FACTOR EQUIVALENCE OF GALOIS MODULES AND REGULATOR CONSTANTS

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Abstract. We compare two approaches to the study of Galois module structures: on the one hand factor equivalence, a technique that has been used by Fröhlich and others to investigate the Galois module structure of rings of integers of number fields and of their unit groups, and on the other hand regulator constants, a set of invariants attached to integral group representations by Dokchitser and Dokchitser, and used by the author, among others, to study Galois module structures. We show that the two approaches are in fact closely related, and interpret results arising from these two approaches in terms of each other. We then use this comparison to derive a factorisability result on higher K-groups of rings of integers, which is a direct analogue of a theorem of de Smit on S-units.

1. Introduction

Let G be a finite group. Factor equivalence of finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-modules is an equivalence relation that is a weakening of local isomorphism. It has been used e.g. in [5, 14, 13] among many other works to derive restrictions on the Galois module structure of rings of integers of number fields and of their units in terms of other arithmetic invariants.

More recently, a set of rational numbers has been attached to any finitely generated $\mathbb{Z}[G]$-module, called regulator constants [7], with the property that if two modules are locally isomorphic, then they have the same regulator constants. These invariants have been used in [2] and in [1] to investigate the Galois module structure of integral units of number fields, of higher K-groups of rings of integers, and of Mordell-Weil groups of elliptic curves over number fields.

It is quite natural to ask whether there is a connection between the two approaches to Galois modules and whether the results of one can be interpreted in terms of the other. It turns out that there is indeed a strong connection, which we shall investigate here. We will begin in the next section by recalling the definitions of factorisability, of factor equivalence, and of regulator constants. We will then establish some purely algebraic results that link factor equivalence and regulator constants. In §3 we will revisit the relevant results of [5, 13, 2, 1] on Galois module structures and will use the link established in §2 to compare them to each other. Finally, in §4 we will use the results of §2 to prove a factorisability result on K-groups of rings of integers that is a direct analogue of [13, Theorem 5.2].

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Throughout the paper, whenever there will be mention of a group G, we will always assume it to be finite. All $\mathbb{Z}[G]$-modules will be assumed to be finitely generated and all representations will be finite-dimensional.

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2. Factorisability and regulator constants

2.1. **Factorisability and factor equivalence.** We will begin by recalling the definition of factorisability and of factor equivalence, and by discussing slight reformulations. This concept first appears in [10] and plays a prominent rôle e.g. in the works of Fröhlich.

Definition 2.1. Let G be a group (always assumed to be finite), and let X be an abelian group, written multiplicatively. A function $f : H \mapsto x \in X$ on the set of subgroups H of G with values in X is *factorisable* if there exists an injection of abelian groups $\iota : X \hookrightarrow Y$ and a function $g : \chi \mapsto y \in Y$ on the irreducible characters of G with values in Y, with the property that

$$\iota(f(H)) = \prod_{\chi \in \mathrm{Irr}(G)} g(\chi)^{\langle \chi, \mathbb{C}[G/H] \rangle}$$

for all $H \leq G$, where $\mathrm{Irr}(G)$ denotes the set of irreducible characters of G, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product of characters.

The definition one often sees in connection with Galois module structures is a special case of this: X is usually taken to be the multiplicative group of fractional ideals of the ring of integers $\mathcal{O}_k$ of some number field k, and Y is required to be the ideal group of $\mathcal{O}_K$ for some finite Galois extension K/k with Galois group G, with ι being the natural map $I \mapsto I\mathcal{O}_K$.

Let us introduce convenient representation theoretic language to concisely rephrase the above definition.

Definition 2.2. The *Burnside ring* $B(G)$ of a group G is the free abelian group on isomorphism classes $[S]$ of finite G-sets, modulo the subgroup generated by elements of the form

$$[S] + [T] - [S \sqcup T],$$

and with multiplication defined by

$$[S] \cdot [T] = [S \times T].$$

Definition 2.3. The *representation ring* $R_{\mathcal{K}}(G)$ of a group G over the field $\mathcal{K}$ is the free abelian group on isomorphism classes $[\rho]$ of finite dimensional $\mathcal{K}$-representations of G, modulo the subgroup generated by elements of the form

$$[\rho] + [\tau] - [\rho \oplus \tau],$$

and with multiplication defined by

$$[\rho] \cdot [\tau] = [\rho \otimes \tau].$$

In the case that $\mathcal{K} = \mathbb{Q}$, which will be the main case of interest, we will omit the subscript and simply refer to the representation ring $R(G)$ of G.

