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PAPER

Instantaneous modulations in time-varying complex optical potentials

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Abstract

We study the impact of a spatially homogeneous yet non-stationary dielectric permittivity on the dynamical and spectral properties of light. Our choice of potential is motivated by the interest in $\mathcal{PT}$-symmetric systems as an extension of quantum mechanics. Because we consider a homogeneous and non-stationary medium, $\mathcal{PT}$ symmetry reduces to time-reversal symmetry in the presence of balanced gain and loss. We construct the instantaneous amplitude and angular frequency of waves within the framework of Maxwell's equations and demonstrate the modulation of light *amplification* and *attenuation* associated with the well-defined temporal domains of *gain* and *loss*, respectively. Moreover, we predict the *splitting* of extrema of the angular frequency modulation and demonstrate the associated shrinkage of the modulation period. Our theory can be extended for investigating similar time-dependent effects with matter and acoustic waves in $\mathcal{PT}$-symmetric structures.

1. Introduction

During the past years, a new class of Hamiltonians has been widely investigated, which extends quantum mechanics from the Hermitian into the non-Hermitian (complex) domain [1]. Despite the lack of Hermiticity, Bender *et al* have shown in their seminal papers that a Hamiltonian can have real eigenspectra if it possesses so-called parity-time ($\mathcal{PT}$) symmetry [2, 3]. Such a symmetry means there is invariance of the theory under parity (spatial) reflection $\mathcal{P}$: $\hat{p} \to -\hat{p}$, $\hat{x} \to -\hat{x}$, and time reflection $\mathcal{T}$: $\hat{p} \to -\hat{p}$ $(t \to -t)$, $\mathrm{i} \to -\mathrm{i}$, $\hat{x} \to \hat{x}$, where $\hat{p}$ and $\hat{x}$ are the momentum and position operators, respectively, while t is the time coordinate and i is the imaginary unit. This combined $\mathcal{PT}$ symmetry leads to subtle changes in the unitary evolution of the system and modification of the inner product [4–7]. As $\mathcal{PT}$ symmetry represents an extension of quantum mechanics, it is nowadays used in various different contexts, such as quantum reflection [8–10] and chaos [11], and has even been generalized to fermionic [12, 13], gyrotropic [14, 15] and magnetic systems [16].

Although the concept of $\mathcal{PT}$ symmetry was originally introduced in quantum mechanical systems, one has found experimental evidence and also a wide range of applications in classical optics. In 2010, Rüter *et al* were the first to realize a $\mathcal{PT}$-optical coupled system that involves well-defined regions with *gain* and *loss* regimes, inherent to the *complex-valued* refractive index [17]. Such an extension of the concept of spacetime reflection into the classical domain stems from the works of El-Ganainy *et al* [18] and Makris *et al* [19, 20], who have employed the similarity between the Schrödinger and a scalar approximation of Maxwell's equations to describe the dynamics of light beams in $\mathcal{PT}$-symmetric optical lattices. There have been further theoretical [21–24] and experimental [25–27] studies dealing with the implementation of the parity-time reversal symmetry in optics especially relevant for the development of new artificial structures and materials (see also the recent review paper [28] and references cited therein).

$\mathcal{PT}$-symmetric structures have mainly been investigated in the spatial domain (that is, for time-independent complex potentials) and little attention has been paid to the study of the non-stationary regime.

The significance of considering the *time-dependent* potentials both in the quantum [29–33] and classical theories [34, 35] arises from the attempt to examine the full time evolution of the system. Recently, a non-Hermitian opto-mechanical structure has been realized experimentally in the temporal domain [36]. Despite the recent works, however, there are no rigorous analytical studies of wave equations with non-stationary complex potentials possessing time reflection symmetry. Given that the energy (frequency) and time are conjugate variables, a successful solution of $\mathcal{PT}$-symmetric time-dependent Maxwell's equations would constitute a complete characterization of the dynamical and spectral features of light.

The purpose of this paper is to study the dynamics of light in time-dependent optical potentials having $\mathcal{PT}$ symmetry. We consider a spatially homogenous system, for which $\mathcal{PT}$ reduces to symmetry under time reversal, albeit in the presence of both gain and loss. In view of this, we calculate both the instantaneous amplitude and angular frequency of waves and show how the complex-valued dielectric permittivity controls the light in the temporal domain. The resulting modulations of amplification and attenuation of the amplitude is demonstrated to be associated with well-defined regimes of gain and loss, respectively. A comparison with the angular frequency modulation by a real permittivity is provided to reveal the impact of $\mathcal{PT}$-symmetric potentials in that we observe (i) splitting of the extrema and (ii) a shrinkage of the frequency modulation period. Moreover, a particular emphasis is placed on studying modulations of light amplification and attenuation for experimentally accessible values of the modulated permittivity.

