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Laplace maps and constraints for a class of third order partial differential operators

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Abstract

We explore the existence of a class of generalised Laplace maps for third order partial differential operators of the form

$$\partial_1\partial_2\partial_3 + a_1\partial_2\partial_3 + a_2\partial_1\partial_3 + a_3\partial_1\partial_2 + a_{12}\partial_3 + a_{23}\partial_1 + a_{13}\partial_2 + a_{123}$$

and related first order 3 systems and show that they require the satisfaction of constraints on the invariants for such operators.

1 Introduction

The classical Laplace maps are transformations of an essentially algebraic nature between second order, linear, partial differential operators. They were developed by Darboux in [4] and can be thought about in a variety of ways with application to several areas. They are (generically invertible) maps between invariants associated with different operators.

The nomenclature surrounding this type of map is neither fixed nor always clear. So it is that the classical maps are sometimes described as Laplace transformations (rather than maps), as belonging to the more general class of *Darboux transformations* and as *intertwining* Laplace transformations (in order to distinguish this approach from other possible avenues of generalization). Such maps may also be considered to act either on the partial differential operators themselves or on the elements of their kernels, the homogeneous solution spaces.

In the theory of integrable systems the Toda lattice [23] has been an influential paradigm. It is related to the theory of Laplace maps in that the maps generate three term recurrence relations on the (indexed) invariants which are exactly the equations of the Toda chain on \mathbb{Z} over \mathbb{R}^2 [26]. The vanishing of an invariant corresponds to the end of the chain and to a factorizable operator. From the kernel of this factorizable case all the invariants on the chain can be generated.

The Laplace maps can also be lifted to transformations of non-linear systems of hydrodynamic type because the Riemann invariants satisfy second order, linear partial differential equations [6, 27].

In geometry the transformations describe maps between immersed surfaces and their accompanying conjugate nets (coordinates), a connection generalised to higher dimensions in [11, 12]. This relation can be developed in discrete geometry also [5, 16].

As geometrical objects the invariants themselves can be derived following the Cartan method of moving frames [20].

From a purely algebraic point of view, Laplace maps have been studied in work on factorization [3, 10, 21, 22, 24, 25]. The general feeling appears to be that the classical instance of Laplace maps is not easily generalizable.

Further classes of generalization include *Darboux transformations of type I* [17, 18, 19] and of *continued type* [9]. A looser and more general notion of *intertwining Laplace transformation* than that presented in this paper can be found in [7].

Generalizations of an even less classical nature involve quasideterminants [14] and supersymmetry or geometry [13, 15, 8].

This paper has two immediate predecessors. The first is [1] where Laplace maps for 3×3 systems are discussed. The current paper will slightly generalise that work. The second is [2] where a classification of certain invariants for a very large class of arbitrary order invariants is given.

Maps of this sort are usually characterised by an intertwining property of the form

$$A^\sigma a = a^\sigma A$$

where one is interested in the kernel of A^σ in relation to that of A :

$$a : \ker A \rightarrow \ker A^\sigma.$$

In this account we describe several classes of intertwining relation for third order partial differential operators in scalar and system form and show that intertwining maps of this type can exist only when certain constraints on the invariants of the operators are satisfied. This is perhaps, compared with the second order case, a disappointing result but it is compatible with the general difficulty of generalisation apparent in the literature. It is also the case that these constraints are not preserved as functions of the transformed invariants so that one may not successively apply the Laplace maps.

Further we compare the results for scalar and system forms. It is not generally true that scalar third order partial differential equations can be written in 3×3 system form nor vice versa. Again, certain conditions on invariants have to be satisfied and we look at the way the Laplace maps correspond under these conditions.

In section 2 we review the classical situation, presenting it in a form convenient to generalisation by introducing invariants as zeroth order differential operators constructed from the natural first and second order operators of the theory.

In the next section a technical trick is introduced which allows us to analyse polynomial identities between differential operators by reducing them to successively lower order identities and we introduce a deformation of the standard Laplace map.

Section 4 attempts to generalise the approach of section 2 to third order differential operators considering two scenarios: firstly first order and then second order intertwining operators. We find that constraints arise in each case, but they are fewer in number for the second order intertwiner. We then review the 3×3 system. Laplace maps for this case have been considered before but we clarify here the role of constraints needed for the Laplace map to function and we discuss the relation between scalar and system forms and their overlap when a Laplace map exists.

Throughout the paper our philosophy is to work with noncommutative polynomials generated by a finite set of (noncommuting) linear differential operators.

In particular our starting point is always a single linear operator of total degree n in derivations $\partial_1, \dots, \partial_n$ but which is of degree one only in each distinct derivation. In [2] these were called *hyperbolic* but this suggests a reality condition that does not feature in the discussion.

Although in that paper systematic processes describe the construction of invariants, the calculations in the current paper are somewhat *ad hoc* and a deeper understanding of their structure would be required for discussion of higher orders to be feasible.

Finally we make some concluding remarks and suggestions for further exploration.

2 The Classical case

2.1 Scalar form

Consider the second order partial differential operator

$$L_{12} = \partial_1 \partial_2 + a_2 \partial_1 + a_1 \partial_2 + a_{12}.$$

The coefficients belong to a differential field with derivations ∂_1 and ∂_2 and we assume no relations (algebraic or differential) between them. We write $a_{i,j}$ for the ∂_j derivative of a_i etc. Note that the operator is symmetric in indices: $L_{12} = L_{21}$.

We are interested in invariants of such operators under transformations of the form: $L_{12} \mapsto L_{12}^g = g^{-1} L_{12} g$ where g is an arbitrary element of the differential field or an extension thereof. Invariants are constructed by defining

$$L_1 = \partial_1 + a_1, \quad L_2 = \partial_2 + a_2$$

and writing down the functions

$$I_{12} = L_{12} - L_1 L_2, \quad I_{21} = L_{12} - L_2 L_1.$$

These are invariants by virtue of being differential functions (zeroth order operators) of the coefficients:

$$\begin{aligned} I_{12}^g &= L_{12}^g - L_1^g L_2^g \\ &= g^{-1} (L_{12} - L_1 L_2) g \\ &= I_{12} \end{aligned}$$

Thus

$$L_{12} = L_1 L_2 + I_{12} = L_2 L_1 + I_{21}$$

where $I_{12} = a_{12} - a_1 a_2 - a_{2,1}$ and $I_{21} = a_{12} - a_1 a_2 - a_{1,2}$.

