salient details, referring the reader to [5], [3], [4], for example, for a full exposition and other references.

I shall write A_1 for the unit ball of a subset A of a normed space, and $\langle x, x' \rangle$ for the value of the functional x' in X' at x in X.

Throughout $\mathcal{A}$ will denote a complex unital Banach algebra (with identity 1).

For $x \in \mathcal{A}$, $\|x\| = 1$, define the *support set* at x

$$D(\mathcal{A}, x) = \{\varphi \in \mathcal{A}' : \langle x, \varphi \rangle = 1 = \|\varphi\|\}.$$

Then for $a \in \mathcal{A}$ define the sets

$$V(\mathcal{A}, a, x) = \{\langle ax, \varphi \rangle : \varphi \in D(\mathcal{A}, x)\}$$

and their union, the *algebra numerical range*

$$V(\mathcal{A}, a) = \bigcup_{x \in \mathcal{A},\ \|x\|=1} V(\mathcal{A}, a, x)\,.$$

For any $a \in \mathcal{A}$ the *spectrum of* a

$$\sigma(a) \subset V(\mathcal{A}, a)\,.$$

The fundamental link between the numerical range of a and the growth of the group $\{e^{ra} : r \in \mathbb{R}\}$ is

Theorem 1.1 *For each* $a \in A$

$$\begin{aligned} \max\{\Re\lambda : \lambda \in V(A, a)\} &= \sup\left\{r^{-1} \log \|e^{ra}\| : r > 0\right\} \\ &= \lim_{r \to 0+} r^{-1}\{\|1 + ra\| - 1\}\,. \end{aligned}$$

Definition 1.2 *An element* $h \in \mathcal{A}$ *is* hermitian *if its algebra numerical range is real: equivalently, if* $\|e^{irh}\| = 1 \quad (r \in \mathbb{R})$*: equivalently, if* $\|1 + irh\| \le 1 + o(r) \quad (\mathbb{R} \ni r \to 0)$.

Remark 1.3 *If* h *is hermitian then the convex hull of the spectrum satisfies*

$$\operatorname{co} \sigma(h) = V(\mathcal{A}, h)\,.$$

Theorem 1.4 (Sinclair's Theorem) $\|h\| = \rho(h)$ *(the spectral radius of* h*) for any hermitian* $h \in \mathcal{A}$.

The now classical numerical range characterisation of C*-algebras is

Theorem 1.5 (Vidav-Palmer Theorem) *Let* $\mathcal{H}$ *be the set of hermitian elements of a complex unital Banach algebra* $\mathcal{A}$*. If* $\mathcal{A} = \mathcal{H} + i\,\mathcal{H}$ *then* $\mathcal{A}$ *is a C*-algebra [under the given norm and the natural involution].*

When X is a complex Banach space and $x \in X$ we define the set of *support functionals* at x as

$$D(x) = \{\phi \in X' : \|\phi\| = 1,\ \langle x,\ \phi\rangle = \|x\|\}$$

and define

$$\Pi_X = \{(x, \phi) \in X \times X' : \|x\| = 1,\, \phi \in D(x)\}$$

ie

$$\Pi_X = \{(x, \phi) \in X \times X' : \langle x,\ \phi\rangle = \|x\| = \|\phi\| = 1\}.$$

The *spatial numerical range* $V(T)$ [one may write $V_X(T)$ to indicate the underlying space explicitly] of the operator T is defined as

$$V(T) = \big\{\langle Tx,\ \phi\rangle : (x, \phi) \in \Pi_X\big\}.$$

Definition 1.6 *An operator* H *on* X *is* hermitian *if its spatial numerical range is real: equivalently, if* $\|e^{irH}\| = 1 \quad (r \in \mathbb{R})$*: equivalently, if* $\|I_X + irH\| \le 1 + o(r) \quad (\mathbb{R} \ni r \to 0)$*: equivalently, if* H *is hermitian in the Banach algebra* $L(X)$*.*
$\mathcal{H}_X$ *is the set of hermitian operators on* X*.*