There is a natural map $B(G) \to R(G)$ that sends a G-set X to the permutation representation $\mathbb{Q}[X]$. Denote its kernel by $K(G)$. By Artin's induction theorem, this map always has a finite cokernel $C(G)$ of exponent dividing $|G|$. Moreover, $C(G)$ is known to be trivial in many special cases, e.g. if G is nilpotent, or a symmetric group. The cokernel $C(G)$ is important when strengthenings of the notion of factorisability are considered, such as F-factorisability, but will not be important for us.

It follows immediately from Definition 2.1 and from standard representation theory that for f to be factorisable, it has to be constant on conjugacy classes of subgroups. There is a bijection between conjugacy classes of subgroups of G and isomorphism classes of transitive G-sets, which assigns to $H \leq G$ the set of cosets G/H with left G-action by multiplication, and to a G-set S the conjugacy class of any point stabiliser $\mathrm{Stab}_G(s)$, $s \in S$. An arbitrary G-set is a disjoint union of transitive G-sets, and so an element of $B(G)$ can be identified with a formal $\mathbb{Z}$-linear combination of conjugacy classes of subgroups of G. So if f is a factorisable function, then it can be thought of as a function on conjugacy classes of subgroups of G, equivalently on transitive G-sets, and then extended linearly to yield a group homomorphism $B(G) \to X$.

Proposition 2.4. *Let $f : B(G) \to X$ be a group homomorphism, where X is an abelian group. The following are equivalent:*

(1) *f is factorisable in the sense of Definition 2.1.*
(2) *There exists an injection $\iota : X \hookrightarrow Y$ of abelian groups such that the composition $\iota \circ f$ factors through the natural map $B(G) \to R_{\mathbb{C}}(G)$.*
(3) *There exists an injection $\iota' : X \hookrightarrow Y'$ such that $\iota' \circ f$ factors through the natural map $B(G) \to R(G)$, i.e. there is a homomorphism $g' : R(G) \to Y'$ that makes the following diagram (whose first row is exact) commute:*

$$\begin{tikzcd} 0 \arrow[r] & K(G) \arrow[r] & B(G) \arrow[r] \arrow[d, "f"'] & R(G) \arrow[r] \arrow[d, "g'"] & C(G) \arrow[r] & 0 \\ & & X \arrow[r, hook, "\iota'"'] & Y'. & & \end{tikzcd}$$

(4) *The homomorphism f is trivial on $K(G) = \ker(B(G) \to R(G))$.*

Proof. The condition (2) is just a reformulation of (1).

Suppose that condition (2) is satisfied, and let us deduce (3). Let g be the map $R_{\mathbb{C}}(G) \to Y$ whose existence is postulated by (2). Define Y' to be the subgroup of Y generated by $\iota(X)$ and by $g(R(G))$, define g' to be the restriction of g to $R(G)$, followed by the inclusion $g(R(G)) \hookrightarrow Y'$, and ι' to be ι, followed by the inclusion $\iota(X) \hookrightarrow Y'$. Then Y', ι', g' satisfy (3).

A brief diagram chase shows that (3) implies (4): since ι', is an injection, $\ker(\iota' \circ f) = \ker f$. So for the diagram in (3) to commute, we must have

$\ker f \geq \ker(B(G) \to R(G)) = K(G)$. Incidentally, exactly the same proof shows also that (2) implies (4).

Finally, the implication (4) $\Rightarrow$(2),(3) follows from two standard facts about abelian groups:

- any abelian group can be embedded into a divisible abelian group,
- and any homomorphism from a subgroup A of an abelian group B to a divisible group D extends to a homomorphism from B to D.