The paper is organized as follows. In section 2 we briefly discuss $\mathcal{PT}$ symmetry in optics by considering space-independent but time-varying modulations of a complex-valued dielectric permittivity. For such a modulated permittivity, symmetric under time-reversal, we construct an analytical solution to Maxwell's equations under the assumption that the modulation rate is much smaller than the wave frequency. This solution is further exploited in section 3 to derive and analyze the instantaneous amplitude and angular frequency of light. Peculiar properties, such as modulations of amplitude amplification and attenuation as well as the splitting of frequency extrema and shrinkage of frequency modulation, are discussed in detail. Finally, conclusions are given in section 4.

2. Solution to Maxwell's equations for space-independent but time-varying dielectric permittivity obeying $\mathcal{PT}$ symmetry

We start with a brief discussion of general properties of classical optical systems possessing $\mathcal{PT}$ symmetry. Building on the formal equivalence of the Schrödinger equation with the paraxial Helmholtz (Maxwell) equation, we identify the complex refractive index $n = \Re(n) + \mathrm{i}\Im(n)$ as the optical potential, the real ($\Re(n)$) and imaginary ($\Im(n)$) parts of which are, correspondingly, even and odd functions of spacetime coordinates to ensure the $\mathcal{PT}$ invariance of the theory [17–20]. Likewise, since for non-magnetic structures the real ($\Re(\varepsilon)$) and imaginary ($\Im(\varepsilon)$) parts of the dielectric permittivity are defined via $\Re(\varepsilon) = [\Re(n)]^2 - [\Im(n)]^2$ and $\Im(\varepsilon) = 2\Re(n)\Im(n)$, the symmetry, $\Re[\varepsilon(x, t)] = \Re[\varepsilon(-x, -t)]$, and anti-symmetry, $\Im[\varepsilon(x, t)] = -\Im[\varepsilon(-x, -t)]$, relations guarantee that the full wave equation, without any approximation, remains invariant under the parity-time transformation [37, 38]. Throughout this work, we place our emphasis on the *temporal* domain and consider a spatially homogeneous yet *time-dependent* dielectric permittivity, $\varepsilon(t)$. For the sake of illustration we choose

$$\varepsilon(t) \equiv \varepsilon(\tau) = \tilde{\varepsilon} + \varepsilon_1 \cos^2(\tau) + \mathrm{i}\varepsilon_2 \sin(2\tau), \tag{1}$$

as a modulated time-dependent optical potential (see figure 1), similar to its spatial counterpart as discussed in [19, 20] [6]. In equation (1), $\tilde{\varepsilon}$ is the background dielectric constant, ε_1 represents the amplitude of the real profile of the potential, whereas ε_2 describes the strength of the gain/loss periodic distribution. Moreover, $\tau \equiv bt$ is a dimensionless time, where $b \geqslant 0$ acts as a scaling factor and indicates the rate (i.e., the frequency) of modulation of the permittivity, that we assume to occur slower than the oscillations of the wave. This is reminiscent of the similar form of modulation adopted in [19, 20] for the spatial case and could be experimentally realized by utilizing opto-mechanical [36] or electro-optical systems [39]. Note that we treat the amplitudes ε_1 and ε_2 as signed quantities in equation (1), though in general the behavior is not symmetric under an exchange of the sign of the amplitudes.

In order to investigate dynamical and spectral properties of light in non-stationary $\mathcal{PT}$-symmetric structures, we derive an exact second order differential equation from Maxwell's equations [40] for the electric displacement vector $\boldsymbol{\mathcal{D}}$,

[6] Note that this choice of the time-varying permittivity, being the analog of the refractive index used in [19] in position space, does not satisfy the standard Kramers–Kronig relations, i.e., the medium is not subject to the causality principle. $\varepsilon(t)$ is valid for all time (that is, does not vanish for negative times) so that *a priori* no restrictions on the past are necessary.

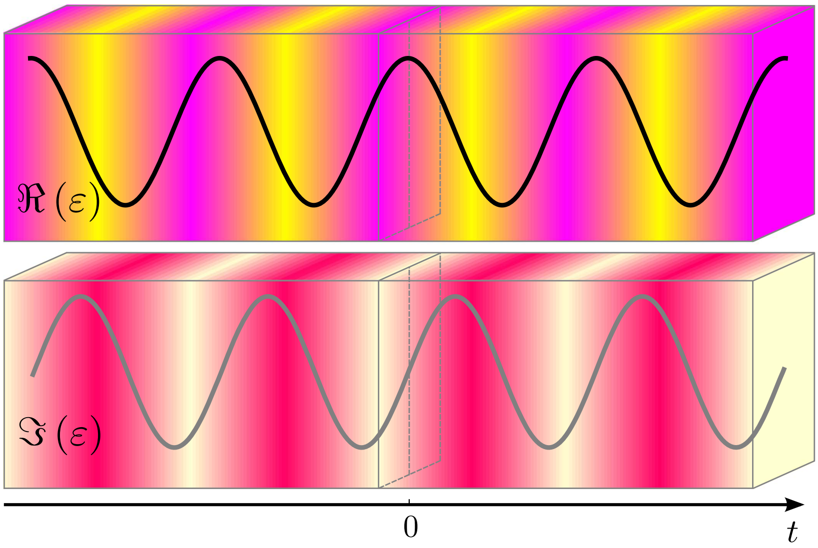

Figure 1. Symmetric real, $\Re(\varepsilon)$, and anti-symmetric imaginary, $\Im(\varepsilon)$, parts of the time-dependent dielectric permittivity.

