On Brauer–Kuroda type relations of S-class numbers in dihedral extensions

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Abstract. Let F/k be a Galois extension of number fields with dihedral Galois group of order $2q$, where q is an odd integer. We express a certain quotient of S-class numbers of intermediate fields, arising from Brauer–Kuroda relations, as a unit index. Our formula is valid for arbitrary extensions with Galois group D_{2q} and for arbitrary Galois-stable sets of primes S, containing the Archimedean ones. Our results have curious applications to determining the Galois module structure of the units modulo the roots of unity of a D_{2q}-extension from class numbers and S-class numbers. The techniques we use are mainly representation theoretic and we consider the representation theoretic results we obtain to be of independent interest.

1. Introduction

Dirichlet [7] was the first to establish a relation between class numbers of a number field and its subfields in 1842: he showed that for a positive integer d that is not a square, the quotient of the class number h of $\mathbb{Q}(\sqrt{d}, \sqrt{-1})$ by the product of the class numbers h_d of $\mathbb{Q}(\sqrt{d})$ and h_{-d} of $\mathbb{Q}(\sqrt{-d})$ is either equal to 1 or 2. In 1950, Brauer [4] and Kuroda [11] independently initiated a systematic study of relations between class numbers in number fields arising from isomorphisms of permutation representations of finite groups. More precisely, if G is a finite group and $\{H_i\}_i$ and $\{H'_j\}_j$ are sets of subgroups such that there is an isomorphism of permutation representations of G,

$$\bigoplus_i \mathbb{Q}[G/H_i] \cong \bigoplus_j \mathbb{Q}[G/H'_j],$$

and if F/K is a Galois extension of number fields with Galois group G, then Artin formalism for Artin L-functions implies that we have an equality of zeta-functions of the corresponding fixed fields:

$$\prod_i \zeta_{F^{H_i}}(s) = \prod_j \zeta_{F^{H'_j}}(s).$$

More generally, if S is any finite G-stable set of places of F containing all the Archimedean ones, then we have an analogous equality of S-zeta functions. Invoking the analytic class

number formula (see e.g. [18], Chapter I, Corollary 2.2) yields the equality

$$\prod_i \frac{h_S(F^{H_i}) R_S(F^{F_i})}{w(F^{H_i})} = \prod_j \frac{h_S(F^{H'_j}) R_S(F^{F'_j})}{w(F^{H'_j})}, \tag{1}$$

where for a number field M, $h_S(M)$, $R_S(M)$, and $w(M)$ denotes the S-class number of M, the S-regulator of M, and the number of roots of unity in M, respectively. See below for precise definitions.

Sometimes, the value of the class number quotient can be given an interpretation in terms of a unit index. In Dirichlet's case, the quantity $2h/(h_d h_{-d})$ is the index in the unit group of $\mathbb{Q}(\sqrt{d}, \sqrt{-1})$ of the subgroup generated by the roots of unity and the unit group of $\mathbb{Q}(\sqrt{d})$. If one wants to make an analogous statement for a general base field and any bi-quadratic extension, then the class number quotient must have the class numbers of all three intermediate quadratic extensions in the denominator and the formula is more complicated due a larger unit rank. A correct formula for bi-quadratic extensions in this more general case, with S equal to the set of Archimedean primes, was only given in 1994 by Lemmermeyer [13].

Our main result is a unit index formula for Galois extensions with Galois group D_{2q} for q any odd integer. Let $\mathcal{O}_M^\times$ denote the units in the ring of integers of a number field M. In 1977, Halter-Koch showed:

Theorem ([10], Section 4). *Let $F/\mathbb{Q}$ be a Galois extension with Galois group D_{2p} for p an odd prime. Let K be the quadratic subfield and $L \ne L'$ be two intermediate extensions of degree p over $\mathbb{Q}$. Let $r(K)$ be the rank of the units in K, which is either 0 or 1. Then*

$$\frac{h(F) p^{r(K)+1}}{h(K) h(L)^2} = [\mathcal{O}_F^\times : \mathcal{O}_L^\times \mathcal{O}_{L'}^\times \mathcal{O}_K^\times].$$

This was generalised to arbitrary base fields by Lemmermeyer in 2005 under a restrictive assumption on the extension:

Theorem ([14], Theorem 2.2). *Let F/k be a Galois extension of number fields with Galois group D_{2p} for p an odd prime, let K be the intermediate quadratic extension and $L \ne L'$ be two intermediate extensions of degree p over k. Assume that F/K is unramified. Let $r(k)$ and $r(K)$ denote the ranks of the unit groups in the respective fields. Then*

$$\frac{h(F) h(k)^2 p^{r(K)+1-r(k)}}{h(K) h(L)^2} = [\mathcal{O}_F^\times : \mathcal{O}_L^\times \mathcal{O}_{L'}^\times \mathcal{O}_K^\times].$$

In 2008, Caputo and Nuccio derived a formula for D_{2q} extensions, where q is any odd integer for certain base fields and certain extension:

Theorem ([5], Theorem 3.4). *Let k be a totally real number field, F/k be a totally imaginary Galois extension with Galois group D_{2q}, where q is an odd integer. Let K be the intermediate quadratic and L, L' be fixed fields of elements σ, σ' of order 2 such that $\sigma\sigma'^{-1}$ is*

of order q. Then

$$\frac{h(F)h(k)^2 q}{h(K)h(L)^2} = \frac{[\mathcal{O}_F^\times : \mathcal{O}_L^\times \mathcal{O}_{L'}^\times \mathcal{O}_K^\times]}{[\mathcal{O}_L^\times \mathcal{O}_{L'}^\times \cap \mathcal{O}_K^\times : \mathcal{O}_k^\times]}.$$

In this paper we complete the study of unit index formulae for dihedral extensions of degree $2q$. We will only state the formula explicitly for D_{2p}, where p is a prime and explain how it is derived for D_{2q} for arbitrary odd integers q since the formula gets unwieldy in the general case, although conceptually not difficult.

Theorem 1.1. *Let F/k be a Galois extension of number fields with Galois group D_{2p} for p an odd prime, let K be the intermediate quadratic extension and L, L' be distinct intermediate extensions of degree p. Let S be a finite $\mathrm{Gal}(F/k)$-stable set of places of F including the Archimedean ones. We write $\mathcal{O}_S^\times$ for S-units, h_S for S-class numbers and r_S for the ranks of S-units. Let $a(F/k, S)$ be the number of primes of k which lie below those in S and whose decomposition group is equal to D_{2p}. Finally, set δ to be 3 if L/k is obtained by adjoining the p-th root of a non-torsion S-unit (thus so is F/K) and 1 otherwise. Then we have*

$$\frac{h_S(F)h_S(k)^2}{h_S(K)h_S(L)^2} = p^{\alpha/2} \times [\mathcal{O}_{S,F}^\times : \mathcal{O}_{S,L}^\times \mathcal{O}_{S,L'}^\times \mathcal{O}_{S,K}^\times],$$

where $\alpha = 2r_S(k) - r_S(K) - \dfrac{r_S(F) - r_S(K)}{p - 1} + a(F/k, S) - \delta$.

Note that all the terms in the exponent of p are very easy to compute in practice (e.g. taking S to be the set of Archimedean places forces $a(F/k, S) = 0$; see Section 6 for more examples).

For arbitrary Galois extensions and sets of subgroups H_i, H_j' giving isomorphic permutation representations, Brauer showed that the class number quotient

$$\prod_i h(F^{H_i})/\prod_j h(F^{H_j'})$$

takes only finitely many values as F ranges over all Galois extensions of K with Galois group G (see [4], Satz 5). He further showed that $\prod_i w(F^{H_i})/\prod_j w(F^{H_j'})$ is a power of 2 (see [4], Section 2) and observed that if p is a prime number not dividing the order of G, then $\mathrm{ord}_p\left(\prod_i R(F^{H_i})/\prod_j R(F^{H_j'})\right) = 0$ ([4], Satz 4, Bemerkung 2). However, there is to date no general formula that explains exactly what values the regulator quotient can take. As a by-product of our calculations, we get the following result in this direction:

Theorem 1.2. *Let G be a finite group, let N be a normal subgroup such that G/N is cyclic, let l be a prime number not dividing the order of N. Let F/K be a Galois extension of number fields with Galois group G and $\{H_i\}$, $\{H_j'\}$ be sets of subgroups yielding an isomorphism of permutation representations as above. Let S be a finite G-stable set of places of F including all the Archimedean ones. Then*

$$\mathrm{ord}_l\left(\prod_i R_S(F^{H_i})/\prod_j R_S(F^{H_j'})\right) = 0,$$

where R_S denotes regulators of S-units, and we have an equality of the l-parts of S-class numbers: $\prod_i h_S(F^{H_i})_l = \prod_j h_S(F^{H'_j})_l$.

We will briefly describe the structure of the paper and the main ideas of the proofs.

Already Brauer pointed out that the regulator quotient is a purely representation theoretic invariant of the $\mathbb{Z}[G]$-module $\mathcal{O}^\times_{S,F}$. This observation was crucial for proving that the regulator quotient takes only finitely many values for a fixed base field and varying Galois extensions F with Galois group G. The main step towards the proof of both Theorem 1.1 and Theorem 1.2 is a representation theoretic description of the regulator quotient. We will provide such a description in Section 2 by linking the regulator quotients to certain invariants, first introduced by Tim and Vladimir Dokchitser in [9] and further explored by the Dokchitser brothers in [8] and by the author in [1] in the context of elliptic curves. These invariants are rational numbers that can be attached to pairs consisting of an integral representation of a group and an isomorphism of permutation representations. We will call these numbers *Dokchitser constants* (deviating from the original name 'regulator constants'). To express the regulator quotients in terms of Dokchitser constants is not entirely trivial and is done in Proposition 2.15.

De Smit [17], Theorem 2.2, has derived a different expression for the regulator quotient, which turns out to be closely related to ours. In Section 3 we will give an alternative definition of the Dokchitser constants and will show how this ties in with de Smit's result. The alternative definition will also be useful to derive some properties of the Dokchitser constants, which will lead to a proof of Theorem 1.2. We think that this section is of independent interest, since it sheds some light on the nature of Dokchitser constants and therefore complements the results of [8].

In Section 4 we turn to Dokchitser constants in dihedral groups. As it turns out, one can compute all the Dokchitser constants for all integral representations of D_{2p}. We should mention that D_{2p} must be regarded as a lucky exception. Although it suffices to determine the Dokchitser constants for indecomposable representations (see Proposition 2.7), a finite group can have infinitely many non-isomorphic indecomposable integral representations and nobody knows how to classify them in general. However, D_{2p} has finitely many and they have been written down explicitly in [12]. Still, it is not clear a priori that their Dokchitser constants can be computed in general since their number grows with p.

In Section 5 we use the properties of Dokchitser constants established in Section 3 to prove Theorem 1.2. We then use the computation of Dokchitser constants for D_{2p} to prove Theorem 1.1 and explain how, using formal properties of Dokchitser constants, one can derive a formula for D_{2q}-extensions for any odd number q. Surprisingly, the generalisation to D_{2q} is rather easy. Because the most general formula would look rather long and obstruct its conceptual simplicity, we will not write it down. Considering S-units instead of just units also introduces very few conceptual difficulties. We note that the way we obtain a unit index formula for D_{2q}-extensions is a completely general procedure to glue unit index formulae together from intermediate extensions. We hope that this will prove useful in the search of unit index formulae in much more general contexts.

In the last section we give various examples. For example, we show how the formulae of Halter-Koch and of Lemmermeyer follow from our Theorem 1.1. We also demonstrate

how our computations can sometimes be used to determine the structure of the Galois module given by the units modulo torsion in a dihedral extension in terms of the class numbers and S-class numbers of the field and its subfields.

We should mention that we use very little number theory in this paper. We need the analytic class number formula (or merely its compatibility with Artin formalism), but unlike the proof of a special case of Theorem 1.1 in [14], we do not need any class field theory.

Notation. Throughout the paper, the following notation will be used for a number field F and for S a finite set of places of F containing the Archimedean ones:

$h_S(F)$ the S-class number of F, i.e. the class number of the ring of all elements of F which are integral at all places not in S;

$w(F)$ the number of roots of unity in F;

$\mathcal{O}_{S,F}^{\times}$ the group of S-units of F;

$r_S(F)$ the $\mathbb{Z}$-rank of $\mathcal{O}_{S,F}^{\times}$, i.e. $|S| - 1$;

$U_S(F)$ the group of S-units modulo torsion; we will often identify units of F with their image in $U_S(F)$, when no confusion can arise;

$R_S(F)$ the S-regulator of F, i.e. the absolute value of the determinant of the square matrix of size $r_S(F)$ whose (i,j)-th entry is $\|\log(u_i)\|_{\mathfrak{p}_j}$, where $\mathfrak{p}_j$ runs through the set of all but one absolute values attached to the places in S and $\{u_1, \ldots, u_{r_S(F)}\}$ is a set of generators of the group of S-units mod torsion;

$\zeta_{F,S}(s)$ the S-zeta function of F, $\zeta_{F,S}(s) = \prod_{\mathfrak{p} \notin S} (1 - \mathrm{N}\mathfrak{p}^{-s})^{-1}$ for $\Re(s) > 1$, the product taken over the places of F not in S and $\mathrm{N}\mathfrak{p}$ denoting the absolute norm of $\mathfrak{p}$.

The normalisations of the absolute values $\| \cdot \|_{\mathfrak{p}}$ attached to places $\mathfrak{p}$ are as follows: If $F_{\mathfrak{p}} = \mathbb{R}$, then the absolute value is just the usual real absolute value. If $F_{\mathfrak{p}} = \mathbb{C}$, then it is the square of the usual absolute value. If $F_{\mathfrak{p}}$ is a p-adic field with residue field of size q, then the attached absolute value is the p-adic absolute value, normalised in such a way that $\|\pi\|_{\mathfrak{p}} = 1/q$ for any uniformiser π in $F_{\mathfrak{p}}$.

When k is a subfield of F, we will write $S|_k$ for the set of places of k lying below those in S. We will often write $h_S(k)$ etc. instead of $h_{S|_k}(k)$.

2. Regulator quotients and Dokchitser constants

In this section we recall the definition of Dokchitser constants from [9] (where they were called regulator constants) and relate them to quotients of regulators of number fields. But first, we introduce a convenient language to talk about the Brauer–Kuroda type relations.