Since f is trivial on $K(G)$, it induces a homomorphism from $B(G)/K(G)$, which is canonically identified with a subgroup of $R(G) \leq R_{\mathbb{C}}(G)$. Now, embed X into a divisible group Y, and extend $f : B(G)/K(G) \to Y$ to a homomorphism $R(G) \hookrightarrow R_{\mathbb{C}}(G) \to Y$. $\square$

Remark 2.5. (1) It follows from the last part of the proof that if X is divisible, then Y' can be taken to be equal to X in Proposition 2.4. Also, if $C(G)$ is trivial, then $B(G)/K(G) \cong R(G)$, and again Y' can be taken to be equal to X.

(2) If X is the group of fractional ideals of a number field k, and if f vanishes on $K(G)$, then Y' can always be taken to be the group of fractional ideals of a suitable Galois extension K/k, so this is not an additional restriction. Indeed, a sufficient condition on Y' is that elements of $B(G)/K(G)$ that are n-divisible in $R(G)$ are mapped under f to elements of X that become n-divisible in Y'. So if X is the group of fractional ideals of a number field k, this condition translates into relative ramification indices of some integral ideals of K being divisible by some integers, and some elements of k having certain n-th roots in K.

Remark 2.6. In [6], the word "representation-theoretic" has been used in place of "factorisable".

Definition 2.7. Let G be a group, and let M, N be two $\mathbb{Z}$-free $\mathbb{Z}[G]$-modules such that there is an isomorphism of $\mathbb{Q}[G]$-modules $M \otimes \mathbb{Q} \cong N \otimes \mathbb{Q}$. Fix an embedding $i : M \to N$ of G-modules with finite cokernel. Then M and N are said to be *factor equivalent*, written $M \wedge N$, if the function $H \mapsto [N^H : i(M^H)]$ is factorisable.

The notion of factor equivalence is independent of the choice of the embedding i, and defines an equivalence relation on the set of $\mathbb{Z}$-free $\mathbb{Z}[G]$-modules. If $M \otimes \mathbb{Z}_p \cong N \otimes \mathbb{Z}_p$ for some prime p, then i can be chosen to have a cokernel of order coprime to p. Indeed, $M \otimes \mathbb{Z}_p \cong N \otimes \mathbb{Z}_p$ if and only if $M \otimes \mathbb{Z}_{(p)} \cong N \otimes \mathbb{Z}_{(p)}$ ([9], see also [11]), and an isomorphism $M \otimes \mathbb{Z}_{(p)} \to N \otimes \mathbb{Z}_{(p)}$ gives rise to an embedding i with cokernel of order coprime to p, by composing it with multiplication by an integer to clear denominators. It follows that two modules that are locally isomorphic at all primes p are factor equivalent.

The above definition is the one usually appearing in the literature, but it will be convenient for us to follow [13] in defining factor equivalence for $\mathbb{Z}[G]$-modules that are not necessarily $\mathbb{Z}$-free:

Definition 2.8. Let G be a group, and let M, N be two $\mathbb{Z}[G]$-modules such that there is an isomorphism of $\mathbb{Q}[G]$-modules $M \otimes \mathbb{Q} \cong N \otimes \mathbb{Q}$. Fix a map

$i : M \to N$ of G-modules with finite kernel and cokernel. Then M and N are said to be *factor equivalent* if the function $H \mapsto [N^H : i(M^H)] \cdot |\ker(i)^H|^{-1}$ is factorisable.

Again, this notion is independent of the choice of the map i, and defines an equivalence relation on the set of $\mathbb{Z}[G]$-modules that weakens the relation of lying in the same genus (where M and N are said to lie in the same genus if $M \otimes \mathbb{Z}_p \cong N \otimes \mathbb{Z}_p$ for all primes p).

2.2. **Regulator constants.** We continue to denote by G an arbitrary (finite) group. We also continue to use the identification between conjugacy classes of subgroups of G and isomorphism classes of transitive G-sets. Under this identification, a general element of $B(G)$ will be written as $\Theta = \sum_{H\leq G} n_H H$ with the sum running over mutually non-conjugate subgroups, and with $n_H \in \mathbb{Z}$. An element of $K(G)$ is such a linear combination with the property that the virtual permutation representation $\bigoplus_H \mathbb{Q}[G/H]^{\oplus n_H}$ is 0. Alternatively, more down to earth, if we write Θ as $\Theta = \sum_i n_i H_i - \sum_j n'_j H'_j$ with all n_i, n'_j non-negative, then Θ is in $K(G)$ if and only if the permutation representations $\bigoplus_i \mathbb{Q}[G/H_i]^{\oplus n_i}$ and $\bigoplus_j \mathbb{Q}[G/H'_j]^{\oplus n'_j}$ are isomorphic.

