GRASSMANN PHASE SPACE METHODS FOR FERMIONS. II. FIELD THEORY

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0.1 Abstract

In both quantum optics and cold atom physics, the behaviour of bosonic photons and atoms is often treated using phase space methods, where mode annihilation and creation operators are represented by c-number phase space variables, with the density operator equivalent to a distribution function of these variables. The anti-commutation rules for fermion annihilation, creation operators suggests the possibility of using anti-commuting Grassmann variables to represent these operators. However, in spite of the seminal work by Cahill and Glauber and a few applications, the use of Grassmann phase space methods in quantum - atom optics to treat fermionic systems is rather rare, though fermion coherent states using Grassmann variables are widely used in particle physics.

This paper presents a phase space theory for fermion systems based on distribution functionals, which replace the density operator and involve Grassmann fields representing anti-commuting fermion field annihilation, creation operators. It is an extension of a previous phase space theory paper for fermions (Paper I) based on separate modes, in which the density operator is replaced by a distribution function depending on Grassmann phase space variables which represent the mode annihilation and creation operators. This further development of the theory is important for the situation when large numbers of fermions are involved, resulting in too many modes to treat separately. Here Grassmann fields, distribution functionals, functional Fokker-Planck equations and Ito stochastic field equations are involved. Typical applications to a trapped Fermi gas of interacting spin 1/2 fermionic atoms and to multi-component Fermi gases with non-zero range interactions are presented, showing that the Ito stochastic field equations are local in these cases. For the spin 1/2 case we also show how simple solutions can be obtained both for the untrapped case and for an optical lattice trapping potential.

1 Introduction

This paper treats many-body fermion systems via a phase space theory. It extends a previous paper (Paper I, [1]) on this topic - in which the modes (or single particle states) that the fermions may occupy were treated separately, to the situation where fermion field creation and annihilation operators are involved rather than operators for the separate modes. This approach is more suitable when large numbers of fermions are involved since the Pauli exclusion principle guarantees that large numbers of modes are needed to accommodate them, resulting in too many modes to treat separately. In the previous paper we have presented phase space theory as being one of a range of methods (see Ref. [1]) used to treat many-body systems [2] in non-relativistic quantum physics. Briefly, for phase space theories based on separate modes the density operator describing the quantum state is represented by a distribution function of phase space variables, which are associated with the mode annihilation and creation operators. Measureable quantities such as quantum correlation functions and Fock state probabilities and coherences are expressed as phase space integrals involving the distribution function and specific functions of the phase space variables describing the measureable quantity. The evolution equation for the density operator is replaced by a Fokker-Planck equation [3] for the distribution function. In turn, the phase space variables are replaced by time dependent stochastic variables, and Ito stochastic equations [4] for these variables are used to give the same results for measured quantities as would be obtained by solving the Fokker-Planck equation. Phase space theory for bosons is now a standard approach, described in many text-books (see for example [5], [6]). For bosonic systems where the mode annihilation and creation operators satisfy commutation relations, the phase space variables are c-numbers. However, in fermionic systems these operators satisfy anti-commutation rules, and one option for developing a phase space theory for fermions is for the phase space variables to be Grassmann variables - since these anti-commute in multiplication. For fermions c-number phase space theories also exist, but these typically involve linking pairs of mode operators with the phase space variables (see Ref. [1] for details).

Phase space theory for fermions based on Grassmann phase space variables was pioneered by Cahill and Glauber [7], although this paper did not treat dynamical evolution problems – either time evolution under a Liouville-von Neumann or master equation or temperature evolution under a Matsubara equation. Evolution was treated in an important later contribution by Plimak, Collett and Olsen [8]. Applications of Grassman phase space theory to specific problems (such as in Refs. [9], [10], [11], [12], [13], [14]) have also been made. The theory presented in Ref. [1] is similar to that in Ref. [8] in using an un-normalised B distribution function, for which the drift term in the Fokker-Planck equation depends linearly on the phase space variables. However, the Ito stochastic equations for stochastic Grassmann variables obtained in Ref. [1] were based on a different approach (developed from a treatment by Gardiner (see pp 95-96 in [4]) for bosons) in which the phase space average at any time of an arbitary function and stochastic average of same function are required to coincide, and

for this to occur the terms in the Ito stochastic equations were related to the drift and diffusion terms in the Fokker-Planck equation for distribution function. The stochastic equations obtained in Ref. [8] are different, and are based on an ansatz requiring the distribution function itself at a final (imaginary) time to be obtained by taking the distribution function at the initial time, replacing the phase variables by time dependent stochastic Grassmann variables related via a stochastic c-number linear transformation to the original (time independent) Grassmann phase variables, and then taking a stochastic average of the stochastic distribution function multiplied by the determinent associated with the stochastic c-number linear transformation. The differences between the approaches in [8] and [1] are fully discussed in the latter paper, along with the important issue of showing how numerical calculations of Fock state populations etc can still be carried out based on Grassmann phase space theory - without there being a need to represent Grassmann variables on the computer.

In phase space theories based on fields rather than separate modes, the density operator describing the quantum state is represented by a distribution functional (functionals and their calculus are described in many textbooks, for example Ref. [15]) involving phase space field functions that are associated with the field annihilation and creation operators. This extension is important even for bosonic cases where although most particles may occupy one or two condensate modes, there are sometimes large numbers of modes that have non-zero occupancy, such as for spatial squeezing in quantum optics [16]. Measureable quantities such as quantum correlation functions and Fock state probabilities and coherences are expressed as phase space functional integrals involving the distribution functional and other functions of the phase space fields specific to the measureable quantity. The evolution equation for the density operator is replaced by a functional Fokker-Planck equation for the distribution functional. In turn, the phase space fields are replaced by time dependent stochastic fields, and Ito stochastic field equations for these stochastic fields are used to give the same results for measured quantities as would be obtained by solving the functional Fokker-Planck equation. Such phase space distribution functional theories have been widely used in the bosonic case - such as for photons [16], where the field functions are c-number fields. The paper by Steel et al [17] sets out the theory for bosonic gases, and there have been many further developments such as including gauge fields [18], hybrid theories [19], [20], [21], [22] involving P and W distributions and double phase spaces, the last allowing for time dependent mode functions. For fermionic systems c-number field based phase space theories have also been formulated, involving c-number field functions either associated with atomic spin field operators [23] or pairs of fermion field operators.

However, in developing a field based phase space theory for fermionic systems, the present paper involves associating the fermion field annihilation and creation operators directly with field functions - for analogous reasons to the separate mode case. The theory involves two independent field functions ψ and ψ^+ which are associated with the field annihilation and creation operators respectively. Again, the anti-commutation rules satisfied by the fermion field op-

erators suggest using field functions with similar anti-commutation properties, and hence the present theory employs Grassmann fields rather than c-number fields. Grassmann fields can be expanded in terms of c-number mode functions, but now the expansion coefficients are Grassmann variables. It is important to note however, that the field and mode approaches are interchangeable, as can be shown via via mode expansions for the Grassmann fields. A Grassmann field approach was also adopted for fermion problems by Plimak, Collett and Olsen [8].

After associating the field annihilation and creation operators with Grassmann field functions, we introduce the Grassmann distribution functionals that replace the density operator, starting from the distribution functions based on modal Grassmann phase space variables. The distribution function for modal phase space variables is equivalent to the distribution functional for phase space fields. Both P and B distribution functionals are considered. We then derive the expressions for measurable quantities such as quantum correlation functions, Fock state probabilities and coherences as Grassmann phase space functional integrals involving the distribution functional and field functions specific to the measureable quantity. A careful derivation is given of the functional Fokker-Planck equation for the distribution functional starting from the Fokker-Planck equation, with expressions for the drift and diffusion terms in the functional Fokker-Planck equations related to the corresponding terms in the original Fokker-Planck equation. The correspondence rules for the field annihilation and creation operators are also established via mode expansions. However, they then can be used to derive functional Fokker-Planck equations directly from the evolution equation for the density operator when the Hamiltonian is expressed in terms of field operators. After defining the time dependent stochastic Grassmann fields that replace the field functions, we then derive the Ito stochastic field equations, giving explicit expressions for the classical and noise fields in terms of the related quantities in the Ito stochastic equations based on modes. An important result establishes the general relationship between the classical and noise field terms in the Ito stochastic field equations and the drift and diffusion terms in the functional Fokker-Planck equation. This relationship is then used to determine the classical and noise terms in the Ito stochastic field equations directly from the functional Fokker-Planck equation without requiring mode expansions. For the B distribution case - also occuring in Plimak et al [8] - we show that the stochastic Grassmann fields at later times are related linearly to these fields at an earlier time, and where the linear relationships only involve (stochastic) c-numbers and not Grassmann fields. This allows numerical calculations based on stochastic Grassmann fields to be carried out. Important issues involved in carrying out numerical calculations for Grassmann fields need to be considered - for example the position dependence of the stochastic Grassmann fields must be represented as points on a spatial grid, and in some cases non-local Ito stochastic field equations must be solved.

Typical applications, first to a trapped Fermi gas of spin $1/2$ fermionic atoms with zero range interactions, and second to multi-component Fermi gases with non-zero range interactions are presented, showing that the Ito stochastic field

equations are local in these cases. For the spin 1/2 case we also show how simple solutions can be obtained both for the untrapped case and for an optical lattice trapping potential.

The plan of this paper is as follows. In Section 2 functional phase space theory for fermions is presented based on Grassmann fields. Section 3 covers functional Fokker-Planck equations and Section 4 sets out the Ito Stochastic field equations, with a SubSection dealing with numerical issues. The final Section 5 presents applications, the first SubSection treating a trapped Fermi gas of spin 1/2 fermionic atoms with zero range interactions and the second SubSection focuses on multi-component Fermi gases with non-zero range interactions. The Appendices set out detailed material too lengthy for the main body of the paper. In Appendix 8 we introduce the field annihilation and creation operators and a typical Hamiltonian for interacting fermions expressed in terms of field operators. Fock states for fermion positions are described. Appendix 9 describes fermion position states, Appendix 10 sets out Grassmann functional calculus, Appendix 11 describes the functional P distribution. Derivations of the Ito stochastic field equations and the functional Fokker-Planck equation for the application to spin 1/2 fermionic atoms are set out in Appendices 12 and 13. The Appendices for the previous paper (Ref. [1]) also contain useful background material - including a description of Grassmann algebra and calculus (Appendix A).

A far more detailed account of Grassmann phase space theory for fermions, covering both the separate mode and the field cases and the relationship to c-number phase space theory for bosons is set out in the recently published book by the present authors entitled *Phase Space Methods for Degenerate Quantum Gases* [15]. Readers are refered to this textbook for a more complete treatment. Here we focus on presenting the most essential results.

2 Functional Phase Space Theory

In systems with a large number of fermions the Pauli exclusion principle implies there must also be a large number of mode involved. In this case the distribution function treatment based on treating mode separately becomes unwieldy and a switch to a treatment avoiding a consideration of separate modes is highly desirable. Field annihilation and creation operators now replace the separate mode operators. In *functional phase space theory* the density operator is now represented by a *distribution functional* rather than a distribution function, and the field operators are represented by *Grassmann fields*, rather than mode operators being represented by Grassmann phase space variables. It should be noted though that the distribution functional is entirely *equivalent* to the distribution function and the two approaches are inter-changeable. If the Grassmann fields are expanded in terms of spatial mode functions with the Grassmann phase space variables as coefficients, then the distribution functional is equivalent to a Grassmann function of the phase space variables. We simply *choose* the distribution functional to be such that the equivalent distribution function is the *same* as the previous distribution function. As previously, distribution functionals of the P type and the un-normalised B type both occur, though here we focus on the latter because of its greater usefulness in numerical work. When using functional phase space theory *functional Fokker-Planck* equations now replace Fokker-Planck equations and can be *derived* either from the Fokker-Planck equations or more simply by establishing *correspondence rules* involving the Grassmann fields and functional derivatives that are equivalent to the original correspondence rules that involved Grassmann phase space variables and Grassmann derivatives. Similarly *Ito stochastic field* equations for *Grassman stochastic fields* replace Ito stochastic equations for stochastic Grassmann phase variables and can be *derived* from these. The Ito stochastic field equations can also be obtained *more directly* from the drift and diffusion terms in the functional Fokker-Planck equation As functional calculus involving Grassmann fields is not widely known, for convenience its key features are set out in Appendix 10.

2.1 Grassmann Fields

In the case of bosons, the field operators $\hat{\Psi}(r)$,$\hat{\Psi}^{\dagger}(r)$ are associated with *c-number fields* $\psi(r), \psi^{+}(r)$. Here r refers to spatial position - which may be in 1D, 2D or 3D. Such fields may be expanded in terms of orthonormal mode functions $\phi_i(r)$ in the form

$$\psi(r) = \sum_i \alpha_i \, \phi_i(r) \tag{1}$$

$$\psi^{+}(r) = \sum_i \phi_i^{*}(r) \alpha_i^{+}. \tag{2}$$

where α_i, α_i^{+} are c-number phase space variables. The mode functions satisfy orthonormality and completeness conditions analogous to (94), (95)

For fermions, the field operators $\hat{\Psi}(r)$,$\hat{\Psi}^\dagger(r)$ are associated with *Grassmann fields* $\psi(r), \psi^+(r)$. Grassmann fields can also be expanded in terms of a suitable orthonormal set of mode functions but now with Grassmann phase space variables g_k or g_k^+ as expansion coefficients. Thus we have for Grassmann functions $\psi(r), \psi^+(r)$ associated with field annihilation and creation operators

$$\psi(r) = \sum_i g_i \, \phi_i(r) \tag{3}$$

$$\psi^+(r) = \sum_i \phi_i^*(r) g_i^+. \tag{4}$$

Again, if $g_i^+ = g_i^*$ then $\psi^+(r) = \psi^*(r)$, the complex conjugate Grassmann field.

The Grassmann fields are odd *Grassmann functions* of the first order. The following results for the expansion coefficients can easily be obtained

$$g_k = \int dr \, \phi_k^*(r) \psi(r) \tag{5}$$

$$g_k^+ = \int dr \, \phi_k(r) \psi^+(r) \tag{6}$$

which has the same form as for c-number fields.

Grassmann fields differ from bosonic fields in that they anti-commute and their square and higher powers are zero. Thus with $\eta(r) = \sum_l h_l \, \phi_l(r)$ a second Grassmann field

$$\psi(r)\eta(s) + \eta(s)\psi(r) = 0 \tag{7}$$

$$(\psi(r))^2 = (\psi(r))^3 = \cdots = 0. \tag{8}$$

These rules follow from the anti-commutation properties of Grassman numbers.