2.1. Relations of permutation representations and Dokchitser constants.

Let G be any finite group. We recall the following standard definitions (see e.g. [6]):

Definition 2.1. The *Burnside ring* of G is defined as the ring of formal $\mathbb{Z}$-linear combinations of isomorphism classes $[X]$ of finite G-sets modulo the relations

$$[X] + [Y] = [X \sqcup Y], \quad [X][Y] = [X \times Y],$$

where $X \sqcup Y$ denotes the disjoint union and $X \times Y$ denotes the Cartesian product.

The set of isomorphism classes of transitive G-sets is in bijection with the set of conjugacy classes of subgroups of G via the map which assigns to the subgroup H the set of co-sets G/H. We will usually represent elements of the Burnside ring as formal sums $\sum_i H_i - \sum_j H_j'$ using this identification.

Definition 2.2. Let A be either $\mathbb{Q}$ or $\mathbb{Z}_{(p)}$, the localisation of $\mathbb{Z}$ at a prime p. The *representation ring* of G over A is the ring of formal $\mathbb{Z}$-linear combinations of isomorphism classes $[M]$ of A-free finite dimensional AG-modules modulo the relations

$$[M] + [N] = [M \oplus N], \quad [M][N] = [M \otimes N].$$

Here and in the rest of the paper, AG denotes the group algebra of the group G over A.

We have a natural map from the Burnside ring to the representation ring that sends a G-set X to the AG-module $A[X]$ with A-basis indexed by the elements of X and the natural G-action. If we take A to be $\mathbb{Q}$, then the image of the Burnside ring in the representation ring has finite index (called the *Artin index* of the group G).

Definition 2.3. We will call an element Θ of the kernel of the above map from the Burnside ring of G to the representation ring over A an *AG-relation*. If $A = \mathbb{Q}$, then we will drop A from the notation and will just say that Θ is a *G-relation*.

The number of isomorphism classes of irreducible rational representations of a finite group G is equal to the number of conjugacy classes of cyclic subgroups of G (see [16], Section 13.1, Corollaire 1). Since, as remarked above, the image of the Burnside ring has full

rank in the representation ring over $\mathbb{Q}$, the lattice of G-relations has rank equal to the number of conjugacy classes of non-cyclic subgroups.

Example 2.4. Let p be an odd prime. The dihedral group D_{2p} with $2p$ elements has one non-cyclic subgroup, namely itself. Decomposing the various permutation representations into irreducible summands, one easily finds that $\Theta = 1 - 2C_2 - C_p + 2D_{2p}$ is a relation. Since it is not divisible by any integer, it must span the $\mathbb{Z}$-lattice of D_{2p}-relations.

We now recall the concept that will be of central importance in this paper:

Definition 2.5. Let G be a finite group, let $\Theta = \sum_i H_i - \sum_j H_j'$ be an AG-relation and let $\mathscr{R}$ be a principal ideal domain such that its field of fractions $\mathscr{K}$ has characteristic not dividing $|G|$. Given an $\mathscr{R}$-free finite rank $\mathscr{R}G$-module Γ such that $\Gamma \otimes \mathscr{K}$ is self-dual, we fix a non-degenerate G-invariant bilinear pairing $\langle \cdot, \cdot \rangle$ on Γ with values in some extension $\mathscr{L}$ of $\mathscr{K}$. For any subgroup H of G, the fixed points Γ^H are also $\mathscr{R}$-free, since $\mathscr{R}$ is a PID, and the pairing is also non-degenerate when restricted to Γ^H by [8], Lemma 2.15. We may thus define the *Dokchitser constant* of Γ with respect to Θ to be

$$\mathscr{C}_\Theta(\Gamma) = \frac{\prod_i \det\left(\frac{1}{|H_i|}\langle \cdot, \cdot \rangle \mid \Gamma^{H_i}\right)}{\prod_j \det\left(\frac{1}{|H_j'|}\langle \cdot, \cdot \rangle \mid \Gamma^{H_j'}\right)} \in \mathscr{L}^\times/(\mathscr{R}^\times)^2,$$

where each inner product matrix is evaluated with respect to some $\mathscr{R}$-basis on the fixed submodule. If the matrix of the pairing on Γ^H with respect to some fixed basis is M, then changing the basis by the change of basis matrix $X \in \mathrm{GL}(\Gamma^H)$ changes the matrix of the pairing to $X^{\mathrm{tr}} MX$. So the Dokchitser constant is indeed a well-defined element of $\mathscr{L}^\times/(\mathscr{R}^\times)^2$.

Convention. From now on, $\mathscr{R}$ will be assumed to be a PID with field of fractions $\mathscr{K}$ of characteristic not dividing $|G|$. We will refer to $\mathscr{R}G$-modules Γ that are free and of finite rank over $\mathscr{R}$ as $\mathscr{R}G$-lattices. We will always assume that $\Gamma \otimes \mathscr{K}$ is self-dual. When we refer to subgroups, we will always mean subgroups up to conjugation unless specifically otherwise stated. So the subgroups H and H' will be treated as being the same if the G-sets G/H and G/H' give the same element of the Burnside ring.

The choice of pairing is not present in the notation of Dokchitser constants and indeed we have:

Theorem 2.6 ([8], Theorem 2.17). *The value of $\mathscr{C}_\Theta(\Gamma)$ is independent of the choice of the pairing.*

In particular, the pairing can always be chosen to be $\mathscr{K}$-valued and so we see that the Dokchitser constant is in fact an element of $\mathscr{K}^\times/(\mathscr{R}^\times)^2$. Note that if $\mathscr{R} = \mathbb{Z}$, then the Dokchitser constant is just a rational number. If $\mathscr{R} = \mathbb{Z}_p$, then at least the p-adic order of the Dokchitser constant is well defined. If on the other hand $\mathscr{R} = \mathbb{Q}$, then the Dokchitser constant is only defined up to rational squares, and if $\mathscr{R} = \mathbb{Q}_p$, then only the parity of the p-adic order is defined. An immediate consequence of Theorem 2.6 is

Proposition 2.7 ([8], Corollary 2.18).   *The Dokchitser constants are multiplicative in Θ and in Γ, i.e.*

$$\mathscr{C}_\Theta(\Gamma \oplus \Gamma') = \mathscr{C}_\Theta(\Gamma)\mathscr{C}_\Theta(\Gamma'),$$

$$\mathscr{C}_{\Theta+\Theta'}(\Gamma) = \mathscr{C}_\Theta(\Gamma)\mathscr{C}_{\Theta'}(\Gamma).$$

Example 2.8.   Take $G = S_3$. There are three irreducible complex representations of S_3, namely the trivial representation $\mathbf{1}$, the one-dimensional sign representation ϵ and a two-dimensional representation ρ, and they are all defined over $\mathbb{Q}$. We saw in Example 2.4 that there is, up to integer multiples, a unique relation

$$1 - 2C_2 - C_3 + 2S_3$$

and it is easy to check that the corresponding Dokchitser constants (with $\mathscr{R} = \mathbb{Q}$) of all three irreducible representations are equal to 3 modulo rational squares. The representations $\mathbf{1}$ and ϵ contain a unique G-invariant $\mathbb{Z}$-lattice each up to isomorphism and their Dokchitser constants (with $\mathscr{R} = \mathbb{Z}$) are $1/3$ and 3, respectively. The two-dimensional representation ρ contains two non-isomorphic G-invariant $\mathbb{Z}$-lattices. Both can be visualised as hexagonal lattices, generated by two shortest distance vectors P and Q, on which the 3-cycles act as rotations by $120°$.
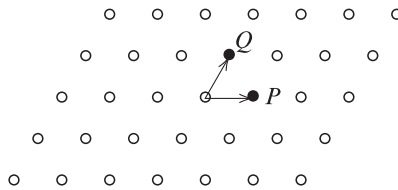
On one, the 2-cycles act by reflection through a shortest distance vector (e.g. through P) and on the other the 2-cycles act by reflection through the long diagonal of the fundamental parallelograms (which are $P + Q$ and its rotations by $120°$ in the sketch). Each one of the two can be embedded into the other G-equivariantly with index 3, but there is no G-equivariant bijection between them. The Dokchitser constants (again with $\mathscr{R} = \mathbb{Z}$) of the two lattices are easily computed to be $1/3$ and 3, respectively.

2.2. Some properties of Dokchitser constants.   We will collect some properties of Dokchitser constants that we will need later. The details can be found in [8]. We first quote a result that shows that, at least for $\mathbb{Q}G$-modules, only *finitely many* primes p can appear in the Dokchitser constants:

Proposition 2.9.   *If $\mathscr{R} = \mathbb{Q}$ or $\mathbb{Q}_p$ and $p \nmid |G|$, then $\operatorname{ord}_p(\mathscr{C}_\Theta(\Gamma))$ is even for any G-relation Θ.*

Proof.   See [8], Corollary 2.28.   □

In Section 3 we will generalise this statement to $\mathscr{R} = \mathbb{Z}$ and $\mathscr{R} = \mathbb{Z}_p$ and we will further restrict for what primes $\operatorname{ord}_p(\mathscr{C}_\Theta(\Gamma))$ can be non-zero.

Relations can be restricted to subgroups, induced from subgroups and lifted from quotients as follows: Let $\Theta = \sum_i H_i - \sum_j H_j'$ be a G-relation.

- **Induction.** If G' is a group containing G, then by transitivity of induction, Θ can be induced to a G'-relation $\Theta{\uparrow}^{G'} = \sum_i H_i - \sum_j H_j'$ of G'.

- **Inflation.** If $G \cong \tilde{G}/N$, then each H_i corresponds to a subgroup $\tilde{H}_i$ of $\tilde{G}$ containing N and similarly for H_j' and, inflating the permutation representations from a quotient, we see that $\tilde{\Theta} = \sum_i \tilde{H}_i - \sum_j \tilde{H}_j'$ is a $\tilde{G}$-relation.

- **Restriction.** If H is a subgroup of G, then by Mackey decomposition Θ can be restricted to an H-relation $\Theta{\downarrow}_H = \sum_i \sum_{g \in H_i \backslash G/H} H \cap H_i^g - \sum_j \sum_{g \in H_j' \backslash G/H} H \cap H_j'^g$.

We have the following compatibility between these operations and the corresponding operations applied to representations Γ:

Proposition 2.10. *Let G be a finite group and Γ an $\mathscr{R}G$-lattice.*

- *If $H < G$ and Θ is an H-relation, then $\mathscr{C}_\Theta(\Gamma{\downarrow}_H) = \mathscr{C}_{\Theta{\uparrow}^G}(\Gamma)$.*

- *If $G \cong \tilde{G}/N$ and Θ is a G-relation with $\tilde{\Theta}$ the lifted relation, then $\mathscr{C}_\Theta(\Gamma) = \mathscr{C}_{\tilde{\Theta}}(\Gamma)$, where Γ can also be regarded as a $\tilde{G}$-representation.*

- *If $G < G'$ and Θ is a G'-relation, then $\mathscr{C}_\Theta(\Gamma{\uparrow}^{G'}) = \mathscr{C}_{\Theta{\downarrow}_G}(\Gamma)$.*

Proof. See [8], Proposition 2.45. ☐

2.3. Quotients of regulators of number fields. We now want to explain the relationship between Dokchitser constants and quotients of regulators in Brauer–Kuroda type relations. Let G be a finite group, $\Theta = \sum_i H_i - \sum_j H_j'$ be a G-relation and F/k be a Galois extension of number fields with Galois group G. Let S be a finite G-stable set of places of F including all the Archimedean ones.

In the definition of the Dokchitser constant we need to fix a pairing on our $\mathbb{Z}G$-lattice, so to turn regulator quotients into Dokchitser constants, we need to find a suitable pairing on $U_S(F)$. It seems tempting therefore to multiply the matrix $(\log\|u_i\|_{\mathfrak{p}_j})$, whose determinant is the S-regulator of the field, with its transpose and to take the pairing of which the resulting matrix will be the Gram matrix. In other words, we would then define the inner product of u_i with u_j by $\sum_{k=1}^{|S|-1} (\log\|u_i\|_{\mathfrak{p}_k} \log\|u_j\|_{\mathfrak{p}_k})$, with the sum running over all but one place in S. This approach does not work because the resulting pairing does not behave well upon restriction to subfields. We need its restriction to a subfield M to be equal to the pairing of that subfield scaled by the degree of F/M. We are very grateful to Samir Siksek for suggesting to us to try instead summing over all places in S, rather than all but one. We also need another slight twist:

Definition 2.11. Let M be a number field and $\mathscr{S}$ be a finite set of places of M including the Archimedean ones. Define the bilinear pairing $\langle \cdot, \cdot \rangle_M$ on $U_{\mathscr{S}}(M)$ by

$$\langle u_i, u_j \rangle_M = \sum_{\mathfrak{P} \in \mathscr{S}} \frac{1}{e_{\mathfrak{P}} f_{\mathfrak{P}}} \log\|u_i\|_{\mathfrak{P}} \log\|u_j\|_{\mathfrak{P}},$$

where $e_{\mathfrak{P}}$ is the absolute ramification index of $\mathfrak{P}$ and $f_{\mathfrak{P}}$ is the degree of its residue field over the prime subfield (defined to be 1 if $\mathfrak{P}$ is Archimedean).

We begin by establishing the non-degeneracy of the pairing and by linking its determinant to the usual $\mathscr{S}$-regulator of a number field.

