A NOTE ON GREEN FUNCTORS WITH INFLATION

ALEX BARTEL AND MATTHEW SPENCER

Abstract. This note is motivated by the problem to understand, given a commutative ring F, which G-sets X, Y give rise to isomorphic $F[G]$-representations $F[X] \cong F[Y]$. A typical step in such investigations is an argument that uses induction theorems to give very general sufficient conditions for all such relations to come from proper subquotients of G. In the present paper we axiomatise the situation, and prove such a result in the generality of Mackey functors and Green functors with inflation. Our result includes, as special cases, a result of Deligne on monomial relations, a result of the first author and Tim Dokchitser on Brauer relations in characteristic 0, and a new result on Brauer relations in characteristic $p > 0$. We will need the new result in a forthcoming paper on Brauer relations in positive characteristic.

1. Introduction

The Burnside ring $\mathrm{B}(G)$ of a finite group G is, as a group, the free abelian group on the set of isomorphism classes of transitive G-sets. Any transitive G-set is isomorphic to a set of cosets G/H for some $H \leq G$, so we may write elements of $\mathrm{B}(G)$ as formal $\mathbb{Z}$-linear combinations of symbols $[G/H]$. Let A be a field of characteristic $p \geq 0$. The representation ring $\mathrm{R}_A(G)$ of a finite group G over A is, as a group, the free abelian group on the set of isomorphism classes of finitely generated indecomposable $A[G]$-modules (not to be confused with the Grothendieck group of the category of finitely generated $A[G]$-modules, which is also sometimes denoted by $\mathrm{R}_A(G)$). For every finite group G there is a natural homomorphism $\mathrm{B}(G) \to \mathrm{R}_A(G)$, which sends the isomorphism class represented by a G-set X to the isomorphism class of the $A[G]$-module $A[X]$ with a canonical A-basis given by the elements of X, and with G acting by permutations on this basis. Let $\mathrm{K}_A(G)$ denote the kernel of this homomorphism. It is easy to see that $\mathrm{K}_A(G)$, as a subgroup of $\mathrm{B}(G)$, only depends on the characteristic of A, and we refer to elements of $\mathrm{K}_A(G)$ as *Brauer relations of G in characteristic p.*

It is an old problem, with many applications in number theory and geometry, to understand the structure of $\mathrm{K}_A(G)$ for all finite groups G. See e.g. [1, §1] for a brief overview of the history of the problem and of some of the applications. The most efficient and, from the point of view of number theoretic and geometric applications, the most useful way of giving a complete characterisation of $\mathrm{K}_A(G)$, not just as an abstract group, but with an explicit description of generators, is to view $\mathrm{K}_A(G)$ as a Mackey functor with inflation. We briefly explain informally what this means, and refer to Section 2 for the formal discussion.

Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK

E-mail addresses: a.bartel@warwick.ac.uk, M.J.Spencer@warwick.ac.uk.

2010 *Mathematics Subject Classification.* 19A22, 20B05, 20B10.

If H is a subgroup of a finite group G, then Brauer relations of H can be induced to Brauer relations of G. Moreover, if $\bar{G}$ is a quotient of a finite group G, then Brauer relations of $\bar{G}$ can be lifted to Brauer relations of G. Let $\mathrm{Imprim}_{\mathrm{K}_A}(G)$ be the subgroup of $\mathrm{K}_A(G)$ generated by all relations that are induced from proper subgroups or lifted from proper quotients, and let $\mathrm{Prim}_{\mathrm{K}_A}(G)$ be the quotient $\mathrm{K}_A(G)/\mathrm{Imprim}_{\mathrm{K}_A}(G)$. If one can give, for every finite group G, generators of $\mathrm{Prim}_{\mathrm{K}_A}(G)$, then one obtains a list of Brauer relations with the property that all Brauer relations in all finite groups are $\mathbb{Z}$-linear combinations of inductions and lifts of relations in this list.

In [1] the structure of $\mathrm{Prim}_{\mathrm{K}_A}(G)$ has been completely determined, in the above sense, in the case when A has characteristic 0. The following theorem was a crucial step towards that result. If q is a prime number, then a group is called *q-quasi-elementary* if it has a normal cyclic subgroup of q-power index. A group is called *quasi-elementary* if it is q-quasi-elementary for some prime number q.

Theorem 1.1 ([1], Theorem 4.3)**.** *Let G be a finite group that is not quasi-elementary. Then:*

(a) *if all proper quotients of G are cyclic, then* $\mathrm{Prim}_{\mathrm{K}_\mathbb{Q}}(G) \cong \mathbb{Z}$*;*
(b) *if q is a prime number such that all proper quotients of G are q-quasi-elementary, and at least one of them is not cyclic, then* $\mathrm{Prim}_{\mathrm{K}_\mathbb{Q}}(G) \cong \mathbb{Z}/q\mathbb{Z}$*;*
(c) *if there exists a proper quotient of G that is not quasi-elementary, or if there exist distinct prime numbers q_1 and q_2 and, for $i = 1$ and 2, a proper quotient of G that is non-cyclic q_i-quasi-elementary, then* $\mathrm{Prim}_{\mathrm{K}_\mathbb{Q}}(G)$ *is trivial.*

Moreover, in all cases, $\mathrm{Prim}_{\mathrm{K}_\mathbb{Q}}(G)$ *is generated by any element of* $\mathrm{K}_\mathbb{Q}(G) \subseteq \mathrm{B}(G)$ *of the form* $[G/G] + \sum_{H \lneq G} a_H[G/H]$, $a_H \in \mathbb{Z}$.

Deligne [7] had proven a similar result on relations between monomial representations, see Theorem 5.1 below.

The main motivation for this paper is to understand $\mathrm{Prim}_{\mathrm{K}_A}(G)$ when A has positive characteristic. To that end, we prove the following characteristic p analogue of Theorem 1.1, which will be used in a forthcoming paper to give a characterisation of $\mathrm{Prim}_{\mathbb{F}_p}(G)$. If p and q are prime numbers, then a group is called *p-hypo-elementary* if it has a normal p-subgroup with cyclic quotient, and it is called a *(p,q)-Dress group* if it has a normal p-subgroup with q-quasi-elementary quotient.

Theorem 1.2. *Let G be a finite group that is not a (p,q)-Dress group for any prime number q. Then:*

(a) *if all proper quotients of G are p-hypo-elementary, then* $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}$*;*
(b) *if q is a prime number such that all proper quotients of G are (p,q)-Dress groups, and at least one of them is not p-hypo-elementary, then* $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}/q\mathbb{Z}$*;*
(c) *if there exists a proper quotient of G that is not a (p,q)-Dress group for any prime number q, or if there exist distinct prime numbers q_1 and q_2 and, for $i = 1$ and 2, a proper quotient of G that is a non-p-hypo-elementary (p,q_i)-Dress group, then* $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ *is trivial.*

Moreover, in all cases, $\mathrm{Prim}_{\mathrm{K}_{F_p}}(G)$ *is generated by any element of* $\mathrm{K}_{\mathbb{F}_p}(G) \subseteq \mathrm{B}(G)$ *of the form* $[G/G] + \sum_{H \lneq G} a_H[G/H]$*,* $a_H \in \mathbb{Z}$*.*

To prove part (b) of Theorem 1.2, we prove an induction theorem for (p, q)-Dress groups, which we believe to be of independent interest. It is a characteristic p analogue of the main Theorem of [8].