2.2 Grassmann Functionals - Basic Idea

The concept of a functional $F[\psi(r)]$ is well-established in the case of c-number fields. Essentially, a functional $F[\psi(r)]$ maps a c-number field $\psi(r)$ onto a c-number, which depends on all the values of $\psi(x)$ over its entire range of r. There has been a widespread use of functionals both in classical and quantum physics (see Ref. [1] for background references), so their properties and calculus involving functional derivatives and integrals will not be presented here. A recent textbook Ref. [15] outlines these key aspects of c-number functionals.

It is important to note that as the value of the function at any point in the range for x is determined uniquely by the expansion coefficients $\{\alpha_k\}$, then the functional $F[\psi(r)]$ must therefore also just depend on the c-number expansion coefficients, and hence may also be viewed as a c-number function $f(\alpha_1, \cdots, \alpha_k, \cdots \alpha_n)$ of the expansion coefficients. This equivalence is useful when functional differentiation and integration are considered.

$$F[\psi(r)] \equiv f(\alpha_1, \cdots, \alpha_k, \cdots \alpha_n) \tag{9}$$

This feature also applies to functionals of the form $F[\psi(r),\psi^{+}(r)]$, which again are equivalent to a c-number function of both sets of expansion coefficients.

$$F[\psi(r),\psi^{+}(r)] \equiv f(\alpha_1,\cdots,\alpha_k,\cdots\alpha_n,\alpha_1^{+},\cdots,\alpha_k^{+},\cdots\alpha_n^{+}) \tag{10}$$

The idea of a functional $F[\psi(r)]$ can be extended to cases where $\psi(r)$ is a Grassmann field rather than a c-number field. Analogous to the situation for c-number functionals, a *Grassmann functional* $F[\psi(r)]$ maps the g-number function $\psi(r)$ onto a Grassmann function that depends on *all* the values of $\psi(r)$ over its entire range. Examples of Grassmann functionals are given in Appendix 10. Grassmann *functional derivatives* and Grassmann *functional integrals* can be defined for Grassmann functionals, following analogous definitions for c-number functionals. These definitions and associated rules for these processes are outlined in Appendix 10.

As for c-number fields, the expansion coefficients $\{g_k\}$ determine the Grassmann field at any point in the range for x, so the functional $F[\psi(r)]$ must therefore also just depend on the g-number expansion coefficients. Hence the Grassmann functional $F[\psi(r)]$ is equivalent to a Grassmann function $f(g_1,\cdots,g_k,\cdots g_n)$ of the expansion coefficients. This equivalence is used to relate Grassmann functional differentiation and integration to ordinary Grassmann differentiation and Grassmann integration with respect to the Grassmann phase space variables.

$$F[\psi(r)] \equiv f(g_1,\cdots,g_k,\cdots g_n) \tag{11}$$

This feature also applies to functionals of the form $F[\psi(r),\psi^{+}(r)]$, which are also equivalent to a Grassmann function of all the expansion coefficients.

$$F[\psi(r),\psi^{+}(r)] \equiv f(g_1,\cdots,g_k,\cdots g_n,g_1^{+},\cdots,g_k^{+},\cdots g_n^{+}) \tag{12}$$

This result is of central importance in the present paper, where we consider un-normalised B Grassmann distribution functionals of fields $\psi(r),\psi^{+}(r)$. The Grassmann functional $B[\psi(r),\psi^{+}(r)]$ is *equivalent* to the previous un-normalised B Grassmann distribution function $B(g_k,g_k^{+})$ of the Grassmann phase space variables g_k,g_k^{+} treated in Ref. [1]

$$B[\psi(r),\psi^{+}(r)] \equiv B(g_k,g_k^{+}) \tag{13}$$

The same symbol will be used for both, but note the [..] for functionals, (..) for functions. There is a similar equivalence for normalised P Grassmann distribution functionals and functions. Essentially, the distribution functional $B[\psi(r),\psi^{+}(r)]$ and the distribution function $B(g_k,g_k^{+})$ are just two different ways of representing the same density operator. This equivalence enables us to use results previously obtained for the separate mode theory in Paper I (Ref. [1]) to derive equivalent results for the field theory approach outlined in the present paper.

2.3 Field Operators - New Notation

It is now useful to change the notation to both allow for the presence of differing internal states $|\alpha\rangle$ for the fermions and to treat field annihilation and creation operators and their mode expansions via unified expressions. The field annihilation, creation operators $\widehat{\psi}_\alpha(r), \widehat{\psi}^\dagger_\alpha(r)$ for internal state $|\alpha\rangle$ will now be denoted $\widehat{\psi}_{\alpha A}(r)$, with $A = 1, 2$ distinguishing annihilation and creation operators and r denoting the spatial position The expansion in terms of mode annihilation, creation operators $\widehat{c}_{\alpha i}, \widehat{c}^\dagger_{\alpha i}$ (denoted $\widehat{c}^A_{\alpha i}$ with $A = 1, 2$ for annihilation and creation operators) and orthnormal mode functions and their conjugates $\phi_{\alpha i}(r), \phi^*_{\alpha i}(r)$ now denoted $\xi^A_{\alpha i}(r)$ with $A = 1, 2$ will now be written as

$$\widehat{\psi}_{\alpha A}(r) = \sum_i \widehat{c}^A_{\alpha i}\, \xi^A_{\alpha i}(r) \tag{14}$$

2.4 Distribution Functionals - B Distribution

To formulate functional phase space theory we first associate the field annihilation and creation operators $\widehat{\psi}_\alpha(r), \widehat{\psi}^\dagger_\alpha(r)$ for each internal state $|\alpha\rangle$ associated with *Grassmann field functions* $\psi_\alpha(r), \psi^+_\alpha(r)$, which in accord with the new notation will be denoted $\psi_{\alpha A}(r)$ with $A = 1, 2$. These can be expanded in terms of mode functions $\xi^A_{\alpha i}(r)$ with Grassmann phase variables $g_{\alpha i}, g^+_{\alpha i}$ (denoted $g^A_{\alpha i}$ with $A = 1, 2$ and $i = 1, 2, .., n_\alpha$) for each of the n_α spatial modes associated with this internal state acting as the expansion coefficients

$$\psi_{\alpha A}(r) = \sum_i g^A_{\alpha i}\, \xi^A_{\alpha i}(r) \tag{15}$$

Clearly these are linear and odd Grassmann fuctions, and the Grassmann fields are equivalent to the Grassmann phase variables.

If we now denote $\psi(r) \equiv \{\psi_\alpha(r), \psi^+_\alpha(r)\}$ and $g \equiv \{g_{\alpha i}, g^+_{\alpha i}\}$ we then represent the density operator $\widehat{\rho}$ by a B distribution functional $B[\psi(r)]$ of the Grassmann fields. As explained above, the distribution functional $B[\psi(r)]$ is equivalent to the previous distribution function $B(g)$.

$$B[\psi(r)] \equiv B(g) \tag{16}$$

Similarly, a P distribution functional $P[\psi(r)]$ is equivalent to the previous P distribution function $P(g)$. The two distribution functionals are related via

$$P[\psi(r)] = B[\psi(r)]\, \exp(+\int dx\, \psi(r)\psi^+(r)) \tag{17}$$

as can easily be established via mode expansions.

2.5 Quantum Correlation Functions

The previous normally ordered quantum correlation functions based on separate modes (see Eq. (27) in Ref. [1]) now involve the field creation and annihilation operators $\hat{\Psi}^\dagger(\mathbf{r}), \hat{\Psi}(\mathbf{r})$ and are of the form

$$\begin{aligned} & G^{(p,q)}(\mathbf{r}_1, \mathbf{r}_2 \cdots \mathbf{r}_p; \mathbf{s}_q \cdots \mathbf{s}_2, \mathbf{s}_1) \\ = \; & \langle \hat{\Psi}(\mathbf{r}_1)^\dagger \cdots \hat{\Psi}(\mathbf{r}_p)^\dagger \hat{\Psi}(\mathbf{s}_q) \cdots \hat{\Psi}(\mathbf{s}_1) \rangle \\ = \; & \mathrm{Tr}(\hat{\Psi}(\mathbf{s}_q) \cdots \hat{\Psi}(\mathbf{s}_1)\, \hat{\rho}\, \hat{\Psi}(\mathbf{r}_1)^\dagger \cdots \hat{\Psi}(\mathbf{r}_p)^\dagger). \end{aligned} \tag{18}$$

where for an N particle system we require $p, q \leq N$ to give a non-zero result. For the fermion case we also require $p-q = 0, \pm 2, \pm 4, \cdots$ due to the SSR. As we will see, this result can be written in terms of phase space functional integrals using the general result relating Grassmann functional integrals and Grassmann phase space integrals in Eq. (170) in Appendix 10.

$$\begin{aligned} & \int D\psi^+ \, D\psi \, F[\psi(r), \psi^+(r)] \\ = \; & \lim_{n\to\infty} \lim_{\epsilon \to 0} \int \cdots \int dg_n^+ \cdots dg_k^+ \cdots dg_1^+ dg_n \cdots dg_k \cdots dg_1 \\ \times \; & f(g_1, \cdots, g_k, \cdots g_n, g_1^+, \cdots, g_k^+, \cdots g_n^+) \end{aligned} \tag{19}$$

where the functional $F[\psi(r), \psi^+(r)]$ and the Grassmann function $f(g, g^+)$ are equivalent.

If we substitute mode expansions (109) for the field operators then using (18) the quantum correlation functions can be expressed in terms of the P distribution functional as follows

$$\begin{aligned} & G^{(p,q)}(\mathbf{r}_1, \mathbf{r}_2 \cdots \mathbf{r}_p; \mathbf{s}_q \cdots \mathbf{s}_2, \mathbf{s}_1) \\ = \; & \sum_{m_q \cdots m_1} \sum_{l_1 \cdots l_p} \phi_{m_q}(\mathbf{s}_q) \cdots \phi_{m_1}(\mathbf{s}_1) \phi_{l_1}^*(\mathbf{r}_1) \cdots \phi_{l_p}^*(\mathbf{r}_p) \mathrm{Tr}(\hat{c}_{m_q} \cdots \hat{c}_{m_1} \hat{\rho}(t) \hat{c}_{l_1}^\dagger \cdots \hat{c}_{l_p}^\dagger). \\ = \; & \sum_{m_q \cdots m_1} \sum_{l_1 \cdots l_p} \phi_{m_q}(\mathbf{s}_q) \cdots \phi_{m_1}(\mathbf{s}_1) \phi_{l_1}^*(\mathbf{r}_1) \cdots \phi_{l_p}^*(\mathbf{r}_p) \\ & \times \int \prod_i dg_i^+ \prod_i dg_i \, (g_{m_q} \cdots g_{m_2} g_{m_1}) P(g, g^+) (g_{l_1}^+ g_{l_2}^+ \cdots g_{l_p}^+) \\ = \; & \int \prod_i dg_i^+ \prod_i dg_i \, (\psi(\mathbf{s}_q) \cdots \psi(\mathbf{s}_1)) \, P(g, g^+) (\psi^+(\mathbf{r}_1) \cdots \psi^+(\mathbf{r}_p)) \\ = \; & \int \int D\psi^+ D\psi \, \psi(\mathbf{s}_q) \cdots \psi(\, \mathbf{s}_1) P[\psi(r), \psi^+(r)] \psi^+(\mathbf{r}_1) \cdots \psi^+(\mathbf{r}_p)) \end{aligned} \tag{20}$$

where in the last step we have replaced the Grassmann phase space integral by the equivalent Grassmann functional integral via (19). Hence we see that the quantum correlation functions involving fermion field operators are given as Grassmann functional integrals in which the Grassmann fields have replaced the field operators.

2.6 Fock State Populations and Coherences

We consider fermion position eigenstates as in Eq.(111)

$$\begin{aligned} |\Phi\{\mathbf{r}\}\rangle &= |\mathbf{r}_1 \cdots \mathbf{r}_p\rangle = \hat{\Psi}_f^\dagger(\mathbf{r}_1) \cdots \hat{\Psi}_f^\dagger(\mathbf{r}_p)|0\rangle \\ |\Phi\{\mathbf{s}\}\rangle &= |\mathbf{s}_1 \cdots \mathbf{s}_p\rangle = \hat{\Psi}_f^\dagger(\mathbf{s}_1) \cdots \hat{\Psi}_f^\dagger(\mathbf{s}_p)|0\rangle \end{aligned} \tag{21}$$

The population for state $|\Phi\{\mathbf{r}\}\rangle$ and the coherence between the state $|\Phi\{\mathbf{r}\}\rangle$ and the state $|\Phi\{\mathbf{s}\}\rangle$ are given by

$$P(\Phi\{\mathbf{r}\}) = Tr(|\Phi\{\mathbf{r}\}\rangle \langle\Phi\{\mathbf{r}\}| \hat{\rho}) \tag{22}$$

$$C(\Phi\{\mathbf{s}\}; \Phi\{\mathbf{r}\}) = Tr(|\Phi\{\mathbf{s}\}\rangle \langle\Phi\{\mathbf{r}\}| \hat{\rho}) \tag{23}$$

Substituting for the field operators from Eqs.(109) and using the results (30) and (31) in Ref. [1] we see that for the fermion position probability can be expressed in terms of the B distribution functional as

$$\begin{aligned} P(\Phi\{\mathbf{r}\}) &= \sum_{l_1,\cdots l_p} \sum_{m_1,\cdots m_p} \phi_{l_1}^*(\mathbf{r}_1) \cdots \phi_{l_p}^*(\mathbf{r}_p)\, \phi_{m_p}(\mathbf{r}_p) \cdots \phi_{m_1}(\mathbf{r}_1) \\ &\quad \times \int d\mathbf{g}^+ d\mathbf{g}\ \ g_{m_p} \cdots g_{m_1}\, B(g, g^+)\, g_{l_1}^+ \cdots g_{l_p}^+ \\ &= \int D\psi^+ D\psi\, \psi(\mathbf{r}_p) \cdots \psi(\mathbf{r}_1)\, B[\psi(r), \psi^+(r)]\, \psi^+(\mathbf{r}_1) \cdots \psi^+(\mathbf{r}_p) \end{aligned} \tag{24}$$

and similarly for the fermion position coherence

$$C(\Phi\{\mathbf{s}\}; \Phi\{\mathbf{r}\}) = \int D\psi^+ D\psi\, \psi(\mathbf{r}_p) \cdots \psi(\mathbf{r}_1) B[\psi(r), \psi^+(r)]\, \psi^+(\mathbf{s}_1) \cdots \psi^+(\mathbf{s}_p) \tag{25}$$

where the phase space integrals have been converted into functional integrals involving the fermion B distribution functional. These results are useful for discussing similtaneous position measurements - note the probability density factors such as $\psi^+(\mathbf{r}_1)\psi(\mathbf{r}_1)$ - and for discussing spatial coherence effects in systems such as Fermi gases. In both cases the result is given as a functional average of products of field functions, rather similar to equivalent classical formulae.

