How far is $\left(1+\dfrac{a}{n}\right)^n$ *from* e^a ?

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Abstract

We present effective upper and lower bounds for the distance from $\left(1+\dfrac{a}{n}\right)^n$ to e^a for an element a of a complex unital Banach algebra and positive integer n. Specifically:

$$\frac{1}{2n}\sup\left\{\left|\Re(z^2)\right| e^{\Re(z)} : z \in \sigma(a)\right\} \ \lesssim_{(2)}\ \left\|e^a - \left(1+\frac{a}{n}\right)^n\right\| \ \le\ \frac{\|a\|^2}{2n}\, e^{\|a\|}$$

where $\sigma(a)$ is the spectrum of a. The symbol $\lesssim_{(p)}$ means *"less than or equal to, up to a term of order n^{-p}"* as discussed below.

1 Introduction — technical preliminaries

The purpose of this paper is to establish asymptotic estimates for the quantity

$$\delta(a,n) \ =\ \left\|e^a - \left(1+\frac{a}{n}\right)^n\right\| \tag{1.1}$$

where a is an element of a Banach algebra $\mathcal{A}$ and n is a (large) positive integer. We tackle this problem in three stages: (i) for $\mathbb{R}$ [§2], (ii) for $\mathbb{C}$ [§3], and (iii) for general $\mathcal{A}$ [§4].

The following notation will be useful in presenting our results.

Notation 1. *For a fixed positive integer p and functions a and b defined on $\mathbb{N}$ the expression $a(n) \gtrsim_{(p)} b(n)$ is shorthand for $a(n) \ge b(n) + O(n^{-p})$: that is,*

$$a(n) - b(n) \ \ge\ \frac{M}{n^p} \qquad (n \ge N)$$

where M and N are constants (N positive) independent of n. The symbols $\lesssim_{(p)}$ and $\simeq_{(p)}$ are defined analogously. These three relations are all transitive.
We shall later extend the use of this notation to instances where the argument n may run through the set of half-integers.

Our treatment depends on the estimates of [2, Corollaries 1.1 & 1.2].

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Proposition 1 (*Essential Estimates*)**.** *Consider real numbers x and t. If $x > 0$ and $t > x$ then*

$$\exp\left(x - \frac{x^2}{2t}\right) \;<\; \left(1 + \frac{x}{t}\right)^t \;<\; \exp\left(x - \frac{x^2}{2(t+x)}\right),$$

while if $x < 0$ and $t > |x|$ we have

$$\exp\left(x - \frac{x^2}{2(t+x)}\right) \;<\; \left(1 + \frac{x}{t}\right)^t \;<\; \exp\left(x - \frac{x^2}{2t}\right).$$

Lemma 1. *The lengths of the 'indeterminacy intervals' of Proposition 1 satisfy*

$$\left.\begin{array}{ll} (x>0) & \exp\left(x - \dfrac{x^2}{2(t+x)}\right) - \exp\left(x - \dfrac{x^2}{2t}\right) \\[2ex] (x<0) & \exp\left(x - \dfrac{x^2}{2t}\right) - \exp\left(x - \dfrac{x^2}{2(t+x)}\right) \end{array}\right\} \;<\; \frac{5}{4}\frac{|x|^3}{t^2}$$

when $t > 2\max\{|x|, x^2\} > 0$.

Proof. When $x > 0$ the interval in question has length

$$\begin{aligned} &\exp\left(-\frac{x^2}{2(t+x)}\right) - \exp\left(-\frac{x^2}{2t}\right) \\ &= \exp\left(-\frac{x^2}{2t}\right)\left\{\exp\left(\frac{x^2}{2}\left[\frac{1}{t} - \frac{1}{t+x}\right]\right) - 1\right\} \\ &= \exp\left(-\frac{x^2}{2t}\right)\left\{\exp\left(\frac{x^2}{2t}\frac{x}{t+x}\right) - 1\right\} \\ &< \exp(r) - 1, \end{aligned}$$

where $r = \dfrac{|x|^3}{2t(t+x)}$. Similarly, when $x < 0$ the interval has length