Lemma 2.12. *Let M be a number field and $\mathscr{S}$ be a finite set of places of M containing all the Archimedean ones. Then we have*

$$\det\left(\langle \cdot, \cdot \rangle_M \mid U_{\mathscr{S}}(M)\right) = \frac{\sum\limits_{\mathfrak{P} \in \mathscr{S}} f_{\mathfrak{P}} e_{\mathfrak{P}}}{\prod\limits_{\mathfrak{P} \in \mathscr{S}} f_{\mathfrak{P}} e_{\mathfrak{P}}} \cdot R_{\mathscr{S}}(M)^2.$$

Proof. Write $\mathscr{S} = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_{r+1}\}$ and define the following matrix:

$$\mathfrak{X} = \begin{pmatrix} \sqrt{f_{\mathfrak{P}_1} e_{\mathfrak{P}_1}} & \dfrac{\log\|u_1\|_{\mathfrak{P}_1}}{\sqrt{f_{\mathfrak{P}_1} e_{\mathfrak{P}_1}}} & \cdots & \dfrac{\log\|u_r\|_{\mathfrak{P}_1}}{\sqrt{f_{\mathfrak{P}_1} e_{\mathfrak{P}_1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{f_{\mathfrak{P}_{r+1}} e_{\mathfrak{P}_{r+1}}} & \dfrac{\log\|u_1\|_{\mathfrak{P}_{r+1}}}{\sqrt{f_{\mathfrak{P}_{r+1}} e_{\mathfrak{P}_{r+1}}}} & \cdots & \dfrac{\log\|u_r\|_{\mathfrak{P}_{r+1}}}{\sqrt{f_{\mathfrak{P}_{r+1}} e_{\mathfrak{P}_{r+1}}}} \end{pmatrix}.$$

Then by the product formula and by the definition of our pairing, we have

$$\mathfrak{X}^{\mathrm{tr}}\mathfrak{X} = \begin{pmatrix} \sum\limits_{\mathfrak{P} \in \mathscr{S}} f_{\mathfrak{P}} e_{\mathfrak{P}} & 0 & \cdots & 0 \\ 0 & \langle u_1, u_1 \rangle_M & \cdots & \langle u_1, u_r \rangle_M \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \langle u_r, u_1 \rangle_M & \cdots & \langle u_r, u_r \rangle_M \end{pmatrix}$$

and so

$$(2) \qquad \det(\mathfrak{X}^{\mathrm{tr}}\mathfrak{X}) = \det\left(\langle \cdot, \cdot \rangle_M \mid U_{\mathscr{S}}(M)\right) \cdot \sum_{\mathfrak{P} \in \mathscr{S}} f_{\mathfrak{P}} e_{\mathfrak{P}}.$$

On the other hand, thanks to the product formula, by multiplying the i-th row of $\mathfrak{X}$ by $\sqrt{f_{\mathfrak{P}_i} e_{\mathfrak{P}_i}}$ for each i and by adding all the rows of the resulting matrix to the last one, we get zeros in the entire bottom row, apart from the bottom left entry, where we get $\sum\limits_{\mathfrak{P} \in \mathscr{S}} f_{\mathfrak{P}} e_{\mathfrak{P}}$. Moreover, the resulting matrix with the first column and the bottom row deleted

has determinant equal to $R_{\mathscr{S}}(M)$. In summary, we see that

$$
(3) \qquad \det(\mathfrak{X}) = \frac{\sum\limits_{\mathfrak{P} \in \mathscr{S}} f_{\mathfrak{P}} e_{\mathfrak{P}}}{\prod\limits_{\mathfrak{P} \in \mathscr{S}} \sqrt{f_{\mathfrak{P}} e_{\mathfrak{P}}}} \cdot R_{\mathscr{S}}(M),
$$

and by combining equations (2) and (3), the claim follows. $\quad\square$

We now return to our previous scenario and explain how the pairing behaves in relations. It is clear that if F/k is a Galois extension of number fields and S is a finite set of places of F containing all the Archimedean ones, then $\langle \cdot, \cdot \rangle_F$ is G-invariant. It also behaves correctly under restriction to subfields:

Lemma 2.13. *Let F/k be a Galois extension, S be a finite Galois stable set of places of F including the Archimedean ones, $L \leqq M$ be subfields of F containing k and u_i, u_j be S-units in $U_S(L)$. Then*

$$
\langle u_i, u_j \rangle_M = [M : L] \langle u_i, u_j \rangle_L.
$$

Proof. This is easy to see by considering each prime of $S|_L$ separately since, with our normalisations of the absolute values, we have

$$
\log\|u\|_{\mathfrak{P}} = e_{\mathfrak{P}/\mathfrak{p}} f_{\mathfrak{P}/\mathfrak{p}} \log\|u\|_{\mathfrak{p}}
$$

for any $u \in U_S(L)$ and for any prime $\mathfrak{P} \in S|_M$ above a prime $\mathfrak{p} \in S|_L$. $\quad\square$

There is, however, a slight caveat in working with units modulo torsion because if F/k is a Galois extension with Galois group G, then the fixed submodule of $U_S(F)$ under a subgroup H of G need not be canonically isomorphic to $U_S(F^H)$. We will need to understand exactly when it is and what the difference is whenever it is not. Write $\mu(M)$ for the group of roots of unity of a number field M. Then, from the short exact sequence

$$
1 \to \mu(F) \to \mathcal{O}_{S,F}^{\times} \to U_S(F) \to 1
$$

we get the usual long exact sequence of group cohomology

$$
1 \to \mu(F)^H \to (\mathcal{O}_{S,F}^{\times})^H \to U_S(F)^H \to \mathrm{H}^1\big(H, \mu(F)\big) \to \mathrm{H}^1\big(H, \mathcal{O}_{S,F}^{\times}\big)
$$

for any subgroup H of G. We see immediately that $(\mathcal{O}_{S,F}^{\times})^H / \mu(F)^H = \mathcal{O}_{S,F^H}^{\times} / \mu(F^H)$ is canonically isomorphic to $U_S(F)^H$ if and only if $\ker\big(\mathrm{H}^1\big(H, \mu(F)\big) \to \mathrm{H}^1\big(H, \mathcal{O}_{S,F}^{\times}\big)\big) = 0$. We have that $f \in \mathrm{H}^1\big(H, \mu(F)\big)$ is in this kernel if and only if there is an S-unit $u \in \mathcal{O}_{S,F}^{\times}$ such that $f(h) = h(u)/u \in \mu(F)$ for all $h \in H$. If f is not a co-boundary itself, then $u \notin \mu(F)$ and u can without loss of generality be taken to be non-torsion. We deduce that F must contain a root of a non-torsion S-unit of an intermediate extension of F/F^H. Conversely, if it does, then defining f as above gives a non-trivial element of the kernel. In summary, we record

Lemma 2.14. *With F/k and S as above, we have, for any subgroup H of the Galois group of F/k, that $U_S(F)^H \cong U_S(F^H)$ if no intermediate extension of F/F^H is obtained by*

adjoining a root of a non-torsion S-unit. In general,

$$[U_S(F)^H : U_S(F^H)] = \#\ker\big(\mathrm{H}^1\big(H, \mu(F)\big) \to \mathrm{H}^1(H, \mathcal{O}_{S,F}^\times)\big).$$

We are now ready to prove the main result of this section, which links regulator quotients to Dokchitser constants.

Proposition 2.15. *Let F/k be a finite Galois extension with Galois group G and let S be a finite G-stable set of places of F including the Archimedean ones. Write*

$$\lambda(H) = \#\ker\big(\mathrm{H}^1\big(H, \mu(F)\big) \to \mathrm{H}^1(H, \mathcal{O}_{S,F}^\times)\big).$$

If $\Theta = \sum\limits_i H_i - \sum\limits_j H_j'$ is a G-relation, then we have

$$\mathscr{C}_\Theta\big(U_S(F)\big) = \frac{\mathscr{C}_\Theta(\mathbf{1})}{\prod\limits_{\mathfrak{p} \in S|_k} \mathscr{C}_\Theta(\mathbb{Z}[G/D_\mathfrak{p}])} \cdot \frac{\prod\limits_i \big(R_S(F^{H_i})/\lambda(H_i)\big)^2}{\prod\limits_j \big(R_S(F^{H_j'})/\lambda(H_j')\big)^2},$$

where, for each $\mathfrak{p} \in S|_k$, $D_\mathfrak{p}$ is a decomposition subgroup of G at $\mathfrak{p}$.

Remark 2.16. In particular, if the decomposition groups at all primes in S are cyclic, which is for example the case if S is the set of Archimedean places, then the associated Dokchitser constants are trivial by [8], Lemma 2.46, and the formula just reads

$$\mathscr{C}_\Theta\big(U_S(F)\big) = \mathscr{C}_\Theta(\mathbf{1}) \cdot \frac{\prod\limits_i \big(R_S(F^{H_i})/\lambda(H_i)\big)^2}{\prod\limits_j \big(R_S(F^{H_j'})/\lambda(H_j')\big)^2}.$$

Proof. For any $H \leqq G$ we have

$$\lambda(H)^2 \cdot \det\left(\frac{1}{|H|} \langle \cdot, \cdot \rangle_F \mid U_S(F)^H\right) \overset{\text{Lemma 2.14}}{=\!=\!=} \det\left(\frac{1}{|H|} \langle \cdot, \cdot \rangle_F \mid U_S(F^H)\right)$$

$$\overset{\text{Lemma 2.13}}{=\!=\!=} \det\big(\langle \cdot, \cdot \rangle_{F^H} \mid U_S(F^H)\big)$$

$$\overset{\text{Lemma 2.12}}{=\!=\!=} \frac{\sum\limits_{\mathfrak{P} \in S|_{F^H}} f_\mathfrak{P} e_\mathfrak{P}}{\prod\limits_{\mathfrak{P} \in S|_{F^H}} f_\mathfrak{P} e_\mathfrak{P}} \cdot R_S(F^H)^2.$$

Since S contains precisely all the places above the places in $S|_k$, we have, for each $\mathfrak{p} \in S|_k$,

$$\sum\limits_{\substack{\mathfrak{P} \in S|_{F^H}, \\ \mathfrak{P} | \mathfrak{p}}} f_\mathfrak{P} e_\mathfrak{P} = \sum\limits_{\substack{\mathfrak{P} \in S|_{F^H}, \\ \mathfrak{P} | \mathfrak{p}}} f_\mathfrak{p} e_\mathfrak{p} f_{\mathfrak{P}/\mathfrak{p}} e_{\mathfrak{P}/\mathfrak{p}} = f_\mathfrak{p} e_\mathfrak{p} [F^H : k]$$

and thus, for any $H \leqq G$ we have

$$\sum\limits_{\mathfrak{P} \in S|_{F^H}} f_\mathfrak{P} e_\mathfrak{P} = [F^H : k] \sum\limits_{\mathfrak{p} \in S|_k} f_\mathfrak{p} e_\mathfrak{p} = \frac{|G|}{|H|} \sum\limits_{\mathfrak{p} \in S|_k} f_\mathfrak{p} e_\mathfrak{p}.$$

Observe that by [8], Example 2.30, the term $\sum\limits_{\mathfrak{p} \in S|_k} e_\mathfrak{p} f_\mathfrak{p}$, being a constant, vanishes in a relation.

Also, for each $\mathfrak{p} \in S|_k$ we have

$$\prod_{\substack{\mathfrak{P} \in S|_{F^H}, \\ \mathfrak{P} | \mathfrak{p}}} f_\mathfrak{P} e_\mathfrak{P} = (f_\mathfrak{p} e_\mathfrak{p})^{\#\{\text{primes of } F^H \text{ above } \mathfrak{p}\}} \cdot \prod_{\substack{\mathfrak{P} \in S|_{F^H}, \\ \mathfrak{P} | \mathfrak{p}}} (f_{\mathfrak{P}/\mathfrak{p}} e_{\mathfrak{P}/\mathfrak{p}}).$$

Now, by [8], Example 2.37, the first of the two factors vanishes in a relation. Also, by [8], Corollary 2.44, the second factor may be replaced by $\mathscr{C}_\Theta(\mathbb{Z}[G/D])$ in a relation, where D is a decomposition subgroup of G at $\mathfrak{p}$. Combining everything we have said, we obtain

$$\mathscr{C}_\Theta\big(U_S(F)\big) = \frac{\prod\limits_i \det\left(\dfrac{1}{|H_i|}\langle\cdot,\cdot\rangle_F \mid U_S(F)^{H_i}\right)}{\prod\limits_j \det\left(\dfrac{1}{|H_j'|}\langle\cdot,\cdot\rangle_F \mid U_S(F)^{H_j'}\right)}$$

$$= \frac{\mathscr{C}_\Theta(\mathbf{1})}{\prod\limits_{\mathfrak{p} \in S|_k} \mathscr{C}_\Theta(\mathbb{Z}[G/D_\mathfrak{p}])} \cdot \frac{\prod\limits_i \big(R_S(F^{H_i})/\lambda(H_i)\big)^2}{\prod\limits_j \big(R_S(F^{H_j'})/\lambda(H_j')\big)^2},$$

as claimed. $\qquad\square$

Now that we know how to turn quotients of regulators of number fields into Dokchitser constants, which are purely representation theoretic invariants, we will establish some properties of Dokchitser constants in the next two sections.

3. An alternative description of Dokchitser constants

The definition of Dokchitser constants that we have given above is somewhat unsatisfactory, since it involves making an arbitrary choice (that of a pairing) on which the result does not depend. It would be nice to have a definition that avoids any arbitrary choices. As a first step in the investigation of the properties of Dokchitser constants, we will provide an alternative definition, which depends on fixing more specific information about the relation (on which the result again does not depend) but not on any choices connected with the representation.

Let $\Theta = \sum\limits_i H_i - \sum\limits_j H_j'$ be an AG-relation. Define the G-sets $S_1 = \bigsqcup\limits_i G/H_i$ and $S_2 = \bigsqcup\limits_j G/H_j'$. Then to say that $\mathbb{Q}[S_1] \cong \mathbb{Q}[S_2]$ is equivalent to saying that there exists an embedding of $\mathbb{Z}G$-lattices

$$\phi : \mathbb{Z}[S_1] \hookrightarrow \mathbb{Z}[S_2]$$

with finite cokernel. Also, to say that $\mathbb{Z}_{(l)}[S_1] \cong \mathbb{Z}_{(l)}[S_2]$ is equivalent to saying that there is such a ϕ with finite cokernel of order coprime to l.

With these remarks in mind, let $\mathscr{R}$ be a PID containing $\mathbb{Z}$, let Γ be an $\mathscr{R}G$-lattice and fix an injection of $\mathscr{R}G$-modules

$$\phi : \mathscr{R}[S_1] \hookrightarrow \mathscr{R}[S_2].$$

By dualising this, we obtain

$$\phi^{\mathrm{tr}} : \mathrm{Hom}(\mathscr{R}[S_2], \mathscr{R}) \hookrightarrow \mathrm{Hom}(\mathscr{R}[S_1], \mathscr{R}).$$

Recall that if Γ is any $\mathscr{R}G$-lattice and if $g \in G$ is represented by the matrix M with respect to some $\mathscr{R}$-basis, then in the action on $\mathrm{Hom}(\Gamma, \mathscr{R})$, g is represented by $(M^{-1})^{\mathrm{tr}}$ with respect to the dual basis. But this is equal to M if M is a permutation matrix and so permutation modules are canonically self-dual. In summary, we have a map

$$\phi^{\mathrm{tr}} : \mathscr{R}[S_2] \hookrightarrow \mathscr{R}[S_1].$$

Applying the functor $\mathrm{Hom}(\_, \Gamma)$ yields

$$\phi^* : \mathrm{Hom}_{\mathscr{R}}(\mathscr{R}[S_2], \Gamma) \to \mathrm{Hom}_{\mathscr{R}}(\mathscr{R}[S_1], \Gamma)$$

and

$$(\phi^{\mathrm{tr}})^* : \mathrm{Hom}_{\mathscr{R}}(\mathscr{R}[S_1], \Gamma) \to \mathrm{Hom}_{\mathscr{R}}(\mathscr{R}[S_2], \Gamma).$$

Upon restricting to the G-invariant subspaces, we obtain maps ϕ_G^* and $(\phi^{\mathrm{tr}})_G^*$ between the corresponding spaces of G-homomorphisms (to avoid index overload, we are abusing notation slightly by not including Γ in the notation of these maps). Since $\mathscr{R}$ is a PID, the spaces of G-homomorphisms are $\mathscr{R}$-free. Also, since $\phi \otimes \mathscr{K}$ and $\phi^{\mathrm{tr}} \otimes \mathscr{K}$ are both isomorphisms, so are $\phi_G^* \otimes \mathscr{K}$ and $(\phi^{\mathrm{tr}})_G^* \otimes \mathscr{K}$. Thus both ϕ_G^* and $(\phi^{\mathrm{tr}})_G^*$ have non-zero determinants.