2.7 Characteristic Functional

For completeness, we also define the characteristic functional $\chi[h(r), h^+(r)]$ via

$$\chi[h(r), h^+(r)] = \mathrm{Tr}(\hat{\Omega}^+[h^+(r)]\ \hat{\rho}\, \hat{\Omega}^-[h(r)]) \tag{26}$$

$$\begin{aligned} \hat{\Omega}^+[h^+(r)] &= \exp i \int dr\, \hat{\Psi}(r) h^+(r) = 1 + i \int dr\, \hat{\Psi}(r) h^+(r) \\ \hat{\Omega}^-[h(r)] &= \exp i \int dx\, h(r) \hat{\Psi}^\dagger(r) = 1 + i \int dr\, h(r) \hat{\Psi}^\dagger(r) \end{aligned} \tag{27}$$

where for the field operators $\hat{\Psi}(r)$ and $\hat{\Psi}^\dagger(r)$ we associate a pair of Grassmann fields $h^+(r)$ and $h(r)$. Note that h, h^+ are both complex, and are not related to each other. Note that $\chi[h(r), h^+(r)]$ only depends on the $h(r), h^+(r)$ and not on their complex conjugates $h^*(r), h^{+*}(r)$. The simplification of the exponentials follows from second and higher powers of the field operators being zero.

The characteristic functional $\chi[h(r), h^+(r)]$ is related to the P distribution functional $P[\psi(r), \psi^+(r)]$ via a phase space functional integral

$$\begin{aligned} & \chi[h(r), h^+(r)] \\ = & \int D\psi^+ D\psi \, \exp i \left\{ \int dr \, \psi(r) h^+(r) \right\} P[\psi(r), \psi^+(r)] \, \exp i \left\{ \int dr \, h(r) \psi^+(r) \right\} \end{aligned} \tag{28}$$

Note the similarity of this relationship to that in Eq. (34) in Ref. [1] for the characteristic and distribution functions. The proof is given in Appendix 11. A derivation of the result (20) for the quantum correlation function can also be carried out starting from the characteristic functional, and involves functional differentiation of functional integrals.

3 Functional Fokker- Planck Equations

In the case of separate modes, the distribution function satisfies a Fokker-Planck equation of the form (see Eq. (49) in Ref. [1])

$$\frac{\partial}{\partial t}B(g,g^{+}) = -\sum_{p=1}^{2n}(A_pB(g,g^{+}))\frac{\overleftarrow{\partial}}{\partial g_p} + \frac{1}{2}\sum_{p,q=1}^{2n}(D_{pq}B(g,g^{+}))\frac{\overleftarrow{\partial}}{\partial g_q}\frac{\overleftarrow{\partial}}{\partial g_p} \quad (29)$$

where A_p is the drift vector and D_{pq} is the diffusion matrix. Explicit expressions for the drift vector and diffusion matrix are set out in Ref. [1] (see Eqs. (50), (51) and (52) therein) for the case where the density operator satisfies a Markovian master equation. However, as Grassmann derivatives can be related to functional derivatives - as in Eqs.(148) and (149) in Appendix 10

$$\frac{\overleftarrow{\partial}}{\partial g_k} = \int dr\phi_k(r)\left(\frac{\overleftarrow{\delta}}{\delta\psi(r)}\right)_r \qquad \frac{\overleftarrow{\partial}}{\partial g_k^{+}} = \int dr\phi_k^{*}(r)\left(\frac{\overleftarrow{\delta}}{\delta\psi^{+}(r)}\right)_r . \quad (30)$$

and as the distribution function and distribution functional are equivalent, we can convert the Fokker-Planck equation into an equivalent functional Fokker-Planck equation.

3.1 Functional Fokker-Planck Equation - B Distribution

In terms of the new notation with $g_p \to g^A_{\alpha i}$, $A_p \to A^A_{\alpha i}$, $D_{pq} \to D^{A\,B}_{\alpha i\,\beta j}$ we find that the *functional Fokker-Planck equation* is

$$\begin{aligned}\frac{\partial}{\partial t}B[\psi] &= -\sum_{\alpha A}\int dr\,(A_{\alpha A}[r]\,B[\psi])\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha A}(r)} \\ &\quad +\frac{1}{2}\sum_{\alpha A,\beta B}\iint dr\,ds\,(D_{\alpha A\,\beta B}[r,s]\,B[\psi])\frac{\overleftarrow{\delta}}{\delta\psi_{\beta B}(s)}\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha A}(r)}\end{aligned} \quad (31)$$

The drift vector and diffusion matrix functionals are given by

$$A_{\alpha A}[\psi(r),r] = \sum_i \xi^A_{\alpha\,i}(r)\,A^A_{\alpha i}(g) \quad (32)$$

$$D_{\alpha A\,\beta B}[\psi(r),r;\psi(s),s] = \sum_{ij}\xi^A_{\alpha\,i}(r)\,D^{A\,B}_{\alpha i\,\beta j}(g)\,\xi^B_{\beta\,j}(s) \quad (33)$$

showing how the Fokker-Planck equation could be turned into its equivalent functional Fokker-Planck equation. Note the spatial integrals, single for the drift term, double for the diffusion term. The drift vector and diffusion matrix can be considered as Grassmann functionals. Fortunately, the diffusion matrix often only depends on one spatial coordinate such as in the case where the zero range approximation applies to the Hamiltonian.

3.2 Correspondence Rules - B Distribution

The functional Fokker-Planck equation may also be obtained more directly via establishing the correspondence rules associated with the field operators. These correspondence rules may be applied directly to the master, Liouville or Matsubara equation for the evolution of the density operator to give the functional Fokker-Planck equation, assuming the Hamiltonian operator and the relaxation operators are expressed in terms of the field operators rather than the mode operators. This approach is used in Section 5.

The correspondence rules are derived from those in Eq. (45) in Ref. [1] for the separate modes distribution function. Noting the mode expansion (14) for the field operators we see that

$$\hat{\rho} \;\Rightarrow\; \sum_i \phi_{\alpha i}(r)\hat{c}_{\alpha i}\,\hat{\rho} \qquad B(g) \Rightarrow \sum_i \phi_{\alpha i}(r) g_{\alpha i}\, B[\psi] \tag{34}$$

$$\hat{\rho} \;\Rightarrow\; \hat{\rho}\sum_i \phi_{\alpha i}(r)\hat{c}_{\alpha i} \qquad B(g) \Rightarrow B[\psi]\sum_i \phi_{\alpha i}(r)\left(+\frac{\overleftarrow{\partial}}{\partial g^+_{\alpha i}}\right) \tag{35}$$

$$\hat{\rho} \;\Rightarrow\; \sum_i \phi^*_{\alpha i}(r)\hat{c}^\dagger_{\alpha i}\,\hat{\rho} \qquad B(g) \Rightarrow \sum_i \phi^*_{\alpha i}(r)\left(+\frac{\overrightarrow{\partial}}{\partial g_{\alpha i}}\right) B[\psi] \tag{36}$$

$$\hat{\rho} \;\Rightarrow\; \hat{\rho}\sum_i \phi^*_{\alpha i}(r)\hat{c}^\dagger_{\alpha i} \qquad B(g) \Rightarrow\; B[\psi]\sum_i \phi^*_{\alpha i}(r) g^+_{\alpha i}. \tag{37}$$

Then if the distribution function $B(g)$ is replaced by the distribution functional $B[\psi(r)]$ we see from Eqs. (138), (140), (142) and (144) in Appendix 10 that the correspondence rules are

$$\hat{\rho} \;\Rightarrow\; \widehat{\psi}_\alpha(r)\,\hat{\rho} \quad B[\psi] \Rightarrow \psi_\alpha(r)\, B[\psi] \qquad \hat{\rho} \Rightarrow \hat{\rho}\,\widehat{\psi}_\alpha(r) \qquad B[\psi] \Rightarrow B[\psi]\,(+\frac{\overleftarrow{\delta}}{\delta\psi^+_\alpha(r)})$$

$$\hat{\rho} \;\Rightarrow\; \widehat{\psi}^\dagger_\alpha(r)\,\hat{\rho} \quad B[\psi] \Rightarrow (+\frac{\overrightarrow{\delta}}{\delta\psi_\alpha(r)})\, B[\psi] \qquad \hat{\rho} \Rightarrow \hat{\rho}\,\widehat{\psi}^\dagger_\alpha(r) \quad B[\psi] \Rightarrow\; B[\psi]\,\psi^+_\alpha(r) \tag{38}$$

Analogous correspondence rules apply to the P distribution functional. For completeness these are set out in Appendix 11

3.3 Case of P Distribution

In the case of the P distribution functional the functional Fokker-Planck has the same general features as that for in (31) for the B distribution. The drift vector and diffusion matrix functionals will of course differ from those for the B distribution, since the correspondence rules (see Appendix 11) are not the same. The relationship between the drift vector and diffusion matrix functionals and the drift vector and diffusion matrix functions for the P distribution is the same as in (32) and (32).

4 Ito Stochastic Field Equations

In Paper I (Ref. [1]) we established Ito stochastic equations that were equivalent to the Fokker-Planck equation by requiring that the phase space average at any time t of an arbitrary function $F(g, g^+)$ and stochastic average of same function $F(\widetilde{g}(t), \widetilde{g}^+(t))$ coincide (see Eq. (53) in Ref. [1]) when the phase space variables g_p are replaced by stochastic variables $\widetilde{g}_p(t)$, and for this to occur the Ito stochastic equations for $\widetilde{g}_p(t)$ must be suitably related to Fokker-Planck equation for distribution function $B(g, g^+, t)$.

$$\left\langle F(g, g^+)\right\rangle_t = \overline{F(\widetilde{g}(t),\widetilde{g}^+(t))} \tag{39}$$

This approach is based on a treatment by Gardiner (see pp 95-96 in [4]) which established the relationship between Fokker-Planck and Ito stochastic equations for bosons.

In the case of separate modes, the stochastic phase variables satisfy Ito stochastic equations given by Eq. (60) in Ref. [1]).

$$\frac{d}{dt}\widetilde{g}_p(t) = C^p(\widetilde{g}(t)) + \sum_a B^p_a(\widetilde{g}(t))\ \Gamma_a(t_+) \tag{40}$$

where $C^p(\widetilde{g}(t))$ and $B^p_a(\widetilde{g}(t))$ are odd Grassmann functions. The quantities $\Gamma_a(t_+)$ are c-number Gaussian-Markoff random noise terms [6]. The C^p and B^p_a terms were shown (see Eqs. (69) and (70) in Ref. [1]) to be related to the drift vector and diffusion matrix in the Fokker-Planck equation via

$$C^p(g, g^+) = -A_p(g, g^+) \tag{41}$$

$$[B(g, g^+)B^T(g, g^+)]_{qp} = D_{qp}(g, g^+) \tag{42}$$

The basic stochastic average properties of the Gaussian-Markoff random noise terms are

$$\begin{aligned}
\overline{\Gamma_a(t_1)} &= 0 \\
\overline{\Gamma_a(t_1)\Gamma_b(t_2)} &= \delta_{ab}\delta(t_1 - t_2) \\
\overline{\Gamma_a(t_1)\Gamma_b(t_2)\Gamma_c(t_3)} &= 0 \\
\overline{\Gamma_a(t_1)\Gamma_b(t_2)\Gamma_c(t_3)\Gamma_d(t_4)} &= \overline{\Gamma_a(t_1)\Gamma_b(t_2)}\,\overline{\Gamma_c(t_3)\Gamma_d(t_4)} + \overline{\Gamma_a(t_1)\Gamma_c(t_3)}\,\overline{\Gamma_b(t_2)\Gamma_d(t_4)} \\
&\quad + \overline{\Gamma_a(t_1)\Gamma_d(t_4)}\,\overline{\Gamma_b(t_2)\Gamma_c(t_3)}
\end{aligned} \tag{43}$$

showing that the stochastic averages of a single Γ is zero and the stochastic average of the product of two Γ's is zero if they are different and delta function correlated in the time difference if they are the same. In addition, the stochastic averages of products of odd numbers of Γ are zero and stochastic averages of products of even numbers of Γ are the sums of products of stochastic averages of pairs of Γ. The Γ_a are listed as $a = 1, 2, \cdots, 2n^2$, where the number of modes is n.

A further important property is that any $F(\widetilde{g}(t))$ and the products of any $\Gamma_a(t_+)$ at later times t_+ are uncorrelated

$$\begin{aligned} & \overline{F(\widetilde{g}(t_1))\Gamma_a(t_2)\Gamma_b(t_3)\Gamma_c(t_4)..\Gamma_k(t_l)} \\ = \; & \overline{F(\widetilde{g}(t_1))}\;\overline{\Gamma_a(t_2)\Gamma_b(t_3)\Gamma_c(t_4)..\Gamma_k(t_l)} \qquad t_1 < t_2, t_3, .., t_l \end{aligned} \tag{44}$$

It was first pointed out in [8] (see also Paper I (Ref. [1]) that for the B distribution the expression for C^p in the Ito stochastic equation can be written as (see Eq. (80) of Ref. [1])

$$C^p(g, g^+) = \sum_{r=1}^{2n} L_r^p \, g_r \tag{45}$$

and in Ref. [1] the expression for B_a^p in the Ito stochastic equation was shown to be given by (see Eq. (75) of Ref. [1])

$$B_a^p(g, g^+) = \sum_{r=1}^{2n} K_{r,a}^p \, g_r \tag{46}$$

where the L_r^p and $^p_{r,a}$ are c-numbers. Both terms involve a linear dependence on the Grassmann phase variables, resulting in a linear Ito stochastic equation for the $\widetilde{g}_p(t)$. We note that if we now introduce Grassmann stochastic fields via the expansions

$$\widetilde{\psi}(r) = \sum_i \widetilde{g}_i \, \phi_i(r) \qquad \widetilde{\psi}^+(r) = \sum_i \phi_i^*(r) \widetilde{g}_i^+. \tag{47}$$

we can easily convert the Ito stochastic equations for $\widetilde{g}_i(t), \widetilde{g}_i^+(t)$ into linear Ito stochastic field equations for $\widetilde{\psi}(r), \widetilde{\psi}^+(r)$.

In view of the key relationship (39) between the phase space integral of an arbitrary Grassmann function and the stochastic average of this function, we can establish a relationship between the phase space functional integral of products of Grassmann fields (as occur in expressions for quantum correlation functions, Fock state populations and coherences) and stochastic averages of the same products of stochastic fields. This result is the basis for numerical calculations based on Grassmann stochastic fields for fermions rather than solving the functional Fokker-Planck equation and calculating functional integrals involving the non-stochastic Grassmann fields in order to determine quantum correlation functions etc. . The proof is similar to that for (20), but in reverse. Using (19) with $F[\psi(r), \psi^+(r)] \equiv \psi(\mathbf{s}_q) \cdots \psi(\ \mathbf{s}_1) P[\psi(r), \psi^+(r)] \psi^+(\mathbf{r}_1) \cdots \psi^+(\mathbf{r}_p)$ and Eqs.