Definition 2.9. An element $\Theta = \sum_H n_H H$ of $K(G)$ is called a *Brauer relation.*

The following invariants of $\mathbb{Z}[G]$-modules were introduced in [7] and used e.g. in [2, 1] to investigate Galois module structures, as we shall review in the next section:

Definition 2.10. Let G be a group and M a $\mathbb{Z}[G]$–module. Let $\langle\cdot,\cdot\rangle : M \times M \to \mathbb{C}$ be a bilinear G–invariant pairing that is non–degenerate on M/tors. Let $\Theta = \sum_{H\leq G} n_H H \in K(G)$ be a Brauer relation. The regulator constant of M with respect to Θ is defined by

$$\mathcal{C}_\Theta(M) = \prod_{H\leq G} \det\left(\frac{1}{|H|}\langle\cdot,\cdot\rangle\big|M^H/\operatorname{tors}\right) \in \mathbb{C}^\times.$$

Here and elsewhere, the abbreviation tors refers to the $\mathbb{Z}$-torsion subgroup.

This is independent of the choice of pairing [6, Theorem 2.17]. As a consequence, $\mathcal{C}_\Theta(M)$ is always a rational number, since the pairing can always be chosen to be $\mathbb{Q}$-valued. It is also immediate that $\mathcal{C}_{\Theta_1+\Theta_2}(M) = \mathcal{C}_{\Theta_1}(M)\mathcal{C}_{\Theta_2}(M)$, so given a $\mathbb{Z}[G]$-module, it suffices to compute the regulator constants with respect to a basis of $K(G)$. In other words, this construction assigns to each $\mathbb{Z}[G]$-module essentially a finite set of rational numbers, one for each element of a fixed basis of $K(G)$.

One can show that if M, N are two $\mathbb{Z}[G]$-modules such that $M \otimes \mathbb{Z}_p \cong N \otimes \mathbb{Z}_p$, then for all $\Theta \in K(G)$ the p-parts of $\mathcal{C}_\Theta(M)$ and $\mathcal{C}_\Theta(N)$ are the same. So, like factor equivalence, regulator constants provide invariants of a $\mathbb{Z}[G]$-module that, taken together, are coarser than the genus.

2.3. **The connection between factor equivalence and regulator constants.** Let M, N be two $\mathbb{Z}[G]$-modules with the property that $M \otimes \mathbb{Q} \cong N \otimes \mathbb{Q}$, let $i : M \to N$ be a map of G-modules with finite kernel and cokernel. Fix a $\mathbb{C}$-valued bilinear pairing $\langle\cdot,\cdot\rangle$ on N that is non-degenerate on

N/tors. The following immediate observation is crucial for linking regulator constants with the notion of factorisability:

$$\begin{array}{rcl}\det\left(\langle\cdot,\cdot\rangle\big|i(M)/\operatorname{tors}\right) & = & [N/\operatorname{tors}:i(M)/\operatorname{tors}]^2\cdot\det\left(\langle\cdot,\cdot\rangle\big|N/\operatorname{tors}\right)\\ & = & \dfrac{[N:i(M)]^2}{|\ker i|^2}\cdot\dfrac{|M_{\operatorname{tors}}|^2}{|N_{\operatorname{tors}}|^2}\cdot\det\left(\langle\cdot,\cdot\rangle\big|N/\operatorname{tors}\right).\end{array}$$

We deduce

Lemma 2.11. *Let M, N be two $\mathbb{Z}[G]$-modules such that $M\otimes\mathbb{Q}\cong N\otimes\mathbb{Q}$, let $\Theta=\sum_H n_H H$ be a Brauer relation. Then*

$$\mathcal{C}_\Theta(M)=\prod_H\left(\frac{[N^H:i(M^H)]}{|\ker(i|_M^H)|}\cdot\frac{|M_{\operatorname{tors}}^H|}{|N_{\operatorname{tors}}^H|}\right)^{2n_H}\cdot\mathcal{C}_\Theta(N)$$

for any map $i:M\to N$ of G-modules with finite kernel and cokernel.