Definition 3.1. Define

$$\mathscr{C}_\Theta'(\Gamma) = \frac{\det(\phi^{\mathrm{tr}})_G^*}{\det \phi_G^*} \in \mathscr{K}^\times / (\mathscr{R}^\times)^2$$

with both determinants computed with respect to the same bases on $\mathrm{Hom}_{\mathscr{R}[G]}(\mathscr{R}[S_i], \Gamma)$, $i = 1, 2$. If we change the basis on $\mathrm{Hom}_{\mathscr{R}[G]}(\mathscr{R}[S_1], \Gamma)$, say, then the quotient changes by the square of the determinant of change of basis, so it really is a well-defined element of $\mathscr{K}^\times / (\mathscr{R}^\times)^2$.

The injection ϕ is not present in the notation and indeed it will turn out that the residue class of $\mathscr{C}_\Theta'(\Gamma)$ modulo $(\mathscr{R}^\times)^2$ only depends on the relation and the module but not on any other choices. The main result of this section is

Theorem 3.2. *Let G be a finite group, $\mathscr{R}$ be a principal ideal domain containing $\mathbb{Z}$ with field of fractions $\mathscr{K}$, $\Theta = \sum_i H_i - \sum_j H_j'$ be an AG-relation, where A is either $\mathbb{Q}$ or $\mathbb{Z}_{(p)}$, and Γ be an $\mathscr{R}G$-lattice. Fix an injection $\phi : \mathscr{R}[S_1] \hookrightarrow \mathscr{R}[S_2]$ and obtain ϕ_G^* and $(\phi^{\mathrm{tr}})_G^*$ as above. Fix*

a G-invariant non-degenerate bilinear pairing $\langle \cdot , \cdot \rangle$ on Γ (which exists because $\Gamma \otimes \mathscr{K}$ is self-dual by convention). Then

$$\frac{\det(\phi^{\mathrm{tr}})_G^*}{\det \phi_G^*} \equiv \frac{\prod_i \det\left(\frac{1}{|H_i|} \langle \cdot , \cdot \rangle \,|\, \Gamma^{H_i}\right)}{\prod_j \det\left(\frac{1}{|H_j'|} \langle \cdot , \cdot \rangle \,|\, \Gamma^{H_j'}\right)} \; \mathrm{mod}(\mathscr{R}^\times)^2.$$

Proof.   Define a pairing $(\cdot , \cdot)_1$ on $\mathrm{Hom}_{\mathscr{R}}(\mathscr{R}[S_1], \Gamma)$ by

$$(f_1, f_2)_1 = \frac{1}{|G|} \sum_{s \in S_1} \langle f_1(s), f_2(s) \rangle$$

and define an analogous pairing $(\cdot , \cdot)_2$ on $\mathrm{Hom}_{\mathscr{R}}(\mathscr{R}[S_2], \Gamma)$. It is immediate that these pairings, when restricted to the spaces of G-homomorphisms, are G-invariant. We first claim that $(\phi^{\mathrm{tr}})^*$ is the adjoint of ϕ^* with respect to these pairings. Indeed, it suffices to check this for any particular choice of bases on $\mathrm{Hom}_{\mathscr{R}}(\mathscr{R}[S_1], \Gamma)$ and on $\mathrm{Hom}_{\mathscr{R}}(\mathscr{R}[S_2], \Gamma)$. Let $S_1 = \{s_1, \ldots, s_n\}$ and choose a basis γ_j, $j = 1, \ldots, r$, of Γ. Define $f_{i,j} \in \mathrm{Hom}_{\mathscr{R}}(\mathscr{R}[S_1], \Gamma)$ by $f_{i,j}(s_i) = \gamma_j$, $f_{i,j}(s) = 0$ for all $s \neq s_i$. Then $f_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, r$, is a basis of $\mathrm{Hom}_{\mathscr{R}}(\mathscr{R}[S_1], \Gamma)$. Fix the analogous basis $f_{i,j}'$ for $\mathrm{Hom}_{\mathscr{R}}(\mathscr{R}[S_2], \Gamma)$, where $S_2 = \{s_1', \ldots, s_n'\}$. We compute

$$(4) \qquad |G| \cdot (f_{i,j}, \phi^* f_{r,t}')_1 = \sum_{s \in S_1} \langle f_{i,j}(s), f_{r,t}'(\phi(s)) \rangle = \langle \gamma_j, f_{r,t}'(\phi(s_i)) \rangle$$

$$= \langle \gamma_j, \phi_{i,r} \gamma_t \rangle = \langle \phi_{i,r} \gamma_j, \gamma_t \rangle = \langle f_{i,j}(\phi^{\mathrm{tr}} s_r'), \gamma_t \rangle$$

$$= \sum_{s' \in S_2} \langle f_{i,j}(\phi^{\mathrm{tr}} s'), f_{r,t}'(s') \rangle = |G| \cdot \left((\phi^{\mathrm{tr}})^* f_{i,j}, f_{r,t}'\right)_2$$

as required. Next, for a subgroup H of G we can identify $\mathrm{Hom}_G(G/H, \Gamma)$ with Γ^H via $f \mapsto f(1)$. We claim that under this identification, we have

$$(5) \qquad \det\left((\cdot , \cdot)_1 \,|\, \mathrm{Hom}_{\mathscr{R}[G]}(\mathscr{R}[S_1], \Gamma)\right) \equiv \prod_i \det\left(\frac{1}{|H_i|} \langle \cdot , \cdot \rangle \,|\, \Gamma^{H_i}\right) \quad \mathrm{mod}(\mathscr{R}^\times)^2$$

and similarly for S_2. Indeed, if for subgroups $H_i \neq H_k$, we have that $\mathscr{R}[G/H_i]$ and $\mathscr{R}[G/H_k]$ are summands of $\mathscr{R}[S_1]$, then an element of $\mathrm{Hom}_{\mathscr{R}[G]}(\mathscr{R}[S_1], \Gamma)$ which is trivial outside of G/H_i is orthogonal to an element which is trivial outside of G/H_k. So it suffices to prove the claim for $S_1 = G/H$. We compute

$$(f_1, f_2)_1 = \frac{1}{|G|} \sum_{s \in G/H} \langle f_1(s), f_2(s) \rangle = \frac{1}{|G|} \sum_{s \in G/H} \langle s \cdot f_1(1), s \cdot f_2(1) \rangle$$

$$= \frac{1}{|G|} \sum_{s \in G/H} \langle f_1(1), f_2(1) \rangle = \frac{1}{|H|} \langle f_1(1), f_2(1) \rangle,$$

which immediately implies the claim. Now, fix bases $v_1, \ldots, v_m$ and $v_1', \ldots, v_m'$ on $\mathrm{Hom}_{\mathscr{R}[G]}(\mathscr{R}[S_1], \Gamma)$ and $\mathrm{Hom}_{\mathscr{R}[G]}(\mathscr{R}[S_2], \Gamma)$, respectively. Then

$$\frac{\prod_i \det\left(\frac{1}{|H_i|}\langle\cdot,\cdot\rangle \mid \Gamma^{H_i}\right)}{\prod_j \det\left(\frac{1}{|H'_j|}\langle\cdot,\cdot\rangle \mid \Gamma^{H'_j}\right)} \overset{\text{by (5)}}{\equiv} \frac{\det\left((v_i,v_j)_1\right)}{\det\left((v'_k,v'_l)_2\right)}$$

$$\equiv \frac{\det\left((v_i,\phi_G^* v'_l)_1\right)/\det(\phi_G^*)}{\det\left(((\phi^{\mathrm{tr}})_G^* v_i, v'_l)_2\right)/\det((\phi^{\mathrm{tr}})_G^*)}$$

$$\overset{\text{by (4)}}{\equiv} \det\left((\phi^{\mathrm{tr}})_G^*\right)/\det(\phi_G^*) \bmod(\mathscr{R}^\times)^2,$$

which concludes the proof. $\square$

Corollary 3.3. *The value of $\mathscr{C}'_\Theta(\Gamma)$ is independent of the choice of ϕ and we have $\mathscr{C}'_\Theta(\Gamma) = \mathscr{C}_\Theta(\Gamma)$ for all G-relations Θ and all $\mathscr{R}G$-lattices Γ.*

Remark 3.4. It is not difficult to prove the independence of ϕ directly, using explicit calculations with the bases $f_{i,j} : s_l \mapsto \delta_{i,l} \cdot \gamma_j$ and $f'_{i,j} : s'_l \mapsto \delta_{i,l} \cdot \gamma_j$ from the above proof. This also gives an alternative proof of the independence of $\mathscr{C}_\Theta(\Gamma)$ of the pairing.

Remark 3.5. It is instructive to compare our alternative definition of Dokchitser constants in conjunction with Proposition 2.15 with the formula for class number quotients derived by de Smit in [17], Theorem 2.2. In his formula, the torsion subgroup of the units is more directly incorporated into the whole expression. However, arbitrary $\mathbb{Z}G$-modules of a given group G are more difficult to classify than $\mathbb{Z}$-free modules and we will need to use the classification from [12] for $G = D_{2p}$ in the next section. That is the main reason why we pass to the quotient modulo torsion first and then recover the torsion separately in the shape of $\lambda(H)$. Another reason to work with Dokchitser constants is that a lot of the computations of Dokchitser constants in the next section will be easier using a pairing rather than an embedding ϕ.

An immediate consequence of the alternative definition is the following:

Lemma 3.6. *Let G be a finite group and Θ be a $\mathbb{Z}_{(l)}G$-relation. Then $\mathrm{ord}_l\left(\mathscr{C}_\Theta(\Gamma)\right) = 0$ for all $\mathbb{Z}G$-lattices Γ.*

Proof. As remarked above, we can find an injection of G-modules $\phi : \mathbb{Z}[S_1] \hookrightarrow \mathbb{Z}[S_2]$ with co-kernel of order coprime to l. For any injection of free abelian groups with finite co-kernel, the order of its co-kernel is equal to the absolute value of its determinant (with respect to any bases). Now, applying $\mathrm{Hom}_G(\_, \Gamma)$ to the tautological short exact sequence

$$0 \to \mathbb{Z}[S_1] \overset{\phi}{\to} \mathbb{Z}[S_2] \to \mathrm{coker}(\phi) \to 0$$

gives the long exact sequence

$$0 \to \mathrm{Hom}_G(\mathbb{Z}[S_2], \Gamma) \overset{\phi_G^*}{\to} \mathrm{Hom}_G(\mathbb{Z}[S_1], \Gamma) \to \mathrm{Ext}^1_{\mathbb{Z}G}\left(\mathrm{coker}(\phi), \Gamma\right).$$

Since $\mathrm{Ext}^1_{\mathbb{Z}G}\left(\mathrm{coker}(\phi), \Gamma\right)$ has no l-torsion, neither does $\mathrm{coker}(\phi_G^*)$. The same goes for $(\phi^{\mathrm{tr}})_G^*$ and the proof is complete. $\square$

Definition 3.7. Let l be a prime number. A finite group is called *l-hypo-elementary* if it has a normal Sylow l-subgroup with cyclic quotient. Equivalently, an *l*-hypo-elementary group is a semi-direct product of an *l*-group acted on by a cyclic group of order coprime to *l*.

Theorem 3.8 (Conlon's induction theorem). *Given any finite group H and any commutative ring $\tilde{R}$ in which every prime divisor of $|H|$ except possibly l is invertible, there exist integers $\alpha_{H'}$ such that some integer multiple of the trivial representation of H over $\tilde{R}$ is equal to $\sum_{H'} \alpha_{H'} \tilde{R}[H/H']$ in the representation ring over $\tilde{R}$, where the sum is taken over l-hypo-elementary subgroups of H.*

A proof can be found e.g. in [6], (80.60). We will use this result to considerably strengthen Proposition 2.9:

Proposition 3.9. *Let G be a finite group, let N be a normal subgroup such that the quotient group $C = G/N$ is cyclic. Let l be a prime not dividing the order of N and let $\mathscr{R} = \mathbb{Z}$ or $\mathbb{Z}_l$. Then*

$$\operatorname{ord}_l\big(\mathscr{C}_\Theta(\Gamma)\big) = 0$$

for all $\mathscr{R}G$-lattices Γ and all G-relations Θ.

Proof. By Lemma 3.6, it suffices to show that every $\mathbb{Q}G$-relation is in fact a $\mathbb{Z}_{(l)}G$-relation. For that, it is enough to show that the rank of the sublattice of $\mathbb{Z}_{(l)}G$-relations is equal to the rank of the lattice of $\mathbb{Q}G$-relations, since the former is saturated in the latter.[1] By Theorem 3.8, the rank of the lattice of $\mathbb{Z}_{(l)}G$-relations is at least equal to the number of conjugacy classes of non-*l*-hypo-elementary subgroups. Explicitly, for each subgroup H of G which is not *l*-hypo-elementary, we get a $\mathbb{Z}_{(l)}G$-relation $\alpha_H H - \sum_{H'} \alpha_{H'} H'$, the sum taken over *l*-hypo-elementary subgroups of H. All relations obtained in this way are clearly linearly independent, since each one contains a unique 'maximal' subgroup that has the property that all other subgroups featuring in the relation are contained in this one. Since the rank of the lattice of $\mathbb{Q}G$-relations is equal to the number of conjugacy classes of non-cyclic subgroups of G, it is enough to show that any *l*-hypo-elementary subgroup of G must by cyclic.