Now suppose there is an element ϕ (in a field extension) such that $L_{12} \phi = 0$. Define $\phi^\sigma = L_2 \phi$ and $\phi^\Sigma = L_1 \phi$. These satisfy the pairs

$$\begin{aligned} L_2 \phi &= \phi^\sigma \\ L_1 \phi^\sigma &= -I_{12} \phi \end{aligned} \tag{1}$$

and

$$\begin{aligned} L_1 \phi &= \phi^\Sigma \\ L_2 \phi^\Sigma &= -I_{21} \phi \end{aligned} \tag{2}$$

The σ - and Σ -Laplace transformed equations are those satisfied by ϕ^σ and ϕ^Σ obtained by eliminating ϕ from the above pairs.

$$\begin{aligned}(L_2^\sigma L_1 + I_{12})\phi^\sigma &= (L_1 L_2^\sigma + I_{12} + [L_2^\sigma, L_1])\phi^\sigma = 0 \\ (L_1^\Sigma L_2 + I_{21})\phi^\Sigma &= (L_2 L_1^\Sigma + I_{21} + [L_1^\Sigma, L_2])\phi^\Sigma = 0\end{aligned}$$

where $L_2^\sigma = I_{12} L_2 I_{12}^{-1}$ and $L_1^\Sigma = I_{21} L_1 I_{21}^{-1}$. This implies transformations on the invariants:

$$\begin{aligned}I_{21}^\sigma &= I_{12} \\ I_{12}^\sigma &= I_{12} + [L_2^\sigma, L_1] \\ I_{21}^\Sigma &= I_{21} + [L_1^\Sigma, L_2] \\ I_{12}^\Sigma &= I_{21}\end{aligned}$$

and correspondingly:

$$L_1^\sigma = L_1, \quad L_2^\Sigma = L_2.$$

These relations are all summarised in the simple intertwining relations

$$\begin{aligned}L_{12}^\sigma L_2 &= L_2^\sigma L_{12} \\ L_{12}^\Sigma L_1 &= L_1^\Sigma L_{12}.\end{aligned}$$

For example, the first implies the following chain of argument.

$$\begin{aligned}(L_2^\sigma L_1^\sigma + I_{21}^\sigma)L_2 &= L_2^\sigma(L_1 L_2 + I_{12}), \\ L_2^\sigma(L_1^\sigma - L_1)L_2 &= L_2^\sigma I_{12} - I_{21}^\sigma L_2.\end{aligned}$$

By looking at leading order terms in differential operators,

$$\begin{aligned}L_1^\sigma &= L_1 \\ I_{21}^\sigma &= I_{12} \\ L_2^\sigma &= I_{12} L_2 I_{12}^{-1}\end{aligned}$$

and finally, since $L_{12}^\sigma = L_{21}^\sigma$,

$$\begin{aligned}I_{12}^\sigma &= I_{21}^\sigma + [L_2^\sigma, L_1^\sigma] \\ &= I_{12} + [L_2^\sigma, L_1] \\ &= 2I_{12} - I_{21} + (\log I_{12})_{,12}.\end{aligned}$$

We can however, also deform the intertwining relation by writing:

$$L_{12}^\sigma L_2' = L_2'^\sigma L_{12}, \quad (3)$$

where the primed operators are monic in ∂_2 still but with coefficients distinct from the unprimed operators. We consider this case at the conclusion of the next section.

2.2 System form

The situation just described can always be represented in system form. We can write the pairs of equations for ϕ^σ and ϕ^Σ as

$$\begin{aligned} \begin{pmatrix} L_2 & -1 \\ I_{12} & L_1 \end{pmatrix} \begin{pmatrix} \phi \\ \phi^\sigma \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} L_2 & I_{21} \\ -1 & L_1 \end{pmatrix} \begin{pmatrix} \phi^\Sigma \\ \phi \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

and the intertwining relations become

$$\begin{pmatrix} L_2^\sigma & -1 \\ I_{12}^\sigma & L_1^\sigma \end{pmatrix} \begin{pmatrix} L_2 & 0 \\ 0 & L_2^\sigma \end{pmatrix} = \begin{pmatrix} L_2^\sigma & 0 \\ [L_2^\sigma, L_1] & L_2^\sigma \end{pmatrix} \begin{pmatrix} L_2 & -1 \\ I_{12} & L_1 \end{pmatrix}$$

and

$$\begin{pmatrix} L_2^\Sigma & I_{21}^\Sigma \\ -1 & L_1^\Sigma \end{pmatrix} \begin{pmatrix} L_1^\Sigma & 0 \\ 0 & L_1 \end{pmatrix} = \begin{pmatrix} L_1^\Sigma & [L_1^\Sigma, L_2] \\ 0 & L_1^\Sigma \end{pmatrix} \begin{pmatrix} L_2 & I_{21} \\ -1 & L_1 \end{pmatrix}$$

3 Methodology

We adopt some ideas from the paper [2] where invariants of arbitrary order differential operators were constructed.

Let I be a set of distinct labels. We associate with I a partial differential operator in derivations ∂_i for $i \in I$:

$$L_I = \sum_{J \subseteq I} a_{I \setminus J} \partial_J.$$

Here the symbol ∂_J is the product of all derivations ∂_j for $j \in J$. All the coefficients a_K for $K \subseteq I$ are algebraically and differentially independent. We associate such an operator with any subset of I in a similar manner.

The coefficients are totally symmetric in their indices so that L_I is a function on the set; that is, L_I is totally symmetric in its indices.

For example if $I = \{1, 2, 3\}$,

$$\begin{aligned} L_{123} &= \partial_1 \partial_2 \partial_3 + a_1 \partial_2 \partial_3 + a_2 \partial_3 \partial_1 + a_3 \partial_1 \partial_2 \\ &\quad + a_{12} \partial_3 + a_{23} \partial_1 + a_{31} \partial_2 + a_{123} \\ L_{12} &= \partial_1 \partial_2 + a_1 \partial_2 + a_2 \partial_1 + a_{12} \\ L_{23} &= \partial_2 \partial_3 + a_2 \partial_3 + a_3 \partial_2 + a_{23} \\ L_{31} &= \partial_3 \partial_1 + a_3 \partial_1 + a_1 \partial_3 + a_{31} \\ L_1 &= \partial_1 + a_1 \\ L_2 &= \partial_2 + a_2 \\ L_3 &= \partial_3 + a_3 \end{aligned}$$

We consider the non-commutative polynomial ring over in these operators over some constants K :

$$K_I = K[\{L_J | J \subseteq I\}].$$

The invariants are the zeroth order differential elements of this ring. For example, in $K_{\{1,2\}}$ both $L_{12} - L_1L_2$ and $L_{21} - L_2L_1$ are zeroth order. Quite generally, for any index set I , because the differential operators transform as $L_J \mapsto g^{-1}L_Jg$, for an arbitrary function g , any polynomial in the L_J , $F(L_J|J \subseteq I)$, transforms similarly: $F \mapsto g^{-1}Fg$. In the case where such an F happens to be a zeroth order operator (i.e. a function), it is therefore invariant.