$$\begin{aligned} &\exp\left(-\frac{x^2}{2t}\right) - \exp\left(-\frac{x^2}{2(t+x)}\right) \\ &= \exp\left(-\frac{x^2}{2(t+x)}\right)\left\{\exp\left(\frac{x^2}{2}\left[\frac{1}{t+x} - \frac{1}{t}\right]\right) - 1\right\} \\ &= \exp\left(-\frac{x^2}{2(t+x)}\right)\left\{\exp\left(\frac{x^2}{2(t+x)}\frac{-x}{t}\right) - 1\right\} \end{aligned}$$

which again, for the same r,

$$< \exp(r) - 1.$$

Now, under our hypotheses, $2(t+x) = t + (t+2x) \geq t$ and therefore $0 < r < \dfrac{|x|}{t}\dfrac{x^2}{t} < \dfrac{1}{4}$. Hence $e^r - 1 \;<\; r + r^2 \;<\; \dfrac{5}{4}r \;<\; \dfrac{5}{4}\dfrac{|x|^3}{t^2}$. □

Corollary 1. *For any real x we have*

$$\left(1+\frac{x}{m}\right)^m \simeq_{(2)} \exp\left(x-\frac{x^2}{2m}\right).$$

Specifically:

$$\left|\left(1+\frac{x}{m}\right)^m - \exp\left(x-\frac{x^2}{2m}\right)\right| < \frac{5}{4}\,\frac{|x|^3}{m^2}\,\exp(x) \tag{1.2}$$

when $x \neq 0$ and $m > 2\max\{|x|, x^2\}$ is a (half-)integer.

2 Real case — distance from $\left(1+\frac{x}{n}\right)^n$ to e^x

When x is real we have $\left(1+\frac{x}{n}\right)^n < e^x$ for any positive integer n and therefore (1.1) simplifies to

$$\delta(x,n) = e^x - \left(1+\frac{x}{n}\right)^n.$$

Lemma 2. *For any real x we have*

$$\exp\left(x-\frac{x^2}{2n}\right) \simeq_{(2)} e^x - \frac{x^2}{2n}\,e^x.$$

Proof. It is straightforward to establish that $1-r+\frac{r^2}{4} < \exp(-r) < 1-r+\frac{r^2}{2}$ for $0<r<1$. Thus, if given $x \neq 0$ and $n > x^2/2$ we define $r = \frac{x^2}{2n}$, then $0<r<1$ and

$$\exp\left(x-\frac{x^2}{2n}\right) - \left(e^x - \frac{x^2}{2n}\,e^x\right) = \exp(x)\Big[\exp(-r) - 1 + r\Big] \in \left(\frac{r^2}{4}, \frac{r^2}{2}\right)\exp(x).$$

That is

$$\frac{x^4}{16n^2}\,e^x < \exp\left(x-\frac{x^2}{2n}\right) - \left(e^x - \frac{x^2}{2n}\,e^x\right) < \frac{x^4}{8n^2}\,e^x \tag{2.1}$$

for $x \neq 0$. □

Theorem 1. *For any real x we have*

$$\delta(x,n) \simeq_{(2)} \frac{x^2}{2n}\,e^x.$$

Specifically, for x real, $\neq 0$, and $n \geq 2\max\{|x|, x^2\}$ we have

$$\left|\delta(x,n) - \frac{x^2}{2n}\,e^x\right| < \frac{|x|^3\,(|x|+10)}{8n^2}\,e^x. \tag{2.2}$$

Proof. Combining Corollary 1 and Lemma 2 gives

$$\left(1+\frac{x}{n}\right)^n \ \simeq_{(2)} \ \exp\left(x-\frac{x^2}{2n}\right) \ \simeq_{(2)} \ e^x - \frac{x^2}{2n}\,e^x$$

(recall that $\simeq_{(2)}$ is transitive) and therefore

$$\delta(x,n) \ = \ e^x - \left(1+\frac{x}{n}\right)^n \simeq_{(2)} \ \frac{x^2}{2n}\,e^x.$$