So take $H = P \rtimes Z \leqq G$, where P is an *l*-group and Z is cyclic of order coprime to *l*. Since l does not divide $|N|$, we have that

$$P \cong P/P \cap N \cong PN/N \leqq G/N$$

is cyclic. Further, since H/P is abelian, the commutator subgroup H' of H must lie in P, so it is an *l*-group. But also, $H' \leqq G' \leqq N$ since G/N is abelian. Therefore $H' = \{1\}$ since l does not divide $|N|$. It follows that H is abelian, $H = P \times C$ and so cyclic. $\square$

---

[1] The lattice of $\mathbb{Z}_{(l)}G$-relations is the kernel of the natural map from the Burnside ring to the representation ring over $\mathbb{Z}_{(l)}$. This map is clearly linear and kernels of linear maps from abelian groups are always saturated.

4. Dokchitser constants in dihedral groups

We will now compute the Dokchitser constants of all $\mathbb{Z}G$-lattices when $G = D_{2p}$ is the dihedral group with $2p$ elements for p an odd prime and Θ is the relation from Example 2.4. By Proposition 2.7, we only need to compute them for indecomposable representations. Nonetheless, the fact that this can be done at all is a piece of good fortune. We will begin by recalling the classification of indecomposable integral representations of D_{2p} from [12].

Fix a primitive p-th root of unity ζ_p in a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Let $\mathbb{Q}(\zeta_p)^+$ be the maximal real subfield of the p-th cyclotomic field and let $\mathcal{O}^+$ be its ring of integers. Let $\{U_i\}$ be a full set of representatives of the ideal class group of $\mathbb{Q}(\zeta_p)^+$ and take $U_1 = U = \mathcal{O}^+$ to represent the principal ideals. Write $G = \langle \sigma, \varpi : \sigma^2 = \varpi^p = (\sigma\varpi)^2 = 1 \rangle$. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\zeta_p)$. Write A_i for the $\mathbb{Z}G$-module $U_i\mathcal{O}$ on which σ acts as complex conjugation and ϖ as multiplication by ζ_p. Let A_i' be the module $(\bar{\zeta}_p - \zeta_p)U_i\mathcal{O}$ with the same G-action. Set $A = A_1$, $A' = A_1'$.

Finally write $\mathbf{1}$ for the one-dimensional trivial $\mathbb{Z}G$-module, ϵ for the one-dimensional module sending σ to -1 and ϖ to 1 and Δ for the two-dimensional module $\mathbb{Z}[G/C_p]$ which is an extension of $\mathbf{1}$ by ϵ. The following is a complete list of non-isomorphic indecomposable $\mathbb{Z}G$-lattices (see [12]):

- $\mathbf{1}$;

- ϵ;

- Δ;

- for each i, A_i;

- for each i, A_i';

- for each i, a non-trivial extension of $\mathbf{1}$ by A_i', denoted by $(A_i', \mathbf{1})$;

- for each i, a non-trivial extension of ϵ by A_i, denoted by (A_i, ϵ);

- for each i, a non-trivial extension of Δ by A_i, denoted by (A_i, Δ);

- for each i, a non-trivial extension of Δ by A_i', denoted by (A_i', Δ);

- for each i, a non-trivial extension of Δ by $A_i \oplus A_i'$, denoted by $(A_i \oplus A_i', \Delta)$.

It is a trivial check that $\mathscr{C}_\Theta(\mathbf{1}) = 1/p$, $\mathscr{C}_\Theta(\epsilon) = p$, $\mathscr{C}_\Theta(\Delta) = 1$.

Lemma 4.1. *The Dokchitser constants of A and of A' are p and $1/p$, respectively.*

Proof. The matrices of σ, ϖ acting on A' on the left with respect to the basis

$$\bar{\zeta}_p - \zeta_p, (\bar{\zeta}_p - \zeta_p)\zeta_p, (\bar{\zeta}_p - \zeta_p)\zeta_p^2, \ldots, (\bar{\zeta}_p - \zeta_p)\zeta_p^{p-2}$$

are

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \reflectbox{$\ddots$} & \reflectbox{$\ddots$} & \reflectbox{$\ddots$} & \vdots \\ 0 & 1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & -1 \\ 0 & \cdots & 0 & 0 & 1 & -1 \end{bmatrix},$$

respectively. It is immediately seen that the same matrices represent the G-action by multiplication on the submodule

$$\langle \varpi^{\frac{p-1}{2}} - \varpi^{\frac{p+1}{2}}, \varpi^{\frac{p+1}{2}} - \varpi^{\frac{p+3}{2}}, \dots, \varpi^{p-1} - 1, 1 - \varpi, \dots, \varpi^{\frac{p-5}{2}} - \varpi^{\frac{p-3}{2}} \rangle_{\mathbb{Z}}$$

of $\mathbb{Z}[G/C_2]$ with respect to the indicated basis. But this is just the submodule

$$\langle 1 - \varpi^i : i \in \{1, \dots, p-1\} \rangle_{\mathbb{Z}}$$

of the permutation lattice $\mathbb{Z}[G/C_2]$. We can choose the standard pairing on the latter which makes the different coset representatives an orthonormal $\mathbb{Z}$-basis. It is easy to see that the fixed sublattices under 1 and under $\langle \sigma \rangle = C_2$ are

$$\langle 1 - \varpi^i : i = 1, \dots, p-1 \rangle_{\mathbb{Z}} \quad \text{and} \quad \left\langle 2 - \varpi^i - \varpi^{p-i} : i = 1, \dots, \frac{p-1}{2} \right\rangle_{\mathbb{Z}},$$

respectively. The subgroup C_p only fixes the trivial lattice. The matrices of the pairing on these modules with respect to the bases indicated are then

$$\begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 2 & 1 \\ 1 & 1 & \cdots & 1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 6 & 4 & 4 & \cdots & 4 \\ 4 & 6 & 4 & \cdots & 4 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 4 & \cdots & 4 & 6 & 4 \\ 4 & 4 & \cdots & 4 & 6 \end{bmatrix}$$

of sizes $p-1$ and $\frac{p-1}{2}$ with determinants p and $2^{\frac{p-1}{2}}p$, respectively, as can be checked by elementary row operations. So, taking into account the normalisation by the sizes of the subgroups, we get

$$\frac{\det\left(\frac{1}{|1|}\langle \cdot, \cdot \rangle \mid A'^{1}\right) \det\left(\frac{1}{|G|}\langle , \rangle \mid A'^{G}\right)^2}{\det\left(\frac{1}{|C_2|}\langle \cdot, \cdot \rangle \mid A'^{C_2}\right)^2 \det\left(\frac{1}{|C_p|}\langle , \rangle \mid A'^{C_p}\right)} = \frac{p}{p^2} = \frac{1}{p}$$

as claimed.

Now consider the $\mathbb{Z}G$-module $\mathbb{Z}[G/C_2] \otimes_\mathbb{Z} \epsilon$ with diagonal G-action. It is now clear from above that A is isomorphic to the submodule of $\mathbb{Z}[G/C_2] \otimes_\mathbb{Z} \epsilon$ given by $\langle 1 - \varpi^i : i = 1, \ldots, p-1 \rangle$. The fixed submodules under 1 and under C_2 are

$$\langle 1 - \varpi^i : i = 1, \ldots, p-1 \rangle \quad \text{and} \quad \left\langle \varpi^i - \varpi^{p-i} : i = 1, \ldots, \frac{p-1}{2} \right\rangle,$$

respectively, and an entirely similar calculation using the same natural pairing as above shows that $\mathscr{C}_\Theta(A) = p$. $\quad \square$

Lemma 4.2. *We have* $(A', \mathbf{1}) \cong \mathbb{Z}[G/C_2]$ *and* $\mathscr{C}_\Theta((A', \mathbf{1})) = 1$.

Proof. Take the $\mathbb{Z}$-basis $1, \varpi, \ldots, \varpi^{p-1}$ for $\mathbb{Z}[G/C_2]$. Then there is the submodule $\left\langle \sum_{i=0}^{p-1} \varpi^i \right\rangle$ isomorphic to $\mathbf{1}$ and the submodule $\langle 1 - \varpi^i : i \in \{1, \ldots, p-1\} \rangle$ isomorphic to A', but their sum is the submodule $\left\{ \sum_i \alpha_i \varpi^i : \sum \alpha_i \equiv 0 \pmod{p} \right\}$, which is an index p sublattice. In fact, $\Gamma = \mathbb{Z}[G/C_2]$ is indecomposable, for if it were decomposable, it would have to decompose as $\widehat{\Gamma_1} \cap \Gamma \oplus \widehat{\Gamma_2} \cap \Gamma$, where $\Gamma \otimes \mathbb{Q} = \widehat{\Gamma_1} \oplus \widehat{\Gamma_2}$ is the decomposition into irreducible rational representations. But these intersections are easily seen to be the sublattices just exhibited. Thus $\mathbb{Z}[G/C_2]$ must be a non-trivial extension of $\mathbf{1}$ by A' and the first claim follows from the classification of integral representations. The Dokchitser constant of $(A', \mathbf{1})$ is therefore trivial by [8], Lemma 2.46. $\quad \square$

Lemma 4.3. *The Dokchitser constants of the remaining lattices in the above list for* $i = 1$ *are as follows*:

- $\mathscr{C}_\Theta((A, \epsilon)) = 1$;

- $\mathscr{C}_\Theta((A, \Delta)) = 1/p$;

- $\mathscr{C}_\Theta((A', \Delta)) = p$;

- $\mathscr{C}_\Theta((A \oplus A', \Delta)) = 1$.

Proof. It is noted in [12], Section 4, that $(A \oplus A', \Delta) \cong \mathbb{Z}[G/1]$ and so $\mathscr{C}_\Theta((A \oplus A', \Delta)) = 1$ by [8], Lemma 2.46.

For the other three lattices, since we only need to determine the p-parts, it suffices to work up to squares of elements with trivial p-valuation so we will work over $\mathbb{Z}_p$ rather than over $\mathbb{Z}$. Write $\widetilde{(A, \epsilon)} = (A, \epsilon) \otimes_\mathbb{Z} \mathbb{Z}_p$ and similarly for the other lattices. Since $\mathbf{1} \oplus \epsilon$ is an index 2 sublattice of Δ, over $\mathbb{Z}_p$ we have $\tilde{\mathbf{1}} \oplus \tilde{\epsilon} \cong \tilde{\Delta}$. Now, $(A, \epsilon) \otimes \epsilon \cong (A', \mathbf{1})$ and so

$$\widetilde{(A, \epsilon)} \oplus \widetilde{(A', \mathbf{1})} \overset{\text{Lemma 4.2}}{=} \mathbb{Z}_p[G/C_2] \otimes (\tilde{\mathbf{1}} \oplus \tilde{\epsilon})$$

$$\cong \mathbb{Z}_p[G/C_2] \otimes \tilde{\Delta}$$

$$\cong \mathbb{Z}_p[G/C_2] \otimes \mathbb{Z}_p[G/C_p]$$

$$\cong \mathbb{Z}_p[G/1],$$

which has trivial Dokchitser constant by [8], Lemma 2.46. By multiplicativity of Dokchitser constants and by Lemma 4.2, $\mathscr{C}_\Theta((\widetilde{A, \epsilon})) = 1$. Similarly, $(\tilde{A}, \Delta) \cong (\tilde{A}, \tilde{\mathbf{1}} \oplus \tilde{\epsilon})$, and since $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbf{1}, A) = 0$ (cf. [12], Lemma 2.1), it is easy to see that

$$(\tilde{A}, \tilde{\mathbf{1}} \oplus \tilde{\epsilon}) \cong \tilde{\mathbf{1}} \oplus (\widetilde{A, \epsilon}).$$

By multiplicativity of Dokchitser constants, we deduce that

$$\mathscr{C}_\Theta((\widetilde{A, \Delta})) = 1/p \in \mathbb{Q}_p^\times / (\mathbb{Z}_p^\times)^2.$$

Also $\mathrm{Ext}^1_{\mathbb{Z}G}(\epsilon, A') = 0$ and

$$(\tilde{A}', \tilde{\mathbf{1}} \oplus \tilde{\epsilon}) \cong \tilde{\epsilon} \oplus (\widetilde{A', \mathbf{1}}),$$

whence

$$\mathscr{C}_\Theta((\widetilde{A, \Delta})) = p \in \mathbb{Q}_p^\times / (\mathbb{Z}_p^\times)^2. \qquad \square$$

Theorem 4.4. *The Dokchitser constants of all the indecomposable integral representations of D_{2p} for p an odd prime are as follows*:

Γ	$\mathbf{1}$	ϵ	Δ	A_i	A'_i	$(A'_i, \mathbf{1})$	(A_i, ϵ)	(A_i, Δ)	(A'_i, Δ)	$(A_i \oplus A'_i, \Delta)$
$\mathscr{C}_\Theta(\Gamma)$	$1/p$	p	1	$p \ \forall i$	$1/p \ \forall i$	$1 \ \forall i$	$1 \ \forall i$	$1/p \ \forall i$	$p \ \forall i$	$1 \ \forall i$

Proof. For $i = 1$ this is Lemma 4.1, Lemma 4.2 and Lemma 4.3. We will show that A_i is isomorphic to A over $\mathbb{Z}_{(2)}$ and over $\mathbb{Z}_{(p)}$ for all i and A'_i is isomorphic to A' over $\mathbb{Z}_{(2)}$ and over $\mathbb{Z}_{(p)}$ for all i (strictly speaking, the isomorphism over $\mathbb{Z}_{(p)}$ would be enough for this theorem by Proposition 3.9). Recall that A_i, A'_i are given by $(\bar{\zeta}_p - \zeta_p)^j U_i \mathcal{O}$ for $j = 0, 1$, respectively, where U_i runs through representatives of the ideal class group of $\mathbb{Q}(\zeta_p)^+$. Take each U_i to be of norm coprime to $2p$. Then A_i is a sublattice of $A = A_1$ of index coprime to $2p$ and the two are therefore isomorphic over $\mathbb{Z}_2$ and over $\mathbb{Z}_p$. Thus they have the same Dokchitser constants. Similarly, A'_i all have the same Dokchitser constants as $A' = A'_1$. $\qquad \square$

The proof of Theorem 4.4 exhibits an important feature of Dokchitser constants, which we will now summarise.