Theorem 1.3. *Let p and q be prime numbers, let G be a (p, q)-Dress group that is not p-hypo-elementary, and let a be an integer. Then there exists an element in* $\mathrm{K}_{\mathbb{F}_p}(G)$ *of the form* $a[G/G] + \sum_{H \lneq G} a_H[G/H]$*,* $a_H \in \mathbb{Z}$ *if and only if* $q|a$*.*

In fact, we deduce Theorems 1.1 and 1.2, as well as Deligne's theorem on monomial relations, as special cases of a general result on kernels of morphisms between Green functors with inflation. This formalism, which is a mix of axiomatisations that have appeared in the literature many times before, see e.g. [12] and [4], will be introduced in Section 2. In Section 3 we recall the concept of primordial groups for a Mackey functor. Our main theorems on kernels of morphisms of Green functors will be proven in Section 4. Section 5 is devoted to concrete applications, and it is there that we prove Theorems 1.1, 1.2, and 1.3.

Acknowledgements. During parts of this project, the first author was partially supported by a Research Fellowship from the Royal Commission for the Exhibition of 1851, and by an EPSRC First Grant, and the second author is supported by an EPSRC Doctoral Grant. We would like to thank these institutions for their financial support. We would also like to thank an anonymous referee for a careful reading of the paper and for numerous helpful suggestions.

Our rings are always assumed to be associative, with a unit element. Let R be a commutative ring. By an R-algebra we mean a ring A equipped with a map $R \to \mathrm{Z}(A)$, where $\mathrm{Z}(A)$ denotes the centre of A. If $\mathfrak{p}$ is a prime ideal of R, then $R_{\mathfrak{p}}$ denotes the localisation of R at $\mathfrak{p}$. In this paper, R will always denote a domain, and U, H, G, and K will always denote finite groups.

2. Mackey and Green functors with inflation

One can find many variations on the theme of Mackey functors in the literature. The axiomatisation that we need is very similar to those of [12, 4].

Definition 2.1. A *global Mackey functor with inflation* (MFI) over R is a collection $\mathcal{F}$ of the following data.

- For every finite group G, $\mathcal{F}(G)$ is an R-module;
- for every monomorphism $\alpha\colon H \hookrightarrow G$ of finite groups, $\mathcal{F}_*(\alpha)\colon \mathcal{F}(H) \to \mathcal{F}(G)$ is a covariant R-module homomorphism (which we think of as induction);
- for every homomorphism $\epsilon\colon H \to G$ of finite groups, $\mathcal{F}^*(\epsilon)\colon \mathcal{F}(G) \to \mathcal{F}(H)$ is a contravariant R-module homomorphism (which we think of as restriction when ϵ is a monomorphism, and as inflation when ϵ is an epimorphism);

satisfying the following conditions.

(MFI 1) Transitivity of induction: for all group monomorphisms $U \overset{\beta}{\hookrightarrow} H \overset{\alpha}{\hookrightarrow} G$, we have $\mathcal{F}_*(\alpha\beta) = \mathcal{F}_*(\alpha)\mathcal{F}_*(\beta)$.

(MFI 2) Transitivity of restriction/inflation: for all group homomorphisms $U \overset{\beta}{\to} H \overset{\alpha}{\to} G$, we have $\mathcal{F}^*(\alpha\beta) = \mathcal{F}^*(\beta)\mathcal{F}^*(\alpha)$.

(MFI 3) For all inner automorphisms $\alpha\colon G \to G$, we have $\mathcal{F}^*(\alpha) = \mathcal{F}_*(\alpha) = 1$.

(MFI 4) For all automorphisms $\alpha\colon G \to G$, we have $\mathcal{F}_*(\alpha) = \mathcal{F}^*(\alpha^{-1})$.

(MFI 5) The Mackey condition: for all pairs of monomorphisms $\alpha\colon H \hookrightarrow G$ and $\beta\colon K \hookrightarrow G$, we have

$$\mathcal{F}^*(\beta)\mathcal{F}_*(\alpha) = \sum_{g \in \alpha(H)\backslash G/\beta(K)} \mathcal{F}_*(\phi_g)\mathcal{F}^*(\psi_g),$$

where ϕ_g is the composition

$$\phi_g\colon \beta(K)^g \cap \alpha(H) \overset{c_g}{\to} \beta(K) \cap {}^g\alpha(H) \hookrightarrow \beta(K) \overset{\beta^{-1}}{\to} K,$$

c_g denoting conjugation by g, and ψ_g is the composition

$$\psi_g\colon \alpha(H) \cap \beta(K)^g \hookrightarrow \alpha(H) \overset{\alpha^{-1}}{\to} H.$$

(MFI 6) Commutativity of induction and inflation: whenever there is a commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\alpha} & G \\ {\scriptstyle\epsilon}\downarrow & & \downarrow{\scriptstyle\delta} \\ \bar{H} & \xrightarrow{\beta} & \bar{G}, \end{array}$$

where ϵ, δ are epimorphisms, and α, β are monomorphisms, we have $\mathcal{F}^*(\delta)\mathcal{F}_*(\beta) = \mathcal{F}_*(\alpha)\mathcal{F}^*(\epsilon)$.

We will often use the following more intuitive notation: if $\mathcal{F}$ is an MFI, and $\alpha\colon H \hookrightarrow G$ is a monomorphism, we will write $\mathrm{Res}_{G/H}$ for $\mathcal{F}^*(\alpha)$, and $\mathrm{Ind}_{G/H}$ for $\mathcal{F}_*(\alpha)$. The suppressed dependence on α and $\mathcal{F}$ will not cause any confusion. Similarly, if $\epsilon\colon G \to \bar{G}$ is an epimorphism with kernel N, we will write $\mathrm{Inf}_{G/N}$ for $\mathcal{F}^*(\epsilon)$.

Definition 2.2. A *Green functor with inflation* (GFI) over R is an MFI $\mathcal{F}$ over R, satisfying the following additional conditions.

(GFI 1) For every finite group G, $\mathcal{F}(G)$ is an R-algebra.

(GFI 2) For every homomorphism $\alpha\colon H \to G$ of finite groups, $\mathcal{F}^*(\alpha)$ is a homomorphism of R-algebras.