Remark 1.7 *An operator on Hilbert space is hermitian (in the numerical range sense) if and only if it is selfadjoint.*

2 Projections

A projection [*ie* an idempotent] p is *nontrivial* if p and $\overline{p}$, the complement of p in the identity, are both nonzero.
An *elementary observation* to be used repeatedly: if p is an idempotent in a unital Banach algebra then, for $z \in \mathbb{C}$,

$$e^{izp} = \overline{p} + e^{iz}p\,.$$

Remark 2.1 *If p is a nonzero* hermitian *projection in a unital Banach algebra then $\|p\| = 1$. This is usually presented as a corollary to Sinclair's Theorem. However it lies nearer the surface: for $\|1 - 2p\| = \|e^{i\pi p}\| \le 1$ and therefore $2\|p\| \le \|1 - (1 - 2p)\| \le 2$.*

From now on the setting is spatial, on a Banach space X.

Remark 2.2 *If E is any projection on a Banach (even on a normed) space X, and if $\phi \in X'$, then for $z \in EX$ we have $\langle z,\, \phi\rangle = \langle Ez,\, \phi\rangle = \langle z,\, E'\phi\rangle$ and so*

$$E'\phi|_{EX} = \phi|_{EX}\,.$$

Thus

$$\|E'\phi\| \ge \|E'\phi\|_{EX} = \|\phi\|_{EX}\,.$$

2.1 Hermitian compressions — contractive projections

Given $(x, \phi) \in \Pi_X$ and a projection E on X it is natural to wonder in what circumstances Ex and $E'\phi$ are mutually supportive (*ie*, whether $\langle Ex,\, E'\phi\rangle = \|Ex\|\|E'\phi\|$). In this direction we have

Lemma 2.3 *Let E be a projection on X. Suppose that $z \in EX$ and $(z, \theta) \in \Pi_{EX}$. Let ϕ be any extension of norm 1 of θ from EX to X. Then $(z, \phi) \in \Pi_X$.*
Conversely, if $z \in EX$ and $(z, \phi) \in \Pi_X$ then $(z, \phi|_{EX}) \in \Pi_{EX}$.
Suppose further that E is contractive. *Then $(Ez, E'\phi) \in \Pi_X$ for any such ϕ. Consequently*

$$V_{EX}(ETE) \subset V_X(T)$$

for any $T \in L(X)$.

Proof. The *first* assertion is clear: for $1 = \langle z,\theta\rangle = \langle z,\phi\rangle$. The converse too is immediate. *Next*, if also E is contractive, we have $1 = \langle z,\, E'\phi\rangle \le \|E'\phi\| \le \|\phi\| = 1$ and therefore $(Ez, E'\phi) \in \Pi_X$.
Consequently

$$\begin{aligned} V_{EX}(ETE) &= \{\langle ETEz,\, \theta\rangle : (z,\theta) \in \Pi_{EX}\} \\ &= \{\langle ETEz,\, \phi\rangle : (z,\phi) \in \Pi_X\} \\ &\subset \{\langle Tz,\, E'\phi\rangle : (z,E'\phi) \in \Pi_X\} \\ &\subset V_X(T). \end{aligned} \qquad \square$$

As an immediate corollary:

Theorem 2.4 *If E is a contractive projection and H is hermitian on X then EHE is hermitian on EX. If H is positive on X then EHE is positive on EX.* $\square$

Remark 2.5 *Not all contractive projections are hermitian. Even the stronger hypothesis $\|E\| = \|\overline{E}\| = 1$ does not compel E to be hermitian — see example cited in [1].*

2.2 Hermitian projections

It has long been known, see [1], that *hermitian* projections can be characterised as follows:

$$\|x\| = \sup\{|\langle Ex,\ \phi\rangle| + \left|\langle \overline{E}x,\ \phi\rangle\right| : \|\phi\| \le 1\}$$

for any $x \in X$: and that, when E is a hermitian projection,

$$\|\lambda Ex + \mu\overline{E}y\| = \|Ex + \overline{E}y\|$$

for any $x, y \in X$ (equivalently, $\lambda E + \mu\overline{E}$ is an isometry) for any $\lambda, \mu \in \mathbb{T}$.