By combining this with Proposition 2.4, we obtain

Corollary 2.12. *Two $\mathbb{Z}[G]$-modules M and N with the property that $M\otimes\mathbb{Q}\cong\mathbb{N}\otimes\mathbb{Q}$ are factor equivalent if and only if*

$$\mathcal{C}_\Theta(M)/\mathcal{C}_\Theta(N)=\prod_H\left(\frac{|M_{\operatorname{tors}}^H|}{|N_{\operatorname{tors}}^H|}\right)^{2n_H}$$

for all Brauer relations $\Theta=\sum_H n_H H$. In particular, if M and N are $\mathbb{Z}$-free and satisfy $M\otimes\mathbb{Q}\cong\mathbb{N}\otimes\mathbb{Q}$, then they are factor equivalent if and only if $\mathcal{C}_\Theta(M)=\mathcal{C}_\Theta(N)$ for all $\Theta\in K(G)$.

3. Galois module structure

We shall now show by way of several examples how Lemma 2.11 and Corollary 2.12 link known results on Galois module structures with each other.

Throughout this section, let K/k be a finite Galois extension of number fields with Galois group G. The ring of integers $\mathcal{O}_K$, and its unit group $\mathcal{O}_K^\times$ are both $\mathbb{Z}[G]$-modules. More generally, if S is any G-stable set of places of K that contains the Archimedean places, then the group of S-units $\mathcal{O}_{K,S}^\times$ of K is a $\mathbb{Z}[G]$-module. It is a long standing and fascinating problem to determine the G-module structure of these groups, e.g. by comparing it to other well-known G-modules or by linking it to other arithmetic invariants.

A starting point is the observation that $\mathcal{O}_K\otimes\mathbb{Q}\cong\mathbb{Q}[G]^{\oplus[k:\mathbb{Q}]}$ as $\mathbb{Q}[G]$-modules. Also, by Dirichlet's unit theorem, $\mathcal{O}_{K,S}^\times\otimes\mathbb{Q}\cong I_{K,S}\otimes\mathbb{Q}$, where

$$I_{K,S}=\ker\left(\mathbb{Z}[S]\to\mathbb{Z}\right),$$

with the map being the augmentation map that sends each $v\in S$ to 1. It is therefore natural to compare the Galois module $\mathcal{O}_K$ to $\mathbb{Z}[G]^{\oplus[k:\mathbb{Q}]}$ and $\mathcal{O}_{K,S}^\times$ to $I_{K,S}$.

3.1. **Additive Galois module structure.** It had been known since E. Noether that $\mathcal{O}_K$ lies in the same genus as $\mathbb{Z}[G]^{\oplus[k:\mathbb{Q}]}$ if and only if K/k is at most tamely ramified. The following is therefore particularly interesting in the wildly ramified case:

Theorem 3.1 ([13], Theorem 3.2, see also [5], Theorem 7 (Additive))**.** *We always have that $\mathcal{O}_K$ is factor equivalent to $\mathbb{Z}[G]^{\oplus[k:\mathbb{Q}]}$.*

We will now give a very short proof of this result in terms of regulator constants. First, note that by Corollary 2.12 the statement is equivalent to the claim that for any $\Theta \in K(G)$, $\mathcal{C}_\Theta(\mathcal{O}_K) = \mathcal{C}_\Theta(\mathbb{Z}[G]^{\oplus[k:\mathbb{Q}]})$. Since regulator constants are multiplicative in direct sums of modules ([6, Corollary 2.18]), and since $\mathcal{C}_\Theta(\mathbb{Z}[G]) = 1$ for all $\Theta \in K(G)$ ([6, Example 2.19]), we have reduced the proof of the theorem to showing that $\mathcal{C}_\Theta(\mathcal{O}_K) = 1$ for all $\Theta \in K(G)$.