More precisely, using (1.2) and (2.1), for $x \neq 0$,

$$-\frac{5}{4}\frac{|x|^3}{n^2}\,e^x \ < \ \left(1+\frac{x}{n}\right)^n - \exp\left(x-\frac{x^2}{2n}\right) \ < \ \frac{5}{4}\frac{|x|^3}{n^2}\,e^x$$

and

$$\frac{x^4}{16n^2}\,e^x \ < \ \exp\left(x-\frac{x^2}{2n}\right) - \left(e^x - \frac{x^2}{2n}\,e^x\right) < \frac{x^4}{8n^2}\,e^x,$$

so, adding, we get

$$\frac{x^4}{16n^2}\,e^x - \frac{5}{4}\frac{|x|^3}{n^2}\,e^x \ < \ \frac{x^2}{2n}\,e^x - \delta(x,n) \ < \ \frac{x^4}{8n^2}\,e^x + \frac{5}{4}\frac{|x|^3}{n^2}\,e^x \ = \ \frac{|x|^3\,(|x|+10)}{8n^2}\,e^x,$$

from which (2.2) follows. □

Remark 1. *This result is more precise than the restriction of Theorem 2 or Theorem 5 to real z.*

3 Complex case – distance from $\left(1+\frac{z}{n}\right)^n$ to e^z

Notation 2. *For $z = x+iy \in \mathbb{C}$ (that is, $x = \Re(z)$ and $y = \Im(z)$) and for $n \in \mathbb{N}$ define $m = \frac{n}{2}$ (so, from now on, m will be a positive (half-)integer). Write*

$$\xi = x + \frac{|z|^2}{4m} \in \mathbb{R} \qquad \& \qquad \zeta = \left|1+\frac{z}{n}\right|^2 \tag{3.1}$$

so that

$$\zeta = 1 + \frac{x}{m} + \frac{|z|^2}{4m^2} = 1 + \frac{\xi}{m}. \tag{3.2}$$

By the triangle inequality

$$\begin{aligned} \delta(z,n) \ &= |e^z - \zeta^m| \ \geq \ \left||e^z| - \left|1+\frac{z}{n}\right|^n\right| \\ &= \ |e^x - \zeta^m|\,. \end{aligned} \tag{3.3}$$

Further, let

$$\omega(m) \ = x + \frac{y^2 - x^2}{4m}. \tag{3.4}$$

Then

$$\xi - \frac{\xi^2}{2m} \quad = \quad x + \frac{y^2 - x^2}{4m} - \frac{x\,|z|^2}{4m^2} - \frac{|z|^4}{32m^3} \;=\; \omega(m) - \eta(m), \tag{3.5}$$

where

$$\eta(m) \quad = \quad \frac{x\,|z|^2}{4m^2} + \frac{|z|^4}{32m^3} \quad = \quad \frac{|z|^2}{32m^3}\Big(8mx + |z|^2\Big) \tag{3.6}$$

has the same sign as x for large m (when $x \neq 0$).

Standing assumption

For the rest of this section z will be a nonzero complex number and m a (half-)integer such that

$$m \geq \max\{1,\ 4\,|z|\,,\ 4\,|z|^2\}. \tag{3.7}$$

Remark 2. *Given $z = x + iy \in \mathbb{C}$ and a positive half-integer $m \geq \max\{1,\ 4\,|z|\,,\ 4\,|z|^2\}$ we have*

$$-\,|z| \;\leq\; x \leq \xi \leq |z| + |z|\,\frac{|z|}{4m} \leq \frac{17}{16}\,|z|\,.$$

Thus

$$\omega \;\leq\; x + \frac{1}{16}, \quad \xi \;\leq x + \frac{1}{16} \quad \& \quad 2\,\max\{|\xi|\,, |\xi|^2\} < m.$$

We establish the main result of this section, Theorem 2, by demonstrating the chain of asymptotic equalities

$$\zeta^m \;\simeq_{(2)}\; \exp\left(\xi - \frac{\xi^2}{2m}\right) \;\simeq_{(2)}\; \exp(\omega(m)) \;\simeq_{(2)}\; e^x\left(1 + \frac{y^2 - x^2}{4m}\right).$$

Lemma 3.