(3), (4) we have

$$
\begin{aligned}
& \int\int D\psi^+ D\psi \, (\psi(\mathbf{s}_q)\cdots\psi(\ \mathbf{s}_1) P[\psi(r),\psi^+(r)]\psi^+(\mathbf{r}_1)\cdots\psi^+(\mathbf{r}_p)) \\
= & \int \prod_i dg_i^+ \prod_i dg_i \, (\psi(\mathbf{s}_q)\cdots\psi(\mathbf{s}_1))\, P(g,g^+)(\psi^+(\mathbf{r}_1)\cdots\psi^+(\mathbf{r}_p)) \\
= & \sum_{m_q\cdots m_1} \sum_{l_1\cdots l_p} \phi_{m_q}(\mathbf{s}_q)\cdots\phi_{m_1}(\mathbf{s}_1)\phi^*_{l_1}(\mathbf{r}_1)\cdots\phi^*_{l_p}(\mathbf{r}_p) \\
& \times \int \prod_i dg_i^+ \prod_i dg_i \, (g_{m_q}\cdots g_{m_2} g_{m_1}) P(g,g^+)(g^+_{l_1} g^+_{l_2}\cdots g^+_{l_p}) \\
= & \sum_{m_q\cdots m_1} \sum_{l_1\cdots l_p} \phi_{m_q}(\mathbf{s}_q)\cdots\phi_{m_1}(\mathbf{s}_1)\phi^*_{l_1}(\mathbf{r}_1)\cdots\phi^*_{l_p}(\mathbf{r}_p) \\
& \times \overline{(\widetilde{g}_{m_q}(t)\cdots\widetilde{g}_{m_2}(t)\widetilde{g}_{m_1}(t))(\widetilde{g}^+_{l_1}(t)\widetilde{g}^+_{l_2}(t)\cdots\widetilde{g}^+_{l_p}(t))} \\
= & \overline{\widetilde{\psi}(\mathbf{s}_q,t)\cdots\widetilde{\psi}(\ \mathbf{s}_1,t)\widetilde{\psi}^+(\mathbf{r}_1,t)\cdots\widetilde{\psi}^+(\mathbf{r}_p,t)}
\end{aligned}
\tag{48}
$$

The step from line two to line three follows from Eq. (39).

4.1 Ito Stochastic Field Equations - B Distribution

In terms of the general notation, the fermionic stochastic fields $\widetilde{\psi}_\alpha(r,t), \widetilde{\psi}^+_\alpha(r,t)$ (denoted $\widetilde{\psi}_{\alpha A}(r,t), A = 1,2$) are defined by the same mode expansion (15) as for the time independent Grassmann fields $\psi_\alpha(r), \psi^+_\alpha(r)$ but now with the Grassmann phase space variables $g_{\alpha i}, g^+_{\alpha i}$ replaced by time dependent Grassman stochastic variables $\widetilde{g}_{\alpha i}, \widetilde{g}^+_{\alpha i}$ (denoted $\widetilde{g}^A_{\alpha i}, A = 1,2$) for each mode.

$$
\widetilde{\psi}_{\alpha A}(r) = \sum_i \widetilde{g}^A_{\alpha i}\, \xi^A_{\alpha i}(r) \tag{49}
$$

The t dependence will be left understood for simplicity of notation.

In terms of the new notation the Ito stochastic equations for the stochastic phase variables are

$$
\frac{d}{dt}\widetilde{g}^A_{\alpha i}(t) = C^A_{\alpha i}(\widetilde{g}) + \sum_a B^{\alpha i\, A}_a(\widetilde{g}(t))\, \Gamma_a(t_+) \tag{50}
$$

We have shown in Paper I (Ref. [1]) that for separate modes the Ito stochastic equations for the $\widetilde{g}_p$ in Eq. (40) (see Eqs.(60) or (63) in Ref. [1]) are equivalent to the Fokker-Planck equation in Eq.(29) for the case of the B distribution function (see Eq. (49) in Ref. [1]), and as this Fokker-Planck equation is equivalent to the functional Fokker-Planck equation in Eq.(31) for the B distribution functional, we can derive the equivalent Ito stochastic field equation for the $\widetilde{\psi}_{\alpha A}(r)$ by starting from the Ito stochastic equations (50) for the $\widetilde{g}^A_{\alpha i}$, multiplying by the mode function $\xi^A_{\alpha i}(r)$ and summing over the modes for the same internal component α.

The Ito stochastic field equations obtained are

$$\frac{\partial}{\partial t}\widetilde{\psi}_{\alpha A}(r) = C_{\alpha A}[\widetilde{\psi}(r),r] + \sum_a B_a^{\alpha A}[\widetilde{\psi}(r),r]\,\Gamma_a(t_+) \quad (51)$$

$$= \left(\frac{\partial}{\partial t}\widetilde{\psi}_{\alpha A}(r)\right)_{class} + \left(\frac{\partial}{\partial t}\widetilde{\psi}_{\alpha A}(r)\right)_{noise} \quad (52)$$

The right side is the sum of a classical field term and a noise field term. The classical and noise field terms involve Grassmann functionals $C_{\alpha A}[\widetilde{\psi}(r),r]$ and $B_a^{\alpha A}[\widetilde{\psi}(r),r]$ defined by

$$C_{\alpha A}[\widetilde{\psi}(r),r] = \sum_i C^A_{\alpha i}(\widetilde{g})\,\xi^A_{\alpha\, i}(r) \quad (53)$$

$$B_a^{\alpha A}[\widetilde{\psi}(r),r] = \sum_i B_a^{\alpha i\, A}(\widetilde{g})\,\xi^A_{\alpha i}(r) \quad (54)$$

Note that the classical field term is also stochastic because it involves a functional of the stochastic fields $\widetilde{\psi}_{\alpha A}(r)$. The noise field term is stochastic, not only for this reason but also because it involves the Gaussian-Markoff noise terms. The derivation of the Ito stochastic field equations is set out in Appendix 12.

We can then show that the relationships between these quantities and the drift (32) and diffusion (33) terms in the functional Fokker-Planck equation are

$$C_{\alpha A}[\widetilde{\psi}(r),r] = -A_{\alpha A}[\widetilde{\psi}(r),r] \quad (55)$$

$$(BB^T)_{\alpha A\,\beta B} = \sum_a B_a^{\alpha A}[\widetilde{\psi}(r),r]\,B_a^{\beta B}[\widetilde{\psi}(s),s]$$

$$= D_{\alpha A\,\beta B}[\widetilde{\psi}(r),r;\widetilde{\psi}(s),s] \quad (56)$$

The first relationship follows from (53) and (41). The second result follows from (54) and (42), the last equation being

$$\sum_a B_a^{\alpha i\, A}(g)\,B_a^{\beta j\, B}(g) = D^{A\,B}_{\alpha i\,\beta j}(g) \quad (57)$$

in the new notation.

In practice mode expansions for fields not needed to derive the Ito stochastic field equations. Having derived the functional Fokker-Planck equations using the correspondence rules for the field operators, we can then simply use the general results we have established in Eqs. (55) and (56) to obtain the classical and noise field quantities $C_{\alpha A}[\widetilde{\psi}(r),r]$ and $B_a^{\alpha A}[\widetilde{\psi}(r),r]$. The classical field quantity is simply $C = -A$ from the drift functional, whilst the noise field quantity requires a factorisation of the diffusion functional D as in $BB^T = D$.

4.2 Form of Drift and Diffusion Functionals

Having obtained the mode based expressions in (53) and (54) for the terms in the Ito stochastic field equations (51) we can develop these expressions further

to express $C_{\alpha A}[\widetilde{\psi}(r), r]$ and $B_a^{\alpha A}[\widetilde{\psi}(r), r]$ in terms of the stochastic fields $\widetilde{\psi}_{\beta A}(s)$. As we will see, the relationship is *linear* but in general will be *non-local*. The linearlity feature follows from the drift vector, diffusion matrix in the Fokker-Planck equation for the $B(g)$ distribution function depending linearly, bilinearly on the Grassmann phase space variables.

In the case of the B distribution the drift terms $C_{\alpha i}^A(\widetilde{g})$ are linear functions of the stochastic phase variables $\widetilde{g}_{\alpha i}^A$, based on Eq. (45) (see Eq. (80) in Ref. [1])

$$C_{\alpha i}^A(\widetilde{g}) = \sum_{\beta\, j} L_{\beta j}^{A\,\alpha i}\, \widetilde{g}_{\beta j}^A \tag{58}$$

so that from (53) and (49)

$$C_{\alpha A}[\widetilde{\psi}(r), r] = \sum_{i\beta\, j} L_{\beta j}^{A\,\alpha i} \int ds\, \xi_{\alpha\, i}^A(r)\, \xi_{\beta\, j}^A(s)^*\, \widetilde{\psi}_{\beta A}(s) \tag{59}$$

which indicates that $C_{\alpha A}[\widetilde{\psi}(r), r]$ is a linear functional of the stochastic fields.

The terms $B_a^{\alpha i\, A}(\widetilde{g})$ related to the diffusion matrix are obtained from Eq. (46) (see Eq. (75) in Ref. [1]) as

$$B_a^{\alpha i\, A}(\widetilde{g}) = \sum_{B\, \beta j} K_{B\, \beta j,\, a}^{A\, \alpha i}\, \widetilde{g}_{\beta j}^B \tag{60}$$

so that from (54) and (49)

$$B_a^{\alpha A}[\widetilde{\psi}(r), r] = \sum_{iB\beta j} K_{B\, \beta j,\, a}^{A\, \alpha i} \int ds\, \xi_{\alpha\, i}^A(r)\, \xi_{\beta\, j}^B(s)^*\, \widetilde{\psi}_{\beta B}(s) \tag{61}$$

which indicates that $B_\alpha^{\alpha A}[\widetilde{\psi}(r), r]$ is a linear functional of the stochastic fields.

As noted above both (59) and (61) give the drift and diffusion field terms at spatial position r in the Ito stochastic field equations as spatial integrals of the stochastic fields at other positions s. Thus in general the ito stochastic field equations are *non-local* - which increases the complexity for numerical calculations. However, as we will see in Section 5 there are important cases where the stochastic field equations remain *local*.

Thus the classical and noise field terms are given by

$$\left(\frac{\partial}{\partial t}\widetilde{\psi}_{\alpha A}(r)\right)_{class} = \sum_{i\beta\, j} L_{\beta j}^{A\,\alpha i} \int ds\, \xi_{\alpha\, i}^A(r)\, \xi_{\beta\, j}^A(s)^*\, \widetilde{\psi}_{\beta A}(s) \tag{62}$$

$$\left(\frac{\partial}{\partial t}\widetilde{\psi}_{\alpha A}(r)\right)_{noise} = \sum_a \sum_{iB\beta j} K_{B\, \beta j,\, a}^{A\, \alpha i} \int ds\, \xi_{\alpha\, i}^A(r)\, \xi_{\beta\, j}^B(s)^*\, \widetilde{\psi}_{\beta B}(s)\, \Gamma_a(t_+) \tag{63}$$

This shows the key result in the stochastic field case that the Ito stochastic field equations are still linear in the stochastic fields, and and thus the fields at time $t+\delta t$ are related linearly to the fields at time t and also the linear transformation (though stochastic due to the Γ_a) only involves c-numbers. This feature still applies even if the equations are non-local.

4.3 Stochastic Averages - Classical and Noise Fields

In the case of the classical field

$$\overline{\left(\frac{\partial}{\partial t}\widetilde{\psi}_{\alpha A}(r,t_{1+})\right)_{class}} = 0 \tag{64}$$

so the stochastic average is zero.

For the noise field - average fluctuations zero, average product two noise $\rightarrow$ diffusion matrix and delta correlated in time

$$\begin{aligned}
&\overline{\left(\frac{\partial}{\partial t}\widetilde{\psi}_{\alpha A}(r,t_{1+})\right)_{noise}} = 0 \\
&\overline{\left(\frac{\partial}{\partial t}\widetilde{\psi}_{\alpha A}(r,t_{1+})\right)_{noise}\left(\frac{\partial}{\partial t}\widetilde{\psi}_{\beta B}(s,t_{2+})\right)_{noise}} \\
= \quad &\overline{D_{\alpha A\,\beta B}(\widetilde{\psi}(r,t_{1,2}),r;\widetilde{\psi}(s,t_{1,2}),s)} \times \delta(t_1-t_2).
\end{aligned} \tag{65}$$

so the stochastic average is zero and the stochastic average of the product of two noise fields are delta correlated in the time difference and given by the stochastic average of the relevant diffusion matrix elements. The stochastic average of the products of odd numbers of noise fields are zero, and for products of even numbers of noise fields the result is the sums of products of pairs of noise fields.

4.4 Case of P Distribution

In the case of the P distribution functional the Ito stochastic field equations have the same general features as that for in (51) for the B distribution. The classical and noise fields will of course differ from those for the B distribution, since the functional Fokker-Planck equations are not the same. The relationship between the classical and noise field quantities and the drift vector and diffusion matrix functionals in the functional Fokker-Planck equation for the P distribution functional is the same as in (55) and (56). However, the classical field term is no longer a linear functional of the stochastic fields.

4.5 Numerical Issues

Analogous to the treatment based on separate modes, the Ito stochastic field equations (51) can be written in the form

$$\begin{aligned}
&\widetilde{\psi}_{\alpha A}(r,t+\delta t) \\
= \quad &\sum_{\beta B}\int ds \left(\begin{array}{c} \delta_{\alpha A,\beta B}\,\delta(r-s) + \delta_{A,B}\sum_{i\,j} L^{A\,\alpha i}_{\beta j}\,\xi^{A}_{\alpha\,i}(r)\,\xi^{A}_{\beta\,j}(s)^*\,\delta t \\ +\sum_{a}\sum_{ij} K^{A\,\alpha i}_{B\,\beta j,\,a}\,\xi^{A}_{\alpha\,i}(r)\,\xi^{B}_{\beta\,j}(s)^*\,\delta w_a(t_+) \end{array}\right)\widetilde{\psi}_{\beta B}(s,t)
\end{aligned} \tag{66}$$

$$= \sum_{\beta B}\int ds\;\Theta_{\alpha A,\,\beta B}(r,s,t_+)\,\widetilde{\psi}_{\beta B}(s,t) \tag{67}$$

where we have used the linearity features (59) and (61) for the Ito terms $C_{\alpha A}[\widetilde{\psi}(r),r]$ and $B_a^{\alpha A}[\widetilde{\psi}(r),r]$ and where the *Wiener stochastic variable* $\widetilde{w}_a(t)$ and its increment are

$$\widetilde{w}_a(t) = \int_{t_0}^{t} dt_1 \Gamma_a(t_1) \qquad \delta\widetilde{w}_a(t_+) = \int_{t+}^{t+\delta t} dt_1\, \Gamma_a(t_1) \tag{68}$$

An important result for the stochastic average of the product of two Wiener increments is

$$\overline{\delta\widetilde{w}_a(t_+)\delta\widetilde{w}_b(t_+)} = \delta_{a,b}\,\delta t \tag{69}$$

The properties of Wiener increments are set out in Eq. (85) in Ref. [1].