The next result is stronger. It seems to be new — I have not seen it recorded. My proof depends on the Vidav-Palmer Theorem.

Theorem 2.6 *If E is a nontrivial hermitian projection on X then, for any $\lambda, \mu \in \mathbb{C}$,*

$$\|\lambda E + \mu\overline{E}\| = \max\{|\lambda|, |\mu|\}\,.$$

Proof. The linear span of E and $\overline{E}$ is a (commutative) C*-algebra on X and therefore

$$\|\lambda E+\mu\overline{E}\|^2 = \|(\lambda E+\mu\overline{E})^*(\lambda E+\mu\overline{E})\| = \|\,|\lambda|^2 E+|\mu|^2\overline{E}\| = \max\{|\lambda|^2, |\mu|^2\}\,. \qquad \square$$

Two subspaces Y and Z of X are *Birkhoff (*also *James) orthogonal*, written $Y \perp Z$, or $Y \perp_{\mathrm{B}} Z$, when $\|y\| \le \|y + z\|$ $(y \in Y, z \in Z)$.

The next result is a special case of the fact that *if H is a positive hermitian in a C*-algebra on X then* $\ker H \perp_B HX$ which may be proved along the lines of the proof that follows, using the fact that $(I + \varepsilon^{-1}H)^{-1}$ is a contraction, relying on the Vidav-Palmer Theorem. It is also a corollary of an even more general result of Crabb & Sinclair [6]: *if* $0 \in \sigma_p(T) \cap \partial V(T)$ *for some* $T \in L(X)$ *then* $\ker T \perp_B TX$. For projections I can offer a particularly elementary proof.

Theorem 2.7 *If E is a hermitian projection on X then $EX \perp_{\mathrm{B}} \overline{E}X$: that is*

$$\|Ex\| \le \|Ex + \overline{E}y\|$$

for any $x, y \in X$.

Proof. Assume, without loss of generality, that E is nontrivial. Note that, for $\varepsilon > 0$,

$$(I + \varepsilon^{-1}\overline{E})^{-1} = E + \frac{\varepsilon}{1+\varepsilon}\overline{E}$$

is hermitian and has spectrum $\left\{\dfrac{\varepsilon}{1+\varepsilon}, 1\right\}$ so must be a contraction. Thus

$$\|Ex + \varepsilon y\| \le \|(I + \varepsilon^{-1}\overline{E})(Ex + \varepsilon y)\| = \|Ex + \overline{E}y + \varepsilon y\|$$

and the result follows on letting $\varepsilon \to 0+$. $\square$

Theorem 2.8 *If E is a hermitian projection on X and $\phi \in X'$ then*

$$\|E'\phi\| = \|\phi\|_{EX} .$$

Proof. By Remark 2.2

$$\|\phi\|_{EX} = \|E'\phi\|_{EX} \leq \|E'\phi\| .$$

Next, for E hermitian and any $x \in X$,

$$\frac{|\langle x,\, E'\phi\rangle|}{\|x\|} = \frac{|\langle Ex,\, \phi\rangle|}{\|Ex + \overline{E}x\|} \leq \frac{|\langle Ex,\, \phi\rangle|}{\|Ex\|}$$

which shows that $\|E'\phi\| \leq \|\phi\|_{EX}$. □

3 Order & the EHE & HPP Problems

An *order relation* on the set of hermitians is defined naturally in terms of numerical range: $H \geq 0$ if and only if $V(H) \subset \mathbb{R}^+$: equivalently, $\sigma(H) \subset \mathbb{R}^+$.