$$\zeta^m \;\simeq_{(2)}\; \exp\left(\xi - \frac{\xi^2}{2m}\right).$$

Proof. Apply (1.2) (with ξ in place of x). Then, bearing in mind Remark 2, and also the arithmetical fact that $\frac{5}{4}\left(\frac{17}{16}\right)^3 e^{\frac{1}{16}} < 1.6$, we have

$$\begin{aligned}\left|\zeta^m - \exp\left(\xi - \frac{\xi^2}{2m}\right)\right| &< \frac{5}{4}\frac{|\xi|^3}{m^2}\exp(\xi) \\ &< \frac{5}{4}\left(\frac{17}{16}\right)^3 \frac{|z|^3}{m^2}\exp\left(x + \frac{1}{16}\right) \;<\; 2\,\frac{|z|^3}{m^2}\exp\left(x\right).\end{aligned}$$

□

Lemma 4.

$$\exp\left(\xi - \frac{\xi^2}{2m}\right) \;\simeq_{(2)}\; \exp(\omega(m)).$$

Proof. According to (3.6), we have

$$
\begin{aligned}
|\eta| &\leq \frac{|z|^3}{4m^2} + \frac{|z|^4}{32m^3} = \frac{|z|^3}{4m^2}\left(1 + \frac{|z|}{8m}\right) \\
&\leq \frac{|z|^3}{4m^2}\left(1 + \frac{1}{2}\right) = \frac{3}{8}\frac{|z|^3}{m^2} \leq \frac{3}{128} < 1, \qquad (3.8)
\end{aligned}
$$

from which $|e^{-\eta} - 1| < 2|\eta|$. Thus, using (3.5), (3.4), and (3.8), together with Remark 2,

$$
\begin{aligned}
\left|\exp\left(\xi - \frac{\xi^2}{2m}\right) - \exp(\omega)\right| &\overset{(3.5)}{\leq} \exp(\omega)\left|e^{-\eta} - 1\right| \overset{(3.4)}{\leq} \left|e^{-\eta} - 1\right| \exp\left(\tfrac{1}{16}\right)\exp(x) \\
&\leq 2|\eta| \exp\left(\tfrac{1}{16}\right)\exp(x) \overset{(3.8)}{\leq} 2\left(\frac{3}{8}\frac{|z|^3}{m^2}\right)\exp\left(\tfrac{1}{16}\right)\exp(x) \\
&\leq \frac{|z|^3}{m^2}\exp(x).
\end{aligned}
$$

□

Lemma 5.

$$
\exp(\omega(m)) \simeq_{(2)} e^x\left\{1 + \frac{y^2 - x^2}{4m}\right\}.
$$

Proof. Recall that $|e^r - 1 - r| \leq r^2 \quad (-1 < r < 1)$. With $r = \dfrac{y^2 - x^2}{4m}$ we have $|r| \leq \dfrac{|z|^2}{4m} \leq \dfrac{1}{16}$, and therefore

$$
\left|\exp\left(\frac{y^2 - x^2}{4m}\right) - 1 - \frac{y^2 - x^2}{4m}\right| \leq \frac{|z|^4}{16m^2},
$$

from which

$$
\left|\exp\left(x + \frac{y^2 - x^2}{4m}\right) - e^x\left(1 + \frac{y^2 - x^2}{4m}\right)\right| \leq \frac{|z|^4}{16m^2}e^x.
$$

□

Theorem 2. *Given $z = x + iy \in \mathbb{C}$ we have*

$$
\delta(z, n) = \left|e^z - \left(1 + \frac{z}{n}\right)^n\right| \gtrsim_{(2)} \frac{|\Re(z^2)|}{2n}e^{\Re(z)}.
$$

Specifically:

$$
\delta(z, n) \geq \frac{|\Re(z^2)|}{2n}e^{\Re(z)} - \frac{12|z|^3}{n^2}\left(1 + \frac{|z|}{48}\right)e^{\Re(z)}
$$

for $n \geq 2\max\{1, 4|z|, 4|z|^2\}$.