$$\Delta\boldsymbol{\mathcal{D}}(\boldsymbol{r},\, t) - \frac{\varepsilon(t)}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{\mathcal{D}}(\boldsymbol{r},\, t) = 0, \tag{2}$$

which is valid for an arbitrary shape of the time-dependent dielectric permittivity [41]. Here, Δ is the Laplace operator and c is the speed of light in vacuum (for the theory of non-stationary electromagnetism see, e.g., [42–44]). Without the modulation rate, i.e., when $b = 0$, the standard linear dispersion relation $\boldsymbol{k} = \omega\sqrt{\varepsilon(0)}\,\hat{\boldsymbol{k}}/c$ holds, where $\varepsilon(0) \equiv \tilde{\varepsilon} + \varepsilon_1$ is the permittivity at $\tau = 0$ and $\hat{\boldsymbol{k}}$ is the unit vector in the direction of propagation. In the presence of modulation (1), both the amplitude and the angular frequency of light undergo a time-dependent modification governed by equation (2). Accounting for this *instantaneous* effect we seek a solution of equation (2) by making the ansatz

$$\boldsymbol{\mathcal{D}}(\boldsymbol{r},\, t) = \hat{\boldsymbol{u}}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,\mathcal{F}(\tau), \tag{3}$$

which reflects the spatial homogeneity of the permittivity. Here, $\hat{\boldsymbol{u}}$ is the unit vector along the polarization direction, while the complex-valued 'amplitude' $\mathcal{F}$ describes the influence of the modulated potential on the light. In the absence of any modulation, we expect to recover the free propagation of light through a uniform medium with a constant dielectric permittivity so that $\mathcal{F} = 1$. Note that a similar (full) wave equation for the *space*-dependent electric field and permittivity is discussed in [37] for describing the so-called $\mathcal{PT}$-symmetric coherent-perfect-absorber laser. Moreover, time reversal and time-dependent wave propagation is studied in various aspects, such as for time-localized perturbations combined with spatial periodicity [45–48] and sigmoidally changing systems with either real or complex permittivity/refractive index [41, 49–52].

Next, we insert the ansatz (3) in equation (2) and obtain a second order linear differential equation for $\mathcal{F}$,

$$\left(\frac{b}{\omega}\right)^2\ddot{\mathcal{F}}(\tau) - 2\mathrm{i}\frac{b}{\omega}\,\dot{\mathcal{F}}(\tau) + \left(\frac{\varepsilon(0)}{\varepsilon(\tau)} - 1\right)\mathcal{F}(\tau) = 0, \tag{4}$$

where 'dot' refers to the derivative with respect to the dimensionless time τ. As our interest is restricted to modulations of the complex dielectric permittivity profile, which are slow when compared to the oscillations of light, we can adopt $b/\omega \ll 1$ and henceforth safely ignore the first term in equation (4). In this approximation, the remaining first order differential equation generally determines the instantaneous angular frequency as

$$\Omega(\tau) = \omega - b\Im\left(\frac{\dot{\mathcal{F}}}{\mathcal{F}}\right) = \frac{\omega}{2}\left[1 + \frac{\varepsilon(0)\Re[\varepsilon(\tau)]}{|\varepsilon(\tau)|^2}\right], \tag{5}$$

for an arbitrary form of the dielectric permittivity[7]. The exact solution of the reduced equation when integrated from the 'initial' time 0 to some instant of time τ leads to

$$\mathcal{F}(\tau) = \exp\left\{\mathrm{i}\frac{\omega\tau}{2b} - \frac{\omega\mathcal{C}}{2b}\mathrm{arctanh}(\mathcal{C}\mathcal{B})\right\}\exp\left\{\frac{\omega\mathcal{C}}{2b}\mathrm{arctanh}[\mathcal{C}(\mathcal{B} - \mathrm{i}\mathcal{A}\tan\tau)]\right\}. \tag{6}$$

This explicitly exhibits the $\mathcal{PT}$ symmetry of the displacement, $\boldsymbol{\mathcal{D}}^{\mathcal{PT}} = \boldsymbol{\mathcal{D}}$. In equation (6), the constant parameters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are introduced for the sake of brevity: $\mathcal{A} \equiv \tilde{\varepsilon}/(\tilde{\varepsilon} + \varepsilon_1) > 0$ carries information about the

[7] This reminds us of the analogous definition of the local wave vector in the spatial domain (see, e.g., [53]).