The result in (66) shows that the stochastic field $\widetilde{\psi}_{\alpha A}(r,t+\delta t)$ at position r at time $t+\delta t$ is related *linearly* to the stochastic fields $\widetilde{\psi}_{\beta B}(s,t)$ at various positions s at time t, and that the quantities involved in this linear relationship are all *c-numbers* - these involving both stochastic variables $\delta w_a(t_+)$ and non-stochastic quantities such as mode functions $\xi^A_{\alpha\, i}(r)$, $\xi^B_{\beta\, j}(s)$ and quantities such as $L^{A\,\alpha i}_{\beta j}$ and $K^{A\,\alpha i}_{B\,\beta j,\,a}$ obtainable from the functional Fokker-Planck equation. This outcome is analogous to that occuring for separate modes and is the basis for possible numerical treatments. Note however the additional complexity due to spatial integration and differing internal components. In practice, the spatial integration process is often not required - the stochastic fields on each side of (66) involving the same spatial position r.

We can then use (67) and the uncorrelation property (44) to derive a result for the stochastic average of products of the $\widetilde{\psi}_{\alpha A}(r,t+\delta t)$ at time $t+\delta t$ showing that it involves stochastic average of products of the $\widetilde{\psi}_{\beta B}(r,t)$ at time t with the same number of factors, and that the relationship between the two sets of stochastic averages is linear - with a transformation matrix that only involves a stochastic average of c-numbers. These c-numbers are of course stochastic as they involve Wiener increments as well as quantities obtained from the drift and diffusion terms in the functional Fokker-Planck equation. We find that

$$\begin{aligned}
&\overline{\widetilde{\psi}_{\alpha A}(r_1,t+\delta t)\widetilde{\psi}_{\beta B}(r_2,t+\delta t)...\widetilde{\psi}_{\sigma S}(r_p,t+\delta t)} \\
= \;& \sum_{\eta R}\int ds_1 \sum_{\theta Q}\int ds_2 \;...\sum_{\omega Z}\int ds_p \\
& \times \left[\Theta_{\alpha A,\,\eta R}(r_1,s_1,t_+)\;\Theta_{\beta B,\,\theta Q}(r_2,s_2,t_+)\,...\Theta_{\sigma S,\,\omega Z}(r_p,s_p,t_+)\right]_{\;stoch\,aver} \\
& \times\overline{\widetilde{\psi}_{\eta R}(s_1,t)\,\widetilde{\psi}_{\theta Q}(s_2,t)\,...\widetilde{\psi}_{\omega Z}(s_p,t)}
\end{aligned} \tag{70}$$

This result is analogous to Eq. (88) in Ref. [1]. In principle, this result enables stochastic averages of products of stochastic Grassmann fields at the end of a small time interval to be determined from the set of such stochastic averages (of the same number of factors) at the beginning of the interval. By dividing a finite time interval up into a number of small intervals the set of stochastic averages of a given number of factors can be evolved from the set at an initial time with the same number of factors to that applying at a later time. Then the

initial set of stochastic averages can be obtained from the initial density operator using the expressions (24) and (25) for initial position Fock state populations and coherences - the phase space functional integrals being equivalent to the stochastic averages in accord with (39) and (170). The calculation is of the same order of complexity as for the separate mode case if the total number of space grid elements is the same as the number of modes involved.

5 Application to Fermion Field Models

5.1 A. Two Component Zero Range Model

As a typical application of fermion field models we consider a trapped Fermi gas of spin 1/2 fermionic atoms with spin conserving collisions of zero range between pairs of atoms .Here there are two distinct internal states corresponding to spin up and spin down atoms, designated by $\alpha = u(\uparrow), d(\downarrow)$, with $-\alpha$ referring to the opposite spin state. The Fermi gas is isolated from the environment so no relaxation effects are involved. This model was also considered by Plimak et al [8].

5.1.1 Hamiltonian

The Hamiltonian is written in terms of the field operators $\hat{\Psi}_\alpha(r)$, $\hat{\Psi}_\alpha(r)^\dagger$ as

$$\begin{aligned}\hat{H}_f &= \int dr \left(\sum_\alpha \frac{\hbar^2}{2m} \nabla\hat{\Psi}_\alpha(r)^\dagger \cdot \nabla\hat{\Psi}_\alpha(\ r) + \sum_\alpha \hat{\Psi}_\alpha(r)^\dagger V_\alpha \hat{\Psi}_\alpha(r) \right. \\ &\qquad \left. + \frac{g_f}{2} \sum_\alpha \hat{\Psi}_\alpha(r)^\dagger \hat{\Psi}_{-\alpha}(r)^\dagger \hat{\Psi}_{-\alpha}(r) \hat{\Psi}_\alpha(r) \right) \qquad (71) \\ &= \widehat{K} + \widehat{V} + \widehat{U} \qquad (72)\end{aligned}$$

and is the sum of kinetic energy, trap potential energy and collision interaction energy terms.

5.1.2 Functional Fokker-Planck Equation - B Distribution

The functional Fokker-Planck equation can be obtained via applying the correspondence rules to the Liouville-non Neumann equation. The contributions can be written as a sum of terms from $\widehat{K}, \widehat{V}, \widehat{U}$. As the derivation requires some complex manipulation it is presented in Appendix 13. In the case considered the functional Fokker-Planck equations can be written in a simpler notation in which the field operators $\hat{\Psi}_\alpha(r)$, $\hat{\Psi}_\alpha(r)^\dagger$ are represented via Grassmann fields $\psi_\alpha(r)$ and $\psi_\alpha^+(r)$ respectively - so in presenting the final equations we will not use the $A = 1, 2$ notation to distinguish $\psi_\alpha(r)$ and $\psi_\alpha^+(r)$ as we did for the general theory.

The kinetic energy term is

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}B[\psi(r)]\right)_K \\
= \;& \frac{i}{\hbar}\int ds\,\left[\left\{\left(\frac{\hbar^2}{2m}\nabla^2\psi_u(s)\,B[\,\psi(r)]\right)\frac{\overleftarrow{\delta}}{\delta\psi_u(s)}\right\}\right. \\
&+\left\{\left(\frac{\hbar^2}{2m}\nabla^2\psi_d(s)\,B[\,\psi(r)]\right)\frac{\overleftarrow{\delta}}{\delta\psi_d(s)}\right\} \\
&-\left\{\left(\frac{\hbar^2}{2m}\nabla^2\psi_u^+(s)\,B[\,\psi(r)]\right)\frac{\overleftarrow{\delta}}{\delta\psi_u^+(s)}\right\} \\
&\left.-\left\{\left(\frac{\hbar^2}{2m}\nabla^2\psi_d^+(s)\,B[\,\psi(r)]\right)\frac{\overleftarrow{\delta}}{\delta\psi_d^+(s)}\right\}\right]
\end{aligned} \tag{73}$$

and only contributes to the drift term.

The trap potential energy term is

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}B[\psi(r)]\right)_V \\
= \;& \frac{i}{\hbar}\int ds\,\left[-\left\{(V_u\psi_u(s)\,B[\,\psi(r)])\,\frac{\overleftarrow{\delta}}{\delta\psi_u(s)}\right\}-\left\{(V_d\psi_d(s)\,B[\,\psi(r)])\,\frac{\overleftarrow{\delta}}{\delta\psi_d(s)}\right\}\right. \\
&\left.+\left\{\left(V_u\psi_u^+(s)\,B[\,\psi(r)]\right)\frac{\overleftarrow{\delta}}{\delta\psi_u^+(s)}\right\}+\left\{\left(V_d\psi_d^+(s)\,B[\,\psi(r)]\right)\frac{\overleftarrow{\delta}}{\delta\psi_d^+(s)}\right\}\right]
\end{aligned} \tag{74}$$

and also only contributes to the drift term.

The fermion-fermion interaction term is

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}B[\psi(r)]\right)_U \\
= \;& \frac{i}{\hbar}\frac{g}{2}\int ds\,\left[\left\{\psi_d(s)\psi_u(s)\,B[\psi(r)]\frac{\overleftarrow{\delta}}{\delta\psi_d(s)}\frac{\overleftarrow{\delta}}{\delta\psi_u(s)}\right\}\right. \\
&+\left\{\psi_u(s)\psi_d(s)\,B[\psi(r)]\frac{\overleftarrow{\delta}}{\delta\psi_u(s)}\frac{\overleftarrow{\delta}}{\delta\psi_d(s)}\right\} \\
&-\left\{\psi_d^+(s)\psi_u^+(s)\,B[\psi(r)]\frac{\overleftarrow{\delta}}{\delta\psi_d^+(s)}\frac{\overleftarrow{\delta}}{\delta\psi_u^+(s)}\right\} \\
&\left.-\left\{\psi_u^+(s)\psi_d^+(s)\,B[\psi(r)]\frac{\overleftarrow{\delta}}{\delta\psi_u^+(s)}\frac{\overleftarrow{\delta}}{\delta\psi_d^+(s)}\right\}\right]
\end{aligned} \tag{75}$$

and is the only contribution to the diffusion term.

5.1.3 Ito Stochastic Field Equations

A straightforward application of the general theory using (55) and (56) to determine the classical and noise field terms gives the Ito stochastic field equations in the form (51). The diffusion matrix is simple to factorise into the Takagi form [28]. After minor algebra the Ito field equations can be put in the form analogous to Eq. (81) in Ref. [1] (see also Eq.(194) in Appendix 12 for the intermediate step). We have

$$\begin{bmatrix} \tilde{\psi}_u(s,t+\delta t) \\ \tilde{\psi}_d(s,t+\delta t) \\ \tilde{\psi}_u^+(s,t+\delta t) \\ \tilde{\psi}_d^+(s,t+\delta t) \end{bmatrix} = \begin{bmatrix} \Theta_{u,u}(t^+) & \Theta_{u,d}(t^+) & 0 & 0 \\ \Theta_{d,u}(t^+) & \Theta_{d,d}(t^+) & 0 & 0 \\ 0 & 0 & \Theta^+_{u,u}(t^+) & \Theta^+_{u,d}(t^+) \\ 0 & 0 & \Theta^+_{d,u}(t^+) & \Theta^+_{d,d}(t^+) \end{bmatrix} \begin{bmatrix} \tilde{\psi}_u(s,t) \\ \tilde{\psi}_d(s,t) \\ \tilde{\psi}_u^+(s,t) \\ \tilde{\psi}_d^+(s,t) \end{bmatrix} \tag{76}$$

where the Θ quantities are given by

$$\begin{aligned} [\Theta(t^+)] &= \begin{bmatrix} 1-\frac{i}{\hbar}\left\{-\frac{\hbar^2}{2m}\nabla^2+V_u\right\}\delta t & \sqrt{\frac{ig}{2\hbar}}\{\delta\widetilde{w}_{u,d}+i\,\delta\widetilde{w}_{d,u}\} \\ \sqrt{\frac{ig}{2\hbar}}\{\delta\widetilde{w}_{u,d}-i\,\delta\widetilde{w}_{d,u}\} & 1-\frac{i}{\hbar}\left\{-\frac{\hbar^2}{2m}\nabla^2+V_d\right\}\delta t \end{bmatrix} \\ [\Theta^+(t^+)] &= \begin{bmatrix} 1+\frac{i}{\hbar}\left\{-\frac{\hbar^2}{2m}\nabla^2+V_u\right\}\delta t & \sqrt{\frac{ig}{2\hbar}}\{\delta\widetilde{w}_{u+,d+}+i\,\delta\widetilde{w}_{d+,u+}\} \\ \sqrt{\frac{ig}{2\hbar}}\{-\delta\widetilde{w}_{u+,d+}+i\,\delta\widetilde{w}_{d+,u+}\} & 1+\frac{i}{\hbar}\left\{-\frac{\hbar^2}{2m}\nabla^2+V_d\right\}\delta t \end{bmatrix} \end{aligned} \tag{77}$$

and involve the Laplacian, the trap potentials, Wiener increments (68) as well as collision and mass parameters from the Hamiltonian. Note that the Ito stochastic field equations are *local* in this case.

This form of the Ito stochastic field equations shows that the Grassmann stochastic fields at time $t+\delta t$ are linearly related to the Grassmann stochastic fields at time t via quantities that only involve c-numbers. The quantities include stochastic Wiener increments as well as the Laplacian and spatially dependent potentials, but nevertheless no Grassmann variables are involved. A similar Grassmann independent feature applied to the situation when separate modes were treated (see Eq.(81) in Ref. [1]), and is a feature restricted to treatments involving the B distribution, as Plimak et al first pointed out [8]. .

The result in (76) (analogous to Eq. (81) for separate modes in Ref. [1]) shows how Grassmann stochastic fields at any time t_f can be related to Grassmann stochastic fields at initial time t_0 via c-number stochastic quantities. The

final stochastic averages of products of Grassmann stochastic fields at initial time t_0 are determined from initial conditions via c-number calculations, just as in the separate modes case.