Proof. Since $m > 2\max\{|\xi|, |\xi|^2\}$, see Remark 2, we can apply Lemmas 3, 4 and 5 to get

$$\begin{aligned}\left|\zeta^m - e^x - \frac{y^2-x^2}{4m}\,e^x\right| &\leq \left|\zeta^m - \exp\left(\xi - \frac{\xi^2}{2m}\right)\right| \\ &\quad + \left|\exp\left(\xi - \frac{\xi^2}{2m}\right) - \exp(\omega)\right| \\ &\quad + \left|e^x\left[\exp\left(\frac{y^2-x^2}{4m}\right) - 1 - \frac{y^2-x^2}{4m}\right]\right| \\ &\leq 2\,\frac{|z|^3}{m^2}\exp(x) + \frac{|z|^3}{m^2}\exp(x) + \frac{|z|^4}{16m^2}\exp(x) \\ &= \frac{M}{m^2}\end{aligned}$$

where $M = 3\,|z|^3\left(1 + \frac{|z|}{48}\right)e^x$ depends only on z. Now

$$\left||\zeta^m - e^x| - \frac{|y^2-x^2|}{4m}\,e^x\right| \leq \left|\zeta^m - e^x - \frac{y^2-x^2}{4m}\,e^x\right| \leq \frac{M}{m^2}$$

and therefore

$$\delta(z,n) \overset{(3.3)}{\geq} |\zeta^m - e^x| \geq \frac{|y^2-x^2|}{4m}\,e^x - \frac{M}{m^2}\,. \qquad \square$$

Remark 3. *The upper bound*

$$\delta(z,n) = \left|e^z - \left(1 + \frac{z}{n}\right)^n\right| \leq \frac{|z|^2}{2n}\,e^{|z|},$$

valid for $z \in \mathbb{C}$ and $n > |z|$, is the scalar variant of Theorem 5 below. Note the presence of the term $e^{|z|}$ in contrast to the e^x of Theorem 1.

When z lies on an (anti)diagonal

Theorem 2 tells us little when $\Re(z^2) = 0$, that is, when z lies on $\mathcal{X}$, the union of the diagonal and antidiagonal in $\mathbb{C}$:

$$\mathcal{X} = \{z \in \mathbb{C} : x^2 = y^2\} = \{z \in \mathbb{C} : \Re(z^2) = 0\}.$$

We can, however, then derive a higher order estimate for the lower bound.

Theorem 3. *Let $z = x + iy\ (\neq 0) \in \mathbb{C}$ with $x^2 = y^2$. Then*

$$\delta(z,n) \gtrsim_{(3)} \frac{|\Re(z)|^3}{2n^2}\,e^{\Re(z)}.$$

Specifically,

$$\left|e^z - \left(1 + \frac{z}{n}\right)^n\right| > \frac{|x|^3}{2n^2}\,e^x - \frac{|x|\,x^3(1+x)}{n^3}\,e^x \qquad (n > 4\,|x|).$$

Proof. Write $l = \dfrac{x}{m}$. Then, by (3.7) and (3.2), we have

$$|\, l| < 1 \qquad \& \qquad \zeta = 1 + \frac{\xi}{m} = 1 + l + \frac{l^2}{2}. \tag{3.9}$$

Let I be the interval with end-points e^l and ζ. Then

$$\begin{aligned} \delta(z, n) &\geq \left|(e^l)^m - \zeta^m\right| \\ &\geq \left|e^l - \zeta\right| \min_{r \in I} \left(\frac{\mathrm{d}}{\mathrm{d}r}\,(r^m)\right) \\ &= m\left|e^l - \zeta\right| \min\left\{\exp\left(\frac{m-1}{m}x\right),\ \zeta^{m-1}\right\}. \end{aligned}$$

Now

$$\begin{aligned} \left|e^l - \zeta\right| &\overset{(3.9)}{=} \left|\frac{l^3}{3!} + \frac{l^4}{4!} + \frac{l^5}{5!} + \dots\right| = \left|\frac{l^3}{3!}\left(1 + \frac{l}{4}\right) + \frac{l^5}{5!}\left(1 + \frac{l}{6}\right) + \dots\right| \\ &\overset{(3.9)}{=} \frac{|\, l|^3}{3!}\left(1 + \frac{l}{4}\right) + \frac{|\, l|^5}{5!}\left(1 + \frac{l}{6}\right) + \dots \geq \frac{3}{4}\frac{|\, l|^3}{3!} = \frac{|\, l|^3}{8} = \frac{|\, x|^3}{8m^3}. \end{aligned}$$