In the paper [2] a large class of invariants is constructed.

We can describe such invariant elements by finding the kernel of a map Θ defined by

$$\Theta(L) = [L, \theta]$$

θ being regarded as an arbitrary function.

Θ acts as a derivation on the ring K_I . We can then take a set of such maps $\{\Theta_i|i \in I\}$ corresponding to differentiation of elements of K_I with respect to indices by choosing $\theta = x_i$.

We illustrate the methodology by a redescription of the classical case. Thus

$$\Theta_1(L_{12}) = L_2, \quad \Theta_1(L_2) = 0, \quad \Theta_1(L_1) = 1,$$

and so on. In particular we see that I_{12} and I_{21} are invariants because

$$\begin{aligned} \Theta_1(L_{12} - L_1L_2) &= L_2 - L_2 = 0, \\ \Theta_2(L_{12} - L_1L_2) &= L_1 - L_1 = 0, \end{aligned}$$

and similarly for I_{21} .

We can use the Θ_i maps to analyse the intertwining relation. By applying Θ_1 and Θ_2 successively we obtain, for the σ case for example, the following non trivial relations:

$$\begin{aligned} L_{12}^\sigma L_2 &= L_2^\sigma L_{12} \\ L_1^\sigma L_2 + L_{12}^\sigma &= L_{12} + L_2^\sigma L_1 \\ 2L_1^\sigma &= 2L_1. \end{aligned}$$

The last tells us that $L_1^\sigma = L_1$ and we then rearrange the second to give $I_{21}^\sigma = I_{12}$. The first equation can then be written

$$(L_{12}^\sigma - L_2^\sigma L_1^\sigma)L_2 = L_2^\sigma(L_{12} - L_1L_2),$$

equivalently

$$L_2^\sigma = I_{12}L_2I_{12}^{-1},$$

and finally

$$I_{12}^\sigma = I_{21}^\sigma + [L_2^\sigma, L_1^\sigma] = I_{12} + [L_2^\sigma, L_1].$$

Returning to the deformed case (3) we may analyse this in the same way. By derivation using Θ_1 and Θ_2 we obtain the tower:

$$\begin{aligned} L_{12}^\sigma L_2' &= L_2'^\sigma L_{12} \\ L_2^\sigma L_2' &= L_2'^\sigma L_2 \\ L_1^\sigma L_2' + L_{12}^\sigma &= L_{12} + L_2'^\sigma L_1 \\ L_2^\sigma + L_2' &= L_2'^\sigma + L_2 \\ L_1^\sigma &= L_1. \end{aligned}$$

From these equations (in reverse order) follow:

$$\begin{aligned}
L_1^\sigma &= L_1 \\
L_2'^\sigma &= L_2^\sigma + \phi \\
L_2' &= L_2 + \phi \\
L_2^\sigma \phi &= \phi L_2 \\
I_{21}^\sigma &= I_{12} - \phi_{,1} \tag{4}
\end{aligned}$$

$$I_{21}^\sigma \phi_{,2} - \phi I_{21,2}^\sigma = \phi^2 (I_{21} - I_{21}^\sigma) \tag{5}$$

Note that this deformation is not equivalent to the undeformed case: we cannot gauge away the ϕ term in the primed operators by writing $L_2' = g^{-1} L_2 g$ without introducing compensating terms in L_{12} and so on.

From equation (5) we obtain, by putting

$$\phi = \frac{I_{21}^\sigma}{\psi}$$

the relation

$$I_{21}^\sigma = I_{21} + \psi_{,2} = I_{21} + \left(\frac{I_{21}^\sigma}{\phi} \right)_{,2}.$$

Then ϕ is determined by the invariants I_{12} and I_{21} via the following equation:

$$I_{12} - I_{21} = \left(\frac{I_{12}}{\phi} \right)_{,2} + \phi_{,1} - (\log \phi)_{,12}.$$

In the limit that ϕ , but not $\phi^{-1}\phi_{,2}$, tends to zero, we recover the classical case:

$$\phi^{-1}\phi_{,2} = I_{12}^{-1} I_{12,2}$$

and $\phi = I_{12}$.

Otherwise this equation looks not easy to solve and is the object of further study.

4 Third order case

In the third order case, $I = \{1, 2, 3\}$ there are six invariants on two labels,

$$I_{ij} = L_{ij} - L_i L_j, \quad i, j \in \{1, 2, 3\}.$$

The Jacobi identity,

$$[L_1, [L_2, L_3]] + [L_2, [L_3, L_1]] + [L_3, [L_1, L_2]] = 0,$$

yields a single identity

$$I_{12,3} + I_{23,1} + I_{31,2} - I_{21,3} - I_{32,1} - I_{13,2} = 0.$$

On three labels we have the set

$$I_{ijk} = L_{ijk} - I_{jk} L_i - I_{ik} L_j - I_{ij} L_k - L_i L_j L_k.$$

Because of the total symmetry of L_{ijk} we may make any of these our choice for one independent invariant. Indeed under transpositions of indices:

$$\begin{aligned} I_{jik} &= I_{ijk} \\ I_{ikj} &= I_{ijk} + (I_{kj} - I_{jk})_{,i} \\ I_{kji} &= I_{ijk} + (I_{ki} - I_{ik})_{,j}. \end{aligned}$$

One checks the invariance of I_{123} by, for example,

$$\Theta_1(I_{123}) = L_{23} - I_{23} - L_2L_3 = 0$$

and likewise with Θ_2 and Θ_3 .

We call the expression of L_{ijk} in terms of invariants and the L_i ,

$$L_{ijk} = L_iL_jL_k + I_{jk}L_i + I_{ik}L_j + I_{ij}L_k + I_{ijk},$$

the *invariant expansion* of L_{ijk} .