(GFI 3) Frobenius reciprocity: for every monomorphism $\alpha\colon H \hookrightarrow G$, and for all $x \in \mathcal{F}(H)$, $y \in \mathcal{F}(G)$, we have

$$\begin{aligned} \mathrm{Ind}_{G/H}(x) \cdot y &= \mathrm{Ind}_{G/H}(x \cdot \mathrm{Res}_{G/H}(y)), \\ y \cdot \mathrm{Ind}_{G/H}(x) &= \mathrm{Ind}_{G/H}(\mathrm{Res}_{G/H}(y) \cdot x). \end{aligned}$$

Definition 2.3. A *morphism* from an MFI (respectively GFI) $\mathcal{F}$ to an MFI (respectively GFI) $\mathcal{G}$ is a collection r of R-module (respectively R-algebra) homomorphisms $r_G : \mathcal{F}(G) \to \mathcal{G}(G)$ for each finite group G, commuting in the obvious way with $\mathcal{F}_*, \mathcal{F}^*, \mathcal{G}_*, \mathcal{G}^*$.

Definition 2.4. Let $\mathcal{F}$ be a GFI over R. A (left) *module* under $\mathcal{F}$ is an MFI $\mathcal{M}$ over R, satisfying the following conditions.

(MOD 1) For every group G, $\mathcal{M}(G)$ is an R-linear (left) $\mathcal{F}(G)$-module, i.e. there is a map $\mathcal{F}(G) \times \mathcal{M}(G) \to \mathcal{M}(G)$ factoring through $\mathcal{F}(G) \otimes_R \mathcal{M}(G)$.

(MOD 2) For every homomorphism $\epsilon\colon H \to G$ of finite groups, and for all $x \in \mathcal{F}(G)$, $y \in \mathcal{M}(G)$, we have
$$\mathcal{M}^*(\epsilon)(x \cdot y) = \mathcal{F}^*(\epsilon)(x) \cdot \mathcal{M}^*(\epsilon)(y).$$

(MOD 3) For every monomorphism $\alpha\colon H \hookrightarrow G$, and for all $x \in \mathcal{F}(H)$, $y \in \mathcal{M}(G)$, we have
$$\mathcal{F}_*(\alpha)(x) \cdot y = \mathcal{F}_*(\alpha)(x \cdot \mathcal{M}^*(\alpha)(y)).$$

Example 2.5. The following are examples of GFIs over $\mathbb{Z}$.

(a) The Burnside ring functor B: for a finite group G, $\mathrm{B}(G)$ is the free abelian group on isomorphism classes $[X]$ of finite G-sets, modulo the relations $[X \sqcup Y] - [X] - [Y]$ for all G-sets X, Y, and with multiplication defined by $[X] \cdot [Y] = [X \times Y]$. Here, B_* is the usual induction of G-sets, and B^* is inflation/restriction of G-sets.

(b) The representation ring functor R_F over a given field F: for a finite group G, $\mathrm{R}_F(G)$ is the free abelian group on isomorphism classes $[V]$ of finitely generated $F[G]$-modules, modulo the relations $[U \oplus V] - [V] - [U]$, and with multiplication defined by $[U] \cdot [V] = [U \otimes_F V]$, with diagonal G-action on the tensor product. As in the previous example, $(\mathrm{R}_F)_*$ is induction of modules, and $(\mathrm{R}_F)^*$ is inflation/restriction.

(c) The monomial ring functor M: for a finite group G, $\mathrm{M}(G)$ is the free abelian group on conjugacy classes of symbols $[H, \lambda]$, where H runs over subgroups of G, and λ runs over complex 1-dimensional representations of H, and with multiplication defined by
$$[H, \lambda] \cdot [K, \chi] = \sum_{g \in H \backslash G / K} [{}^gH \cap K, \mathrm{Res}_{{}^gH/({}^gH \cap K)}\, {}^g\lambda \cdot \mathrm{Res}_{K/({}^gH \cap K)}\, \chi].$$
If $\alpha : U \hookrightarrow G$ is a monomorphism, $[H, \lambda] \in \mathrm{M}(U)$, and $[K, \chi] \in \mathrm{M}(G)$, then
$$\mathrm{M}_*(\alpha)([H, \lambda]) = [\alpha(H), \lambda \circ \alpha^{-1}],$$
$$\mathrm{M}^*(\alpha)([K, \chi]) = \sum_{g \in \alpha(U) \backslash G / K} [\alpha^{-1}(\alpha(U) \cap {}^gK), \mathrm{Res}_{{}^gK/(\alpha(U) \cap {}^gK)}\, {}^g\chi \circ \alpha].$$

Every GFI is a module under itself, called the (left) *regular module*. We also have the obvious notions of sub-MFIs, sub-GFIs, and submodules.

Definition 2.6. A *left ideal of a GFI* is a sub-MFI that is also a submodule of the left regular module.

Definition 2.7. Let $r : \mathcal{F} \to \mathcal{G}$ be a morphism of MFIs over R. Its *kernel* $\mathcal{K}$ is defined as follows: for every finite group G, we define $\mathcal{K}(G) = \ker(r(G)\colon \mathcal{F}(G) \to \mathcal{G}(G))$; for every homomorphism $\epsilon\colon H \to G$ of groups, we define $\mathcal{K}^*(\epsilon) = \mathcal{F}^*(\epsilon)|_{\mathcal{F}(G)}$; and for every monomorphism $\alpha\colon H \to G$ of groups, we define $\mathcal{K}_*(\alpha) = \mathcal{F}_*(\alpha)|_{\mathcal{F}(H)}$. The *image* of a morphism is defined analogously. Let $\mathcal{F}$ be a sub-MFI (respectively an ideal) of the MFI (respectively GFI) $\mathcal{G}$. The quotient $\mathcal{Q} = \mathcal{G}/\mathcal{F}$ is defined as follows: for every finite group G, we define $\mathcal{Q}(G) = \mathcal{G}(G)/\mathcal{F}(G)$; for every homomorphism $\epsilon\colon H \to G$, we define $\mathcal{Q}^*(\epsilon) = \mathcal{G}^*(\epsilon) \pmod{\mathcal{F}(H)}$; and for every monomorphism $\alpha\colon H \to G$, we define $\mathcal{Q}_*(\alpha) = \mathcal{G}_*(\alpha) \pmod{\mathcal{F}(G)}$.

The proof of the following is routine and will be omitted.