Further, recalling the definitions (3.1) and (3.2), we see that $\xi > x$. Moreover, $0 < 1 + \dfrac{x}{m} < 1 + \dfrac{\xi}{m} = \zeta$ for $m > |x|$. Thus, by Proposition 1,

$$\begin{aligned} \zeta^{m-1} &> \left(1 + \frac{x}{m}\right)^{\frac{m-1}{m}} \\ &> \exp\left(\frac{m-1}{m}\left(x - \frac{x^2}{2m}\right)\right) \\ &> \exp\left(x - \frac{x}{m} - \frac{x^2}{2m}\right) \qquad\qquad if \quad m > x > 0, \end{aligned}$$

while

$$\begin{aligned} \zeta^{m-1} &> \exp\left(\frac{m-1}{m}\left(x - \frac{x^2}{2(m+x)}\right)\right) \\ &> \exp\left(x - \frac{x}{m} - \frac{x^2}{m}\right) \qquad\qquad if \quad m > -2x > 0. \end{aligned}$$

Hence

$$\min\left\{\exp\left(\frac{m-1}{m}x\right),\ \zeta^{m-1}\right\} > \exp\left(x - \frac{x}{m} - \frac{x^2}{m}\right)$$

and therefore

$$\begin{aligned} \delta(z, n) &> \frac{|x|^3}{8m^2}\exp\left(x - \frac{x}{m} - \frac{x^2}{m}\right) \qquad (m > 2\,|x|) \\ &\geq \frac{|x|^3}{8m^2}\, e^x \left(1 - \frac{x(1+x)}{m}\right). \end{aligned}$$

□

Supremum of $\{\delta(z,n) : z \in K\}$

Note that

$$\delta(z,n) \;\le\; |e^z| + \left(1+\frac{\xi}{m}\right)^m \;\le\; e^x + e^{\xi} = e^x\left(1+e^{\frac{|z|^2}{4m}}\right) \le e^x\left(1+e^{|z|}\right),$$

showing that the following is a good definition:

Definition 1. *Given a bounded subset K of $\mathbb{C}$ define*

$$\Delta(K,n) = \sup\{\delta(z,n) : z \in K\}$$

for each $n \in \mathbb{N}$.

Combining the results of Theorems 2 and 3 yields

Theorem 4. *Let K be a bounded subset of $\mathbb{C}$. Then, asymptotically, to within a term of the order n^{-2}, and uniformly over K,*

$$\Delta(K,n) \;\gtrsim_{(2)}\; \frac{1}{2n}\sup\left\{\left|\Re(z^2)\right| e^{\Re(z)} : z \in K\right\}.$$

If $K \subset \mathcal{X}$ then, asymptotically, to within a term of order n^{-3}, and uniformly over K,

$$\Delta(K,n) \;\gtrsim_{(3)}\; \frac{1}{2n^2}\sup\left\{\left|\Re(z)\right|^3 e^{\Re(x)} : z \in K\right\}.$$

4 How far is $\left(1+\frac{a}{n}\right)^n$ from e^a in a Banach algebra?

Upper bound for $\delta(a,n)$

Theorem 5. *Let $(\mathcal{A}, \|\cdot\|)$ be a real or complex norm-unital Banach algebra and $a \in \mathcal{A}$ with $a \neq 0$. Then, for every integer $n > \|a\|$,*

$$\begin{aligned}\delta(a,n) \;=\; \left\|e^a - \left(1+\frac{a}{n}\right)^n\right\| \;&\le\; e^{\|a\|} - \left(1+\frac{\|a\|}{n}\right)^n \\ &<\; e^{\|a\|}\left[1-\exp\left(-\frac{\|a\|^2}{2n}\right)\right] \\ &<\; \frac{\|a\|^2}{2n}\, e^{\|a\|}.\end{aligned}$$

Proof. Following [1, §8], for every $n \in \mathbb{N}$ we define the partial sums and powers

$$s_n = \sum_{k=0}^{n}\frac{a^k}{k!}, \quad \sigma_n = \sum_{k=0}^{n}\frac{\|a\|^k}{k!}, \quad b_n = \left(1+\frac{a}{n}\right)^n, \quad \beta_n = \left(1+\frac{\|a\|}{n}\right)^n. \tag{4.1}$$

Then

$$\lim_{n\to\infty} s_n = \exp(a) = \lim_{n\to\infty} b_n \quad \& \quad \lim_{n\to\infty}\sigma_n = \exp(\|a\|) = \lim_{n\to\infty}\beta_n. \tag{4.2}$$