If we choose the pairing on $\mathcal{O}_K$ defined by

$$\langle a, b\rangle = \sum_\sigma \sigma(a)\sigma(b)$$

with the sum running over all embeddings $\sigma : K \hookrightarrow \mathbb{C}$, then the determinants on $\mathcal{O}_K^H$, $H \le G$, appearing in the definition of regulator constants are nothing but the absolute discriminants Δ_{K^H}. The fact that these vanish in Brauer relations follows immediately from the conductor-discriminant formula.

3.2. **Multiplicative Galois module structure.** As we have mentioned above, it is natural to compare $\mathcal{O}_{K,S}^\times$ with $I_{K,S}$, since they span isomorphic $\mathbb{Q}[G]$-modules. For $H \le G$, let $S(K^H)$ denote the set of places of K^H below those in S, and let $h_S(K^H)$ denote the S-class number of K^H.

Theorem 3.2 ([13], Theorem 5.2, see also [5], Theorem 7 (Multiplicative))**.** *Fix an embedding $i : I_{K,S} \hookrightarrow \mathcal{O}_{K,S}^\times$ of G-modules with finite cokernel. For $\mathfrak{p} \in S(K^H)$, let $f_\mathfrak{p}$ be its residue field degree in K/K^H, define*

$$n(H) = \prod_{\mathfrak{p}\in S(K^H)} f_\mathfrak{p}, \quad l(H) = \mathrm{lcm}\{f_\mathfrak{p} \mid \mathfrak{p} \in S(K^H)\}.$$

Then the function

$$H \mapsto [\mathcal{O}_{K^H,S}^\times : i(I_{K,S})^H]\frac{n(H)}{h_S(K^H)l(H)}$$

is factorisable.

As in the additive case, we want to understand and to reprove this theorem in terms of regulator constants. More specifically, we will show it to be equivalent to

Theorem 3.3 ([2], Proposition 2.15 and equation (1))**.** *For $\mathfrak{p} \in S(k)$, let $D_\mathfrak{p}$ be the decomposition group of a prime $\mathfrak{P} \in S$ above $\mathfrak{p}$ (well-defined up to conjugacy). For any Brauer relation $\Theta = \sum_H n_H H \in K(G)$, we have*

$$\mathcal{C}_\Theta(\mathcal{O}_{K,S}^\times) = \frac{\mathcal{C}_\Theta(\mathbf{1})}{\prod_{\mathfrak{p}\in S(k)} \mathcal{C}_\Theta(\mathbb{Z}[G/D_\mathfrak{p}])} \prod_H \left(\frac{w(K^H)}{h_S(K^H)}\right)^{2n_H},$$

where $w(K^H)$ denotes the number of roots of unity in K^H, i.e. the size of the torsion subgroup of $\mathcal{O}_{K^H,S}^{\times}$.

Note that since $I_{K,S}$ is torsion free and $I_{K,S} \hookrightarrow \mathcal{O}_{K,S}^{\times}$ is injective, Proposition 2.4 and Lemma 2.11 imply that Theorem 3.2 is equivalent to the following statement: for any Brauer relation $\Theta = \sum_H n_H H$,

$$\mathcal{C}_\Theta(\mathcal{O}_{K,S}^{\times}) = \mathcal{C}_\Theta(I_{K,S}) \prod_H \left(\frac{w(K^H)n(H)}{h_S(K^H)l(H)} \right)^{2n_H}.$$

The equivalence of Theorems 3.2 and 3.3 will therefore be established if we show that

$$\mathcal{C}_\Theta(I_{K,S}) = \frac{\mathcal{C}_\Theta(\mathbf{1})}{\prod_{\mathfrak{p}\in S(k)} \mathcal{C}_\Theta(\mathbb{Z}[G/D_\mathfrak{p}])} \prod_H \left(\frac{l(H)}{n(H)} \right)^{2n_H}.$$