Lemma 2.8. (a) *Let $r\colon \mathcal{F} \to \mathcal{G}$ be a morphism of MFIs over R. Then its kernel is a sub-MFI of $\mathcal{F}$, and its image is a sub-MFI of $\mathcal{G}$.*
(b) *Let $r\colon \mathcal{F} \to \mathcal{G}$ be a morphism of GFIs over R. Then its kernel is an ideal of $\mathcal{F}$, and its image is a sub-GFI of $\mathcal{G}$.*
(c) *Let $\mathcal{F}$ be a sub-MFI of an MFI $\mathcal{G}$. Then the quotient $\mathcal{G}/\mathcal{F}$ is an MFI.*
(d) *Let $\mathcal{F}$ be an ideal of a GFI $\mathcal{G}$. Then $\mathcal{G}/\mathcal{F}$ is a GFI.*

Example 2.9. The following are some motivating examples for this work.
(a) There is a GFI morphism $m'_{\mathbb{C}}\colon \mathrm{M} \to \mathrm{R}_{\mathbb{C}}$, sending, for every finite group G, a symbol $[H, \lambda] \in \mathrm{M}(G)$ to $\mathrm{Ind}_{G/H}\,\lambda \in \mathrm{R}_{\mathbb{C}}(G)$. The kernel of $m'_{\mathbb{C}}$ was investigated by, among many others, Langlands [10], Deligne [7], Snaith [11], Boltje [3], and Boltje–Snaith–Symonds [5].
(b) Let F be a field. There is a GFI morphism $m_F\colon \mathrm{B} \to \mathrm{R}_F$, which maps, for every finite group G, a G-set X to the permutation module $F[X]$ over F. Its kernel K_F is the MFI of *Brauer relations over F*. In [1], an explicit description of generators of this MFI is given in the case when F is a field of characteristic 0. The primary motivation for this note is to give a similarly explicit description when F is a field of positive characteristic.

3. Primordial groups

If S is a commutative R-algebra, and $\mathcal{F}$ an MFI (respectively GFI) over R, then $S \otimes_R \mathcal{F}$, defined in the obvious way, is an MFI (respectively GFI) over S. If $R = \mathbb{Z}$, then we will suppress any mention of R, and will just say "$\mathcal{F}$ is a MFI (respectively GFI)". Throughout the rest of the paper, Q will denote the field of fractions of R. For a prime ideal $\mathfrak{p}$ of R, we will write $\mathcal{F}_{\mathfrak{p}}$ for $R_{\mathfrak{p}} \otimes_R \mathcal{F}$, and $\mathcal{F}_Q$ for $Q \otimes_R \mathcal{F}$.

Notation 3.1. Let $\mathcal{F}$ be an MFI, and let $\mathcal{X}$ be a class of groups closed under isomorphisms. For every finite group G, we define the following R-submodules of $\mathcal{F}(G)$:

$$\begin{aligned}
\mathcal{I}_{\mathcal{F},\mathcal{X}}(G) &= \sum_{H \leq G,\, H \in \mathcal{X}} \mathrm{Ind}_{G/H}\,\mathcal{F}(H),\\
\mathcal{I}_{\mathcal{F}}(G) &= \sum_{H \lneq G} \mathrm{Ind}_{G/H}\,\mathcal{F}(H).
\end{aligned}$$

Definition 3.2. Let $\mathcal{F}$ be an MFI and let G be a finite group. We say that G is *primordial* for $\mathcal{F}$ if either G is trivial, or $\mathcal{F}(G) \neq \mathcal{I}_{\mathcal{F}}(G)$. We denote the class of all primordial groups for $\mathcal{F}$ by $\mathcal{P}(\mathcal{F})$.

Remark 3.3. Let $\mathcal{F}$ be an MFI.
(a) Suppose that $\mathcal{X}$ is a class of finite groups that is closed under isomorphisms and under taking subgroups, with the property that for every finite group G, we have $\mathcal{F}(G) = \mathcal{I}_{\mathcal{F},\mathcal{X}}(G)$. Then it is shown in [13,

Theorem 2.1] that $\mathcal{X}$ contains the closure of $\mathcal{P}(\mathcal{F})$ under taking all subgroups.

(b) Suppose that $\mathcal{F}$ is a GFI. Then it follows from axiom (GFI 3) that G is primordial for $\mathcal{F}$ if and only if $1_{\mathcal{F}(G)} \notin \mathcal{I}_{\mathcal{F}}(G)$. It easily follows from this and from axioms (GFI 2) and (MFI 6) that $\mathcal{P}(\mathcal{F})$ is closed under quotients.

Example 3.4. (a) Every finite group is primordial for the Burnside ring functor B, and also for $\mathrm{B}_{\mathbb{Q}}$. Indeed, no non-zero multiple of the identity element of $\mathrm{B}(G)$ can be contained in the image of induction from proper subgroups. Similarly, every finite group is primordial for the monomial ring functor M, and also for $\mathrm{M}_{\mathbb{Q}}$.

(b) Recall from Example 2.5 (b) the representation ring functor $\mathrm{R}_{\mathbb{C}}$. It follows from Brauer's induction theorem [2, Theorem 5.6.4] that $\mathcal{P}(\mathrm{R}_{\mathbb{C}})$ is contained in the class of elementary groups, i.e. of direct products of finite cyclic groups by p-groups. Moreover, it is a theorem of Green [9] that in fact $\mathcal{P}(\mathrm{R}_{\mathbb{C}})$ consists precisely of the elementary groups.

(c) Recall from Example 2.9 (a) the GFI morphism $m'_{\mathbb{C}}\colon \mathrm{M} \to \mathrm{R}_{\mathbb{C}}$ from the monomial ring functor to the complex representation ring functor. It follows from Brauer's induction theorem that $(m'_{\mathbb{C}})_G$ is surjective for every finite group G, so by the previous example, $\mathcal{P}(\mathrm{Im}\, m'_{\mathbb{C}})$ consists precisely of the elementary groups.

(d) Recall from Example 2.9 (b) the GFI morphism $m_{\mathbb{Q}}\colon \mathrm{B} \to \mathrm{R}_{\mathbb{Q}}$. Let q be a prime number. Solomon's induction theorem implies that $\mathcal{P}(\mathrm{Im}(m_{\mathbb{Q}})_q)$ is contained in the class of q-quasi-elementary groups, i.e. of semidirect products $C \rtimes U$, with C finite cyclic and U a q-group. Moreover, it is a theorem of Dokchitser [8] that if G is q-quasi-elementary, then the trivial character of G is not in the image of induction of trivial characters from proper subgroups, so $\mathcal{P}(\mathrm{Im}(m_{\mathbb{Q}})_q)$ is precisely the class of all q-quasi-elementary groups.

(e) Let $m_{\mathbb{Q}}$ be as above. It follows from Artin's induction theorem [2, Theorem 5.6.1] that $\mathcal{P}(\mathrm{Im}(m_{\mathbb{Q}})_{\mathbb{Q}})$ is the class of finite cyclic groups.

(f) Let p be a prime number, and let $m_{\mathbb{F}_p}\colon \mathrm{B} \to \mathrm{R}_{\mathbb{F}_p}$ be as in Example 2.9 (b). Dress's induction theorem [1, Theorem 9.4] implies that $\mathcal{P}(\mathrm{Im}\, m_{\mathbb{F}_p})$ is contained in the class of all groups that are (p,q)-Dress groups for some prime number q. We will show in Theorem 5.3 that the trivial representation of a (p,q)-Dress group is not in the image of induction of trivial representations from proper subgroups, so in fact, $\mathcal{P}(\mathrm{Im}\, m_{\mathbb{F}_p})$ is precisely the class of all finite groups that are (p,q)-Dress groups for some prime number q.