Referring to Theorem 1.1 we see that

$$H \text{ is positive if and only if } \begin{array}{lll} \|e^{(-s+ir)H}\| & \leq 1 & \quad for \quad r \in \mathbb{R},\ s \geq 0 \\ \|1 - sH\| & \leq 1 + o(s) & \quad as \quad s \to 0+ \end{array}$$

If H is positive then $H \leq I$ if and only if $\|H\| \leq 1$.

Also, $0 \leq E \leq I$ for any hermitian projection. This is clear from Remark 1.3: and can also be seen immediately from the fact that

$$\|I - sE\| = \|(1-s)E + \overline{E}\| \leq 1$$

for $0 \leq s \leq 1$.

The *natural ordering on projections* is that $E \leq F$ if and only if $EF = FE = E$. If so, $\sigma(F - E) \subset \mathbb{R}^+$ (commutative spectral theory), and then $E \leq F$ in the numerical range sense (*cf* Remark 1.3).

3.1 Hermitian compressions

Theorem 3.1 *Let E be a hermitian projection and H a hermitian operator on X. Then*

1 *EHE is hermitian on EX.*

2 *If, moreover, H is positive on X then EHE is positive on EX.*

Proof. *Either* cite Theorem 2.4 (remembering that hermitian projections are contractive) *or* apply the inequality $\|E + irEHE\| \leq \|I + irH\| \leq 1 + o(r)$ when H is hermitian, and the inequality $\|E - sEHE\| \leq \|I - sH\| \leq 1 + o(s)$ $(s \to 0+)$ when H is also positive. □

Corollary 3.2 *Let E be a hermitian projection on X and H a positive hermitian on X with $\|H\| \leq 1$. If $E \leq H$ then $E = EHE$.*

Proof. For then, since $E \leq H \leq I$, we have $E(E)E \leq EHE \leq E(I)E$ (the theorem asserts this just on EX: but E and EHE are both 0 on $\overline{E}X$). □

3.2 Order on the projections

The next result strengthens [1, Theorem 2.17].

Theorem 3.3 *Suppose that E and F are hermitian projections on X and that $E \leq F$ in the numerical range sense. Then $EF = FE = E$.*

Proof. By the corollary $E = EFE$. A quick calculation shows that $(EF - FE)^3 = 0$, and since $i(EF - FE)$ is hermitian we have $EF = FE$ by Sinclair's Theorem (so both equal E). □

3.3 The EHE & HPP Problems

Theorem 3.1 does not answer the question of whether EHE is hermitian on all of X, and not just on EX. I call this *the EHE Problem.*

The EHE Problem is an instance of a more general one, *the Hermitian Patching Problem* or *HPP*: namely, if H is hermitian in $L(EX)$, and if K is hermitian in $L(\overline{E}X)$, does it follow that $H \oplus K$ is hermitian on X?

4 Ultrahermitian projections

Consider the following two properties that may hold for a projection E. Note that they are symmetrical in E and its complement $\overline{E}$. First,

$$(U1) \qquad \|Ex\|\|E'\phi\| + \|\overline{E}x\|\|\overline{E}'\phi\| \leq \|x\|\|\phi\|$$

for $x \in X$, $\phi \in X'$: and, second,

$$(U2) \qquad \|EAE + \overline{E}B\overline{E}\| \leq 1$$

for any contractions $A, B \in L(X)$.

Hermitian projections on Hilbert spaces have both these properties, as is easy to check.

Lemma 4.1 *If E is a projection and satisfies either (U1) or (U2) then E and $\overline{E}$ are contractions.*

Proof. Suppose (U1). Given x choose a ϕ to support Ex: *ie* $\langle Ex, \phi\rangle = \|Ex\| = \|\phi\| = 1$. Then $\|E'\phi\| \geq \|\phi\|_{EX} = 1$, by Remark 2.2, so $\|Ex\| \leq \|Ex\|\|E'\phi\| \leq \|x\|\|\phi\| = \|x\|$, by (U2).