5.1.4 Case of Free Fermi Gas

In the case of the free Fermi gas where the trap potential is zero, a useful development of the result (76) can be obtained via introducing spatial Fourier transforms. In the case of the stochastic fields the stochastic Fourier fields $\tilde{\phi}_\alpha(k,t)$, $\tilde{\phi}^+_\alpha(k,t)$ are defined by

$$\begin{aligned}\tilde{\psi}_\alpha(s,t) &= \int \frac{dk}{(2\pi)^{3/2}}\,\exp(ik\cdot s)\,\tilde{\phi}_\alpha(k,t)\\ \tilde{\psi}^+_\alpha(s,t) &= \int \frac{dk}{(2\pi)^{3/2}}\,\exp(-ik\cdot s)\,\tilde{\phi}^+_\alpha(k,t)\end{aligned} \tag{78}$$

and the inverse equations. The equations for $\tilde{\psi}_\alpha(s,t)$ and $\tilde{\psi}^+_\alpha(s,t)$ become stochastic equations for the Fourier transforms $\tilde{\phi}_\alpha(k,t)$ and $\tilde{\phi}^+_\alpha(k,t)$. We then have

$$\begin{bmatrix} \tilde{\phi}_u(k,t+\delta t)\\ \tilde{\phi}_d(k,t+\delta t)\\ \tilde{\phi}^+_u(k,t+\delta t)\\ \tilde{\phi}^+_d(k,t+\delta t)\end{bmatrix} = \begin{bmatrix} F_{u,u}(t) & F_{u,d}(t) & 0 & 0\\ F_{d,u}(t) & F_{d,d}(t) & 0 & 0\\ 0 & 0 & F^+_{u+,u+}(t) & F^+_{u+,d+}(t)\\ 0 & 0 & F^+_{d+,u+}(t) & F^+_{d+,d+}(t)\end{bmatrix}\begin{bmatrix} \tilde{\phi}_u(k,t)\\ \tilde{\phi}_d(k,t)\\ \tilde{\phi}^+_u(k,t)\\ \tilde{\phi}^+_d(k,t)\end{bmatrix} \tag{79}$$

where the sub-matrices are

$$\begin{aligned}[F(t)] &= \begin{bmatrix} 1-\frac{i}{\hbar}\left\{\frac{\hbar^2}{2m}k^2\right\}\delta t & \sqrt{\frac{ig}{2\hbar}}\{\delta\widetilde{w}_{u,d}+i\delta\widetilde{w}_{d,u}\}\\ \sqrt{\frac{ig}{2\hbar}}\{\delta\widetilde{w}_{u,d}-i\delta\widetilde{w}_{d,u}\} & 1-\frac{i}{\hbar}\left\{\frac{\hbar^2}{2m}k^2\right\}\delta t\end{bmatrix}\\ [F^+(t)] &= \begin{bmatrix} 1+\frac{i}{\hbar}\left\{\frac{\hbar^2}{2m}k^2\right\}\delta t & \sqrt{\frac{ig}{2\hbar}}\{\delta\widetilde{w}_{u+,d+}+i\delta\widetilde{w}_{d+,u+}\}\\ \sqrt{\frac{ig}{2\hbar}}\{-\delta\widetilde{w}_{u+,d+}+i\delta\widetilde{w}_{d+,u+}\} & 1+\frac{i}{\hbar}\left\{\frac{\hbar^2}{2m}k^2\right\}\delta t\end{bmatrix}.\end{aligned} \tag{80}$$

which though stochastic no longer involve the Laplacian operator, which is replaced by the c-number k^2. The equations can then be solved numerically.

The field operators $\widehat{\psi}_\alpha(r)$ and $\widehat{\psi}^\dagger_\alpha(r)$ themselves may be replaced by momentum field operators $\widehat{\phi}_\alpha(k)$ and $\widehat{\phi}^\dagger_\alpha(k)$ using equations analogous to (78). The

creation field momentum operators $\widehat{\phi}_{\alpha}^{\dagger}(k)$ create a fermionic atom with internal state α and having momentum k, and these are the continuum versions of the mode creation operators $\widehat{c}_{k\,\alpha}$ discussed in Section 6 of Paper I ([1]). Allowing for the difference between box normalisation and delta function normalisation of the free space mode functions, the application of the approach in the present section to the simple four mode model in Section 6 of Paper I ([1]) yields the same results as before for the coherence between momentum Fock states.

5.1.5 Case of Optical Lattice

In the case where the Fermi gas is trapped in an optical lattice the trap potential is periodic, and a useful development of the result (76) can be obtained via introducing transforms.based on Bloch functions. The stochastic field functions in terms of Bloch functions $\chi_{\alpha}^{k,a}(\mathbf{r})$, where k would range over the Brillouin zone and a would list the different bands. Such functions obey an eigenvalue equation of the form

$$\left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{\alpha}\right\}\chi_{\alpha}^{k,a}(r) = \hbar\omega_{\alpha}^{k,a}\chi_{\alpha}^{k,a}(r). \tag{81}$$

This enables the Laplacian and the trap potential to be eliminated in favour of a c-number, the Bloch energy $\hbar\omega_{\alpha}^{k,a}$. A solution via Bloch function transforms can be developed, and applied in numerical work.

5.2 B. Multi-Component Fields with Non-Zero Range Interactions

We can also treat the more general case of fermion fields with several components, where the fields have non-zero range interactions as well as each being coupled to external potentials. Many situations in cold Fermi gases can be studied using this model. Here there are several distinct internal spin states designated by α. The Fermi gas is isolated from the environment so no relaxation effects are involved. All atoms have the same mass m.

5.2.1 Hamiltonian

The Hamiltonian is written in terms of the field operators $\hat{\Psi}_{\alpha}(r)$, $\hat{\Psi}_{\alpha}(r)^{\dagger}$ as

$$\begin{aligned}\hat{H}_f &= \int dr \left(\sum_{\alpha}\frac{\hbar^2}{2m}\nabla\hat{\Psi}_{\alpha}(r)^{\dagger}\cdot\nabla\hat{\Psi}_{\alpha}(\,r) + \sum_{\alpha\beta}\hat{\Psi}_{\alpha}(r)^{\dagger}\,V_{r;r}^{\alpha;\beta}\,\hat{\Psi}_{\beta}(r)\right.\\ &\qquad \left. + \frac{1}{2}\int ds \sum_{\alpha\beta;\gamma\delta}\hat{\Psi}_{\alpha}(r)^{\dagger}\hat{\Psi}_{\beta}(s)^{\dagger}\,V_{rs;rs}^{\alpha\beta;\gamma\delta}\,\hat{\Psi}_{\delta}(s)\hat{\Psi}_{\gamma}(r)\right) \qquad (82)\\ &= \widehat{K} + \widehat{V} + \widehat{U} \qquad (83)\end{aligned}$$

and is the sum of kinetic energy, one body interaction energy and two body interaction energy terms. For simplicity the quantities $V_{r;r}^{\alpha;\beta}$ and $V_{rs;rs}^{\alpha\beta;\gamma\delta}$ are

real. The one body interaction energy terms describe both the interaction of the fermions with trapping potentials ($\alpha = \beta$) as well as interactions with coupling potentials ($\alpha \neq \beta$). Thus a fermion at position r with internal component α is changed into a fermion at position r but with different component β. The quantities $V_{r;r}^{\alpha;\beta}$ reflect the position dependence of these trapping potential and coupling interactions. In the two body interaction energy terms a fermion at position r and internal component γ interacts with a fermion at position s and internal component δ, resulting in the fermion at position r changing into internal component α and the fermion at position s changing into internal component β. The quantities $V_{rs;rs}^{\alpha\beta;\gamma\delta}$ reflect the position dependences of these two body interactions, including there being a non-zero range involved - hence the double spatial integral over both r and s.

By interchanging r and s and from the hermitiancy of the one and two body energy terms several symmetry relations can be established. We find that

$$V_{r;r}^{\alpha;\beta} = V_{r;r}^{\beta;\alpha} \tag{84}$$

$$V_{rs;rs}^{\alpha\beta;\gamma\delta} = V_{sr;sr}^{\beta\alpha;\delta\gamma} \qquad V_{rs;rs}^{\alpha\beta;\gamma\delta} = V_{rs;rs}^{\gamma\delta;\alpha\beta} \tag{85}$$

5.2.2 Functional Fokker-Planck Equation - B Distribution

In the case considered the functional Fokker-Planck equations can be wriiten in a simpler notation in which the field operators $\hat{\Psi}_\alpha(r)$, $\hat{\Psi}_\alpha(r)^\dagger$ are represented via Grassmann fields $\psi_\alpha(r)$ and $\psi_\alpha^+(r)$ respectively - here we will not use the $A = 1, 2$ notation to distinguish $\psi_\alpha(r)$ and $\psi_\alpha^+(r)$ as we did previously. The functional Fokker-Planck equations can be obtained via the application of the correspondence rules in (38). They are given by

$$\begin{aligned}
\frac{\partial}{\partial t}B[\psi] &= -\sum_\alpha \int dr\, (A_\alpha[r]\, B[\psi]) \frac{\overleftarrow{\delta}}{\delta\psi_\alpha(r)} - \sum_\alpha \int dr\, (A_\alpha^+[r]\, B[\psi]) \frac{\overleftarrow{\delta}}{\delta\psi_\alpha^+(r)} \\
&\quad + \frac{1}{2}\sum_{\alpha,\beta} \iint dr\, ds\, (D_{\alpha\,\beta}[r,s]\, B[\psi]) \frac{\overleftarrow{\delta}}{\delta\psi_\beta(s)} \frac{\overleftarrow{\delta}}{\delta\psi_\alpha(r)} \\
&\quad + \frac{1}{2}\sum_{\alpha,\beta} \iint dr\, ds\, (D_{\alpha\,\beta}^{+;+}[r,s]\, B[\psi]) \frac{\overleftarrow{\delta}}{\delta\psi_\beta^+(s)} \frac{\overleftarrow{\delta}}{\delta\psi_\alpha^+(r)}
\end{aligned} \tag{86}$$

where the drift vector and diffusiom matrix elements are given by

$$\begin{aligned}
A_\alpha[r] &= +\frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla^2\psi_\alpha(r) + \sum_\beta V_{r;r}^{\alpha;\beta}\psi_\beta(r)\right) \\
A_\alpha^+[r] &= -\frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla^2\psi_\alpha^+(r) + \sum_\beta V_{r;r}^{\alpha;\beta}\psi_\beta^+(r)\right)
\end{aligned} \tag{87}$$

and

$$D_{\alpha\,\beta}[r,s] = -\frac{i}{\hbar}\left(\sum_{\gamma\delta}\psi_\gamma(r)\,V^{\alpha\beta;\gamma\delta}_{rs;rs}\,\psi_\delta(s)\right)$$
$$D^{+;+}_{\alpha\,\beta}[r,s] = +\frac{i}{\hbar}\left(\sum_{\gamma\delta}\psi^+_\gamma(r)\,V^{\alpha\beta;\gamma\delta}_{rs;rs}\,\psi^+_\delta(s)\right) \qquad (88)$$

Note that there are no cross terms $D^{+;-}_{\alpha\,\beta}[r,s]$ or $D^{-;+}_{\alpha\,\beta}[r,s]$. The drift tems depend linearly on the Grassmann fields, whilst the diffusion terms depend bilinearly on the Grassmann fields. Furthermore, the Grassmann fields $\psi_\alpha(r)$ and $\psi^+_\alpha(r)$ are not coupled to each other, so the Ito stochastic field equations for $\widetilde{\psi}_\alpha(r)$ do not couple to those for $\widetilde{\psi}^+_\alpha(r)$ - so this simplifies the theory. The same situation applied previously to the two component zero range interaction case treated above.

We also see that the diffusion matrix is *anti-symmetric*.

$$D_{\alpha\,\beta}[r,s] = -D_{\beta\,\alpha}[s,r] \qquad D^{+;+}_{\alpha\,\beta}[r,s] = -D^{+;+}_{\beta\,\alpha}[s,r] \qquad (89)$$

This result is easily established using the symmetry properties (85) for the $V^{\alpha\beta;\gamma\delta}_{rs;rs}$ together with the anti-commutation of the Grassmann field functions.

5.2.3 Ito Stochastic Field Equations

We first must find matrices B and B^+ such that $D = BB^T$ and $D^{+;+} = B^+(B^+)^T$. This can be carried out via a generalisation of the proceedure in Section 5 of Paper I [1] for the separate modes case. We write $D_{\alpha\,\beta}[r,s]$ and $D^{+;+}_{\alpha\,\beta}[r,s]$ in the forms

$$D_{\alpha r;\beta s} = \sum_{\gamma\delta} Q^{\alpha r;\,\beta s}_{\gamma r;\,\delta s}\,\psi_\gamma(r)\psi_\delta(s) \qquad D^{+;+}_{\alpha r;\beta s} = \sum_{\gamma\delta}(Q^+)^{\alpha r;\,\beta s}_{\gamma r;\,\delta s}\,\psi^+_\gamma(r)\psi^+_\delta(s)$$

where

$$(Q)^{\alpha r;\,\beta s}_{\gamma r;\,\delta s} = -\frac{i}{\hbar}V^{\alpha\beta;\gamma\delta}_{rs;rs} \qquad (Q^+)^{\alpha r;\,\beta s}_{\gamma r;\,\delta s} = +\frac{i}{\hbar}V^{\alpha\beta;\gamma\delta}_{rs;rs}$$

are new matrices created from the two body interaction terms $V^{\alpha\beta;\gamma\delta}_{rs;rs}$. These matrices Q and Q^+ are both *symmetric* - which follows from the antisymmetry of D and $D^{+;+}$. The matrices Q and Q^+ are such that the rows are listed by the double symbol $\alpha r, \gamma r$ and the columns by the double symbol βs,δs. The proceedure is analogous to the creation of the matrix elements $Q^{p;q}_{r;s}$ in Section **??** where the rows were listed by the double symbol p, r and columns listed as q, s. However, in this field theory situation we have implicitly discretised the spatial positions first.

As in Section 5 of Paper I [1] we can then use Takagi factorisation [28] of the symmetric matrices Q and Q^+ to write

$$(Q)^{\alpha r;\,\beta s}_{\gamma r;\,\delta s} = \sum_a K^{\alpha r}_{\gamma r;\,a}\,K^{\beta s}_{\delta s;\,a} \qquad (Q^+)^{\alpha r;\,\beta s}_{\gamma r;\,\delta s} = \sum_b (K^+)^{\alpha r}_{\gamma r;\,b}\,(K^+)^{\beta s}_{\delta s;\,b} \qquad (90)$$

and then it follows that with

$$B_a^{\alpha r} = \sum_\gamma K^{\alpha r}_{\gamma r;\, a}\, \psi_\gamma(r) \qquad (B^+)_b^{\alpha r} = \sum_\gamma (K^+)^{\alpha r}_{\gamma r;\, b}\, \psi^+_\gamma(r) \tag{91}$$

Note that K and K^+ are inter-related via $K^+ = iK$, since $KK^T = -(K^+)(K^+)^T$.