4. The primitive quotient

In this section, we prove our main theorems on kernels of morphisms of GFIs. The main results of the section are Theorem 4.6, 4.7, and 4.9.

Lemma 4.1. *Let $m\colon \mathcal{F} \to \mathcal{G}$ be a morphism of GFIs over a ring R with kernel $\mathcal{K}$, and let G be a finite group. Then the following are equivalent:*

(i) *the group G is not primordial for* $\mathrm{Im}\, m$*;*

(ii) *for each proper subgroup H of G, there exists an element $x_H \in \mathcal{F}(H)$ such that $x = 1_{\mathcal{F}(G)} + \sum_{H \lneq G} \mathrm{Ind}_{G/H}(x_H) \in \mathcal{K}(G)$.*

Proof. By Remark 3.3 (b), G is not primordial for $\mathrm{Im}\, m$ if and only if

$$m_G(1_{\mathcal{F}(G)}) \in \sum_{H \lneq G} \mathrm{Ind}_{G/H}(m_H(\mathcal{F}(H))) = m_G \sum_{H \lneq G} \mathrm{Ind}_{G/H}(\mathcal{F}(H)).$$

This is equivalent to the existence of elements $x_H \in \mathcal{F}(H)$ for $H \lneq G$ such that $x = 1_{\mathcal{F}(G)} + \sum_{H \lneq G} \mathrm{Ind}_{G/H}(x_H) \in \mathcal{K}(G)$. □

Definition 4.2. Let G be a finite group, let $\mathcal{F}$ be a GFI over R, and let $\mathcal{M}$ be a module under $\mathcal{F}$. Let $D(G)$ be an R-subalgebra of the centre of $\mathcal{F}(G)$. Define the set of *D-imprimitive elements* of $\mathcal{M}(G)$ by

$$\mathrm{Imprim}_{\mathcal{M},D}(G) = D(G) \cdot \left(\sum_{H \lneq G} \mathrm{Ind}_{G/H} \mathcal{M}(H) + \sum_{1 \neq N \lhd G} \mathrm{Inf}_{G/N} \mathcal{M}(G/N) \right).$$

This is an R-submodule of $\mathcal{M}(G)$. Define the *D-primitive quotient* of $\mathcal{M}(G)$ to be the quotient of R-modules

$$\mathrm{Prim}_{\mathcal{M},D}(G) = \mathcal{M}(G) / \mathrm{Imprim}_{\mathcal{M},D}(G).$$

When $D(G)$ is generated by $1_{\mathcal{F}(G)}$ over R, we will drop it from the notation.

Notation 4.3. For the rest of the section, we put ourselves in the following situation. We fix a morphism $m : \mathcal{F} \to \mathcal{G}$ of GFIs over a domain R with the property that $\mathcal{F}(H)$ is R-torsion free for all finite groups H, and we let $\mathcal{K}$ denote its kernel. Recall from Lemma 2.8 that $\mathcal{K}$ is an ideal of $\mathcal{F}$. Further, we fix a finite group G, and an R-subalgebra $D(G)$ of the centre of $\mathcal{F}(G)$. Assume for the rest of the section that the R-module $\mathcal{F}(G)$ is generated by $\mathcal{I}_{\mathcal{F}}(G)$ and $D(G)$.

Lemma 4.4. *Under the hypotheses of Notation 4.3, let $\mathcal{M}$ be any module under $\mathcal{F}$, and let x be any element of $\mathcal{M}(G)$. Then the R-submodule of $\mathcal{M}(G)$ generated by $D(G) \cdot \mathcal{I}_{\mathcal{M}}(G)$ and $D(G) \cdot x$ is an $\mathcal{F}(G)$-submodule.*

Proof. Let Θ be an element of the R-module $D(G) \cdot \mathcal{I}_{\mathcal{M}}(G) + D(G) \cdot x$, and let $\alpha \in \mathcal{F}(G)$. If $\alpha = \mathrm{Ind}_{G/H}\, y$ for some $y \in \mathcal{F}(H)$, where H is a proper subgroup of G, then by property (MOD 3), $\alpha \cdot \Theta = \mathrm{Ind}_{G/H}(y \cdot \mathrm{Res}_{G/H}\, \Theta) \in \mathcal{I}_{\mathcal{M}}(G)$. If, on the other hand, $\alpha \in D(G)$, then $\alpha \cdot \Theta \in D(G) \cdot \mathcal{I}_{\mathcal{M}}(G) + D(G) \cdot x$ by definition. Since $\mathcal{F}(G)$ is assumed to be generated by $\mathcal{I}_{\mathcal{F}}(G)$ and by $D(G)$, it follows that $\alpha \cdot \Theta \in D(G) \cdot \mathcal{I}_{\mathcal{M}}(G) + D(G) \cdot x$ for all $\alpha \in \mathcal{F}(G)$. □

Lemma 4.5. *Under the hypotheses of Notation 4.3, suppose that the equivalent conditions of Lemma 4.1 are satisfied for m and G, and let $x \in \mathcal{K}(G)$ be an element of the form $x = 1_{\mathcal{F}(G)} + \sum_{H \lneq G} \mathrm{Ind}_{G/H}(x_H)$, where $x_H \in \mathcal{F}(H)$. Then*

$$\mathcal{K}(G) = D(G) \cdot \mathcal{I}_{\mathcal{K}}(G) + D(G) \cdot x.$$

Proof. Let $I = D(G) \cdot \mathcal{I}_{\mathcal{K}}(G) + D(G) \cdot x \subseteq \mathcal{K}(G)$. We claim that $\mathcal{K}(G) \subseteq I$. Let $y \in \mathcal{K}(G)$. Lemma 4.4 implies that I is an ideal of $\mathcal{F}(G)$. Since we have $x \in D(G) \cdot x \subseteq I$, it follows that $y \cdot x \in I$. Also,

$$y \cdot x - y = \sum_{H \lneq G} y \cdot \mathrm{Ind}_{G/H}(x_H) = \sum_{H \lneq G} \mathrm{Ind}_{G/H}(\mathrm{Res}_{G/H}(y) \cdot x_H)$$

is in $\mathcal{I}_{\mathcal{K}}(G)$, and therefore in I. It follows that $y = y \cdot x + (y - y \cdot x) \in I$. Thus $\mathcal{K}(G) \subseteq I$, and the proof is complete. □

Theorem 4.6. *Under the hypotheses of Notation 4.3, suppose that there is a non-trivial normal subgroup N of G such that G/N is not primordial for* $\operatorname{Im} m$. *Then* $\operatorname{Prim}_{\mathcal{K},D}(G)$ *is trivial.*