real potential, whereas $\mathcal{B} \equiv \varepsilon_2/(\tilde{\varepsilon} + \varepsilon_1)$ amounts to the complex-valued permittivity, being the signature of the gain/loss mechanism. Both $\mathcal{A}$ and $\mathcal{B}$, combined with $\mathcal{C} \equiv 1/\sqrt{\mathcal{A} + \mathcal{B}^2} > 0$, demonstrate the significance of the real and imaginary parts of the permittivity in the instantaneous character of light. Note that any modulation vanishes if $\mathcal{B} = 0$, $\mathcal{A} = \mathcal{C} = 1$ ($\varepsilon_1 = \varepsilon_2 = 0$) and/or $b = 0$ so that we recover the anticipated free propagation of light, as also $\lim_{b\to 0}\mathcal{F} = 1$. The solution (6) holds for a class of potentials of the form (1), which is fully determined on choosing $\mathcal{A}$, $\mathcal{B}$ and one of the constants in the potential, say $\tilde{\varepsilon}$. In addition, to mark out the range of variation of these parameters, we insert the (approximate) solution (6) into the exact equation (4) and estimate the error for a given modulation frequency b and angular frequency ω numerically. As can be easily checked, the relative error (that is, the term $(b/\omega)^2\ddot{\mathcal{F}}/\mathcal{F}$) does not exceed 0.13 for $\mathcal{A} \in [0.7, 1.7]$ and $\mathcal{B} \in [-0.8, 0.8]$ for [8] a ratio of $b/\omega = 0.01$. Moreover, other analytical solutions for $\mathcal{PT}$-symmetric (quantum mechanical) potentials can be found in [54, 55].

3. Instantaneous characteristics of $\mathcal{PT}$-modulated electromagnetic waves

The solution (6) allows us to fully characterize the dynamics of waves in non-stationary complex potentials. Indeed, by decoupling the real and imaginary parts of the time-dependent component of the electric displacement, $\mathrm{e}^{-\mathrm{i}\omega t}\mathcal{F}(\tau) = |\boldsymbol{\mathcal{D}}(\tau)|\ \mathrm{e}^{\mathrm{i}\Phi(\tau)}$, we obtain direct access to the profile of the instantaneous amplitude $|\boldsymbol{\mathcal{D}}(\tau)|$

$$|\boldsymbol{\mathcal{D}}(\tau)|^2 = |\mathcal{F}(\tau)|^2 = \exp\{-\omega\mathcal{C}b^{-1}\mathrm{arctanh}(\mathcal{C}\mathcal{B})\} \times \exp\left\{\frac{\omega\mathcal{C}}{2b}\mathrm{arctanh}\frac{2\mathcal{B}}{\mathcal{C}(\mathcal{A} + 2\mathcal{B}^2 + \mathcal{A}^2\tan^2\tau)}\right\} \tag{7}$$

and also to that of the instantaneous phase

$$\Phi(\tau) = -\frac{\omega\tau}{2b} - \frac{\omega\mathcal{C}}{4b}\left(\arctan\frac{\mathcal{A}\mathcal{C}\tan\tau}{1 + \mathcal{B}\mathcal{C}} + \arctan\frac{\mathcal{A}\mathcal{C}\tan\tau}{1 - \mathcal{B}\mathcal{C}}\right),$$

the first derivative of which yields the profile (5) of the instantaneous angular frequency, $\Omega = -b\dot{\Phi}$, as one would expect. Its explicit form expressed in terms of parameters $\mathcal{A}$ and $\mathcal{B}$ is

$$\Omega(\tau) = \frac{\omega}{2}\left[1 + \frac{\mathcal{A}\sin^2\tau + \cos^2\tau}{(\mathcal{A}\sin^2\tau + \cos^2\tau)^2 + \mathcal{B}^2\sin^2(2\tau)}\right]. \tag{8}$$

Again, note that when the $\mathcal{PT}$ modulation is 'switched off', the relations $|\boldsymbol{\mathcal{D}}|^2 = 1$, $\Phi = -\omega t$ and $\Omega = \omega$ are obtained (see [56] for main study methods of signals whose frequency content changes in time). Such a modulation, moreover, is different from frequency conversion occurring as a result of temporal switching (see, e.g., [43, 57] and [58] for electromagnetic and acoustic waves, respectively).

For a complete description of the instantaneous amplitude and angular frequency, and in order to reveal their specific properties, we determine the extrema of (7) and (8) at the stationary points $t_0 \in (-\infty, \infty)$. They are given implictly by the equations

$$b\mathcal{A}^2\,|\varepsilon|^2|\dot{\boldsymbol{\mathcal{D}}}|^2 = -\mathcal{B}\omega\tilde{\varepsilon}^2\,|\boldsymbol{\mathcal{D}}|^2\sin(2\tau) = 0,\ 2\mathcal{A}^4\,|\varepsilon|^4\dot{\Omega} = \omega\tilde{\varepsilon}^2\sin(2\tau)\{(1 - \mathcal{A})\mathcal{A}^2[\Re(\varepsilon)]^2 + 4\tilde{\varepsilon}^2\mathcal{B}^2(\mathcal{A}\sin^4\tau - \cos^4\tau)\} = 0. \tag{9}$$

