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SIMPLICITY OF TWISTS OF ABELIAN VARIETIES

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ABSTRACT. We give some easy necessary and sufficient criteria for twists of abelian varieties by Artin representations to be simple.

1. INTRODUCTION

Let $A/k$ be an abelian variety over a field, let $R \leq \text{End}(A)$ be a commutative ring of endomorphisms of $A$ (here and in the sequel, we regard the abelian varieties as schemes over a base, and this is also the category in which our morphisms will live; in particular, $\text{End}(A)$ denotes endomorphisms of $A$ defined over $k$; the same remark applies to statements like “$A$ is principally polarised”, etc.), and let $K/k$ be a finite Galois extension with Galois group $G$. Let $\Gamma$ be an $R[G]$-module, together with an isomorphism $\psi : R^n \to \Gamma$ for some $n$. Attached to this data is the so-called twist of $A$ by $\Gamma$, denoted by $B = \Gamma \otimes_R A$, which is an abelian variety over $k$ with the property that the base change $B_K = B \times_k K$ is isomorphic to $(A_K)^n$.

As soon as $n > 1$, $B$ is, by its very definition, never absolutely simple. But it can be simple over $k$, and to know when this is the case is important for some applications, see e.g. [4]. If $A'$ is a proper abelian subvariety of $A$, then $\Gamma \otimes_R A'$ is a proper abelian subvariety of $\Gamma \otimes_R A$. Similarly, if $\Gamma' \leq \Gamma$ is an $R$-free $R[G]$-submodule of strictly smaller $R$-rank, then $\Gamma' \otimes_R A$ is isogenous to a proper abelian subvariety of $\Gamma \otimes_R A$. The purpose of this note is to point out that, under some mild additional hypotheses (and in particular over number fields in the generic case, when $\text{End}_{\bar{k}}(A) \cong \mathbb{Z}$), these are the only two ways in which $B$ can fail to be simple.

As a concrete example, we mention the following generalisation of Howe’s analysis [4]:

**Theorem 1.1.** Let $A/k$ be a simple abelian variety of dimension 1 or 2 over a number field, let $p$ be an odd prime number and let $K/k$ be a Galois extension with Galois group $G$ of order $p$. If $A$ is not absolutely simple or not principally polarised, assume that $p > 3$. Let $I$ be the augmentation ideal in $\mathbb{Z}[G]$, i.e. the kernel of the map $\mathbb{Z}[G] \to \mathbb{Z}$, $g \mapsto 1 \forall g \in G$. Then $I \otimes_{\mathbb{Z}} A$ is simple if and only if $\text{End}(A) \otimes \mathbb{Q}$ does not contain the quadratic subfield of $\mathbb{Q}(\mu_p)$.

**Remark 1.2.** If $p = 2$, then $I \otimes_{\mathbb{Z}} A$ is a quadratic twist of $A$, and so also simple if $A$ is. Since, for all $p$, $I \otimes \mathbb{Q}$ is the unique non-trivial irreducible $\mathbb{Q}[G]$-module, the theorem completely deals with simplicity of those twists of elliptic curves and of principally polarised absolutely simple abelian surfaces that are trivialised by a cyclic prime degree extension.

**Remark 1.3.** By computing the endomorphism ring of $I \otimes \mathbb{Q}$ as a $\mathbb{Q}[G]$-module, Howe [4] showed part of one implication in the case $\text{dim}(A) = 1$:

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he proved that if \( E/k \) is a non-CM elliptic curve, then \( I \otimes \mathbb{Z} \) is simple. In the proof of the theorem that we present, one does not need to know the endomorphism ring of \( I \otimes \mathbb{Q} \) to deduce the result for elliptic curves; one does, however, need to know it to prove the statement for abelian surfaces.

The same technique yields uniform statements for higher dimensional abelian varieties, where the restriction on \( p \) depends on the dimension of the variety:

**Theorem 1.4.** Fix an integer \( d \). There exists an integer \( p_0 \) such that for all number fields \( k \), all simple abelian varieties \( A/k \) of dimension \( d \), all primes \( p > p_0 \), and all Galois extensions \( K/k \) with cyclic Galois group \( G \) of order \( p \), the twist \( I \otimes \mathbb{Z} A \) is simple if and only if \( \text{End}(A) \otimes \mathbb{Q} \) does not contain a subfield of \( \mathbb{Q}(\mu_p) \) other than \( \mathbb{Q} \). Here, \( I \) is, as in Theorem 1.1, the augmentation ideal in \( \mathbb{Z}[G] \).