Suppose (U2). Choose $A = I$ and $B = 0$. □

Theorem 4.2 *(U1) and (U2) are equivalent.*

Proof. Suppose that E has (U1) and that A and B are contractions. Then, for any $x \in X$ and $\phi \in X'$

$$\begin{aligned} \left|\langle (EAE + \overline{E}B\overline{E})x,\ \phi\rangle\right| &\leq \|EAEx\|\|E'\phi\| + \|\overline{E}B\overline{E}x\|\|\overline{E}'\phi\| \\ &\leq \|Ex\|\|E'\phi\| + \|\overline{E}x\|\|\overline{E}'\phi\| \\ &\leq \|x\|\|\phi\| \end{aligned}$$

which establishes (U2).

Conversely, assuming (U2), choose y, z in X_1 and ψ, χ in X_1'. Define contractions

$$A : x \mapsto \langle x,\ \psi\rangle y, \qquad B : x \mapsto \langle x,\ \chi\rangle z\,.$$

Then given $x \in X$ and $\phi \in X'$ we have

$$\left|\langle (EAE + \overline{E}B\overline{E})x,\ \phi\rangle\right| \leq \|x\|\|\phi\|$$

ie

$$\left|\langle Ex,\ \psi\rangle\langle y,\ E'\phi\rangle + \langle \overline{E}x,\ \chi\rangle\langle z,\ \overline{E}'\phi\rangle\right| \leq \|x\|\|\phi\|\,.$$

On choosing y, z, ψ, χ suitably [ψ to support Ex, χ to support $\overline{E}x$, y to support $E'\phi$ approximately, z to support $\overline{E}'\phi$ approximately] we find that

$$\|Ex\|\|E'\phi\| + \|\overline{E}x\|\|\overline{E}'\phi\| \leq \|x\|\|\phi\|\,. \qquad \Box$$

Theorem 4.3 *If E is a projection satisfying either of the equivalent conditions (U1) or (U2) then E is hermitian.*

Proof. For then

$$\|I + irE\| = \|(1+ir)E + \overline{E}\| \leq |1 + ir| = 1 + o(r)$$

as $r \to 0+$, by (U2). $\Box$

Definition 4.4 *An* ultrahermitian projection *on a Banach space X is an idempotent in $L(X)$ that satisfies either of the equivalent conditions (U1) or (U2).*

For ultrahermitian projections there is a result stronger than Lemma 2.3.

Theorem 4.5 *Let E be an ultrahermitian projection on X. If $(x, \phi) \in \Pi_X$, and also $Ex \neq 0$ and $E'\phi \neq 0$, then $\left(\dfrac{Ex}{\|Ex\|}, \dfrac{E'\phi}{\|E'\phi\|}\right) \in \Pi_X$.*

Proof. Work from (U1). For $(x, \phi) \in \Pi_X$ we have

$$\begin{aligned} 1 &= \langle x,\ \phi\rangle = \langle Ex,\ E'\phi\rangle + \langle \overline{E}x,\ \overline{E}'\phi\rangle \\ &\leq \|Ex\|\|E'\phi\| + \|\overline{E}x\|\|\overline{E}'\phi\| \leq 1 \end{aligned}$$

from which it follows that $\langle Ex,\ E'\phi\rangle = \|Ex\|\|E'\phi\|$ and thus $\left(\dfrac{Ex}{\|Ex\|}, \dfrac{E'\phi}{\|E'\phi\|}\right) \in \Pi_X$. $\Box$

4.1 The Hermitian Patching Problem

This has a positive resolution for *ultrahermitian* E.

Theorem 4.6 *Suppose that E is an* ultrahermitian *projection on X, that H is hermitian in $L(EX)$ and that K is hermitian in $L(\overline{E}X)$. Then $H \oplus K$ is hermitian on X. Moreover, if H and K are both positive then so is $H \oplus K$.*