From here we immediately recognize that both quantities have extrema at $\tau_0 = \pi m/2$ provided that m is an integer whose even values, $m = 2\mathcal{N}$ (with $\mathcal{N}$ being an integer), result in $|\boldsymbol{\mathcal{D}}(\pi\mathcal{N})|^2 = 1$ and $\Omega(\pi\mathcal{N}) = \omega$. In contrast, the odd values, $m = 2\mathcal{N} + 1$, give rise to

$$|\boldsymbol{\mathcal{D}}(\pi/2 + \pi\mathcal{N})|^2 = \exp\{-\omega\mathcal{C}b^{-1}\mathrm{arctanh}(\mathcal{C}\mathcal{B})\}, \tag{10}$$

$$\Omega(\pi/2 + \pi\mathcal{N}) = \omega(1 + \mathcal{A})/(2\mathcal{A}), \tag{11}$$

as obtained from equations (7) and (8), respectively. Equation (10) leads to the maximum possible amplitude amplification ($\mathcal{B} < 0$) and attenuation ($\mathcal{B} > 0$) correspondingly linked to the well-defined temporal domains of gain and loss. A feature of our complex and $\mathcal{PT}$-symmetric potentials is the presence of additional extremal values at

$$\sin^2\tau_0' = (1 - 8\mathcal{A}\mathcal{B}^2/\sqrt{D})/(1 - \mathcal{A}) \tag{12}$$

when the discriminant $D \equiv 16\mathcal{A}\mathcal{B}^2[4\mathcal{B}^2 - (1 - \mathcal{A})^2]$ of the quadratic polynomial (in $\sin^2\tau$) appearing in the curly parentheses in equation (9) is positive, that is when $2|\mathcal{B}| > |1 - \mathcal{A}|$. The threshold, however, does not guarantee that the additional extrema are to be found at real values of τ_0'. For this we need to specify the value of

[8] The choice for the ratio of modulation and wave frequencies is to estimate the upper value of b/ω, for which our approximation is accurate. The effects, as proposed below, remain the same for smaller values of b/ω.

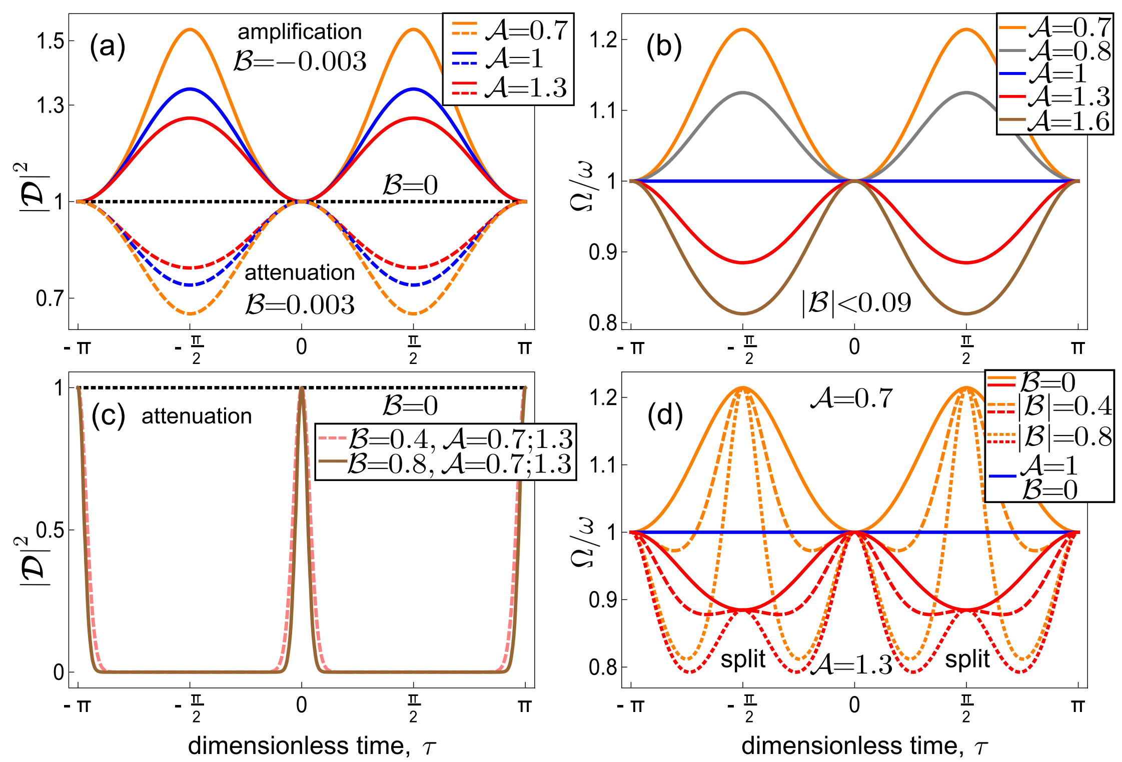

Figure 2. Instantaneous amplitude (left panel, equation (7)) and angular frequency (right panel, equation (5) or (8)) of light modulated by the $\mathcal{PT}$-symmetric dielectric permittivity ε, equation (1), for different values of $\mathcal{A}$ and $\mathcal{B}$. $|\mathcal{B}| < |1 - \mathcal{A}|/2$ corresponds to the upper panel (below the threshold), $|\mathcal{B}| > |1 - \mathcal{A}|/2$ to the lower one (above the threshold).