Proof. By Lemma 4.1, applied to the quotient G/N, there exists an element $z = 1_{\mathcal{F}(G/N)} + \sum_{H/N \lneq G/N} \operatorname{Ind}_{(G/N)/(H/N)}(x_H) \in \mathcal{K}(G/N)$. Since N is non-trivial, the inflation $x = \operatorname{Inf}_{G/N} z$ is contained in $\operatorname{Imprim}_{\mathcal{K},D}(G)$. It follows from Lemma 4.5 that $\mathcal{K}(G) = D(G) \cdot \mathcal{I}_{\mathcal{K}}(G) + D(G) \cdot x \subseteq \operatorname{Imprim}_{\mathcal{K},D}(G)$, as claimed. □

Theorem 4.7. *Under the hypotheses of Notation 4.3, suppose that G is non-trivial, and that* $\operatorname{Prim}_{\mathcal{K},D}(G)$ *is non-trivial. Then G is an extension of the form $1 \to S^d \to G \to H \to 1$, where S is a finite simple group, and H is primordial for* $\operatorname{Im} m$.

Proof. By the existence of chief series, there exists a normal subgroup of G that is isomorphic to S^d, where S is a finite simple group, and $d \geq 1$ is an integer. By Theorem 4.6, the quotient G/S^d is primordial for $\operatorname{Im} m$. □

Assumption 4.8. In addition to the assumptions of Notation 4.3, we now assume that:

- the ring R is a Euclidean domain;
- for every normal subgroup N of G, the inflation map $\operatorname{Inf}_{G/N} \colon \mathcal{F}(G/N) \to \mathcal{F}(G)$ is injective;
- for every quotient G/N, the R-module $\mathcal{F}(G/N)$ is generated by $\mathcal{I}_{\mathcal{F}}(G/N)$ and 1. In particular, the subalgebra $D(G)$ will be assumed to be generated by $1_{\mathcal{F}(G)}$ over R, and will now be dropped from the notation.

Theorem 4.9. *Under the hypotheses of Notation 4.3 and Assumption 4.8, suppose that G is primordial for $\mathcal{F}_Q$ and not primordial for* $\operatorname{Im} m$. *Let $\mathfrak{a}$ be the ideal of R generated by all those $a \in R$ for which there exists a proper quotient G/N and an element $a1_{\mathcal{F}(G/N)} + y \in \mathcal{K}(G/N)$ with $y \in \mathcal{I}_{\mathcal{F}}(G/N)$. Then* $\operatorname{Prim}_{\mathcal{K}}(G)$ *is isomorphic to $R/\mathfrak{a}$ and is generated by the image of any element of the form $x = 1_{\mathcal{F}(G)} + \sum_{H \lneq G} \operatorname{Ind}_{G/H} x_H \in \mathcal{K}(G)$.*

Proof. By Lemma 4.5, the quotient $\operatorname{Prim}_{\mathcal{K}}(G)$ is generated by any $x \in \mathcal{K}(G)$ of the form $x = 1_{\mathcal{F}(G)} + \sum_{H \lneq G} \operatorname{Ind}_{G/H} x_H$, where $x_H \in \mathcal{F}(H)$. Since by assumption G is primordial for $\mathcal{F}_Q$, Remark 3.3 (b) implies that $ax \notin \mathcal{I}_{\mathcal{K}}(G)$ for any non-zero $a \in R$. It also follows from the same remark and from the assumptions 4.3 and 4.8 that any element of $\mathcal{K}(G)$ can be uniquely written as $a1_{\mathcal{F}(G)} + y$, where $a \in R$ and $y \in \mathcal{I}_{\mathcal{F}}(G)$, and analogously for any element of $\mathcal{K}(G/N)$ for every normal subgroup N of G. We deduce that the annihilator $\mathfrak{a} \subseteq R$ of $x + \operatorname{Imprim}_{\mathcal{K}}(G) \in \operatorname{Prim}_{\mathcal{K}}(G)$ is generated, as an R-module, by all those $a \in R$ for which there exists a non-trivial normal subgroup N of G and an element $a1_{\mathcal{F}(G/N)} + y \in \mathcal{K}(G/N)$, where $y \in \mathcal{I}_{\mathcal{F}}(G/N)$. Moreover, we then have $\operatorname{Prim}_{\mathcal{K}}(G) \cong R/\mathfrak{a}$, as claimed. □

Corollary 4.10. *Under the hypotheses of Theorem 4.9, if all proper quotients of G are primordial for $(\operatorname{Im} m)_Q$, then* $\operatorname{Prim}_{\mathcal{K}}(G)$ *is isomorphic to R.*

Proof. Since all proper quotients G/N are primordial for $(\operatorname{Im} m)_Q$, Remark 3.3 (b) implies that the ideal $\mathfrak{a}$ of Theorem 4.9 is zero. □

Corollary 4.11. *Under the hypotheses of Theorem 4.9, suppose that there exists a prime ideal $\mathfrak{p}$ of R such that for every prime ideal $\mathfrak{q} \neq \mathfrak{p}$ there exists a proper quotient of G that is not primordial for $(\operatorname{Im} m)_{\mathfrak{q}}$. Then $\operatorname{Prim}_{\mathcal{K}}(G) \cong R/\mathfrak{p}^n$, where n is the smallest non-negative integer for which there exists a proper quotient G/N and an element $a1_{\mathcal{F}(G/N)} + y \in \mathcal{K}(G/N)$ with $a \in \mathfrak{p}^n \setminus \{0\}$ and $\operatorname{Inf}_{G/N} y \in \mathcal{I}_{\mathcal{F}}(G)$.*

Proof. Let $\mathfrak{q} \neq \mathfrak{p}$ be a prime ideal of R. By Lemma 4.1, applied to the map $\mathcal{F}_{\mathfrak{q}} \to \mathcal{G}_{\mathfrak{q}}$ and to a proper quotient $G/N \notin \mathcal{P}((\operatorname{Im} m)_{\mathfrak{q}})$, there exists $a \in \mathfrak{a}$ that is not in $\mathfrak{q}$, where $\mathfrak{a}$ is the ideal of Theorem 4.9. Since R is a Euclidean domain, this implies that $\mathfrak{a} = \mathfrak{p}^n$ for some integer $n \geq 0$. □

Corollary 4.12. *Under the hypotheses of Theorem 4.9, suppose that for every non-zero prime ideal $\mathfrak{p}$ of R there exists a proper quotient of G that is not primordial for $(\operatorname{Im} m)_{\mathfrak{p}}$. Then $\operatorname{Prim}_{\mathcal{K}}(G)$ is trivial.*

Proof. Let $\mathfrak{p}$ be any non-zero prime ideal. By Lemma 4.1, applied to the map $\mathcal{F}_{\mathfrak{p}} \to \mathcal{G}_{\mathfrak{p}}$ and to a proper quotient $G/N \notin \mathcal{P}((\operatorname{Im} m)_{\mathfrak{p}})$, there exists $a \in \mathfrak{a}$ that is not in $\mathfrak{p}$, where $\mathfrak{a}$ is the ideal of Theorem 4.9. Since R is a Euclidean domain, it follows that $1 \in \mathfrak{a}$. □

5. Applications

In this section we explicate the results of Section 4 in the case of monomial relations and of Brauer relations. The main new results are on Brauer relations in positive characteristic, but we also show how to derive some known results on monomial relations and on Brauer relations in characteristic 0 from the formalism of GFIs. In particular, we prove Theorems 1.1, 1.2, and 1.3 from the introduction. The following result, although not explicitly stated, is proved in [7] along the way to a complete classification of monomial relations in soluble groups.