Similarly concrete results can be obtained for twists by other representations, and we give several more examples in the same vein in the last section.

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### 2. Endomorphisms of twists of abelian varieties

In this section we begin by recalling (see [9, §III.1.3]) the definition of a twist of an abelian variety by an Artin representation, and then give sufficient conditions for the endomorphism ring of such a twist to be an integral domain, equivalently for the twist to be simple. We strongly recommend [6] for a very thorough treatment of twists of abelian varieties, and, more generally, of commutative algebraic groups.

Let \( Y/k \) be an abelian variety, and \( K/k \) a finite Galois extension with Galois group \( G \). A \( K/k \)-form of \( Y \) is a pair \((X, f)\), where \( X/k \) is an abelian variety, and \( f : Y_K \to X_K \) is an isomorphism, defined over \( K \). There is an obvious notion of isomorphism between such pairs, and the set of isomorphism classes of \( K/k \)-forms of \( Y \) is in bijection with the pointed set \( H^1(G, \text{Aut} Y_K) \), where the \( G \)-action on \( \text{Aut} Y_k \) is given by \( \phi^\sigma = \sigma \circ \phi \circ \sigma^{-1} \) for \( \sigma \in G \) and \( \phi \in \text{Aut}(Y_K) \). The bijection is given by assigning to a \( K/k \)-form \((X, f)\) the cocycle represented by \( \sigma \mapsto f^{-1} f^\sigma \), where, as before, \( f^\sigma \) is defined to be \( \sigma \circ f \circ \sigma^{-1} \).

Now, suppose that \( A/k \) is an abelian variety, and \( R \leq \text{End}(A) \) a commutative ring. With \( K/k \) and \( G \) as above, let \( \Gamma \) be an \( R[G] \)-module, together with an \( R \)-module isomorphism \( \psi : R^n \to \Gamma \) for some \( n \in \mathbb{N} \). Then the map \( \alpha_p : \sigma \mapsto \psi^{-1} \psi^\sigma = \psi^{-1} \circ \sigma \circ \psi \in \text{GL}_n(R) \leq \text{Aut}_K A^n \) defines a cocycle in \( H^1(G, \text{Aut}(A_K)^n) \). Indeed, note that since \( G \) acts trivially on automorphisms of \( A^n \) that are defined over \( k \), as is the case for \( \text{GL}_n(R) \leq \text{Aut}(A_K)^n \),

\(^1\)we adhere to the common convention that the superscript for the action is written on the right, even though this is actually a left action.
1-cocycles whose image lies in GL$_n(R)$ are simply group homomorphisms. The twist $B$ of $A$ by $\Gamma$, written $B = \Gamma \otimes_R A$ is, by definition, the $K/k$-form of $A^n$ corresponding to the cocycle $a_{\Gamma}$.

We now come to the endomorphism ring of $B$. Our aim is to find criteria for $B$ to be simple, equivalently for $\text{End}(B)$ to be a division ring. In theory, one can easily describe $\text{End}(B)$ in terms of the $G$-module structure of $\text{End}_K(A)$ and $\text{End}_R(\Gamma)$:

**Lemma 2.1.** There is an isomorphism

$$\text{End}(\Gamma \otimes_R A) \cong (\text{End}_R(\Gamma) \otimes \text{End}_K(A))^G.$$  

**Proof.** This immediately follows from [6, Proposition 1.6], by noting that the absolute Galois group of $k$ acts on $\Gamma$ through the quotient $G$. \hfill \Box

However, in the most general form, this description is not easy to use for determining when the right hand side of the equation is a division ring. On the other hand, generically the situation is much better.

**Assumption 2.2.** For the rest of this section, assume that $\text{End}(A) = \text{End}(A_K)$. Since we are interested in criteria for $B$ to be simple, we will also assume from now on that $A$ itself is simple, therefore so is $A_K$ by the previous assumption.

**Remark 2.3.** This assumption is generically satisfied over number fields in the following sense: fix an abelian variety $A$ over a number field $k$, and a Galois group $G$. A result of Ribet and Silverberg \cite{10, 7} says that, given any subring $O \subseteq \text{End}_{\bar{k}}(A)$ there exists a unique minimal extension $L_O/k$ such that $O \subseteq \text{End}_{L_O}(A)$. So $\text{End}_K(A) = \text{End}(A)$ whenever $K \cap L_S = k$.