$\mathcal{B}$ further, albeit the precise criterion depends on the value of $\mathcal{A}$. For $0 < \mathcal{A} < 1$, the criterion for additional extrema is given by $2|\mathcal{B}| > \sqrt{1 - \mathcal{A}}$ determined by the constraint $\sin^2 \tau_0' > 0$, whereas for $\mathcal{A} > 1$, the criterion changes to $2|\mathcal{B}| > \sqrt{\mathcal{A}^2 - \mathcal{A}}$ following from the condition $\sin^2 \tau_0' < 1$. For $\mathcal{A} = 1$, the two criteria are equivalent, and additional extrema can be expected from equation (9) for any value of $\mathcal{B}$ at $\tau_0' = (2\mathcal{N} + 1)\pi/4$.

The physical reason for this difference is the sign of the modulation in the real part of the dielelectric permittivity in equation (1). For $0 < \mathcal{A} < 1$ the modulation of the real part is positive, that is, the real part varies between $\tilde{\varepsilon}$ and $\tilde{\varepsilon} + \varepsilon_1$, which gives rise to an increase of the instantaneous frequency as shown in figure 2(b). For $\mathcal{A} > 1$ the signed amplitude ε_1 is negative and the permittivity varies between $\tilde{\varepsilon} - |\varepsilon_1|$ and $\tilde{\varepsilon}$, which leads to a decrease. In both cases the position of the additional extrema is aptly described by equation (12), which gives rise to a *splitting* of the modulation extrema (as shown in figure 2(d)) that would otherwise be determined by equation (11) in the case when the instantaneous angular frequency is modulated by the conventional real permittivity. The value of the instantaneous angular frequency at these additional extrema for a modulation by the imaginary permittivity beyond the threshold is then given by

$$\Omega(\tau_0') = \omega\, \mathcal{C}^2(1 + 5\mathcal{A} + 4\mathcal{B}^2 + \sqrt{D}/(4\mathcal{B}^2))/8. \tag{13}$$

It is important to note that even though changing the time origin in equation (1) does affect the formal definition of time-reversal symmetry, the observed effects remain the same with the only modification that the symmetry point is shifted for modulations of both the instantaneous amplitude, equation (7), and angular frequency, equation (8).

Time-periodic $\mathcal{PT}$-symmetric optical potentials feature unusual, though expected modulations of the instantaneous properties of the light. Figure 2 illustrates the evolution of these quantities, possessing time-reversal symmetry ($|\mathcal{D}(\tau)|^2 = |\mathcal{D}(-\tau)|^2, \Omega(\tau) = \Omega(-\tau)$), against the dimensionless time τ for various values of $\mathcal{A}$ and $\mathcal{B}$. Figure 2(a) shows that the sign of $\mathcal{B}$ determines whether the overall dynamics give rise to amplification ($\mathcal{B} < 0$) or attenuation ($\mathcal{B} > 0$), despite the fact that in both cases there are equal periods of gain and loss, occurring, however, with different strengths. In addition to these amplitude variations, figure 2(b) shows modulations of the instantaneous angular frequency towards either higher ($0 < \mathcal{A} < 1$) or lower ($\mathcal{A} > 1$) frequencies, compared to ω. For small values of $|\mathcal{B}|$, the instantaneous angular frequency remains mostly unaltered (figure 2(b)—the additional maxima for $\mathcal{A} = 1$ at $\tau_0' = (2\mathcal{N} + 1)\pi/4$ are too small to discern on this scale) and only the amplitude experiences a modulation. Larger values of $\mathcal{B}$ lead to a vigorous modulation of the amplitude attenuation (figure 2(c)) and to a pronounced modification of the frequency