Using the binomial expansions for b_n and β_n, and introducing the notation

$$\lambda_k(n) = \frac{1}{k\,!}\Big(1 - \frac{n!}{n^k(n-k)!}\Big)$$

for every integer k such that $0 \le k \le n$, we have

$$s_n - b_n = \sum_{k=0}^{n} \underbrace{\frac{1}{k!}\left(1 - \frac{n!}{n^k}\frac{1}{(n-k)!}\right)}_{=\lambda_k(n)\ge 0} a^k = \sum_{k=0}^{n} \lambda_k(n)\, a^k \tag{4.3}$$

and

$$\sigma_n - \beta_n = \sum_{k=0}^{n} \lambda_k(n) \|a\|^k \tag{4.4}$$

with obviously all $\lambda_k(n) \ge 0$, where

$$\lambda_0(n) \equiv \lambda_1(n) \equiv 0 \quad (n \ge 1) \qquad \text{and} \qquad \lambda_2(n) \equiv \frac{1}{2n} \quad (n \ge 2)\,.$$

Using the triangle inequality we estimate

$$\|e^a - b_n\| = \|(e^a - s_n) + (s_n - b_n)\| \le \|e^a - s_n\| + \|s_n - b_n\|\,.$$

Now, considering (4.1)–(4.2),

$$\|e^a - s_n\| = \left\| \sum_{k=n+1}^{\infty} \frac{a^k}{k!} \right\| \le \sum_{k=n+1}^{\infty} \frac{\|a\|^k}{k!} = e^{\|a\|} - \sigma_n,$$

and, referring to (4.3) and (4.4),

$$\|s_n - b_n\| = \left\| \sum_{k=0}^{n} \lambda_k\, a^k \right\| \le \sum_{k=0}^{n} \lambda_k\, \|a\|^k = \sigma_n - \beta_n\,.$$

Consequently

$$\|e^a - b_n\| \le \big(e^{\|a\|} - \sigma_n\big) + (\sigma_n - \beta_n) = e^{\|a\|} - \beta_n\,.$$

Therefore, from (4.1), we obtain the estimate

$$\left\| e^a - \Big(1 + \frac{a}{n}\Big)^n \right\| \;\le\; e^{\|a\|} - \left(1 + \frac{\|a\|}{n}\right)^n$$

which, see Proposition 1,

$$< \; e^{\|a\|} \; - e^{\|a\|} \, \exp\left(- \frac{\|a\|^2}{2n} \right)$$

and this

$$< \; \frac{\|a\|^2}{2n}\, e^{\|a\|}$$

since $1 - e^{-r} < r$ $(r \in \mathbb{R})$. □

Asymptotic lower bound for $\delta(a,n)$

We write, as usual, $\sigma(a)$ for the *spectrum* and $|a|_\sigma$ for the *spectral radius* of an element a of a complex norm-unital Banach algebra.

Theorem 6. *Let $\mathcal{A}$ be a norm-unital complex Banach algebra. Then*

$$\begin{aligned}\delta(a,n) = \left\| e^a - \left(1+\frac{a}{n}\right)^n \right\| \quad &\geq \quad \left| e^a - \left(1+\frac{a}{n}\right)^n \right|_\sigma \\ &\geq \quad \Delta(\sigma(a),n) \\ &\gtrsim_{(2)} \quad \frac{1}{2n} \sup\left\{ \left|\Re(z^2)\right| e^{\Re(z)} : z \in \sigma(a) \right\}\end{aligned}$$

for any $a \in \mathcal{A}$ and positive integer n.
Thus, if a is not quasinilpotent, $\left\| e^a - \left(1+\frac{a}{n}\right)^n \right\|$ tends to 0 no faster than $O(n^{-1})$ unless $\sigma(a)$ is contained in $\mathcal{X}$ in which case the rate of convergence is no faster than $O(n^{-2})$.