We will discuss the system form in a moment. However, it should be emphasised that scalar and system forms are not interchangeable at order three unlike the order two case. We are generally able to write neither the scalar form as a 3×3 system nor such a system in scalar form.

4.1 Scalar form

4.1.1 First order intertwiner

In seeking to generalise the Laplace maps to the third order case we consider intertwining relations of the form

$$L_{123}^{\sigma_i}L_i = L_i^{\sigma_i}L_{123}.$$

For simplicity of notation we will consider $i = 1$ and write $\sigma_1 = \sigma$.

This definition of the first order intertwiner is a natural, formal generalization of the classical case. It would imply that if $\phi \in \ker L_{123}$ then $L_i\phi \in \ker L_{123}^{\sigma_i}$.

Apply the operators $\Theta_3, \Theta_2, \Theta_1$ and Θ_1^2 to obtain the tower of equations

$$L_{123}^{\sigma}L_1 = L_1^{\sigma}L_{123} \tag{6}$$

$$L_{12}^{\sigma}L_1 = L_1^{\sigma}L_{12} \tag{7}$$

$$L_{13}^{\sigma}L_1 = L_1^{\sigma}L_{13} \tag{8}$$

$$L_{23}^{\sigma}L_1 + L_{123}^{\sigma} = L_1^{\sigma}L_{23} + L_{123} \tag{9}$$

$$2L_{23}^{\sigma} = 2L_{23} \tag{10}$$

Equations (7) and (8) are Laplace maps of the earlier kind (with the label 1 replacing 2) so that we immediately deduce:

$$\begin{aligned} L_2^{\sigma} &= L_2 \\ L_3^{\sigma} &= L_3 \\ I_{13}^{\sigma} &= I_{31} \\ I_{12}^{\sigma} &= I_{21} \\ L_1^{\sigma} &= I_{31}L_1I_{31}^{-1} = I_{21}L_1I_{21}^{-1} \\ I_{21}^{\sigma} - I_{21} &= I_{21} - I_{12} + (\log I_{21})_{,12} \\ I_{31}^{\sigma} - I_{31} &= I_{31} - I_{13} + (\log I_{31})_{,13}. \end{aligned} \tag{11}$$

Equation (10) is consistent with the above and requires in addition,

$$I_{23}^\sigma = I_{23}, \quad I_{32}^\sigma = I_{32}.$$

Equation (11) on the other hand implies a constraint on the invariants, a condition that requires satisfaction, if the Laplace map is to exist. The condition is

$$\left(\frac{I_{31}}{I_{21}}\right)_{,1} = 0.$$

If we restrict attention to the differential ring in the coefficients of the differential operators, i.e. we do not employ any extension, then, up to a multiplicative scalar, we conclude that

$$I_{21} = I_{31}.$$

Unfortunately this relation is not preserved under the Laplace map:

$$\begin{aligned} I_{21}^\sigma - I_{31}^\sigma &= I_{12}^\sigma + [L_1^\sigma, L_2] - I_{13}^\sigma - [L_1^\sigma, L_3] \\ &= I_{21} - I_{31} + [L_1 - (\log I_{21})_{,1}, L_2] - [L_1 - (\log I_{31})_{,1}, L_3] \\ &= 2I_{21} - I_{12} - 2I_{31} + I_{13} + (\log I_{31})_{,13} - (\log I_{21})_{,12} \\ &= I_{13} - I_{12} + ((\log I_{31})_{,3} - (\log I_{21})_{,2})_{,1} \\ &\neq 0 \end{aligned}$$

We seek to understand if further constraints arise from equations (9) and (6).

Dealing with equation (9) first, we can write it, using the expansion in invariants, as an expression relating I_{123}^σ to I_{321} :

$$\begin{aligned} L_{123}^\sigma - L_1^\sigma L_{23}^\sigma &= L_{123} - L_{23}L_1 \\ L_1^\sigma L_2 L_3 + I_{21}L_3 + I_{31}L_2 &= L_1 L_2 L_3 + I_{12}L_3 + I_{13}L_2 \\ + I_{23}L_1^\sigma + I_{123}^\sigma - L_1^\sigma(I_{23} + L_2 L_3) &+ (L_{23} - L_2 L_3)L_1 + I_{123} \\ I_{21}L_3 + I_{31}L_2 - I_{23,1} + I_{123}^\sigma &= [L_1, L_2 L_3] + I_{12}L_3 + I_{13}L_2 + I_{123} \\ I_{31}L_2 - I_{23,1} + I_{123}^\sigma &= L_2(I_{31} - I_{13}) + I_{13}L_2 + I_{123} \\ I_{123}^\sigma &= I_{123} + I_{31,2} - I_{13,2} + I_{23,1} \\ &= I_{321} + I_{23,1}. \end{aligned}$$

Finally we deal with equation (6).

Once more using the invariant expansion of L_{123} and L_{123}^σ :

$$\begin{aligned} &(L_1^\sigma L_2^\sigma L_3^\sigma + I_{23}^\sigma L_1^\sigma + I_{13}^\sigma L_2^\sigma + I_{12}^\sigma L_3^\sigma + I_{123}^\sigma)L_1 \\ &= L_1^\sigma(L_2 L_3 L_1 + I_{23}L_1 + I_{31}L_2 + I_{21}L_3 + I_{231}) \end{aligned}$$

Incorporating what we have learnt already,

$$\begin{aligned} &(L_1^\sigma L_2 L_3 + I_{23}L_1^\sigma + I_{31}L_2 + I_{21}L_3 + I_{321} + I_{23,1})L_1 \\ &= L_1^\sigma(L_2 L_3 L_1 + I_{23}L_1 + I_{31}L_2 + I_{21}L_3 + I_{231}) \\ &= L_1^\sigma L_2 L_3 L_1 + L_1^\sigma I_{23}L_1 + I_{31}L_1 L_2 + I_{21}L_1 L_3 + L_1^\sigma I_{231} \end{aligned}$$

and we undertake the following manipulations

$$\begin{aligned} I_{21}[L_3, L_1] + I_{31}[L_2, L_1] + (I_{321} + I_{23,1})L_1 &= [L_1^\sigma, I_{23}]L_1 + L_1^\sigma I_{231} \\ I_{21}I_{13} - 2I_{21}I_{31} + I_{31}I_{12} &= L_1^\sigma I_{231} - I_{231}L_1 \quad (12) \end{aligned}$$

Summarising: the existence of a Laplace map of the form $L_{123}^\sigma L_1 = L_1^\sigma L_{123}$ requires conditions on the form of the third order operator which can be expressed as the constraints on invariants arising from the three different expressions, (11) and (12), for L_1^σ in terms of L_1 . These are succinctly written as the pair

$$\begin{aligned} I_{31}I_{231,1} - I_{31,1}I_{231} &= I_{31}(I_{21}I_{13} - 2I_{21}I_{31} + I_{31}I_{12}) \\ I_{21}I_{231,1} - I_{21,1}I_{231} &= I_{21}(I_{21}I_{13} - 2I_{21}I_{31} + I_{31}I_{12}) \end{aligned}$$

or, if the choice $I_{21} = I_{31}$ is made

$$\left(\frac{I_{231}}{I_{21}}\right)_{,1} = I_{13} + I_{12} - 2I_{21}.$$

These conditions amount to differential constraints on the coefficients a_i , a_{ij} and a_{ijk} .