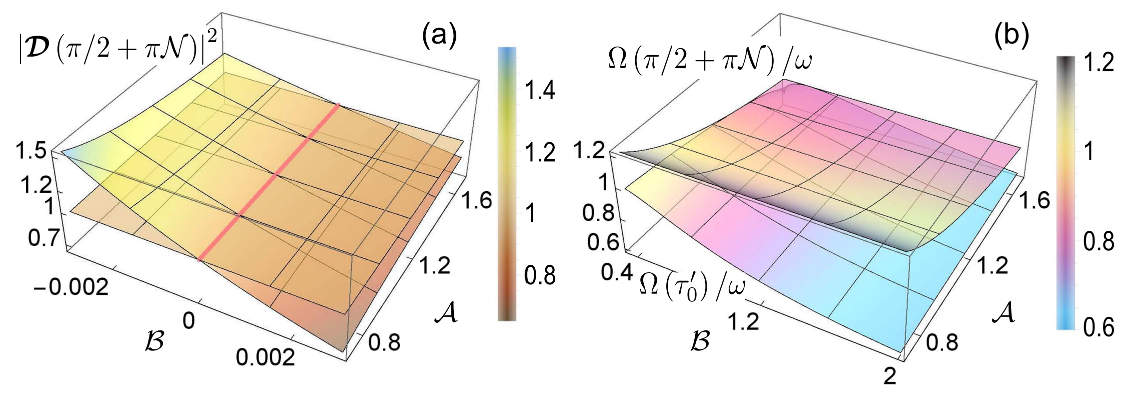

Figure 3. Extrema of the modulus squared of the instantaneous amplitude ((a), equation (10)) and angular frequency ((b), equations (11)–(13)). The comparison is made between modulations of light via a real permittivity profile (given by the unit surface in (a) and the upper surface in (b)) and $\mathcal{PT}$-symmetric permittivity profiles (given by the surface crossing unity in (a) and the lower surface in (b)).

modulation profile (figure 2(d)). Unlike the ordinary modulation of angular frequency, where the modulation rate is commensurate with the scaling factor (i.e., the modulation frequency b) of the real potential [59], in our $\mathcal{PT}$-symmetric structure the extrema of frequency modulation experience a *split* beyond the threshold $2|\mathcal{B}| = |1 - \mathcal{A}|$, as depicted in figure 2(d). The global extremum turns into a local one and two new global extrema appear on either side such that now they occur with a *shrunken* period and the troughs of curves are shifted towards the lower frequency domain, quantitatively determined via equation (13). The values of the instantaneous amplitude and angular frequency at the regular and additional extrema are shown in figure 3 as functions of $\mathcal{A}$ and $\mathcal{B}$, where a comparison is made between the $\mathcal{PT}$- and real-potential induced surfaces. As the gain/loss parameter $\mathcal{B}$ changes its sign from negative to positive, the extrema of the instantaneous amplitude descend from the region of amplification ($|\mathcal{D}(\pi/2 + \pi\mathcal{N})|^2 > 1$) to the region of attenuation ($|\mathcal{D}(\pi/2 + \pi\mathcal{N})|^2 < 1$) for all values of $\mathcal{A}$ (figure 3(a)). By comparison, the unit surface indicates the absence of the amplitude modulation when the imaginary part of the permittivity is 'switched off', that is for $\mathcal{B} = 0$, which marks the line along which both surfaces intersect. In contrast, as seen from figure 3(b), the extrema of the instantaneous angular frequency differ from unity even if the imaginary part of the permittivity is zero. While the $\mathcal{B}$-*in*dependent surface designated by $\Omega(\pi/2 + \pi\mathcal{N})/\omega$ describes the extrema of the instantaneous angular frequency as modulated only by the real part of the permittivity, the surface $\Omega(\tau_0')/\omega$ always lies below $\Omega(\pi/2 + \pi\mathcal{N})/\omega$ and represents the split extrema due to the imaginary part of ε.

Until now, we have discussed how the $\mathcal{PT}$ modulation of the dielectric permittivity affects the instantaneous characteristics of light for various values of $\mathcal{A}$ and $\mathcal{B}$, as allowed for within our approximation. Since the time-dependent permittivity in experimental situations can be modulated with the amplitude much less than through the background dielectric constant (see, e.g., [59]), it is sensible to discuss the impact of the permittivity in the dynamics of light for those values of parameters which are currently available especially in $\mathcal{PT}$-coupled waveguide devices. If we consider a modulation of the dielectric permittivity with the real $|\varepsilon_1| \ll \tilde{\varepsilon}$ and imaginary $|\varepsilon_2| \ll \tilde{\varepsilon}$ parts and keep terms up to first order of $|\varepsilon_1|/\tilde{\varepsilon}$ and $|\varepsilon_2|/\tilde{\varepsilon}$ in the parameters $\mathcal{A} \approx 1 - \varepsilon_1/\tilde{\varepsilon}$, $\mathcal{B} \approx \varepsilon_2/\tilde{\varepsilon}$ and $\mathcal{C} \approx 1 + \varepsilon_1/(2\tilde{\varepsilon})$, we can reduce equation (7) to an experimentally accessible form

$$|\mathcal{D}(\tau)|^2 \approx \exp\left\{\frac{\omega}{2b}\left[-2\mathrm{arctanh}\frac{\varepsilon_2}{\tilde{\varepsilon}} + \mathrm{arctanh}\frac{2\varepsilon_2\cos^2\tau}{\tilde{\varepsilon}}\right]\right\}. \tag{14}$$