Theorem 5.1 (Deligne–Langlands, [7]). *Let $\mathrm{K}'_{\mathbb{C}}$ be the kernel of the morphism of GFIs $m'_{\mathbb{C}}\colon \mathrm{M} \to \mathrm{R}_{\mathbb{C}}$ as in Example 2.9* (a)*. Let G be a finite group that has a non-trivial normal subgroup N such that G/N is not elementary. Let $D(G)$ be generated over $\mathbb{Z}$ by symbols $[G, \lambda]$, as λ runs over isomorphism classes of 1-dimensional representations of G. Then $\operatorname{Prim}_{\mathrm{K}'_{\mathbb{C}},D}(G)$ is trivial.*

Proof. By Example 3.4 (c), the primordial groups for $\operatorname{Im} m'_{\mathbb{C}}$ are precisely the elementary groups, and every group is primordial for $\mathrm{M}_{\mathbb{Q}}$ (see Example 3.4 (a)). It easily follows that the assumptions of Notation 4.3 are satisfied for this morphism of GFIs and this choice of $D(G)$. The result therefore follows from Theorem 4.6. □

Theorem 5.2 (Bartel–Dokchitser, [1]). *Let $\mathrm{K}_{\mathbb{Q}}$ be the kernel of the morphism of GFIs $m_{\mathbb{Q}}\colon \mathrm{B} \to \mathrm{R}_{\mathbb{Q}}$ as in Example 2.9* (b)*, and let G be a finite group that is not quasi-elementary. Then:*

(a) *if all proper quotients of G are cyclic, then $\operatorname{Prim}_{\mathrm{K}_{\mathbb{Q}}}(G) \cong \mathbb{Z}$;*

(b) *if q is a prime number such that all proper quotients of G are q-quasi-elementary, and at least one of them is not cyclic, then* $\mathrm{Prim}_{\mathrm{K}_{\mathbb{Q}}}(G) \cong \mathbb{Z}/q\mathbb{Z}$*;*

(c) *if there exists a proper quotient of G that is not quasi-elementary, or if there exist distinct prime numbers q_1 and q_2 and, for $i = 1$ and 2, a proper quotient of G that is non-cyclic q_i-quasi-elementary, then* $\mathrm{Prim}_{\mathrm{K}_{\mathbb{Q}}}(G)$ *is trivial.*

Moreover, in all cases, $\mathrm{Prim}_{\mathrm{K}_{\mathbb{Q}}}(G)$ *is generated by any element of* $\mathrm{K}_{\mathbb{Q}}(G) \subseteq \mathrm{B}(G)$ *of the form* $[G/G] + \sum_{H \lneq G} a_H[G/H]$, $a_H \in \mathbb{Z}$.

Proof. By Example 3.4 (e), $\mathcal{P}((\mathrm{Im}\, m_{\mathbb{Q}})_{\mathbb{Q}})$ is the class of cyclic groups. Let q be a prime number. By Example 3.4 (d), $\mathcal{P}(\mathrm{Im}\, m_{\mathbb{Q}})$ is the class of quasi-elementary groups, and $\mathcal{P}((\mathrm{Im}\, m_{\mathbb{Q}})_q)$ is the class of q-quasi-elementary groups. Moreover, if U is a non-cyclic q-quasi-elementary group, then by [8], there exists an element of $\mathrm{K}_{\mathbb{Q}}(U) \subseteq \mathrm{B}(U)$ of the form $q[U/U]+\sum_{H \lneq U} a_H[U/H]$. Since every finite group is primordial for B, the hypotheses of Theorem 4.9 are satisfied. Part (a) of the theorem therefore follows from Corollary 4.10. Finally, note that if q_1 and q_2 are distinct prime numbers, then a finite group is both q_1-quasi-elementary and q_2-quasi-elementary if and only if it is cyclic. Parts (b) and (c) of the theorem therefore follow from Corollaries 4.11 and 4.12, respectively. □

Fix a prime number p. The rest of the section is devoted to the kernel $\mathrm{K}_{\mathbb{F}_p}$ of the morphism of GFIs $m_{\mathbb{F}_p} \colon \mathrm{B} \to \mathrm{R}_{\mathbb{F}_p}$ as in Example 2.9 (b).

First, we prove Theorem 1.3, which is a characteristic p analogue of the main result of [8]. We recall the statement.

Theorem 5.3. *Let q be a prime number, let G be a (p, q)-Dress group that is not p-hypo-elementary, and let a be an integer. Then $a[G/G] \in \mathcal{I}_{\mathrm{Im}\, m_{\mathbb{F}_p}}(G)$ if and only if $q|a$.*

Proof. Since G is a (p, q)-Dress group, it is an extension of a q-group U by a normal p-hypo-elementary subgroup $N = P \rtimes C$, where P is a p-group and C is cyclic of order coprime to pq.

First we prove that if $a[G/G] \in \mathcal{I}_{\mathrm{Im}\, m_{\mathbb{F}_p}}(G)$, then $q|a$. Suppose that there exist integers a_H for $H \lneq G$ such that

$$a\mathbb{F}_p[G/G] = \sum_{H \lneq G} a_H \mathbb{F}_p[G/H] \in \mathrm{R}_{\mathbb{F}_p}(G),$$

where the sum runs over representatives of conjugacy classes of subgroups of G, and where $\mathbb{F}_p[G/H] \in \mathrm{R}_{\mathbb{F}_p}(G)$ denotes the linear permutation module $\mathrm{Ind}_{G/H}\, \mathbf{1}_H$ over $\mathbb{F}_p$. By restricting to the normal p-hypo-elementary subgroup N, we find that

$$a\mathbb{F}_p[N/N] = \sum_{H \lneq G} a_H \sum_{g \in G/HN} \mathbb{F}_p[N/N \cap gHg^{-1}]. \tag{5.4}$$

By Conlon's Induction Theorem [6, Lemma 81.2], p-hypo-elementary groups are primordial for $\mathrm{Im}\, m_{\mathbb{F}_p}$, so the coefficient of $\mathbb{F}_p[N/N]$ on the right hand side of equation 5.4 must be equal to a:

$$a = \sum_{N \leq H \lneq G} a_H \cdot \#(G/H).$$

But for every $H \lneq G$ that contains N, the quantity $\#(G/H)$ is divisible by q, so a is divisible by q, as claimed.