4.1.2 Deformed first order intertwiner

It is natural to ask if we can escape the constraints by considering the deformed relation:

$$L_{123}^\sigma L_1' = L_1'^\sigma L_{123} \quad (13)$$

$$L_{12}^\sigma L_1' = L_1'^\sigma L_{12} \quad (14)$$

$$L_{13}^\sigma L_1' = L_1'^\sigma L_{13} \quad (15)$$

$$L_{23}^\sigma L_1' + L_{123}^\sigma = L_1'^\sigma L_{23} + L_{123} \quad (16)$$

$$L_{23}^\sigma = L_{23} \quad (17)$$

From (14) and (15) we will obtain

$$\begin{aligned} I_{12}^\sigma &= I_{21} - \phi_{,2} \\ \left(\frac{I_{12}^\sigma}{\phi}\right)_{,1} &= I_{12}^\sigma - I_{12} \\ I_{13}^\sigma &= I_{31} - \phi_{,3} \\ \left(\frac{I_{13}^\sigma}{\phi}\right)_{,1} &= I_{13}^\sigma - I_{13}. \end{aligned}$$

Apparently we do not thereby escape constraints since the integrability conditions on ϕ require

$$(I_{12}^\sigma - I_{21})_{,3} = (I_{13}^\sigma - I_{31})_{,2} \quad (18)$$

and, writing

$$\frac{I_{12}^\sigma}{I_{13}^\sigma} = \frac{I_{12}^\sigma}{\phi} \frac{\phi}{I_{13}^\sigma},$$

we get

$$\left(\frac{I_{12}^\sigma}{I_{13}^\sigma}\right)_{,1} = \frac{I_{12}^\sigma I_{13} - I_{12} I_{13}^\sigma}{(I_{13}^\sigma)^2} \phi$$

and hence

$$\phi = \frac{I_{12,1}^\sigma I_{13} - I_{12} I_{13,1}^\sigma}{I_{12}^\sigma I_{13} - I_{12} I_{13}^\sigma}.$$

As before there is a limit in which ϕ but not $\frac{\phi_{;1}}{\phi}$ tend to zero and this corresponds to the previous relations: $I_{12}^\sigma = I_{21}$ etc.

Again, study of this situation is postponed to another time.

4.1.3 Second order intertwiner

A second possibility for implementing a type of Laplace transform is to consider a second order intertwiner,

$$L_{123}^\sigma L_{12} = L_{12}^\sigma L_{123} \quad (19)$$

We proceed as before and indeed it turns out that this imposes fewer constraints on the invariants of L_{123} .

Θ_3 yields only trivial identities. The tower of identities obtained by applying Θ_1 and Θ_2 is:

$$L_{123}^\sigma L_{12} = L_{12}^\sigma L_{123} \quad (20)$$

$$L_{23}^\sigma L_{12} + L_{123}^\sigma L_2 = L_2^\sigma L_{123} + L_{12}^\sigma L_{23} \quad (21)$$

$$L_{13}^\sigma L_{12} + L_{123}^\sigma L_1 = L_1^\sigma L_{123} + L_{12}^\sigma L_{13} \quad (22)$$

$$L_{23}^\sigma L_2 = L_2^\sigma L_{23} \quad (23)$$

$$L_{13}^\sigma L_1 = L_1^\sigma L_{13} \quad (24)$$

$$L_3^\sigma L_{12} + L_{23}^\sigma L_1 + L_{13}^\sigma L_2 + L_{123}^\sigma = L_{123} + L_2^\sigma L_{13} + L_1^\sigma L_{23} + L_{12}^\sigma L_3 \quad (25)$$

Again equations (23) and (24) are of the classical form for a Laplace transformation and they give us

$$L_3^\sigma = L_3$$

$$I_{23}^\sigma = I_{32}$$

$$I_{13}^\sigma = I_{31}$$

$$L_2^\sigma = I_{32} L_2 I_{32}^{-1}$$

$$L_1^\sigma = I_{31} L_1 I_{31}^{-1}.$$

In particular, no constraint arises at this level.

Equation (25) is dealt with using the invariant expansions

$$L_{123}^\sigma = L_1^\sigma L_2^\sigma L_3 + I_{32} L_1^\sigma + I_{31} L_2^\sigma + I_{12}^\sigma L_3 + I_{123}^\sigma$$

$$L_{123} = L_3 L_2 L_1 + I_{32} L_1 + I_{31} L_2 + I_{21} L_3 + I_{321}$$

and

$$L_{13} = L_3 L_1 + I_{31}, \quad L_{23} = L_3 L_2 + I_{32}, \quad L^\sigma = L_1^\sigma L_2^\sigma + I_{12}^\sigma,$$

$$L_{23}^\sigma = L_2^\sigma L_3 + I_{32}, \quad L_{13}^\sigma = L_1^\sigma L_3 + I_{31}, \quad L_{12} = L_2 L_1 + I_{21}.$$

The result is an expression for I_{123}^σ :

$$I_{123}^\sigma = I_{321} + I_{32,1} - I_{21,3} + I_{31,2}.$$

We anticipate that equations (21) and (22) will give expressions for I_{12}^σ and I_{21}^σ and no constraints.