Definition 4.5. Given a finite group G and a principal ideal domain $\mathscr{R}$, two $\mathscr{R}G$-lattices M and N are said to lie in the same *genus* if $M \otimes \mathscr{R}_{\mathfrak{p}} \cong N \otimes \mathscr{R}_{\mathfrak{p}}$ as $\mathscr{R}_{\mathfrak{p}}G$-modules for all completions $\mathscr{R}_{\mathfrak{p}}$ at prime ideals $\mathfrak{p}$ of $\mathscr{R}$. This is clearly an equivalence relation.

The two main conceptual steps in the proof of Theorem 4.4 can be summarised as:

Theorem 4.6. *The Dokchitser constants of an $\mathscr{R}G$-lattice only depend on its genus.*

Proposition 4.7. *There exist at most ten genera of indecomposable $\mathbb{Z}D_{2p}$-lattices. Each genus has a representative of the kind considered in Lemma* 4.1, *Lemma* 4.2 *and Lemma* 4.3.

Our goal is to translate the Dokchitser constants into a certain index. To that end, we now turn to the calculation of the index in Γ of the submodule generated by the various fixed submodules. The calculation is fairly similar to those of the Dokchitser constants but exhibits some new features. We will not give it in full detail but will give enough examples to show the main techniques. The result is summarised in the following table, where C_2 and C_2' are two distinct subgroups of D_{2p} isomorphic to the cyclic group of order 2:

Γ	$\mathbf{1}$	ϵ	Δ	A_i	A_i'	$(A_i', \mathbf{1})$	(A_i, ϵ)	(A_i, Δ)	(A_i', Δ)	$(A_i \oplus A_i', \Delta)$
$[\Gamma : \Gamma^{C_2} + \Gamma^{C_2'} + \Gamma^{C_p}]$	1	1	1	$1\ \forall i$	$p\ \forall i$	$1\ \forall i$	$p\ \forall i$	$p\ \forall i$	$1\ \forall i$	$p\ \forall i$

The assertions are clear for the first three modules in the list. For the others, we begin by noting that the index is an invariant of the genus. More precisely, the l-part of the index for Γ is equal to the l-part of the index for $\Gamma \otimes \mathbb{Z}_{(l)}$ for any prime l.

Lemma 4.8. *Let* $\Gamma = A$. *Then* $\Gamma = \Gamma^{C_2} + \Gamma^{C_2'} + \Gamma^{C_p}$.

Proof. We have already noted in the proof of Lemma 4.1 that A is isomorphic to the submodule of $\mathbb{Z}[G/C_2] \otimes_{\mathbb{Z}} \epsilon$ given by $\langle 1 - \varpi^i : i = 1, \ldots, p - 1 \rangle$ and that the submodule fixed by $\langle \sigma \rangle$ is

$$\left\langle \varpi^i - \varpi^{p-i} : i = 1, \ldots, \frac{p-1}{2} \right\rangle.$$

It follows that the submodule fixed by, say, $\langle \varpi^{-1} \sigma \varpi \rangle$ is given by

$$\left\langle \varpi^{-1}(\varpi^i - \varpi^{p-i}) : i = 1, \ldots, \frac{p-1}{2} \right\rangle = \left\langle \varpi^i - \varpi^{p-(i+2)} : i = 0, \ldots, \frac{p-3}{2} \right\rangle.$$

These two are easily seen to generate A. For example, by alternatingly summing elements from them, one can obtain

$$1 - \varpi = (1 - \varpi^{-2}) + (\varpi^{-2} - \varpi^2) + (\varpi^2 - \varpi^{-4}) + \cdots + (\varpi^{p-3} - \varpi)$$

and similarly for all the other generators of A. $\square$

The proof for $\Gamma = (A', \mathbf{1}) = \mathbb{Z}[G/C_2]$ is very similar in spirit and we will omit it.

Lemma 4.9. *Let* $\Gamma = (A \otimes A', \Delta) \cong \mathbb{Z}[G/1]$. *Then* $[\Gamma : \Gamma^{C_2} + \Gamma^{C_2'} + \Gamma^{C_p}] = p$.

Proof. The fixed submodules of $\mathbb{Z}[G/1]$ under $\langle \sigma \rangle$, under $\langle \varpi \sigma \varpi^{-1} \rangle$ and under $\langle \varpi \rangle$, respectively, are immediately seen to be

$$\langle \varpi^i + \sigma \varpi^i : i = 0, \ldots, p - 1 \rangle, \quad \langle \varpi^i + \sigma \varpi^{i+2} : i = 0, \ldots, p - 1 \rangle \quad \text{and} \quad \left\langle \sum_{i=0}^{p-1} \varpi^i, \sum_{i=0}^{p-1} \sigma \varpi^i \right\rangle$$

and it is easy to check that together these submodules generate the kernel of the surjective map

$$\mathbb{Z}[G/1] \to \mathbb{Z}/p\mathbb{Z}, \quad \left(\sum_i \alpha_i \sigma \varpi^i + \sum_j \beta_j \varpi^j\right) \mapsto \sum_i \alpha_i - \sum_j \beta_j \mod p.$$

This kernel is of index p in $\mathbb{Z}[G/1]$ and the claim is established. $\square$

A similar proof, which we omit, shows the same for $\Gamma = A'$.

There are now several ways to finish the calculation. For example, one can note that to compute the p-part of the indices, we can localise everything at p and use multiplicativity of indices in direct sums. The p-parts of all the remaining indices then follow from the direct sum decompositions of the proof of Lemma 4.3.

We note that, by inspection, the quantity

$$\mathscr{I}(\Gamma) = \mathscr{C}_\Theta(\Gamma) \cdot [\Gamma : \Gamma^{C_2} + \Gamma^{C_2'} + \Gamma^{C_p}]^2$$

only depends on the rational representation $\Gamma \otimes \mathbb{Q}$ and not on the lattice itself. We deduce

Lemma 4.10. *Write $\Gamma \otimes \mathbb{Q} = \Lambda$. Let $\mathbf{1}$, ϵ and ρ denote the irreducible rational representations of D_{2p}, where ρ is $(p-1)$-dimensional.[2] Denote by $\langle\Lambda, \cdot\rangle$ the number of copies of a given irreducible rational representation in Λ, analogous to the inner product of complex characters. Then we have $\mathscr{I}(\mathbf{1}) = 1/p$, $\mathscr{I}(\epsilon) = p$, $\mathscr{I}(A_i) = \mathscr{I}(A_i') = p$ for all i, and for any Γ, we have $\mathscr{I}(\Gamma) = p^{\langle\Lambda, \epsilon\rangle + \langle\Lambda, \rho\rangle - \langle\Lambda, \mathbf{1}\rangle}$.*

This identity will be crucial in proving Theorem 1.1.

5. Class number relations—main results

In this section we will collect the results obtained so far to prove the main theorems.

5.1. Possible values of regulator quotients. We will begin by establishing Theorem 1.2.

Theorem 5.1. *Let G be a finite group, let N be a normal subgroup such that G/N is cyclic, let l be a prime number not dividing the order of N. Let F/K be a Galois extension of number fields with Galois group G and $\Theta = \sum_i H_i - \sum_j H_j'$ be a G-relation. Let S be a finite G-stable set of places of F including all the Archimedean ones. Recall the notation*

$$\lambda(H) = \#\ker\left(\mathrm{H}^1\left(H, \mu(F)\right) \to \mathrm{H}^1(H, \mathcal{O}_{S,F}^\times)\right)$$

---

[2] The abuse of notation in using the same letters for the one-dimensional rational representations and integral lattices in them is very mild, since there is a unique integral lattice up to isomorphism in each of the two representations.

for $H \leqq G$. Then

$$\mathrm{ord}_l\left(\prod_i \lambda(H_i)/\prod_j \lambda(H_j')\right) = 0.$$

Proof. For a subgroup H of G, define $\tau_F(H)$ as

$$\tau_F(H) = \ker\left(\mathrm{H}^1(H, \mu(F)) \rightarrow \mathrm{H}^1(H, \mathcal{O}_{S,F}^\times)\right),$$

so that $\lambda(H) = \#\tau_F(H)$. The inflation-restriction exact sequence gives us the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^1\big(H/(H\cap N), \mu(F^{H\cap N})\big) & \longrightarrow & \mathrm{H}^1(H, \mu(F)) & \longrightarrow & \mathrm{H}^1(H\cap N, \mu(F)) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{H}^1\big(H/(H\cap N), \mathcal{O}_{S,F^{H\cap N}}^\times\big) & \longrightarrow & \mathrm{H}^1(H, \mathcal{O}_{S,F}^\times) & \longrightarrow & \mathrm{H}^1(H\cap N, \mathcal{O}_{S,F}^\times),
\end{array}
$$

where the commutativity is obvious on the level of co-cycles. Hence we get the exact sequence

$$0 \rightarrow \tau_{F^N}(HN/N) \rightarrow \tau_F(H) \rightarrow \tau_F(H\cap N).$$

But the l-part of $\lambda(H\cap N)$ is trivial since l does not divide $|H\cap N|$, and so we see that

$$\lambda(H)[l^\infty] = \lambda(HN/N)[l^\infty] = \#\tau_{F^N}(HN/N)[l^\infty].$$

Since G/N is cyclic and therefore has no non-trivial relations, and by applying [8], Theorem 2.36 (q) with $\phi_{G/N}(HN/N) = \lambda(HN/N)[l^\infty]$, $\lambda(H)[l^\infty]$ vanishes in relations and we are done. $\square$

Corollary 5.2. *Under the hypotheses of Theorem 5.1, we have*

$$\mathrm{ord}_l\left(\prod_i R_S(F^{H_i})/\prod_j R_S(F^{H_j'})\right) = 0.$$

Proof. This is an immediate consequence of Proposition 2.15, Proposition 3.9 and the above theorem. $\square$

Corollary 5.3. *Under the hypotheses of Theorem 5.1, we have an equality of the l-parts of class numbers*:

$$\prod_i h_S(F^{H_i})_l = \prod_j h_S(F^{H_j'})_l.$$

Proof. We only need to establish that the l-part of the quotient $\prod_i w(F^{H_i})/\prod_j w(F^{H_j'})$ is trivial. If $l \neq 2$, then this is true in general as observed by Brauer [4], Section 2. If $l = 2$, then $w(F^H) = w(F^{HN})$ and the latter vanishes in relations by exactly the same argument as in the proof of Theorem 5.1. $\square$

Remark 5.4. One could also deduce both corollaries directly, without using Theorem 5.1, from the work of Boltje and of Bley and Boltje on Mackey functors (see e.g. [3], Corollary 2.4, or [2]) combined with the proof of Proposition 3.9.

5.2. Unit index formula for D_{2p}-extensions. We will first prove Theorem 1.1 and then explain how to deduce a formula for D_{2q}, where q is any odd integer. We will not actually write down the formula for D_{2q} because it is less enlightening when it is written out than its conceptual idea. The interested reader should have no difficulties in writing it down for any specific case. Let F/k be a Galois extension of number fields with Galois group $G = D_{2p}$ for p an odd prime. Let K be the intermediate quadratic extension and $L \neq L'$ be two intermediate extensions of degree p over k. As in the previous section, denote by ϖ an element of order p in G and let σ be the involution that fixes L. Let Θ be the relation from Example 2.4 and let S be a G-stable set of primes of F. Our main tool is the compatibility statement between Artin formalism and the analytic class number formula given by equation (1). We will first show that in our case, $\dfrac{w(F)w(k)^2}{w(K)w(L)^2} = 1$. Indeed, since the extension L/k is not Galois, it cannot be obtained by adjoining roots of unity. Since it has no intermediate extensions, we see that $w(L) = w(k)$. Similarly, if F was obtained from K by adjoining roots of unity, then adjoining these same roots to k would give an extension of degree p or $2p$. But the former is not possible by what we have just said and the latter would imply that F/k is abelian. So $w(F) = w(K)$ and our claim follows.

Set Γ to be the Galois module $U_S(F)$ given by the S-units of F modulo torsion and write $\Lambda = \Gamma \otimes \mathbb{Q}$. We will now invoke Proposition 2.15. Note that in our situation, the only subgroup of G for which $\mathscr{C}_\Theta(\mathbb{Z}[G/H]) \neq 1$ is G itself and that

$$\mathscr{C}_\Theta(\mathbb{Z}[G/G]) = \mathscr{C}_\Theta(\mathbf{1}) = 1/p.$$

So

$$\prod_{\mathfrak{p} \in S|_k} \mathscr{C}_\Theta(\mathbb{Z}[G/D_\mathfrak{p}]) = p^{-\#\{\mathfrak{p} \in S|_k : D_\mathfrak{p} = G\}}.$$

Set $a(F/k, S) = \#\{\mathfrak{p} \in S|_k : D_\mathfrak{p} = G\}$. Then we have, using the notation from Proposition 2.15 and from Lemma 4.10,

$$(6) \quad \frac{h_S(F)h_S(k)^2}{h_S(K)h_S(L)^2} \overset{\text{equation (1)}}{=} \frac{R_S(K)R_S(L)^2}{R_S(F)R_S(k)^2}$$

$$\overset{\text{Proposition 2.15}}{=} \left(\mathscr{C}_\Theta(\mathbf{1}) \Big/ \left(\mathscr{C}_\Theta(\Gamma) \prod_{\mathfrak{p} \in S|_k} \mathscr{C}_\Theta(\mathbb{Z}[G/D_\mathfrak{p}]) \right) \right)^{1/2} \cdot \frac{\lambda(C_p)\lambda(C_2)^2}{\lambda(\mathbf{1})\lambda(G)^2}$$

$$= \left(p^{a(F/k, S)-1} / \mathscr{C}_\Theta(\Gamma) \right)^{1/2} \cdot \frac{\lambda(C_p)\lambda(C_2)^2}{\lambda(\mathbf{1})\lambda(G)^2}$$

$$\overset{\text{Lemma 4.10}}{=} \big(p^{a(F/k, S)-1}$$

$$\cdot [\Gamma : \Gamma^{C_2} + \Gamma^{C_2'} + \Gamma^{C_p}]^2 / p^{\langle \Lambda, \epsilon \rangle + \langle \Lambda, \rho \rangle - \langle \Lambda, \mathbf{1} \rangle} \big)^{1/2}$$

$$\times \frac{\lambda(C_p)\lambda(C_2)^2}{\lambda(\mathbf{1})\lambda(G)^2}$$

$$= \left(p^{r_S(k)-(r_S(K)-r_S(k))-(r_S(F)-r_S(K))/(p-1)+a(F/k,S)-1}\right)^{1/2}$$

$$\times \left[\Gamma : \Gamma^{C_2} + \Gamma^{C_2'} + \Gamma^{C_p}\right] \cdot \frac{\lambda(C_p)\lambda(C_2)^2}{\lambda(1)\lambda(G)^2}.$$

Recall that $\lambda(H) = [U_S(F)^H : U_S(F^H)] = \#\ker\big(\mathrm{H}^1(H,\mu(F)) \to \mathrm{H}^1(H,\mathscr{O}_{S,F}^\times)\big)$. As we have discussed before Lemma 2.14, this is trivial for all $H \leqq G$ if neither F/L nor F/K is obtained by adjoining a root of a non-torsion S-unit.