**Notation 2.4.** The following notation will be retained throughout the paper:

- $K/k$ — a Galois extension of fields with Galois group $G$;
- $A/k$ — a simple abelian variety;
- $S = \text{End}(A)$;
- $R \leq S$ — a commutative subring;
- $\Gamma$ — an $R$-free $R[G]$-module;
- $B = \Gamma \otimes_R A$ — the twist of $A$ by $\Gamma$, which is an abelian variety over $k$;
- $D = S \otimes_{\mathbb{Z}} \mathbb{Q}$ — a division algebra;
- $F = R \otimes_{\mathbb{Z}} \mathbb{Q}$ — a field contained in $D$;

Under Assumption 2.2, Lemma 2.1 becomes

$$\text{End}(B) \cong \text{End}_{R[G]}(\Gamma) \otimes_R S. \quad (2.5)$$

In general, it is a subtle question with a rich literature when the tensor product of two division rings over a common subring is a division ring. But for a generic polarised abelian variety, $S = \mathbb{Z}$. More generally, if $S$ is commutative, Schur’s Lemma furnishes an elementary answer to the question of simplicity of $B$:

**Proposition 2.6.** Assume, in addition to Assumption 2.2, that $S$ is commutative, i.e. that $D$ is a field. Then $B$ is simple if and only if $\Gamma \otimes_R D$ is a simple $D[G]$-module.
Proof. The twist $B$ is simple if and only if $\text{End}(B)$ is a division ring, if and only if

$$\text{End}(B) \otimes \mathbb{Q} \cong \text{End}_{R[G]}(\Gamma) \otimes_R D$$

is a division algebra. It is an elementary computation that when $S$ is commutative, $\text{End}_{R[G]}(\Gamma) \otimes_R D$ is precisely the endomorphism ring of the $D[G]$-module $\Gamma \otimes_R D$, the isomorphism given by

$$\text{End}_{R[G]}(\Gamma) \otimes_R D \to \text{End}_{D[G]}(\Gamma \otimes_R D),$$
$$\alpha \otimes f \mapsto (\gamma \otimes g \mapsto \alpha(\gamma) \otimes fg).$$

We deduce that, by Schur’s Lemma, $B$ is simple if and only if $\Gamma \otimes_R D$ is a simple $D[G]$-module. □

There is slightly different way of phrasing this discussion, which is closer to Howe’s original proof. Since $A_K$ is assumed to be simple, $S$ is a division ring, and $\text{End}_K(A^n) \cong M_n(S)$, the $n$-by-$n$ matrix ring over $S$. Since the base change of $B$ to $K$ is isomorphic to $(A_K^n)$, any endomorphism of $B$ gives rise to an endomorphism of $(A_K^n)$, i.e., an element of $M_n(S)$. Conversely, it is easy to characterise the elements of $M_n(S)$ that descend to endomorphisms of $B$:

**Proposition 2.7** ([1], Proposition 2.1). An element of $M_n(S)$ descends to an endomorphism of $B$ if and only if it commutes with all elements of the image of $G$ under the cocycle $a_G : G \to \text{GL}_n(R) \leq \text{GL}_n(S)$.

Now, we merely need to observe that, as we remarked above, the cocycle $a_G$ is in fact nothing but the group homomorphism $G \to \text{Aut}(\Gamma)$ with respect to an $R$-basis on $\Gamma$. The commutant of its image in $M_n(S)$ is the intersection of $M_n(S)$ with the commutant of the image of $a_G$ in $M_n(D)$, where $D = S \otimes \mathbb{Q}$ is, as in Proposition 2.6, assumed to be a field. Moreover, since for any $x \in M_n(D)$, some integer multiple of $x$ lies in $M_n(S)$, the commutant of $a_G$ in $M_n(S)$ is a division ring if and only if its commutant in $M_n(D)$ is a division algebra. By Schur’s Lemma, the latter is the case if and only if $\Gamma \otimes_R D$ is simple.