Proof. Without loss of generality we may assume that $\mathcal{A}$ is commutative. Then

$$\sigma(a) = \{\phi(a) : \phi \in \Phi(\mathcal{A})\}$$

where $\Phi(\mathcal{A})$ is the set of *characters (nontrivial multiplicative functionals)* on $\mathcal{A}$.
Now, for any $\phi \in \Phi(\mathcal{A})$ and any n (recall that the norm dominates the spectral radius),

$$\begin{aligned}\left| e^{\phi(a)} - \left(1+\frac{\phi(a)}{n}\right)^n \right| \quad &= \quad \left| \phi\left(e^a - \left(1+\frac{a}{n}\right)^n \right) \right| \\ &\leq \quad \left| e^a - \left(1+\frac{a}{n}\right)^n \right|_\sigma \\ &\leq \quad \left\| e^a - \left(1+\frac{a}{n}\right)^n \right\|.\end{aligned}$$

The rest follows from Theorem 4. □

Remark 4. *The first inequality in Theorem 5 is sharp: we have equality if a is a positive real in $\mathbb{C}$.*
Theorem 6 too is sharp: if $a \neq 0$ but $a^2 = 0$ then $e^a = \left(1+\frac{a}{n}\right)^n$ for $n \geq 1$.

Hermitian elements of a Banach algebra

We refer to [1, §10] for the background on numerical range in Banach algebras. Recall that an element h of a complex norm-unital Banach algebra is *hermitian* if its *algebra numerical range* is real: equivalently, if $\left\| e^{irh} \right\| = 1$ $(r \in \mathbb{R})$: equivalently, if $\|1 + irh\| \leq 1 + o(r)$ $(\mathbb{R} \ni r \to 0)$. In analogy to Theorem 1, combining Theorems 5 & 6:

Theorem 7. *Let h be a hermitian element of a complex unital Banach algebra. Then*

$$\frac{\|h\|^2}{2n}\, e^{-\|h\|} \lesssim_{(2)} \left\| e^h - \left(1+\frac{h}{n}\right)^n \right\| \leq \frac{\|h\|^2}{2n}\, e^{\|h\|}.$$

When $\|h\| \in \sigma(h)$, then, more precisely,

$$\frac{\|h\|^2}{2n}\, e^{\|h\|} \;\lesssim_{(2)}\; \left\| e^h - \left(1+\frac{h}{n}\right)^n \right\| \;\le\; \frac{\|h\|^2}{2n}\, e^{\|h\|}.$$

Proof. For such an h we have $\sigma(h) \subset \mathbb{R}$ and $\|h\| = |h|_\sigma$: so then at least one of $\|h\|$ and $-\|h\|$ belongs to $\sigma(h)$. Thus, referring to Theorem 6,

$$\sup\left\{ \left|\Re(z^2)\right| e^{\Re(z)} : z \in \sigma(a) \right\} \;\ge\; \|h\|^2\, e^{-\|h\|} \qquad \textit{if} \;\; -\|h\| \in \sigma(h)$$

and

$$\sup\left\{ \left|\Re(z^2)\right| e^{\Re(z)} : z \in \sigma(a) \right\} \;\ge\; \|h\|^2\, e^{\|h\|} \qquad \textit{if} \;\; \|h\| \in \sigma(h). \qquad \square$$

Remark 5. *Our upper estimate (Theorem 5) for the size of $e^a - \left(1+\frac{a}{n}\right)^n$ holds for an element of a real or complex norm-unital Banach algebra, while for the lower bound (Theorem 6) we require the algebra to be complex. Qualitatively speaking, this is no essential restriction because: (i) if a Banach algebra $\mathcal{A}$ has a unit 1 whose norm is $\neq 1$ we can construct an equivalent Banach algebra norm on $\mathcal{A}$ for which $\|1\| = 1$; and (ii) if $\mathcal{A}$ has no unit we can adjoin a unit and renorm so that this unit has norm 1; and (iii) if $\mathcal{A}$ is a Banach algebra over the real field we can embed it isometrically in a complex Banach algebra (see [1, §§3, 4 & 13]). Note that the expression $e^a - \left(1+\frac{a}{n}\right)^n$ can be interpreted as the limit of the sequence $\sum_{k=1}^{N} \left\{ \frac{a^k}{k!} - \binom{n}{k} \frac{a^k}{n^k} \right\}$ as $N \to \infty$ in any Banach algebra (even if it is neither unital nor complex).*

References

[1] F. F. Bonsall & J. Duncan, Complete Normed Algebras, Springer (1973).

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