This is just a linear algebra computation that we will not carry out in full detail, since it is a combination of the computations of [13] and [2]. Indeed, it is shown in [13] that under the embedding

$$\mathbb{Z}[S(K^H)] \hookrightarrow \mathbb{Z}[S], \quad \mathfrak{p} \mapsto \sum_{\mathfrak{q}\in S, \mathfrak{q}|\mathfrak{p}} f_\mathfrak{p}\mathfrak{q} \tag{3.4}$$

we have $[(I_{K,S})^H : I_{K^H,S}] = \frac{n(H)}{l(H)}$. So, instead of computing

$$\mathcal{C}_\Theta(I_{K,S}) = \prod_H \det \left(\frac{1}{|H|} \langle \cdot, \cdot \rangle \big| (I_{K,S})^H \right)^{n_H}$$

for a suitable choice of pairing $\langle \cdot, \cdot \rangle$ on $I_{K,S}$, we may compute

$$\prod_H \det \left(\frac{1}{|H|} \langle \cdot, \cdot \rangle \big| I_{K^H,S} \right)^{n_H}, \tag{3.5}$$

where $I_{K^H,S}$ is identified with a submodule of $I_{K,S}$ as in (3.4). To do that, we note that for any $H \leq G$, $I_{K^H,S}$ is generated by $\mathfrak{p}_1 - \mathfrak{p}_i$, $\mathfrak{p}_i \in S(K^H)\backslash\{\mathfrak{p}_1\}$ for any fixed $\mathfrak{p}_1 \in S(K^H)$, and that there is a natural G-invariant non-degenerate pairing on $I_{K,S}$ that makes the canonical basis of $\mathbb{Z}[S]$ orthonormal. It is now a straightforward computation, which has essentially been carried out in [2], to show that the quantity (3.5) is equal to

$$\frac{\mathcal{C}_\Theta(\mathbf{1})}{\prod_{\mathfrak{p}\in S(k)} \mathcal{C}_\Theta(\mathbb{Z}[G/D_\mathfrak{p}])},$$

as required.

4. K-groups of rings of integers

As another illustration of the connection we have established, we will give an easy proof of an analogue of [13, Theorem 5.2] for higher K-groups of rings of integers. The main ingredient will be the compatibility of Lichtenbaum's conjecture on leading coefficients of Dedekind zeta functions at negative integers with Artin formalism, as proved in [4].

Let $n \geq 2$ be an integer. Let $S_1(F)$, respectively $S_2(F)$ denote the set of real embeddings, respectively of representatives from each pair of complex conjugate embeddings of a number field F, and denote their cardinalities by $r_1(F)$, respectively $r_2(F)$. Denote $S_2(F) \cup S_2(F)$ by $S_\infty(F)$. It is shown in [3] that the ranks of the higher K-groups or rings of integers are as follows:

$$\mathrm{rk}(K_{2n-1}(\mathcal{O}_F)) = \begin{cases} r_1(F)+r_2(F), & n \text{ odd} \\ r_2(F), & n \text{ even.} \end{cases}$$

Let K/k be a finite Galois extension with Galois group G, and let $S_r(K/k)$ denote the set of real places of k that become complex in K. For $\mathfrak{p} \in S_r(K/k)$, let $\epsilon_\mathfrak{p}$ denote the non-trivial one-dimensional $\mathbb{Q}$-representation of the decomposition group $D_\mathfrak{p}$, which has order 2.

By Artin's induction theorem, a rational representation of a finite group is determined by the dimensions of the fixed subrepresentations under all subgroups of G. It therefore follows that we have the following isomorphisms of Galois modules:

$$\begin{aligned} K_{2n-1}(\mathcal{O}_K)\otimes\mathbb{Q} &\cong \mathbb{Q}[S_\infty(K)] \\ &\cong \bigoplus_{\mathfrak{p}\in S_\infty(k)} \mathbb{Q}[G/D_\mathfrak{p}] \quad \text{if } n \text{ is odd, and} \qquad (4.1) \\ K_{2n-1}(\mathcal{O}_K)\otimes\mathbb{Q} &\cong \bigoplus_{\mathfrak{p}\in S_r(K/k)} \mathrm{Ind}_{G/D_\mathfrak{p}}\,\epsilon_\mathfrak{p} \oplus \bigoplus_{\mathfrak{p}\in S_2(k)} \mathbb{Q}[G] \\ &\cong \bigoplus_{\mathfrak{p}\in S_r(K/k)} \mathbb{Q}[G]/\mathbb{Q}[G/D_\mathfrak{p}] \oplus \bigoplus_{\mathfrak{p}\in S_2(k)} \mathbb{Q}[G] \quad \text{if } n \text{ is even.} \qquad (4.2) \end{aligned}$$