Another example in which equation (2.5) can be completely analysed is when $D = S \otimes \mathbb{Q}$ is a quaternion algebra over $F = R \otimes \mathbb{Q}$. In that case, a theorem of Risman [8] asserts that if $D'$ is any division algebra over $F$, then $D \otimes_F D'$ has zero-divisors if and only if $D'$ contains a splitting field for $D$. So we immediately deduce:

**Proposition 2.8.** Assume, in addition to Assumption 2.2, that $D$ is a quaternion algebra over $F = R \otimes \mathbb{Q}$. Then $B$ is simple if and only if $\text{End}_{F[G]}(\Gamma \otimes F)$ contains no splitting field of $D$.

A generalisation in a slightly different direction is the special case that $L = \text{End}_{R[G]}(\Gamma) \otimes \mathbb{Q}$ is a field:

**Proposition 2.9.** Assume, in addition to Assumption 2.2, that $L$ is a field. Suppose also that $R$ is contained in the centre of $\text{End}(A)$. Then $B$ is simple if and only if $L$ intersects every splitting field of $D$ in $F = R \otimes \mathbb{Q}$.
Proof. This follows from the general theory of division algebras, see e.g. [1, §74A]. Indeed, let $Z$ be the centre of $D$. If $L \cap Z \neq F$, then certainly $L \otimes_F D$ is not a division algebra, since $L \otimes_F Z$ is not a field. Suppose that $L \cap Z = F$, so that $L \otimes_F Z$ is a field. Then $L \otimes_F D$ is a simple algebra with centre $L \otimes_F Z$. The dimension of $D$ over $F$ is equal to the dimension of $L \otimes_F D$ over $L$, and their respective dimensions over their centres are therefore also equal. So $L$ intersects a splitting field of $D$ in a field that is bigger than $F$ if and only if the index of $L \otimes_F D$ is smaller than that of $D$ if and only if $L \otimes_F D$ has zero divisors. □

3. Consequences

We first explain how to deduce Theorem 1.1 from Propositions 2.6 and 2.8.

Let $G$ be cyclic of odd prime order $p$. Recall that $I \leq \mathbb{Z}[G]$ is defined to be the augmentation ideal in $\mathbb{Z}[G]$, $I = \ker(\sum_{g \in G} n_g \mapsto \sum_{g \in G} n_g)$. The complexification $I \otimes \mathbb{C}$ is isomorphic to the direct sum of all non-trivial simple $\mathbb{C}[G]$-modules, which are all Galois conjugate. It is therefore easy to see that $I \otimes_{\mathbb{Z}} \mathbb{Q}$ is a simple $\mathbb{Q}[G]$-module, and that moreover, given any number field $D$, $I \otimes_{\mathbb{Z}} D$ is reducible if and only if $D$ intersects $\mathbb{Q}(\mu_p)$ non-trivially.

First, let $A/k$ be an elliptic curve over a number field. Then $\text{End}(A) \otimes \mathbb{Q}$ is a field, and the fact that $\text{End}(A) = \text{End}(A_K)$ for an odd degree extension $K/k$ follows from classical CM theory, see e.g. [5, Chapter 3]. Thus, the dimension 1 case of Theorem 1.1 follows from Proposition 2.6.

The dimension 2 case is more subtle. Let $A/k$ be an absolutely simple abelian surface over a number field. Then $\text{End}(A_{\bar{k}}) \otimes \mathbb{Q}$ is one of the following:

(1) $\mathbb{Q}$,
(2) a real quadratic number field,
(3) a CM field of degree 4,
(4) an indefinite quaternion algebra over $\mathbb{Q}$.

We first claim that in all four cases, $\text{End}(A) = \text{End}(A_K)$ for an odd degree extension $K/k$. This is clear in case 1, and in case 3 this follows from classical CM theory, see e.g. [5, Chapter 3]. For case 2, observe that the absolute Galois group of $k$ acts on $\text{End}(A_{\bar{k}}) \otimes \mathbb{Q}$ by $\mathbb{Q}$-algebra automorphisms. If the endomorphism algebra is a quadratic field, then the action factors through a quotient of $\text{Gal}(\bar{k}/k)$ of index at most 2, which proves the claim. Finally, case 4 is handled by [2, Theorem 1.3].