Proof. For $r \in \mathbb{R}$ we have

$$\|I_X + ir[H \oplus K]\| = \|E(E + irH)E + \overline{E}(\overline{E} + irK)\overline{E}\| = 1 + o(r)$$

by (U2).
If, moreover, both H and K are positive, then, as above,

$$\|I_X - s[H \oplus K]\| = \|E(E - sH)E + \overline{E}(\overline{E} - sK)\overline{E}\| = 1 + o(s)$$

as $s \to 0+$, again by (U2). □

Corollary 4.7 *Suppose that E is an ultrahermitian projection on X, and that H and K are hermitian on X. Then $EHE \oplus \overline{E}K\overline{E}$ is hermitian on X. Moreover, if H and K are both positive then so is $EHE \oplus \overline{E}K\overline{E}$.*

Theorem 3.1 can be strengthened for ultrahermitian projections, so resolving the *EHE Problem*. One need only note, in pedantic style, that $EHE = EHE|_{EX} \oplus 0|_{\overline{E}X}$.

Corollary 4.8 *Let E be an ultrahermitian projection and H a hermitian operator on X. Then*

1 *EHE is hermitian on X. Hence*
2 *EH is hermitian if E and H commute.*
3 *If, moreover, H is positive on X then EHE is positive on X.*

4.2 Ultrahermitian decompositions

As remarked above, the condition for ultrahermiticity is symmetric in E and $\overline{E}$. This symmetry is, perhaps, better to be understood in greater generality.
Consider a finite family $\mathcal{E} = (E_j)$ of mutually orthogonal projections whose sum is I_X:

$$\oplus E_j = I\,.$$

As before, there are two properties $\mathcal{E}$ may have:

$(UD1)$
$$\sum \|E_j x\| \|E_j' \phi\| \le \|x\| \|\phi\|$$

for $x \in X$, $\phi \in X'$: and, second,

$(UD2)$
$$\|\sum E_j A_j E_j\| \le 1$$

for any contractions $A_j \in L(X)$.

It is straightforward, as before, to show that all the E_j are hermitian projections, subject to either (UD1) or (UD2); and that these two properties are equivalent.

One might call such a family $\mathcal{E}$ an *ultrahermitian decomposition of the identity.*

If $\mathcal{F} = (F_k)$ is another such family, *compatible with* $\mathcal{E}$ (in the sense that each E_j commutes with each F_k) then their common refinement $\mathcal{G} = (G_{jk})$ is again an ultrahermitian decomposition of the identity, where $G_{jk} = E_j F_k$.

Induced projections

Each such $\mathcal{E}$ defines a linear projection

$$P_{\mathcal{E}} : L(X) \to L(X) : A \mapsto \sum E_j A E_j$$

under which $\mathcal{H}_X$ is invariant. ($P_{\mathcal{E}}$ is not hermitian.) The restriction of $P_{\mathcal{E}}$ to $\mathcal{H}_X$ is *monotone* on $\mathcal{H}_X$.

If $\mathcal{E}$ and $\mathcal{F}$ are compatible ultrahermitian decompositions and $\mathcal{G}$ is their common refinement as above, then

$$P_{\mathcal{G}} = P_{\mathcal{E}} P_{\mathcal{F}} = P_{\mathcal{F}} P_{\mathcal{E}} \, .$$

4.3 Examples of ultrahermitian projections

The hermitian operators on most common spaces have been catalogued: see, for example, [2]. There are not many, apart from patchings of 'real diagonal operators' (except of course on Hilbert spaces), and the projections among them are all ultrahermitian.

It would be interesting to find a hermitian projection that is not ultrahermitian in a nontrivial way.

4.3.1 l^p type decompositions/patchings

If E is a projection on X and if (*cf* [4, §15])

$$\|x\|^p = \|Ex\|^p + \|\overline{E}x\|^p$$

for all $x \in X$ (for some p in the range $[1, \infty]$) then E is hermitian, and even ultrahermitian. For, given contractions A and B,

$$\begin{aligned} \|(EAE + \overline{E}B\overline{E})x\|^p &= \|EAEx\|^p + \|\overline{E}B\overline{E}x\|^p \\ &\leq \|Ex\|^p + \|\overline{E}x\|^p \\ &= \|x\|^p \, . \end{aligned}$$

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