Write (21) as

$$L_{123}^\sigma L_2 - L_{12}^\sigma L_{23} = L_2^\sigma L_{123} - L_{23}^\sigma L_{12}.$$

Using the expansions

$$L_{123}^\sigma = L_{12}^\sigma L_3^\sigma + I_{13}^\sigma L_2^\sigma + I_{23}^\sigma L_1^\sigma + I_{123}^\sigma$$

$$L_{123} = L_3 L_{12} + I_{32} L_1 + I_{31} L_2 + I_{321} - I_{21,3}$$

the relation simplifies to

$$(I_{123}^\sigma - I_{31,2})L_2 - L_2^\sigma(I_{321} - I_{21,3}) = L_{12}^\sigma I_{32} - I_{32} L_{12} + L_2^\sigma I_{32} L_1 - I_{32} L_1^\sigma L_2.$$

Then

$$\begin{aligned} (I_{123}^\sigma - I_{31,2})L_2 - L_2^\sigma(I_{321} - I_{21,3}) &= L_{12}^\sigma I_{32} - I_{32} L_{12} + I_{32} L_2 L_1 - I_{32} L_1^\sigma L_2 \\ &= (L_1^\sigma L_2^\sigma + I_{12}^\sigma)I_{32} - I_{32} I_{21} - I_{32} L_1^\sigma L_2 \\ &= I_{32,1} L_2 + (I_{12}^\sigma - I_{21})I_{32}. \end{aligned}$$

Define

$$J_{321} = I_{321} - I_{21,3} = I_{123}^\sigma - I_{31,2} - I_{32,1}.$$

Then

$$L_2^\sigma J_{321} - J_{321} L_2 = (I_{21} - I_{12}^\sigma)I_{32} \quad (26)$$

and from equation (22), by transposition of 1 and 2,

$$L_1^\sigma J_{312} - J_{312} L_1 = (I_{12} - I_{21}^\sigma)I_{31}. \quad (27)$$

Note that, following from the properties of I_{ijk} under transposition

$$J_{321} = J_{312}.$$

Using the relations $L_2^\sigma I_{32} = I_{32} L_2$ and $L_1^\sigma I_{31} = I_{31} L_1$ we get expressions for I_{21}^σ and I_{12}^σ :

$$\begin{aligned} I_{12}^\sigma &= I_{21} - \left(\frac{J_{321}}{I_{32}} \right)_{,2} \\ I_{21}^\sigma &= I_{12} - \left(\frac{J_{312}}{I_{31}} \right)_{,1}. \end{aligned}$$

Finally we must analyse the relation (20) itself:

$$L_{123}^\sigma L_{12} = L_{12}^\sigma L_{123}.$$

At this point all the I^σ invariants are defined in terms of the I invariants as are the L_i^σ so the most we can hope for here is a consistent set of relations but we might expect, instead, constraints to arise.

We use the same invariant expansions as in the previous calculations to arrive at,

$$(I_{31} L_2^\sigma + I_{32} L_1^\sigma + I_{123}^\sigma) L_{12} = L_{12}^\sigma (I_{32} L_1 + I_{31} L_2 + J_{321}).$$

We can use the J_{321} relations (26) and (27) to move the L_1^σ and L_2^σ operators from left to right to obtain

$$\begin{aligned} I_{123}^\sigma L_{12} &= I_{32,1} L_2 L_1 - I_{32} L_1^\sigma + I_{12}^\sigma I_{32} L_1 \\ &\quad + I_{31,2} L_1 L_2 - I_{31} L_2^\sigma I_{12} + I_{21}^\sigma I_{31} L_2 \\ &\quad + J_{321} L_1 L_2 + I_{31} (I_{12} - I_{21}^\sigma) L_2 + I_{12}^\sigma J_{321} \\ &\quad + L_1^\sigma (I_{21} - I_{12}^\sigma) I_{32} \end{aligned}$$

Rearranging we obtain the relations between invariants

$$J_{321}I_{12} = (I_{12}^\sigma I_{32})L_1 - L_1^\sigma(I_{12}^\sigma I_{32}) \\ + (I_{31}I_{12})L_2 - L_2^\sigma(I_{31}I_{12}).$$

Since all the objects with σ superfix are already defined, this relation constitutes a constraint on the invariants. Again this amounts to a differential constraint on the coefficients a_i , a_{ij} and a_{ijk} .

4.2 System form

In [1] Laplace type maps are developed for third order systems of the form

$$\begin{pmatrix} L_1 & h_{12} & h_{13} \\ h_{21} & L_2 & h_{23} \\ h_{31} & h_{32} & L_3 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where $L_i = \partial_i + h_{ii}$. In this section we provide a slight generalization of those results.

The set of transformations under which invariants are defined are taken to be conjugations of the matrix differential operator by diagonal matrices of three arbitrary functions

$$G = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix}$$

and the invariants are

$$\begin{aligned} (12) &= h_{12}h_{21} \\ (23) &= h_{23}h_{32} \\ (31) &= h_{31}h_{13} \\ (123) &= h_{12}h_{23}h_{31} \\ (132) &= h_{13}h_{32}h_{21} \\ [12] &= h_{11,2} - h_{22,1} + \frac{1}{2}(\log \frac{h_{12}}{h_{21}})_{,12} \\ [13] &= h_{11,3} - h_{33,1} + \frac{1}{2}(\log \frac{h_{13}}{h_{31}})_{,13} \\ [23] &= h_{22,3} - h_{33,2} + \frac{1}{2}(\log \frac{h_{23}}{h_{32}})_{,23}. \end{aligned}$$

The (ij) and (ijk) are totally symmetric. The $[ij]$ are antisymmetric. Note that this is not an independent set. In fact

$$(12)(23)(31) = (123)(132),$$

and

$$[12]_{,3} + [23]_{,1} + [31]_{,2} = \frac{1}{2}(\log \frac{(123)}{(132)})_{,123}.$$

A number of variant canonical forms are possible. For example, choosing $g_{,1} + h_{11}g = 0$ and $g_1 = g_{,1}$, $g_2 = gh_{21}$, $g_3 = gh_{21}h_{32}$ we obtain

$$G^{-1} \begin{pmatrix} L_1 & h_{12} & h_{13} \\ h_{21} & L_2 & h_{23} \\ h_{31} & h_{32} & L_3 \end{pmatrix} G = \begin{pmatrix} \partial_1 & (12) & (132) \\ 1 & \partial_2 + I_2 & (23) \\ \frac{(13)}{(132)} & 1 & \partial_3 + I_3 \end{pmatrix} \quad (28)$$

where

$$I_{2,1} = [21] + \frac{1}{2}(\log(12))_{,12},$$

and

$$I_{3,1} = [31] + (\log(132))_{,13} - \frac{1}{2}(\log(13))_{,13}.$$

The *deformed* intertwining relation generalises that for the 2×2 case. For example,