It remains to compute $\lambda(H)$ for all $H \leq G$ and to compare the Γ-index in equation (6) with the actual unit index appearing in Theorem 1.1. Since the roots of unity $\mu(F)$ are contained in $\mathscr{O}_{S,K}^\times$ as remarked above, we have

$$(7) \quad [\mathscr{O}_{S,F}^\times : \mathscr{O}_{S,L}^\times \mathscr{O}_{S,L'}^\times \mathscr{O}_{S,K}^\times] = [\mathscr{O}_{S,F}^\times/\mu(F) : \mathscr{O}_{S,L}^\times \mathscr{O}_{S,L'}^\times \mathscr{O}_{S,K}^\times/\mu(F)]$$

$$= \Big[\mathscr{O}_{S,F}^\times/\mu(F) : \big(\mu(F)\mathscr{O}_{S,L}^\times/\mu(F)\big)$$

$$\times \big(\mu(F)\mathscr{O}_{S,L'}^\times/\mu(F)\big)\big(\mu(F)\mathscr{O}_{S,K}^\times/\mu(F)\big)\Big]$$

$$= \Big[\mathscr{O}_{S,F}^\times/\mu(F) : \big(\mathscr{O}_{S,L}^\times/\mu(F) \cap \mathscr{O}_{S,L}^\times\big)$$

$$\times \big(\mathscr{O}_{S,L'}^\times/\mu(F) \cap \mathscr{O}_{S,L'}^\times\big)\big(\mathscr{O}_{S,K}^\times/\mu(F) \cap \mathscr{O}_{S,K}^\times\big)\Big]$$

$$= \Big[\mathscr{O}_{S,F}^\times/\mu(F) : \big(\mathscr{O}_{S,L}^\times/\mu(L)\big)\big(\mathscr{O}_{S,L'}^\times/\mu(L')\big)\big(\mathscr{O}_{S,K}^\times/\mu(K)\big)\Big]$$

$$= [U_S(F) : U_S(F^{C_2})U_S(F^{C_2'})U_S(F^{C_p})],$$

where for $H \leqq G$, $\mathscr{O}_{S,F^H}^\times/\mu(F^H)$ is identified with its image in $\mathscr{O}_{S,F}^\times/\mu(F)$ under the obvious inclusion map. Thus, to compare the Γ-index with the unit index, we need to compute

$$i(F/k,S) = [U_S(F)^{C_2}U_S(F)^{C_2'}U_S(F)^{C_p} : U_S(F^{C_2})U_S(F^{C_2'})U_S(F^{C_p})].$$

We will consider various different cases. The remaining computations are summarised in the following two lemmata:

Lemma 5.5. *We have $\lambda(1) = 1$; $\lambda(C_2) \in \{1,2\}$; $\lambda(C_p) \in \{1,p\}$ with $\lambda(C_p) = p$ if and only if $F = K(\sqrt[p]{u})$ for a non-torsion S-unit $u \in \mathscr{O}_{S,K}^\times$; $\lambda(G) \in \{1,2,p,2p\}$ with $\lambda(G)$ divisible by 2 if and only if $\lambda(C_2) = 2$ and divisible by p if and only if $L = k(\sqrt[p]{u'})$ for a non-torsion S-unit $u' \in \mathscr{O}_{S,k}^\times$.*

Proof. It is clear that $\lambda(1) = 1$. Since, for any subgroup $H \leq G$, $\lambda(H)$ is the order of a subgroup of $\mathrm{H}^1(H,\mu(F))$ and since this cohomology group is cyclic and annihilated by $|H|$, we deduce that $\lambda(H)$ divides $|H|$ for $H \in \{C_2, C_p, G\}$. The 2-part of the λ-quotient vanishes by Theorem 5.1.

By definition, an element of $U_S(F)^{C_p}/U_S(F^{C_p})$ of order p is represented by a non-torsion S-unit $v \in \mathscr{O}_{S,F}^\times\backslash\mu(F)\mathscr{O}_{S,K}^\times$ such that $\varpi(v) = \zeta v$ and $v^p = \bar{\zeta}x$ for $\zeta,\bar{\zeta} \in \mu(F)$ and $x \in K$. But $\mu(F) \subset \mathscr{O}_{S,K}^\times$, so these conditions are equivalent to $v \in \mathscr{O}_{S,F}^\times\backslash\mathscr{O}_{S,K}^\times$, $v^p \in K$. Thus, $\lambda(C_p) = p$ if and only if $F = K(\sqrt[p]{u})$, where u is a non-torsion S-unit in $\mathscr{O}_{S,K}^\times$.

Also, an element of $U_S(F)^G/U_S(F^G)$ of order p is represented by an S-unit $v' \in \mathscr{O}_{S,F}^\times \backslash \mu(F)\mathscr{O}_{S,k}^\times$ such that $\varpi(v') = \zeta_1 v'$, $\sigma(v') = \zeta_2 v'$ and $v'^p = \bar{\zeta}x$ for $\zeta_1, \zeta_2, \bar{\zeta} \in \mu(F)$ and $x \in \mathscr{O}_{S,k}^\times$. If $L = k(\sqrt[p]{u'})$ for a non-torsion S-unit $u' \in \mathscr{O}_{S,k}^\times$, then the conditions are satisfied for $v' = \sqrt[p]{u'}$, so in this case $\lambda(G)$ is divisible by p. Conversely, let $v' \in \mathscr{O}_{S,F}^\times \backslash \mu(F)\mathscr{O}_{S,k}^\times$ represent an element of order p and let ζ_1, ζ_2, $\bar{\zeta}$ and x be as above. We need to find $\tilde{v} \in \mathscr{O}_{S,F}^\times \backslash \mu(F)\mathscr{O}_{S,k}^\times$ satisfying the same conditions, but with $\tilde{v}^p \in \mathscr{O}_{S,k}^\times$ (and not merely in $\mu(F)\mathscr{O}_{S,k}^\times$). Consider $\tilde{v} = \mathrm{Norm}_{F/L}(v') = v'\sigma(v') = \zeta_2 v'^2$. Clearly, it is fixed by G up to roots of unity, since v' is, and also $v'^p \in \mu(F)\mathscr{O}_{S,k}^\times \subseteqq \mathscr{O}_{S,K}^\times$ implies that $\tilde{v}^p \in \mathscr{O}_{S,k}^\times$. So we only need to show that $\tilde{v} \notin \mu(F)\mathscr{O}_{S,k}^\times$. But if $\tilde{v} \in \mu(F)\mathscr{O}_{S,k}^\times \subseteqq \mathscr{O}_{S,K}^\times$, then in fact $\zeta_2 v'^2 = \tilde{v} \in K \cap L = k$, so $v'^2 \in \mu(F)\mathscr{O}_{S,k}^\times$, contradicting the assumption that v' represents an element of order p in $U_S(F)^G/U_S(F^G)$. $\quad\square$

Lemma 5.6. *The index*

$$i(F/k, S) = [U_S(F)^{C_2} U_S(F)^{C_2'} U_S(F)^{C_p} : U_S(F^{C_2}) U_S(F^{C_2'}) U_S(F^{C_p})]$$

is equal to p if $F = K(\sqrt[p]{u})$ for a non-torsion S-unit $u \in \mathscr{O}_{S,K}^\times$ and L is not obtained by adjoining a non-torsion S-unit to k, and is 1 otherwise.

Proof. The statement clearly holds if $\lambda(H) = 1$ for all $H \leqq G$. Next, by the previous lemma, any non-trivial element in $U_S(F)^{C_2}/U_S(F^{C_2})$ can be represented by an element of $U_S(F)^G$. In particular, this representative is fixed by C_p up to roots of unity, so gives an element of $U_S(F)^{C_p}/U_S(F^{C_p})$. Since this latter group has no 2-torsion, we deduce that any non-trivial class in $U_S(F)^{C_2}/U_S(F^{C_2})$ is represented by an element of $U_S(F^{C_p})$ and so the index $i(F/k, S)$ is never divisible by 2.

Similarly, by the same lemma, if $L = k(\sqrt[p]{u'})$ for a non-torsion S-unit $u' \in \mathscr{O}_{S,k}^\times$, then a generator of $U_S(F)^{C_p}/U_S(F^{C_p})$ can be represented by an element of $U_S(F)^G$, which then gives an element of $U_S(F)^{C_2}/U_S(F^{C_2})$. This group has no p-torsion, so the generator of $U_S(F)^{C_p}/U_S(F^{C_p})$ is represented by an element of $U_S(F^{C_2})$ and the index $i(F/k, S)$ is not divisible by p in this case, hence trivial.

Finally, suppose that $F = K(\sqrt[p]{u})$ for a non-torsion S-unit $u \in \mathscr{O}_{S,K}^\times$, but that L is not obtained from k in this way. We will show that then, $\sqrt[p]{u}$ represents a non-trivial coset of

$$U_S(F)^{C_2} U_S(F)^{C_2'} U_S(F)^{C_p} / U_S(F^{C_2}) U_S(F^{C_2'}) U_S(F^{C_p}),$$

necessarily of order p in this quotient. Assume for a contradiction that $\sqrt[p]{u} = u_L u_{L'} u_K$, where $u_M \in \mathscr{O}_{S,M}^\times$ for $M = L, L', K$. Recall, that ϖ denotes an element of G of order p. Let ζ_p be a primitive p-th root of unity in K, which must exist since otherwise F/K would not be Galois. Since

$$\zeta_p u_L u_{L'} u_K = \zeta_p \sqrt[p]{u} = \varpi(\sqrt[p]{u}) = u_K \varpi(u_L u_{L'}),$$

we may replace $\sqrt[p]{u}$ by $\sqrt[p]{u}/u_K$ and assume without loss of generality that $u_K = 1$. Consider the images w, w_L and $w_{L'}$ of $\sqrt[p]{u}$, u_L and $u_{L'}$, respectively, in $\mathscr{O}_{S,F}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathscr{O}_{S,F}^\times$ is regarded as a $\mathbb{Z}$-module by virtue of being a finitely generated abelian group. Recall the notation $\mathbf{1}$, ϵ and ρ of Lemma 4.10 for the irreducible rational representations of D_{2p}. Since w is fixed by C_p and since the C_p-invariant subspace of ρ is trivial (this is true for the complex irreducible two-dimensional representations, of which ρ is the sum), the projection of w

onto the ρ-isotypical component of $\mathcal{O}_{S,F}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$ is trivial. Since the C_2-invariant subspace and the C_2'-invariant subspace of ρ are linearly independent (as can again be seen on the level of the complex irreducible summands of ρ), the projections of w_L and of $w_{L'}$ onto the ρ-isotypical component must also be trivial. But also, the C_2-invariant subspace of the ϵ-isotypical component is trivial, so we deduce that w_L is in the **1**-component, in other words that G acts on u_L by multiplying by roots of unity. So either $u_L \in k$ and thus $u \in L'$, or L is obtained from k by obtaining the p-th root of a non-torsion S-unit, both possibilities contradicting the assumptions. $\square$

Combining the two lemmata with equations (6) and (7) completes the proof of Theorem 1.1.

5.3. A formula for D_{2q} for q any odd integer.

Throughout this subsection we fix the following notation:

Notation. In this subsection we will drop the subscript S from $\mathcal{O}_{S,M}^{\times}$ and write $\mathcal{O}_M^{\times}$ instead. The set-up is as follows:

q $\qquad \prod\limits_{i=1}^{n} p_i$ for p_i odd primes, not necessarily distinct;

G $\qquad$ dihedral group with $2q$ elements, $D_{2q} = \langle a, b \,|\, a^q = b^2 = (ab)^2 = 1 \rangle$;

F/k $\quad$ a Galois extension of number fields with Galois group G;

K $\qquad F^{\langle a \rangle}$;

L $\qquad F^{\langle b \rangle}$;

L' $\qquad F^{\langle ab \rangle}$;

S $\qquad$ Galois stable set of places of F including the Archimedean ones.

For each $j \in \{0, \ldots, n\}$ define

ε_j $\qquad \prod\limits_{i=1}^{j} p_i$ (1 if $j = 0$);

C_j $\qquad \langle a^{\varepsilon_j} \rangle$, the unique subgroup of $\langle a \rangle$ of index $\prod\limits_{i=1}^{j} p_i$ (1 if $j = 0$);

D_j, D_j' $\quad$ the dihedral groups generated by C_j and b and by C_j' and b, respectively.

With this notation, $F^{C_j}/F^{D_{j-1}}$ is an intermediate Galois extension with Galois group D_{2p_j} for $j \in \{1, \ldots, n\}$ and so Theorem 1.1 applies to this extension. By taking the product of the unit index formula over $j = 1, \ldots, n$, we obtain that

$$(8) \qquad \frac{h_S(F)h_S(k)^2}{h_S(K)h_S(L)^2} = \prod_{j=1}^{n} (p_j^{\alpha_j} \times [\mathcal{O}_{F^{C_j}}^{\times} : \mathcal{O}_{F^{C_{j-1}}}^{\times} \mathcal{O}_{F^{D_j}}^{\times} \mathcal{O}_{F^{D_j'}}^{\times}]),$$

where α_j are the corresponding exponents of p_j from Theorem 1.1.