If $A/\bar{k}$ is isogenous to a product of elliptic curves, then there are more possibilities for the structure of $\text{End}(A)$, which have been classified in [3, Theorem 4.3]. It follows from this classification that if $\text{End}(A) \otimes \mathbb{Q}$ is a division algebra, then it is still either isomorphic to $\mathbb{Q}$ or a quadratic field or a quaternion algebra, and that moreover $\text{End}(A) = \text{End}(A_K)$ for any extension $K/k$ of degree coprime to 6. So the dimension 2 case of Theorem 1.1 follows from Proposition 2.6 when $\text{End}(A) \otimes \mathbb{Q}$ is a field, and from Proposition 2.8 when it is a quaternion algebra, which covers all possible cases.

To deduce Theorem 1.4 from Proposition 2.9, we use a result of Silverberg, which we will rephrase slightly for our purposes: for any fixed $d$, there exists
a bound $b$ depending only on $d$ (specifically, $b = 4(9d)^4d$ is enough), such that for all abelian varieties over number fields $A/k$ of dimension $d$, and all extensions $K/k$ of prime degree greater than $b$, $\End(A) = \End(A_K)$. Theorem [4.3] is an immediate consequence of this result together with Proposition [2.6] because $\End_{\mathbb{Q}[G]}(\Gamma \otimes \mathbb{Q}) \cong \mathbb{Q}(\mu_p)$.

Proposition [2.6] has an application to questions of simplicity of Weil restrictions of scalars. If $A/k$ is a simple abelian variety, and $K/k$ is a finite Galois extension with Galois group $G$, then the Weil restriction of scalars $W_{K/k}(A_K)$ is never simple, since there is a surjective trace map $W_{K/k}(A_K) \to A$. Its kernel is, up to isogeny, precisely the twist $I \otimes Z_A$, where $I$ is the augmentation ideal in $\mathbb{Z}[G]$. The following is therefore an immediate consequence of Proposition [2.6]:

**Corollary 3.1.** Let $A/k$ be an abelian variety with $\End(\overline{A}_k) = \mathbb{Z}$. Let $K/k$ be a finite Galois extension with Galois group $G$. The kernel of the trace map $W_{K/k}(A_K) \to A$ is simple over $k$ if and only if $G$ has prime order.

**Proof.** Cyclic groups of prime order are precisely the finite groups with only two rational irreducible representations, i.e. those for which $I \otimes \mathbb{Q}$ is a simple $\mathbb{Q}[G]$-module. □

If $K/k$ is Galois with dihedral Galois group $G$ of order $D_{2p}$, $p$ an odd prime, then there is a unique intermediate quadratic extension $k' = k(\sqrt{d})/k$, and for any abelian variety $A/k$, $W_{K/k}(A_K) \sim A \times A_d \times X^2$, where $A_d$ is the quadratic twist of $A$ by $k'/k$. The remaining factor $X$ (up to isogeny) is the twist of $A$ by a lattice in the $(p - 1)$-dimensional irreducible rational representation $\rho$ of $G$, which is the sum of all the two-dimensional complex representations of $G$.

**Corollary 3.2.** Let $E/k$ be an elliptic curve over a number field, $K/k, X$ as above. Then $X$ is simple.

**Proof.** The values of each irreducible two-dimensional character of $G$ generate the maximal real subfield $\mathbb{Q}(\mu_p)^+$ of the $p$-th cyclotomic field, and they are all Galois conjugate over $Q$. They will therefore remain conjugate over any imaginary quadratic field, so the conclusion holds even when $E$ has CM. □

We conclude with an amusing example of a “symplectic twist”. Let $E/k$ be an elliptic curve over a number field, let $K/k$ be Galois with Galois group $Q_8$, the quaternion group. There are three intermediate quadratic fields, and correspondingly, the Weil restriction $W_{K/k}(E_K)$ has, up to isogeny, four factors $E, E_1, E_2, E_3$ that are quadratic twists of $E$. Write $W_{K/k}(E_K) \sim E \times E_1 \times E_2 \times E_3 \times H$.

**Corollary 3.3.** Let $K/k, E/k, H$ be defined as above. Then $H$ is simple, unless $E$ has CM by an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ with $d$ equal to the sum of three squares, in which case $H$ is isogenous to a product of two isomorphic simple factors.

**Proof.** The factor $H$ is (up to isogeny) the twist of $E$ by two copies of the standard representation of $Q_8$. The endomorphism algebra of this representation is isomorphic to Hamilton’s quaternions, which is split by precisely
the imaginary quadratic fields \( \mathbb{Q}(\sqrt{-d}) \) for which \( d \) is the sum of three squares.

\[ \square \]

References


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