$$\begin{aligned} & \begin{pmatrix} L_1^\sigma & h_{12}^\sigma & h_{13}^\sigma \\ h_{21}^\sigma & L_2^\sigma & h_{23}^\sigma \\ h_{31}^\sigma & h_{32}^\sigma & L_3^\sigma \end{pmatrix} \begin{pmatrix} \partial_1 + u_{11} & h_{12} - h_{12}^\sigma & h_{13} - h_{13}^\sigma \\ 0 & \partial_1 + u_{22} & 0 \\ 0 & 0 & \partial_1 + u_{33} \end{pmatrix} \\ &= \begin{pmatrix} \partial_1 + l_{11} & 0 & 0 \\ h_{21}^\sigma - h_{21} & \partial_1 + l_{22} & 0 \\ h_{31}^\sigma - h_{31} & 0 & \partial_1 + l_{33} \end{pmatrix} \begin{pmatrix} L_1 & h_{12} & h_{13} \\ h_{21} & L_2 & h_{23} \\ h_{31} & h_{32} & L_3 \end{pmatrix} \end{aligned}$$

from which follow

$$\begin{aligned} l_{22} &= u_{22} \\ l_{33} &= u_{33} \\ L_2^\sigma &= L_2 \\ L_3^\sigma &= L_3 \\ h_{23}^\sigma &= h_{23} \\ h_{32}^\sigma &= h_{32} \end{aligned}$$

and

$$\begin{aligned} u_{22} &= h_{11} - (\log h_{21})_{,1} + (12)^\sigma \alpha = h_{11}^\sigma + (\log h_{12}^\sigma)_{,1} + (12)\alpha, \\ u_{33} &= h_{11} - (\log h_{31})_{,1} + (13)^\sigma \beta = h_{11}^\sigma + (\log h_{13}^\sigma)_{,1} + (13)\beta. \end{aligned}$$

Here α and β are functions of invariants satisfying,

$$\frac{\alpha}{\beta} = \frac{(132)^\sigma(13)}{(132)(12)^\sigma} = \frac{(123)(13)^\sigma}{(123)^\sigma(12)}$$

in terms of which the Laplace map takes the form

$$\begin{aligned} (12)^\sigma - (12) &= [12] - \frac{1}{2}(\log(12))_{,12} + ((12)^\sigma \alpha)_{,2} \\ (13)^\sigma - (13) &= [13] - \frac{1}{2}(\log(13))_{,13} + ((13)^\sigma \beta)_{,3} \\ (23)^\sigma - (23) &= 0 \\ [12]^\sigma - [12] &= -\frac{1}{2}(\log((12)(12)^\sigma \alpha^2))_{,12} \\ [13]^\sigma - [13] &= -\frac{1}{2}(\log((13)(13)^\sigma \beta^2))_{,13} \\ [23]^\sigma - [23] &= 0 \\ (132)^\sigma - (132) &= (23) \left(\log \frac{(12)}{(123)} \right)_{,1} - \alpha(12)^\sigma + \beta(13)^\sigma \\ (123)^\sigma - (123) &= (23) \left(\log \frac{(13)}{(132)} \right)_{,1} + \alpha(12)^\sigma - \beta(13)^\sigma. \end{aligned}$$

The Laplace maps respect the differential identities satisfied by the antisymmetric invariants.

$$\begin{aligned}
[12]^\sigma_{,3} + [23]^\sigma_{,1} + [31]^\sigma_{,2} &= [12]_{,3} + [23]_{,1} + [31]_{,2} + \frac{1}{2} \left(\log \frac{(13)(13)^\sigma \beta^2}{(12)(12)^\sigma \alpha^2} \right)_{,123} \\
&= \frac{1}{2} \left(\log \frac{(123)(13)(13)^\sigma \beta^2}{(132)(12)(12)^\sigma \alpha^2} \right)_{,123} \\
&= \frac{1}{2} \left(\log \frac{(123)^\sigma}{(132)^\sigma} \right)_{,123}. \tag{29}
\end{aligned}$$

The other algebraic constraint on the symmetric invariants

$$(123)^\sigma (132)^\sigma = (12)^\sigma (23)^\sigma (31)^\sigma$$

provides a second (complicated) condition, not identically satisfied, on α and β , thus determining them.

Note that in [1] the simplifying choice $u_{22} = u_{33}$ has been made which leads to the relation $\alpha = \beta$. It was not appreciated at that time that such a choice leads to constraints on the invariants. As is seen above allowing $u_{22} \neq u_{33}$ leads to more complexity. This is currently being explored.

4.3 Scalarizability

Because the differential operator of the 3×3 system is defined over a non-commutative ring one cannot, in general, reduce to a single scalar equation as might over a field. But it is clear that the vanishing of any of the h_{ij} is a sufficient condition to do so and we say that in such a case the system is *scalarizable*.

For example, if $h_{23} = 0$ then we can write down a scalar equation for ϕ_2 of the form

$$(\tilde{L}_3 \tilde{L}_1 \tilde{L}_2 - \tilde{L}_3(12) - (13)\tilde{L}_2 + (132))\tilde{\phi}_2 = 0,$$

where

$$\begin{aligned}
\tilde{L}_1 &= L_1 \\
\tilde{L}_2 &= h_{21}^{-1} L_2 h_{21} \\
\tilde{L}_3 &= h_{13} L_3 h_{13}^{-1} \\
\tilde{\phi}_2 &= h_{21}^{-1} \phi_2.
\end{aligned}$$

By comparing with the invariant expansion

$$\tilde{L}_3 \tilde{L}_1 \tilde{L}_2 + \tilde{I}_{32} \tilde{L}_1 + \tilde{I}_{31} \tilde{L}_2 + \tilde{I}_{12} L_3 + \tilde{I}_{312}$$

the corresponding invariants I_{ij} and I_{123} are seen to be

$$\begin{aligned}
\tilde{I}_{32} &= 0 \\
\tilde{I}_{23} &= [23] + (\log(132))_{,23} \\
\tilde{I}_{31} &= -(13) \\
\tilde{I}_{13} &= -(13) + [13] + \frac{1}{2}(\log(13))_{,13} \\
\tilde{I}_{12} &= -(12) \\
\tilde{I}_{21} &= -(12) + [21] + \frac{1}{2}(\log(12))_{,12} \\
\tilde{I}_{312} &= (132) - (12)_{,3}
\end{aligned}$$

using $(23) = 0$.