Before investigating the unit index, we will give a more conceptual explanation of this formula. We have the G-relation

$$\Theta = 1 - 2C_2 - C_{p^n} + 2G.$$

As in the case of D_{2p}, the corresponding quotient of numbers of roots of unity $\dfrac{w(F)w(k)^2}{w(K)w(L)^2}$ is trivial, because if F contains a root of unity, then adjoining this root to k gives an abelian Galois extension of k which must therefore be contained in K. Thus $w(F) = w(K)$ and $w(L) = w(k)$. So, using equation (1), we see that

$$\frac{h_S(F)h_S(k)^2}{h_S(K)h_S(L)^2} = \frac{R_S(K)R_S(L)^2}{R_S(F)R_S(k)^2}$$

and Proposition 2.15 implies that

$$\frac{h_S(F)h_S(k)^2}{h_S(K)h_S(L)^2} = \left(\frac{\mathscr{C}_\Theta(\mathbf{1})}{\mathscr{C}_\Theta(\Gamma) \cdot \prod\limits_{\mathfrak{p} \in S|_k} \mathscr{C}_\Theta(\mathbb{Z}[G/D_\mathfrak{p}])} \right)^{1/2} \frac{\lambda(C_{p^n})\lambda(C_2)^2}{\lambda(\mathbf{1})\lambda(G)^2}.$$

This time, we do not have a classification of all indecomposable integral representations of G at our disposal (in fact the number of their isomorphism classes is infinite when q is not cube-free). Instead, to replace the Dokchitser constant by a unit index, we break up the Dokchitser constant into Dokchitser constants of D_{2p_i}-representations and then use Lemma 4.10. We begin by an obvious lemma:

Lemma 5.7. *Let G be any finite group, $\Theta = \sum\limits_{i \in I} n_i H_i$ be any G-relation with n_i non-zero integers and Γ be any $\mathscr{R}$-free $\mathscr{R}G$-module. Set $H = \bigcap\limits_{i \in I} H_i$. Then*

$$\mathscr{C}_\Theta(\Gamma) = \mathscr{C}_\Theta(\Gamma^H).$$

Proof. This is clear from the definition of Dokchitser constants, since elements of Γ that are not fixed by any of the subgroups occurring in the relation do not contribute to the Dokchitser constant. $\square$

For each integer $j \in \{1, \dots, n\}$ we have the G-relation $\Theta_j = C_j - 2D_j - C_{j-1} + 2D_{j-1}$. We see immediately that $\Theta = \sum\limits_{j=1}^n \Theta_j$ and so by Proposition 2.7 we have

$$\mathscr{C}_\Theta(\Gamma) = \prod_{j=1}^n \mathscr{C}_{\Theta_j}(\Gamma).$$

For each j, Θ_j is induced from the corresponding relation in D_{j-1} and so by Proposition 2.10 we have $\mathscr{C}_{\Theta_j}(\Gamma) = \mathscr{C}_{\Theta_j}(\Gamma\!\downarrow_{D_{j-1}})$, where on the right-hand side Θ_j is viewed as a D_{j-1}-relation. Moreover, by Lemma 5.7 we have $\mathscr{C}_{\Theta_j}(\Gamma) = \mathscr{C}_{\Theta_j}\big((\Gamma\!\downarrow_{D_{j-1}})^{C_j}\big)$. Now, $(\Gamma\!\downarrow_{D_{j-1}})^{C_j}$ can be considered as a $(D_{j-1}/C_j \cong D_{2p_j})$-module $\bar{\Gamma}_j$, and since Θ_j is in fact lifted from the D_{2p_j}-quotient of D_{j-1}, we have from Proposition 2.10

$$\mathscr{C}_\Theta(\Gamma) = \prod_{j=1}^n \mathscr{C}_{\Theta_j}(\bar{\Gamma}_j),$$

where each factor is now a Dokchitser constant in D_{2p_j}. Applying Lemma 4.10 and the discussion of the D_{2p} case recovers equation (8).

Ideally, we would like to replace the product of the unit indices by the index

$$[\mathcal{O}_F^\times : \mathcal{O}_K^\times \mathcal{O}_L^\times \mathcal{O}_{L'}^\times].$$

However, the right-hand side of equation (8) depends on more than this one index and some correction terms will be necessary.

Write $K_j = F^{C_j}$, $L_j = F^{D_j}$ and $L'_j = F^{D'_j}$, so that for example $K_0 = K$, $K_n = F$, $L_0 = k$ and $L_n = L$:
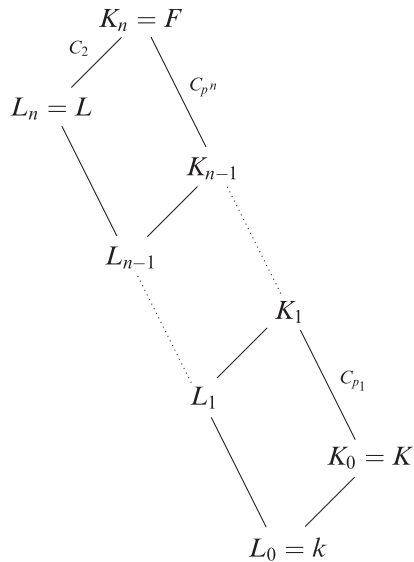
First, note that for any group X and any normal subgroups Y and Z, we have

$$|X/Y| = |YZ/Y| \cdot |X/YZ| = |Z/(Y \cap Z)| \cdot |X/Z/YZ/Z|,$$

provided all the quotients are finite. Applying this in step † below with $X = \mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_1}^\times$, $Y = \mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_0}^\times$ and $Z = \mathcal{O}_{K_1}^\times$, we get

$$[\mathcal{O}_F^\times : \mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_0}^\times] = [\mathcal{O}_F^\times : \mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_1}^\times] \times [\mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_1}^\times : \mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_0}^\times]$$

$$\overset{\dagger}{=} [\mathcal{O}_F^\times : \mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_1}^\times] \times [\mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_1}^\times \cap \mathcal{O}_{K_1}^\times : \mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_0}^\times \cap \mathcal{O}_{K_1}^\times]$$

$$\times [\mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_1}^\times / \mathcal{O}_{K_1}^\times : \mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_0}^\times \mathcal{O}_{K_1}^\times / \mathcal{O}_{K_1}^\times]$$

$$= [\mathcal{O}_F^\times : \mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_1}^\times] \times [\mathcal{O}_{K_1}^\times : \mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_0}^\times \cap \mathcal{O}_{K_1}^\times]$$

$$= [\mathcal{O}_F^\times : \mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_1}^\times] \times [\mathcal{O}_{K_1}^\times : \mathcal{O}_{L_1}^\times \mathcal{O}_{L'_1}^\times \mathcal{O}_{K_0}^\times] / [\mathcal{O}_{L_n}^\times \mathcal{O}_{L'_n}^\times \mathcal{O}_{K_0}^\times \cap \mathcal{O}_{K_1}^\times : \mathcal{O}_{L_1}^\times \mathcal{O}_{L'_1}^\times \mathcal{O}_{K_0}^\times].$$

Repeating this inductively yields

$$\prod_{j=1}^{n}[\mathcal{O}_{K_j}^{\times}:\mathcal{O}_{L_j}^{\times}\mathcal{O}_{L_j'}^{\times}\mathcal{O}_{K_{j-1}}^{\times}] = [\mathcal{O}_F^{\times}:\mathcal{O}_{L_n}^{\times}\mathcal{O}_{L_n'}^{\times}\mathcal{O}_{K_0}^{\times}] \times \prod_{j}[\mathcal{O}_{L_n}^{\times}\mathcal{O}_{L_n'}^{\times}\mathcal{O}_{K_{j-1}}^{\times}\cap\mathcal{O}_{K_j}^{\times}:\mathcal{O}_{L_j}^{\times}\mathcal{O}_{L_j'}^{\times}\mathcal{O}_{K_{j-1}}^{\times}].$$

Finally, substituting this in equation (8) gives the sought for unit index formula.

6. Examples

We first derive some easy consequences of Theorem 1.1:

Corollary 6.1. *Let $F/\mathbb{Q}$ be a Galois extension with Galois group D_{2p} for p an odd prime. Let K be the quadratic subfield and let L and L' be distinct intermediate extensions of degree p over $\mathbb{Q}$. Let $r(K)$ be the rank of the units in K, which is either 0 or 1. Then we have*

$$\frac{h(F)p^{r(K)+1}}{h(K)h(L)^2} = [\mathcal{O}_F^{\times}:\mathcal{O}_K^{\times}\mathcal{O}_L^{\times}\mathcal{O}_{L'}^{\times}].$$

This is the formula derived by Halter-Koch in [10].

Proof. In this case, S consists of the Archimedean primes and none of them have decomposition group D_{2p}. Moreover, F/K cannot be obtained by adjoining a p-th root of a fundamental unit, since for that K has to contain the p-th roots of unity and have unit rank 1, which is impossible. Finally, $r(F) = p(r(K)+1) - 1 = p \cdot r(K) + p - 1$. So formula (6) simplifies to the stated form. $\square$

Corollary 6.2. *Let F/k be a Galois extension of number fields with Galois group $G = D_{2p}$ for p an odd prime, let K be the intermediate quadratic extension and let L and L' be distinct intermediate extensions of degree p. Let S be a G-stable set of primes of F including the Archimedean ones such that their decomposition groups do not contain C_p. Also assume that F/K is not obtained by adjoining a p-th root of a non-torsion S-unit. Then*

$$\frac{h_S(F)h_S(k)^2 p^{r_S(K)+1-r_S(k)}}{h_S(K)h_S(L)^2} = [\mathcal{O}_{S,F}^{\times}:\mathcal{O}_{S,L}^{\times}\mathcal{O}_{S,L'}^{\times}\mathcal{O}_{S,K}^{\times}].$$

The condition that $F \neq K(\sqrt[p]{u})$ for a non-torsion S-unit u of K is for example satisfied when K does not contain the p-th roots of unity or when F/K is unramified at p. In particular, the corollary applies when F/K is unramified, so this includes the case considered by Lemmermeyer in [14], Theorem 2.2.

Proof. We again have that $r_S(F) = p \cdot r_S(K) + p - 1$ since all the places in S are assumed to split in F/K, and the claim is a direct consequence of formula (6). $\square$

In particular cases we can use the classification of integral representations of D_{2p} to express the Galois structure of the units modulo torsion in terms of the class number quotient. This has been explored when the base field is $\mathbb{Q}$ and S contains only the Archimedean place, e.g. in [15]. We will give some more examples in the more general setting.

Example 6.3.   Let k be a real quadratic field and let F/k be a Galois extension with Galois group $G = D_{2p}$. As before, let K be the intermediate quadratic extension and let L be an intermediate extension of degree p and take Γ to be the integral G-representation given by the units of F modulo torsion (or more precisely their usual logarithmic embedding into $\mathbb{R}^{r(F)+1}$). Assume that F/K is not obtained by adjoining a p-th root of a non-torsion unit of K. Further, assume for simplicity that K is totally complex. Then, $r(k) = r(K) = 1$ and $r(F) = 2p - 1$. So the $\mathbb{Q}G$-representation given by $\Gamma \otimes \mathbb{Q}$ contains one copy of the trivial representation and two copies of the $(p - 1)$-dimensional irreducible representation. Using the notation from Section 4, we have the following possible $\mathbb{Z}G$-module structures for Γ together with the corresponding class number quotients:

Γ	$\dfrac{h(F)h(k)^2}{h(K)h(L)^2}$
$A_i \oplus A_i \oplus \mathbf{1}$	$1/p$
$A_i \oplus A_i' \oplus \mathbf{1}$	1
$A_i' \oplus A_i' \oplus \mathbf{1}$	p
$A_i \oplus (A_i', \mathbf{1})$	$1/p$
$A_i' \oplus (A_i', \mathbf{1})$	1

where the values of the class number quotients follow from Proposition 2.15 and the computation of Dokchitser constants in Section 4. In particular, we see that if the class number quotient is p, then this determines the genus of the integral representation Γ. We remind the reader that by the classification of integral representations, the number of the representations A_i in the same genus is equal to the class number of $\mathbb{Q}(\zeta_p)^+$. This is known to be 1 for $p \leq 67$ and conjectured to be 1 for $p \leq 157$ (this conjecture is implied by the generalised Riemann hypothesis), so for 'small' p the class number quotient can sometimes completely determine the Galois module structure of the units modulo torsion.

If K is not totally complex, then the same kind of analysis applies, but the rank of the units of F is larger and there are more possibilities to consider.

Example 6.4.   In the previous example we have seen how, using our general result, we can apply Moser's reasoning from [15] to base fields, different from $\mathbb{Q}$. We will now show how the generalisation to S-units can be useful to complement Moser's results. Let $F/\mathbb{Q}$ be a D_{2p}-extension with K, L and Γ as above. If K is imaginary, then $r(K) = 0$ and $\Gamma \otimes \mathbb{Q}$ only contains one copy of the irreducible $(p - 1)$-dimensional representation. By the classification of integral representations and the computation of their Dokchitser constants in Section 4, we see that the class number quotient is either 1 or $1/p$ and in either case it determines the genus of Γ. However, when K is real, we have the following possibilities for Γ together with the corresponding class number quotients:

number	Γ	$\dfrac{h(F)h(k)^2}{h(K)h(L)^2}$
(1)	$A_i \oplus A_i \oplus \epsilon$	$1/p^2$
(2)	$A_i \oplus A_i' \oplus \epsilon$	$1/p$
(3)	$A_i' \oplus A_i' \oplus \epsilon$	1
(4)	$A_i \oplus (A_i, \epsilon)$	$1/p$
(5)	$A_i' \oplus (A_i, \epsilon)$	1

We see that if the class number quotient is $1/p^2$, then the genus of Γ is again determined (and therefore the whole Galois module structure of Γ is determined if $p \leqq 67$, as remarked in the previous example). However, if the class number quotient is 1 or $1/p$, then we are left with two possibilities. But sometimes, looking at S-class numbers can resolve the ambiguity. Let q be a prime number which is inert or ramified in $K/\mathbb{Q}$ and ramified in F/K. Let S consist of the infinite places of F and the places above q. Let Γ_S be the Galois module given by the S-units of F modulo torsion. Then $\Gamma_S \otimes \mathbb{Q}$ contains one copy of the trivial representation, one copy of the non-trivial one-dimensional representation and two copies of the irreducible $(p-1)$-dimensional representation. Also, Γ_S contains Γ as a saturated sublattice and the possible Galois module structures of Γ_S restrict the possibilities for Γ. For example, if the S-class number quotient is $1/p$, then writing out the list of possibilities for Γ_S (there are 16) we see that Γ is given either by number (1) or by (2) and the two have different class number quotients. Here is a concrete example: Let F be the splitting field of the irreducible cubic polynomial

$$f(x) = x^3 - 34x - 6.$$

The Galois group of $F/\mathbb{Q}$ is S_3 and the class number quotient is $1/3$. Thus, the Galois module structure of the units of F modulo roots of unity is either (2) or (4) from the above list. Now, let S consist of the infinite places of F and the unique place above 2. Then, the S-class number quotient is also $1/3$ and so the Galois module structure of the units of F modulo the roots of unity must be the one numbered (2) in the list.

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