As seen, the instantaneous amplitude no longer depends on the value of the real part of the permittivity modulation given by ε_1, but only on the gain/loss distribution strength ε_2, the sign of which suitably defines the temporal domains of gain ($\varepsilon_2 < 0$) and loss ($\varepsilon_2 > 0$), as also shown in figure 4(a). As the magnitude of the absolute value of ε_2 increases, the amplitude modulations become more pronounced. For a given $|\varepsilon_2|/\tilde{\varepsilon} = 0, 003$, for instance, one can clearly distinguish between the increase of ~35% and decrease of ~25% in the amplitude modulation extrema (the black solid and dashed curves). Such an asymmetry in the gain and loss domains is due to the fact that equation (14) changes its form under the change of the sign of ε_2. In contrast to the amplitude modulations, that distinctly occur already for small values of ε_2, the modulations of the instantaneous angular frequency are driven only by the real profile since the gain/loss strength contributes with the second-order term in the denominator of equation (5). However, if we allow for a strong $\mathcal{PT}$ coupling by setting $|\varepsilon_2|/\tilde{\varepsilon} < 1$, but keep the same restriction for the real amplitude, $|\varepsilon_1|/\tilde{\varepsilon} \ll 1$, the instantaneous frequency will be modulated beyond the threshold, and therefore, the split of extrema and associated shrunken periods can clearly

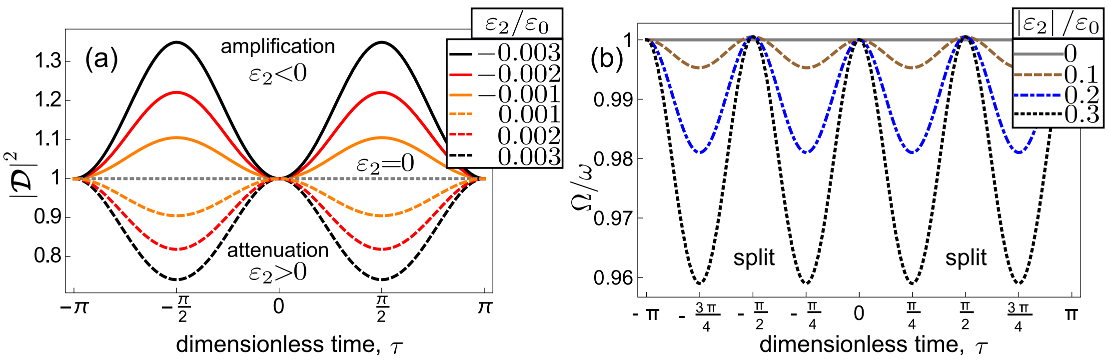

Figure 4. Instantaneous amplitude ((a), equation (14)) and frequency ((b), equation (5)) of light modulated by the $\mathcal{PT}$-symmetric dielectric permittivity ε, equation (1), for the small value of the amplitude ε_1 of the real profile ($|\varepsilon_1|/\tilde{\varepsilon} = 0, 001$) and for selected values of the gain/loss strength ε_2.

be seen at $\tau_0' = (2\mathcal{N} + 1)\pi/4$ since $A \approx 1$, as demonstrated in figure 4(b). Moreover, when $\varepsilon_2/\tilde{\varepsilon} = 0.3$, the change in the instantaneous frequency rises by 4%, as compared to the initial frequency ω.

4. Conclusions

In conclusion, we have examined the impact of the time-dependent $\mathcal{PT}$-symmetric dielectric permittivity in the dynamical and spectral features of light. In our formalism, we have shown that the $\mathcal{PT}$ modulation of light is associated with the well-defined temporal domains of gain ($\mathcal{B} < 0$) and loss ($\mathcal{B} > 0$). We have also determined two different criteria for the splitting of the extrema of the angular frequency modulation to occur and we have demonstrated the shrinkage of the modulation period. Both the split and shrinkage are general, inherent features of time-dependent complex potentials. A direct manifestation of time reflection symmetry in our particular non-stationary structure is always evident. Such effects warrant more detailed future investigations with different time-dependent $\mathcal{PT}$-symmetric potentials. It is true that while the predicted curves on figure 4 could be observed experimentally with techniques available nowadays, the curves on figure 2 are likely to only be accessed in future.

We should mention that our choice of the permittivity suggests that the causality principle is not fulfilled, i.e., the parameters of the permittivity do not follow conventional dispersion constraints, as represented by the Kramers–Kronig relations. Given the known inconsistencies with the Kramers–Kronig relations for $\mathcal{PT}$-symmetric systems [60–63] and other artificial metamaterials with complex-valued permittivity [64–68], analogous relations must be constructed for various types of $\mathcal{PT}$-symmetric time-varying dielectric permittivities, relaxing the strict assumptions made for causality.

Although the theory developed here can be readily expanded for studying similar $\mathcal{PT}$-induced effects for acoustic [52, 58, 69, 70] waves, it also has indirect implications for time-dependent coupling in mechanical systems [71, 72]. The consideration of the analogous theory for modified $\mathcal{PT}$ symmetries [73–75] would be of great interest.

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