Finally we would like to see that the system Laplace map corresponds with a scalar one. It suffices to check that the invariants transform in the same way. The restriction $(23) = 0$ is consistent with (but not equivalent to) the choice $\alpha = \beta = 0$ which follows from the assumption $u_{11} = h_{11}$ and $l_{11} = h_{11}^\sigma$, corresponding to the undeformed 3×3 system. This in turn implies

$$\begin{aligned}
\partial_1 + u_{22} &= h_{21} L_1 h_{21}^{-1} \\
&= (h_{12}^\sigma)^{-1} L_1^\sigma h_{12}^\sigma \\
\partial_1 + u_{33} &= h_{31} L_1 h_{31}^{-1} \\
&= (h_{13}^\sigma)^{-1} L_1^\sigma h_{13}^\sigma
\end{aligned} \tag{30}$$

and consequently

$$(h_{12}^\sigma h_{21}) L_1 (h_{12}^\sigma h_{21})^{-1} = (h_{13}^\sigma h_{31}) L_1 (h_{13}^\sigma h_{31})^{-1},$$

or

$$h_{12}^\sigma h_{21} = h_{13}^\sigma h_{31},$$

or, in invariant terms,

$$\frac{(123)^\sigma}{(13)^\sigma} = \frac{(123)}{(12)}.$$

We impose the limit $\alpha, \beta \rightarrow 0$ and $\frac{\alpha}{\beta} = 1$ on what follows.

Consider the \tilde{I}_{13}^σ and \tilde{I}_{31}^σ invariants arising from the first order scalar Laplace map. From the results on the first order intertwiners,

$$\begin{aligned}
\tilde{I}_{13}^\sigma &= \tilde{I}_{31}^\sigma = -(13) \\
\tilde{I}_{31}^\sigma &= 2\tilde{I}_{31} - \tilde{I}_{13} + (\log \tilde{I}_{31})_{,13} \\
-(13)^\sigma &= -2(13) + (13) - [13] - \frac{1}{2}(\log(13))_{,13} + (\log(13))_{,13} \\
&= -(13) - [13] + \frac{1}{2}(\log(13))_{,13}.
\end{aligned}$$

which rearranges to

$$(13)^\sigma - (13) = [13] - \frac{1}{2}(\log(13))_{,13}$$

as expected.

For the antisymmetric invariant we have,

$$\begin{aligned} -(13)^\sigma + [13]^\sigma + \frac{1}{2}(\log(13)^\sigma)_{13} &= -(13) \\ [13]^\sigma &= (13)^\sigma - (13) - \frac{1}{2}(\log(13)^\sigma)_{13} \\ [13]^\sigma - [13] &= -\frac{1}{2}(\log(13)^\sigma(13))_{13} \end{aligned}$$

The I_{12} relation follows simply from:

$$\begin{aligned} \tilde{I}_{12}^\sigma &= \tilde{I}_{21} \\ -(12)^\sigma &= -(12) + [21] + \frac{1}{2}(\log(12))_{,12} \\ (12)^\sigma - (12) &= [12] - \frac{1}{2}(\log(12))_{,12} \end{aligned}$$

as expected and so on.

The condition that $h_{23} = 0$ implies $(23) = 0$ and for the system Laplace map this gives

$$(132)^\sigma = (132)$$

which we wish to show is equivalent to

$$I_{123}^\sigma = I_{321} + I_{23,1},$$

the Laplace map result for the scalar operator. Rewriting the system relation using the permutation relations on three index objects,

$$\begin{aligned} \tilde{I}_{312}^\sigma - \tilde{I}_{12,3}^\sigma &= \tilde{I}_{312} - \tilde{I}_{12,3} \\ \tilde{I}_{132}^\sigma - \tilde{I}_{12,3}^\sigma &= \tilde{I}_{312} - \tilde{I}_{12,3} \\ \tilde{I}_{123}^\sigma + \tilde{I}_{32,1}^\sigma - \tilde{I}_{23,1}^\sigma - \tilde{I}_{12,3}^\sigma &= \tilde{I}_{321} + \tilde{I}_{12,3} - \tilde{I}_{21,3} - \tilde{I}_{12,3} \\ \tilde{I}_{123}^\sigma + \tilde{I}_{32,1}^\sigma - \tilde{I}_{23,1}^\sigma - \tilde{I}_{21,3}^\sigma &= \tilde{I}_{321} - \tilde{I}_{21,3} \\ \tilde{I}_{123}^\sigma &= \tilde{I}_{321} + \tilde{I}_{23,1} + \tilde{I}_{32,1} \\ &= \tilde{I}_{321} + \tilde{I}_{23,1} \end{aligned}$$

as desired, since $\tilde{I}_{32} = 0$.

4.3.1 Conclusion

This paper has examined in detail a natural generalization of the classical Laplace map on invariants of second order, partial differential operators to third order operators of both scalar and system forms and compared them in the overlap where a system is scalarizable. The basic observation is that such intertwining maps of scalar operators can only exist where prior conditions exist on the invariants. It is shown by example that these can be compatible with the scalarizability of a system for which more general (unconstrained) Laplace maps exist.

There is a further generalization of the notion of Laplace map which would allow us to avoid the constraints in the case of the second order intertwining

relation. We call this a *weak Laplace map*: Consider the inhomogeneous partial differential equation

$$L_{123}\phi = \psi$$

and suppose that the intertwining relation holds only up to an unspecified function, K :

$$L_{12}^\sigma L_{123} - L_{123}^\sigma L_{12} = K.$$

Then the conditions (21) to (25) hold but not (20) and the relations between the I^σ and the I are as in the last section but without the constraint (28). This allows us to define the weak Laplace map:

$$\begin{aligned} L_{12}^\sigma L_{123}\phi &= L_{12}^\sigma \psi \\ L_{123}^\sigma L_{12}\phi &= L_{12}^\sigma \psi - K\phi \\ L_{123}^\sigma \phi^\sigma &= \psi^\sigma \end{aligned}$$

where,

$$\phi \mapsto \phi^\sigma = L_{12}\phi, \quad \psi \mapsto \psi^\sigma = L_{12}^\sigma \psi - K\phi.$$

There seems to be some scope here for further work.

Apart from the study of weak maps and the deformed intertwining relations, other natural questions include:

- What are the general constraints on the invariants of a 3×3 (or higher order) system that make it scalarizable?
- What conditions need to be satisfied by the invariants of a scalar operator in order for it to be written in system form?
- Does the second order intertwining map also correspond to a Laplace map of a 3×3 system?

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