Hence using the general results (51), (55), (56) and the particular formulae for the present case (87) and (91) the Ito stochastic field equations are given by

$$\begin{aligned}
\delta\widetilde{\psi}_\alpha(r) &\equiv \widetilde{\psi}_\alpha(r, t+\delta t) - \widetilde{\psi}_\alpha(r,t) \\
&= -\frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla^2\widetilde{\psi}_\alpha(r,t) + \sum_\beta V^{\alpha;\beta}_{r;r}\widetilde{\psi}_\beta(r,t)\right)\delta t \\
&\quad + \sum_a \sum_\gamma K^{\alpha r}_{\gamma r;\, a}\, \widetilde{\psi}_\gamma(r,t)\, \delta\widetilde{W}_a(t_+)
\end{aligned} \tag{92}$$

and

$$\begin{aligned}
\delta\widetilde{\psi}^+_\alpha(r) &\equiv \widetilde{\psi}^+_\alpha(r, t+\delta t) - \widetilde{\psi}^+_\alpha(r,t) \\
&= +\frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla^2\widetilde{\psi}^+_\alpha(r,t) + \sum_\beta V^{\alpha;\beta}_{r;r}\widetilde{\psi}^+_\beta(r,t)\right)\delta t \\
&\quad + \sum_b \sum_\gamma (K^+)^{\alpha r}_{\gamma r;\, b}\, \widetilde{\psi}^+_\gamma(r,t)\, \delta\widetilde{W}_b(t_+)
\end{aligned} \tag{93}$$

Once again the Ito stochastic field equations are purely local and the Ito stochastic fields at time $t + \delta t$ are related linearly to those at time t via quantities that only involve c-numbers - those these are stochastic due to the Wiener noise increments $\delta\widetilde{W}_{a,b}(t_+)$. We also note that the equations for $\widetilde{\psi}_\alpha(r)$ and $\widetilde{\psi}^+_\alpha(r)$ are not coupled. Depending on the number of component fields and on the number of spatial grid points needed to represent the non-zero range two body terms $V^{\alpha\beta;\gamma\delta}_{rs;rs}$, the numbers of Wiener increments needed could be very large. However, at least in principle the Ito stochastic field equations are numerically solvable

6 Summary and Conclusions

A phase space theory for fermions has been presented for fermion systems based on distribution functionals, which replace the density operator and involve Grassmann fields representing anti-commuting fermion field annihilation, creation operators. This paper is an extension of a previous paper (Paper I, Ref. [1]) for fermion systems based on separate modes, in which the density operator is replaced by a distribution function depending on Grassmann phase space variables which represent the mode annihilation and creation operators. The distribution functional involving Grassmann fields and the distribution function involving Grassmann phase space variables are equivalent - being two alternative ways of representing the density operator, since the Grassmann fields can be expanded in terms of orthonormal mode functions with the Grassmann phase variables as expansion coefficients. This field theory extension is important in the case when large numbers of fermions are involved, since the Pauli exclusion principle results in too many modes to treat separately. Quantum correlation functions, Fock state populations and coherences are given as phase space Grassmann functional integrals. Functional Fokker-Planck equations for the distribution functional have been obtained both starting from the Fokker-Planck equations for the distribution function, and also based on the correspondence rules for the effect of fermion field annihilation, creation operators on the density operator. Ito stochastic field equations are derived, both starting from the Ito stochastic equations for stochastic phase space variables and also directly from the functional Fokker-Planck equation. The Ito stochastic field equations can be non-local, but in many cases they are local. The classical and noise fields shown to be determined from the drift and diffusion terms in functional Fokker-Planck equation. For the B distribution functional case the stochastic Grassmann fields at a later time are related linearly to those at an earlier time via c-number quantities involving Wiener increments and other terms in the Ito stochastic field equations. This makes numerical calculations possible, with the stochastic averages of products Grassmann fields at a later time being linearly related to such products (of the same order) at an initial time. These initial time stochastic averages are obtainable from the initial conditions. Applications of the theory to a trapped interacting Fermi gas is presented, both for a two component case with zero range interactions and multi-component cases with non-zero range interactions. The Ito stochastic field equations are local in both the zero range and finite range interaction cases treated.

It will be of interest to apply the theory to various topics in degenerate Fermi gases. The utility of the theory could first be tested on some well-understood fermion systems that have been treated by other methods, such determining the size of a Cooper pair or treating Feshbach resonance in Fermi gases or deriving the two fluid hydrodynamuic equations for Fermi liquids. Some further development of the formalism could be worthwhile, such as obtaining formulae for two-time quantum correlation functions, as these are linked to fermionic excitations. Some work on this already exists [12]. Degenerate quantum gases involving both fermions and bosons often occur, so expanding the formalism

to include both is desirable, and again some work has already been done [13], [15]. Also, extending the phase space method to include stochastic gauges could be worthwhile to facilitate numerical calculations, and here some work on this for both bosons [29] and fermions [30] using c-number phase space theory has been carried out. It is clearly desirable to carry out numerical applications on fermion systems to fully test out the Grassmann phase space theory.

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8 Appendix A - Systems of Identical Fermions - Key Concepts and Notation

In this Appendix we set out the basic concepts and notation used in this paper. More details on these items are set out in Ref. [1].

8.1 Modes, Annihilation and Creation Operators, Fock States.

In this paper we consider system of identical fermions, which in quantum-atom optics could be fermionic atoms or molecules. The fermions will be associated with position and momentum operators describing their centre of mass motion, but also may exist in different internal states such as those with differing hyperfine quantum numbers. The approach used here is that of second quantisation (see for example [2], [24]), which is based on the existence of *modes* (or single particle states) $|\phi_i\rangle$ - listed in a prescribed order $1, 2, .., i, .., n$ - which the identical fermions may occupy. Modes associated with different internal states $|\,\alpha\rangle$ may be designated $|\phi_{\alpha i}\rangle = \left|\phi_i^{(\alpha)}\right\rangle |\,\alpha\rangle$ in which $\left|\phi_i^{(\alpha)}\right\rangle$ describes the centre of mass motion, but for the moment we will ignore the internal state since experiments can be carried out with all fermions in the same internal state. The position representation of the mode $|\phi_i\rangle$ is $\phi_i(\mathbf{r}) = \langle \mathbf{r}\,|\,\phi_i\rangle$ are is referred to as the mode function. The modes are chosen to be orthonormal and complete.

$$\int d\mathbf{r}\,\phi_i^*(\mathbf{r})\phi_j(\mathbf{r}) = \delta_{i;j} \qquad \langle \phi_i\,|\,\phi_j\rangle = \delta_{i;j} \tag{94}$$

$$\sum_i \phi_i^*(\mathbf{r})\phi_i(\mathbf{r}') = \boldsymbol{\delta}(\mathbf{r}-\mathbf{r}') \qquad \sum_i |\phi_i\rangle\langle\phi_i| = \widehat{1} \tag{95}$$

The Pauli exclusion principle precludes the existence of multi-fermion states where more than one fermion occupies any particular mode. Multi-fermion *Fock states* (which are represented in first quantisation as anti-symmetrised products of the occupied single particle state) can be specified by just stating the number of fermions $\nu_i = 0, 1$ that occupy each mode $|\phi_i\rangle$. Fock states can be written in second quantisation form by introducing fermion *creation* and *annihilation operators*. These are denoted $\hat{c}_i^\dagger$, $\hat{c}_i$ and satisfy the anticommutation rules

$$\begin{aligned} &\{\hat{c}_i, \hat{c}_j^\dagger\} \equiv \hat{c}_i\hat{c}_j^\dagger + \hat{c}_j^\dagger\hat{c}_i = \delta_{ij}, \\ &\{\hat{c}_i, \hat{c}\} = 0 = \{\hat{c}_i^\dagger, \hat{c}_j^\dagger\}. \end{aligned} \tag{96}$$

We note that $\hat{c}_i^2 = (\hat{c}_i^\dagger)^2 = \widehat{0}$ as may also be seen from the anti-commutation rules. This similarity to the Grassmann variables is the reason why Grassmann phase space variables are chosen for fermions. The Fock states in second quantisation form are

$$|\nu\rangle = (\hat{c}_{\phi_1}^\dagger)^{\nu_1}(\hat{c}_{\phi_2}^\dagger)^{\nu_2}\cdots(\hat{c}_{\phi_n}^\dagger)^{\nu_n}|0\rangle = \left|\Phi_{\{\nu\}}\right\rangle \tag{97}$$

where the occupancy notation $\nu \equiv \{\nu_1, \nu_2, .., \nu_i, ..\nu_n\}$ includes all the modes. The vacuum state $|0\rangle$ is an extra state introduced in second quantisation and represents the situation in which no fermions are present. The convention in (97) is that the $\hat{c}^\dagger_{\phi_i}$ are arranged in order with $\hat{c}^\dagger_{\phi_1}$ on the left and $\hat{c}^\dagger_{\phi_n}$ on the right, and where $(\hat{c}^\dagger_{\phi_i})^0 = \hat{1}$, $(\hat{c}^\dagger_{\phi_i})^1 = \hat{c}^\dagger_{\phi_i}$. The modes are listed in the prescribed order to avoid duplication of the Fock states. The Fock states are orthogonal and normalised

$$\langle \nu | \xi \rangle = \delta_{\nu_1 \xi_1} \cdots \delta_{\nu_n \xi_n} \tag{98}$$

and are eigenstates of the fermion *number operator* $\widehat{N} = \sum_i \widehat{n}_i$, where $\widehat{n}_i = \hat{c}^\dagger_i \hat{c}_i$ is the number operator for mode $|\phi_i\rangle$. It is easy to show from the anticommutation rules that the eigenvalues for the fermionic $\widehat{n}_i$ are restricted to $0, 1$. We have

$$\widehat{N}\,|\nu\rangle = N\,|\nu\rangle \tag{99}$$

$$N = \sum_i \nu_i \tag{100}$$

The Fock states form a basis for a Hilbert space, as we will see.

From the anticommutation rules we can then see that

$$\begin{aligned} \hat{c}_i |\nu_1; \cdots; 0; \cdots, \nu_n\rangle &= 0 \\ \hat{c}_i |\nu_1; \cdots; 1; \cdots, \nu_n\rangle &= (-1)^{\eta_i} |\nu_1; \cdots; 0; \cdots, \nu_n\rangle \end{aligned} \tag{101}$$

where $(-1)^{\eta_i} = +1$ or -1 according to whether there are an even or odd number of modes listed preceding the mode $|\phi_i\rangle$ which are occupied ($\eta_i = \sum_{j<i} \nu_j$).

8.2 Quantum States and Super-selection Rules

The general physical *pure state* $|\Psi\rangle$ for N identical fermions can be written as a quantum superposition of the basis Fock states.

$$|\Phi\rangle_N = \sum_{\nu_1 \cdots \nu_n} B_N(\nu) |\nu\rangle \tag{102}$$

$$\sum_\nu |B_N(\nu)|^2 = 1, \tag{103}$$

where the sum runs over the occupancy numbers $\nu \equiv \{\nu_1 \cdots \nu_n\}$, the $B_N(\nu)$ are complex coefficients and where $\sum_{i=1}^n \nu_i = N$. Note that the sum is only over occupation numbers, subject to the constraint that the total occupancy is N. The last equation ensures that the state is normalised. If we consider all the states $|\nu\rangle$ arranged in order of total occupancy $\sum_i \nu_i = 0, 1, 2, 3 \cdots$ a heirarchy of basis states for identical particle systems with various total particle numbers $N = 0, 1, 2 \cdots$ can be listed. It is then convenient to mathematically define a

Hilbert space (*Fock space*) describing identical fermion systems with all possible total particle numbers N containing all superpositions of the form

$$|\Phi\rangle = \sum_N C_N |\Phi\rangle_N = \sum_{\nu_1 \cdots \nu_n} B(\nu)|\nu\rangle \tag{104}$$

where the C_N are complex coefficients. In the last expression the restriction $\sum_{i=1}^n \nu_i = N$ does not apply to the $B(\nu)$. Such a state $|\Phi\rangle$ is not physical, as *super-selection rules* (SSR) [25] do not allow superpositions of states with differing total particle numbers (see [26], [27] and references therein for recent discussions on SSR). However states of this form are useful mathematically, even though the pure physical states are restricted to subspaces of this Hilbert space. An example of non-physical states are the *fermion coherent states* (see Ref. [1] for their definition and properties), where not only are there a linear combinations of Fock states with differing fermion number, but also the expansion coefficients are Grassmann numbers instead of c-numbers.

However, the physical states for identical particle systems are not restricted to pure states of the form (102). Even when the number of particles N is prescribed, for *closed systems* there are *mixed states* described by a quantum density operator $\hat{\rho}_N$ of the form

$$\hat{\rho}_N = \sum_\nu \sum_\xi \rho_N(\nu, \xi)|\nu\rangle\langle\xi|. \tag{105}$$

rather than a state vector. The complex coefficients $\rho_N(\nu, \xi)$ are the density matrix elements. The requirements that the density operator is Hermitian, $\hat{\rho}_N = \hat{\rho}_N^\dagger$, has unit trace $\mathrm{Tr}(\hat{\rho}_N) = 1$, and for a mixed state satisfies the condition $\mathrm{Tr}(\hat{\rho}_N^2) < 1$, lead to well-known constraints on the density matrix elements. For pure states we can write $\hat{\rho}_N = |\Phi\rangle_N\langle\Phi|_N$ and $\mathrm{Tr}\hat{\rho}_N^2 = 1$ and in this case the density matrix elements are $\rho_N(\nu, \xi) = B_N(\nu)B_N^*(\xi)$.

For *open systems* (which particles can enter or leave), physical states can be prepared in which the number of identical fermions is not prescribed. A generalisation of the system state is required to incorporate such cases where the system composition is indefinite. Such states are mixed and also described by a density operator $\hat{\rho}$. However, super-selection rules require that the density operator can only be of the form

$$\hat{\rho} = \sum_N f_N \hat{\rho}_N \tag{106}$$

where $\sum_N f_N = 1, f_N \geq 0$, which only involves component density operators $\hat{\rho}_N$ for systems with specified total fermion numbers N, each weighted by real, positive f_N. The density operator satisfies the earlier requirements for mixed states. Pure states have only one $f_N = 1$ and all others vanish. More general density operators such as

$$\hat{\rho} = \sum_N \sum_M \rho_{N,M} |\Phi\rangle_N\langle\Phi|_M \tag{107}$$

with non-zero $\rho_{N,M}$ do not represent physical states, even though they are mathematical operators in the general mathematical Hilbert space. The physically allowed density operators both for pure or mixed states all commute with the number operator

$$[\widehat{N}, \hat{\rho}] = 0 \tag{108}$$

As pointed out in Ref. [1], the super-selection rule leads to restrictions on the phase space distribution function, and a brief discussion of its origin is presented in Ref. [1].

8.3 Field Operators and Hamiltonian

In systems which contain a large number of modes, the distribution function treatment becomes unwieldy and a switch to a treatment avoiding a consideration of separate modes is highly desirable. The system is then described in terms of *field operators* $\hat{\Psi}^\dagger(\mathbf{r}), \hat{\Psi}(\mathbf{r})$, where $\mathbf{r}$ is the particle position. The field operators may be defined via mode expansions with the mode annihilation and creation operators being the expansion coefficients

$$\hat{\Psi}(\mathbf{r}) = \sum_i \hat{c}_i \phi_i(\mathbf{r}), \quad \hat{\Psi}^\dagger(\mathbf{r}) = \sum_i \hat{c}_i^\dagger \phi_i^*(\mathbf{r}). \tag{109}$$

These operators are associated with the creation, destruction of fermionic particles at particular positions. For the field operators our fundamental discrete mode anticommutation relations are replaced by

$$\begin{aligned} \{\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')\} &= \delta(\mathbf{r} - \mathbf{r}') \\ \{\hat{\Psi}(\mathbf{r}), \hat{\Psi}(\mathbf{r}')\} &= 0 = \{\hat{\Psi}^\dagger(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')\} \end{aligned} \tag{110}$$

which follow from Eq. (96) and the mode completeness result Eq. (95).