We are thus led to compare, using the machine of factorisability, the Galois module structure of $K_{2n-1}(\mathcal{O}_K)$ with $\mathbb{Z}[S_\infty(K)]$ when n is odd, and with

$$\bigoplus_{\mathfrak{p}\in S_r(K/k)} \mathrm{Ind}_{G/D_\mathfrak{p}}(\epsilon_\mathfrak{p}) \oplus \bigoplus_{\mathfrak{p}\in S_2(k)} \mathbb{Z}[G]$$

when n is even. Here and elsewhere, we write $\epsilon_\mathfrak{p}$ interchangeably for the rational representation and for the unique (up to isomorphism) $\mathbb{Z}$-free $\mathbb{Z}[D_\mathfrak{p}]$-module inside it.

Theorem 4.3. *Let K/k be a finite Galois extension of number fields with Galois group G, let $n \geq 2$ be an integer. Then the function*

$$H \mapsto \frac{[K_{2n-1}(\mathcal{O}_K)^H : i(M)^H]}{|K_{2n-2}(\mathcal{O}_{K^H})|}$$

is factorisable at all odd primes, where

$$\begin{aligned} M &= \mathbb{Z}[S_\infty(K)] \cong \bigoplus_{\mathfrak{p}\in S_\infty(k)} \mathbb{Z}[G/D_\mathfrak{p}] \quad \textit{if } n \textit{ is odd, and} \\ M &= \bigoplus_{\mathfrak{p}\in S_r(K/k)} \mathrm{Ind}_{G/D_\mathfrak{p}}(\epsilon_\mathfrak{p}) \oplus \bigoplus_{\mathfrak{p}\in S_2(k)} \mathbb{Z}[G] \quad \textit{if } n \textit{ is even,} \end{aligned}$$

and where $i : M \hookrightarrow K_{2n-1}(\mathcal{O}_K)$ is any inclusion of G-modules.

Proof. Proposition 2.4 and Lemma 2.11 imply that the assertion of the theorem is equivalent to the claim that for any Brauer relation $\Theta = \sum_H n_H H$,

$$\begin{aligned} 1 \quad &=_{2'} \quad \prod_H \frac{[K_{2n-1}(\mathcal{O}_K)^H : i(M)^H]^{2n_H}}{|K_{2n-2}(\mathcal{O}_{K^H})|^{2n_H}} \\ &=_{2'} \quad \frac{\mathcal{C}_\Theta(M)}{\mathcal{C}_\Theta(K_{2n-1}(\mathcal{O}))} \cdot \prod_H \left(\frac{|K_{2n-1}(\mathcal{O}_K)^H_{\mathrm{tors}}|}{|K_{2n-2}(\mathcal{O}_{K^H})|} \right)^{2n_H}, \end{aligned}$$

where $=_{2'}$ means that the two sides have the same p-adic valuation for all odd primes p.

Now, for any odd prime p and any subgroup $H \leq G$, we have

$$(K_{2n-1}(\mathcal{O}_K) \otimes \mathbb{Z}_p)^H \cong K_{2n-1}(\mathcal{O}_{K^H}) \otimes \mathbb{Z}_p.$$

This is a consequence of the Quillen–Lichtenbaum conjecture (see e.g. [8, Proposition 2.9 and the discussion preceding it]), which is known to follow from the Bloch–Kato conjecture, which in turn is now a theorem of Rost, Voevodsky, and Weibel [12, 15, 16]. Moreover, it follows from [4] (see [1, equation (2.6)]) that

$$\prod_H \left(\frac{|K_{2n-1}(\mathcal{O}_{K^H})_{\mathrm{tors}}|}{|K_{2n-2}(\mathcal{O}_{K^H})|} \right)^{2n_H} =_{2'} \mathcal{C}_\Theta(K_{2n-1}(\mathcal{O}_K)).$$

Putting this together, we see that the assertion of the theorem is equivalent to the claim that $\mathcal{C}_\Theta(M) =_{2'} 1$ for all Brauer relations Θ. But $\mathcal{C}_\Theta(M) = 1$ (not just up to powers of 2) by [6, Corollary 2.18 and Proposition 2.45 (2)], and because cyclic groups have no non-trivial Brauer relations. □

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