Now we show that $q[G/G] \in \mathcal{I}_{\operatorname{Im} m_{\mathbb{F}_p}}(G)$. First, we treat a special case: assume that P is the trivial group, so that $G \cong C \rtimes U$ is non-cyclic q-quasi-elementary, where C is cyclic of order coprime to pq. Assume further that either $p \neq q$, or U acts faithfully on C. By [8], there exists an element $x = q[G/G] + \sum_{H \lneq G} a_H[G/H] \in \mathrm{K}_{\mathbb{Q}}(G)$. By Artin's Induction Theorem [2, Theorem 5.6.1], this is equivalent to the statement that there exists an $x \in \mathrm{K}_{\mathbb{Q}}(G)$ as above such that for all cyclic subgroups $H \leq G$, we have $f_H(x) = 0$, where $f_H \colon \mathrm{B}(G) \to \mathbb{Z}$ is defined on a G-set X as the number of fixed points $\#X^H$. But under the hypotheses on G, the cyclic subgroups of G are precisely the p-hypo-elementary subgroups of G. By Conlon's Induction Theorem [6, Lemma 81.2], the above statements are therefore equivalent to the existence of an element $x = q[G/G] + \sum_{H \lneq G} a_H[G/H] \in \mathrm{K}_{\mathbb{F}_p}(G)$, as required.

Now, we deduce the general case. Given a non-p-hypo-elementary (p,q)-Dress group G, let $\tilde{G} = G/P$. This is a non-cyclic q-quasi-elementary group, $\tilde{G} = C \rtimes U$, where U is a q-group, and C is cyclic of order coprime to pq. Let K be the kernel of the action of U on C. If $K = U$ and $p = q$, then $\tilde{G} \cong C \times U$, and G is p-hypo-elementary, contradicting the assumptions. Otherwise, $\bar{G} = \tilde{G}/K$ is as in the special case above, so there exists an element $x = q[\bar{G}/\bar{G}] + \sum_{H \lneq \bar{G}} a_H[\bar{G}/H] \in \mathrm{K}_{\mathbb{F}_p}(\bar{G})$. Taking the inflation of x to G yields the desired element of $\mathrm{K}_{\mathbb{F}_p}(G)$, and the proof is complete. □

Corollary 5.5. *Let q be a prime number. Then $\mathcal{P}((\operatorname{Im} m_{\mathbb{F}_p})_q)$ is the class of (p,q)-Dress groups.*

Proof. By Dress's Induction Theorem in the version as stated in [1, Theorem 9.4], and by Remark 3.3 (a), all primordial groups for $(\operatorname{Im} m_{\mathbb{F}_p})_q$ are (p,q)-Dress groups. The reverse inclusion follows from Theorem 5.3. □

Theorem 5.6. *Let G be a finite group that is not a (p,q)-Dress group for any prime number q. Then:*

(a) *if all proper quotients of G are p-hypo-elementary, then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}$;*
(b) *if q is a prime number such that all proper quotients of G are (p,q)-Dress groups, and at least one of them is not p-hypo-elementary, then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}/q\mathbb{Z}$;*
(c) *if there exists a proper quotient of G that is not a (p,q)-Dress group for any prime number q, or if there exist distinct prime numbers q_1 and q_2 and, for $i = 1$ and 2, a proper quotient of G that is a non-p-hypo-elementary (p,q_i)-Dress group, then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ is trivial.*

Moreover, in all cases, $\mathrm{Prim}_{\mathrm{K}_{F_p}}(G)$ is generated by any element of $\mathrm{K}_{\mathbb{F}_p}(G) \subseteq \mathrm{B}(G)$ of the form $[G/G] + \sum_{H \lneq G} a_H[G/H]$, $a_H \in \mathbb{Z}$.

Proof. By Conlon's Induction Theorem [6, Lemma 81.2], $\mathcal{P}((\operatorname{Im} m_{\mathbb{F}_p})_{\mathbb{Q}})$ is the class of p-hypo-elementary groups. Let q be a prime number. By Corollary 5.5, $\mathcal{P}((\operatorname{Im} m_{\mathbb{F}_p})_q)$ is the class of (p,q)-Dress groups, and $\mathcal{P}(\operatorname{Im} m_{\mathbb{F}_p})$ is the class of all groups that are (p,q')-Dress groups for some prime number q'. Moreover, if U is a non-p-hypo-elementary (p,q)-Dress group, then by Theorem 5.3, there exists an element of $\mathrm{K}_{\mathbb{F}_p}(U) \subseteq \mathrm{B}(U)$ of the form $q[U/U] + \sum_{H \lneq U} a_H[U/H]$. Part (a) of the theorem follows from Corollary 4.10. Finally, note that if q_1 and q_2 are distinct prime numbers, then a finite group is both a (p, q_1)-Dress group and a (p, q_2)-Dress group if and only if it is p-hypo-elementary. Parts (b) and (c) of the theorem therefore follow from Corollaries 4.11 and 4.12, respectively. □

References

[1] A. Bartel and T. Dokchitser, Brauer relations in finite groups, J. European Math. Soc. 17 (2015), 2473–2512.
[2] D. J. Benson, Representations and Cohomology I, Cambridge University Press, 1991.
[3] R. Boltje, A canonical Brauer induction formula, Astérisque 181–182 (1990), 31–59.
[4] R. Boltje, A general theory of canonical induction formulae, J. Algebra 206 (1998), 293–343.
[5] R. Boltje, V. Snaith, and P. Symonds, Algebraicisation of explicit Brauer induction, J. Algebra 148 (1992), 504–527.
[6] C. W. Curtis and I. Reiner, Methods of Representation Theory, Volume II, John Wiley and Sons, New York, 1994.
[7] P. Deligne. Les constantes des équations fonctionnelles des fonctions L, In Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math. 349, Springer Verlag, 1973, 501–597.
[8] T. Dokchitser, Solomon's induction in quasi-elementary groups, J. Group Theory 14 (2011), 49–51.
[9] J. A. Green, On the converse of a theorem of R. Brauer. Proc. Cambridge Philos. Soc. 51 (1955), 237–239.
[10] R. P. Langlands, On the functional equation of the Artin l-functions, Yale University preprint, 1970.
[11] V. Snaith, Explicit Brauer induction, Invent. Math. 94 (1988), 455–478.
[12] P. Webb, Two classifications of simple Mackey functors with applications to group cohomology and the decomposition of classifying spaces, J. Pure and Applied Algebra 88 (1993), 265–304.
[13] J. Thevenaz, Some remarks on G-functors and the Brauer morphism, J. reine angew. Math. 384 (1988), 24–56.