The field creation operators create a particle at a particular position in the given internal state. For a set of distinct particle positions $\mathbf{r}_1, \mathbf{r}_2 \cdots \mathbf{r}_N$, where to avoid duplication an ordering convention is invoked, so that $\mathbf{r}_1 < \mathbf{r}_2 < \cdots < \mathbf{r}_N$. The Fock state $|\mathbf{r}_1 \cdots \mathbf{r}_N\rangle$ which has one fermion at each of the positions $\mathbf{r}_1, \mathbf{r}_2 \cdots \mathbf{r}_N$ is given by

$$|\mathbf{r}_1 \cdots \mathbf{r}_N\rangle = \hat{\Psi}^\dagger(\mathbf{r}_1) \cdots \hat{\Psi}^\dagger(\mathbf{r}_N)|0\rangle \tag{111}$$

where the normalisation and orthogonality conditions for the position states involve delta functions

$$\langle \mathbf{r}_1 \cdots \mathbf{r}_N | \, \mathbf{s}_1 \cdots \mathbf{s}_N \rangle = \delta(\mathbf{r}_1 - \mathbf{s}_1) \cdots \delta(\mathbf{r}_N - \mathbf{s}_N). \tag{112}$$

The proof of these results is given in Appendix 9.

The vacuum state $|0\rangle$, which contains no particles in any mode, is associated with the *vacuum projector* $|0\rangle\langle 0|$. This projector can be written using *normally*

ordered forms of products of exponential operators based on number operators. The definition of normal ordering is set out in Ref. [1]. We have

$$\begin{aligned} |0\rangle\langle 0| &= \mathcal{N}\left(\prod_i \exp(-\hat{c}_i^\dagger \hat{c}_i)\right) \\ &= \mathcal{N}\left[\exp\left(-\sum_i \hat{c}_i^\dagger \hat{c}_i\right)\right] = \mathcal{N}\left[\exp\left(-\int d\mathbf{r}\, \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})\right)\right] \end{aligned} \tag{113}$$

in terms of mode or field operators .

Field annihilation and creation operators do not represent physical quantities and have the effect of changing N particle physical states into $N \pm 1$ particle physical states. Physical quantities can be constructed from the field operators, however, which always have the same number of creation and annihilation operators. For example, in terms of field operators a typical *Hamiltonian* for a fermion system with spin $\frac{1}{2}$ fermions of mass m is

$$\begin{aligned} \hat{H}_f &= \int d\mathbf{r} \left(\sum_\alpha \frac{\hbar^2}{2m} \nabla \hat{\Psi}_\alpha(\mathbf{r})^\dagger \cdot \nabla \hat{\Psi}_\alpha(\ \mathbf{r}) + \sum_\alpha \hat{\Psi}_\alpha(\mathbf{r})^\dagger V_\alpha \hat{\Psi}_\alpha(\mathbf{r}) \right. \\ &\qquad \left. + \frac{g_f}{2} \sum_\alpha \hat{\Psi}_\alpha(\mathbf{r})^\dagger \hat{\Psi}_{-\alpha}(\mathbf{r})^\dagger \hat{\Psi}_{-\alpha}(\mathbf{r}) \hat{\Psi}_\alpha(\mathbf{r}) \right) \end{aligned} \tag{114}$$

where $\alpha = -\frac{1}{2}, +\frac{1}{2}$ lists two internal states with $M_f = -\frac{1}{2}, +\frac{1}{2}$ and the zero range approximation is used for interactions between the particles and V_f is the trapping potential. The field operators are now generalised to allow for internal states. The fermion-fermion interaction is required to be between fermions of opposite spin because $\hat{\Psi}_\alpha^2(\mathbf{r}) = 0$. For bosons strong s wave interactions occur, but for fermions only weak p wave collisions occur unless the fermions have opposite spin, because the Pauli principle prevents two fermions with the same spin from being at the same position. Typical Hamiltonians for interacting fermions can also be expressed in terms of mode annihilation and creation operators (see Eqs. (37), (38) and (39) in Ref. [1]).

9 Appendix B - Fermion Position States

To show that the field creation operators do indeed create a particle at a particular position we note that the single particle state $|\phi_i\rangle$ can be expressed in terms of Dirac delta function normalised *position eigenstates* $|\mathbf{r}\rangle$ via

$$\begin{aligned} |\phi_i\rangle &= \int d\mathbf{r}\ |\mathbf{r}\rangle \langle \mathbf{r}\,|\phi_i\rangle = \int d\mathbf{r}\ |\mathbf{r}\rangle\, \phi_i(\mathbf{r}) \\ \langle \mathbf{r}\,|\mathbf{r}'\rangle &= \boldsymbol{\delta}(\mathbf{r}-\mathbf{r}') \end{aligned} \tag{115}$$

where the mode function $\phi_i(\mathbf{r})$ is the position representation of $|\phi_i\rangle$. Using completeness it then follows that

$$|\mathbf{r}\rangle = \sum_i \phi_i^*(\mathbf{r})|\phi_i\rangle \tag{116}$$

gives the position eigenstates in terms of the single particle states $|\phi_i\rangle$. The sum is over all modes.

We now consider a set of distinct particle positions $\mathbf{r}_1, \mathbf{r}_2 \cdots \mathbf{r}_N$. As in the case of the single particle modes an ordering convention is invoked to avoid duplication. Thus $\mathbf{r}_1 < \mathbf{r}_2 < \cdots < \mathbf{r}_N$. In first quantisation we introduce the state $|\Pi\rangle$ obtained from modes $l_1, l_2, ..., l_N$ by taking the product of $\phi^*_{l_1}(\mathbf{r}_1) \cdots \phi^*_{l_N}(\mathbf{r}_N)$ with the first quantisation form of the Fock state $|l_1 \cdots l_N\rangle$ in which the modes are in any order, and then summing over all the $l_1, l_2, ..., l_N$. Note that terms in which the same mode appears twice or more will be incuded, but as these are zero due to the Pauli principle this is not a problem. Substituting for $|l_1 \cdots l_N\rangle$ using the anti-symmetrising operator $\mathcal{A}$, which involves the sum of all $N!$ permutation operators for the N particles, each permuation being multiplied by $+1, -1$ according to whether is is even,odd, and then using (116).we find that

$$\begin{aligned} |\Pi\rangle &= \sum_{l_1,\cdots,l_N} \phi^*_{l_1}(\mathbf{r}_1) \cdots \phi^*_{l_N}(\mathbf{r}_N)|l_1 \cdots l_N\rangle \\ &= \sum_{l_1,\cdots,l_N} \phi^*_{l_1}(\mathbf{r}_1) \cdots \phi^*_{l_N}(\mathbf{r}_N)(\mathcal{A})|l_1(1)\rangle \cdots |l_N(N)\rangle \\ &= (\mathcal{A})|\mathbf{r}_1(1)\rangle \cdots |\mathbf{r}_N(N)\rangle \\ &= |\mathbf{r}_1 \cdots \mathbf{r}_N\rangle \end{aligned} \tag{117}$$

which is a symmetrised state $|\mathbf{r}_1 \cdots \mathbf{r}_N\rangle$ in which there is one particle at $\mathbf{r}_1$, a second at $\mathbf{r}_2$, $\cdots$, an Nth at $\mathbf{r}_N$. But in second quantisation the state $|\Pi\rangle$ can be written

$$\begin{aligned} |\Pi\rangle &= \sum_{l_1,\cdots,l_N} \phi^*_{l_1}(\mathbf{r}_1) \cdots \phi^*_{l_N}(\mathbf{r}_N)|l_1 \cdots l_N\rangle \\ &= \sum_{l_1,\cdots,l_N} \phi^*_{l_1}(\mathbf{r}_1) \cdots \phi^*_{l_N}(\mathbf{r}_N)\hat{c}^\dagger_{l_1} \cdots \hat{c}^\dagger_{l_P}|0\rangle \\ &= \hat{\Psi}^\dagger(\mathbf{r}_1) \cdots \hat{\Psi}^\dagger(\mathbf{r}_N)|0\rangle \end{aligned} \tag{118}$$

where again for terms where a given mode appears twice or more, such terms are zero because they involve repeated mode creation operators and therefore can be included with no change in $|\Pi\rangle$. Also if the $l_1, l_2, ..., l_N$ are not in conventional order, the result for distinct occupied modes $|l_1 \cdots l_N\rangle = \hat{c}^\dagger_{l_1} \cdots \hat{c}^\dagger_{l_P}|0\rangle$ still applies, since both sides can be changed to conventional order by multiplying by $+1$ or -1. The last expression for $|\Pi\rangle$ is obtained using the expression (109) for the field creation operators. Comparing the two results for $|\Pi\rangle$ we see that the symmetrized state $|\mathbf{r}_1 \cdots \mathbf{r}_N\rangle$ is given by

$$|\mathbf{r}_1 \cdots \mathbf{r}_N\rangle = \hat{\Psi}^\dagger(\mathbf{r}_1) \cdots \hat{\Psi}^\dagger(\mathbf{r}_N)|0\rangle \tag{119}$$

Thus the field creation operators do in fact create particles at particular positions, ordered as $\mathbf{r}_1 < \mathbf{r}_2 < \cdots < \mathbf{r}_N$. The normalisation condition for the position states are

$$\langle \mathbf{r}_1 \cdots \mathbf{r}_N | \, \mathbf{s}_1 \cdots \mathbf{s}_N \rangle = \delta(\mathbf{r}_1 - \mathbf{s}_1) \cdots \delta(\mathbf{r}_N - \mathbf{s}_N). \tag{120}$$

can be obtained using the anti-commutation rules.

10 Appendix C - Grassmann Functional Calculus

In this Appendix the main features of Grassmann functional calculus will be set out. For simplicity we will only consider functionals of a single Grassmann field $\psi(x)$ or a pair of Grassmann fields $\psi(x), \psi^+(x)$. The variable x refers to position, which may be in 1D, 2D or 3D.

10.1 Examples of Grassman-Number Functionals

A somewhat trivial application of the Grassmann functional concept is to express a Grassmann field $\psi(y)$ as a functional $F_y[\psi(x)]$ of $\psi(x)$

$$\begin{aligned} F_y[\psi(x)] &\equiv \psi(y). && (121) \\ &= \int dx\, \delta(y-x)\, \psi(x) && (122) \end{aligned}$$

Here this specific functional involves the Dirac delta function as a kernel.

Another example involves the spatial derivative $\bigtriangledown_y \psi(y)$ which may also be expressed as a functional $F_{\nabla y}[\psi(x)]$

$$\begin{aligned} F_{\nabla y}[\psi(x)] &\equiv \bigtriangledown_y \psi(y) && (123) \\ &= \int dx\, \delta(y-x)\ \bigtriangledown_x \psi(x) \\ &= -\int dx\, \{\bigtriangledown_x \delta(y-x)\}\, \psi(x) \\ &= \int dx\, \{\bigtriangledown_y \delta(y-x)\}\, \psi(x) && (124) \end{aligned}$$

Here the functional involves $\bigtriangledown_y \delta(y-x)$ as a kernel.

A functional is said to be linear if

$$F[c_1\psi_1(x) + c_2\psi_2(x)] = c_1 F[\psi_1(x)] + c_2 F[\psi_2(x)] \qquad (125)$$

where c_1, c_2 are constants. Both the Grassmann function $\psi(y)$ and its spatial derivative are linear functionals.

An example of a non-linear Grassmann functional is

$$\begin{aligned} F_{\chi(2)} &= \left(\int dx\, \chi^*(x)\, \psi(x) \right)^2 \\ &= \int\int dx\, dy\, \chi^*(x)\, \chi^*(y)\, \psi(x)\, \psi(y) && (126) \end{aligned}$$

10.2 Functional Differentiation for Grassmann Fields

The left functional derivative $\frac{\overrightarrow{\delta}}{\delta\psi(x)}F[\psi(x)]$ is defined by

$$F[\psi(x)+\delta\psi(x)] \approx F[\psi(x)] + \int dx\, \delta\psi(x) \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}F[\psi(x)]\right)_x \tag{127}$$

where $F[\psi(x)+\delta\psi(x)]-F[\psi(x)]$ is evaluated correct to the first order in a change $\delta\psi(x)$ in the Grassmann field. Note that the idea of smallness does not apply to the Grassmann field change $\delta\psi(x)$. In Eq.(127) the left side is a functional of $\psi(x)+\delta\psi(x)$ and the first term on the right side is a functional of $\psi(x)$. Both these quantities and the second term on the right side are Grassmann functions. The latter term is a functional of the Grassmann field $\delta\psi(x)$ and thus the functional derivative must be a Grassmann function of x, hence the subscript x. In most situations this subscript will be left understood. Note that $\Delta F = F[\psi(x)+\delta\psi(x)]-F[\psi(x)]$ will in general involve terms that are non-linear in $\delta\psi(x)$. For the example in Eq.(126) $\Delta F = \int\int dx\, dy\, \chi^*(x)\, \chi^*(y)\, \delta\psi(x)\, \delta\psi(y)$ and here the functional derivative is zero.

In addition we note that the Grassmann function $\delta\psi(x)$ does not necessarily commute with the g-function $\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}F[\psi(x)]\right)_x$. This therefore means that there is also a right functional derivative $\left(F[\psi(x)]\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right)_x$ defined by

$$F[\psi(x)+\delta\psi(x)] \approx F[\psi(x)] + \int dx \left(F[\psi(x)]\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right)_x \delta\psi(x) \tag{128}$$

We emphasise again: the functional derivative of a g-number functional is a Grassmann function, not a functional. The specific examples below illustrate this feature.

For functionals of the form $F[\psi(x),\psi^+(x)]$ we have similar expressions for the left and right functional derivatives but now with respect to either $\psi(x)$ or $\psi^+(x)$

$$F[\psi(x),\psi^+(x)+\delta\psi^+(x)] \tag{129}$$

$$\approx F[\psi(x),\psi^+(x)] + \int dx\delta\psi^+(x) \left[\frac{\overrightarrow{\delta}}{\delta\psi^+(x)}F[\psi(x),\psi^+(x)]\right]_x$$

$$\approx F[\psi(x),\psi^+(x)] + \int dx \left[F[\psi(x),\psi^+(x)]\frac{\overleftarrow{\delta}}{\delta\psi^+(x)}\right]_x \delta\psi^+(x). \tag{130}$$

Finally, higher order functional derivatives can be defined by applying the basic definitions to lower order functional derivatives.

10.3 Examples of Grassmann Functional Derivatives

For the case of the functional $F_y[\psi(x)]$ in Eq.(122) that gives the function $\psi(y)$. As

$$\begin{aligned} F_y[\psi(x)+\delta\psi(x)] - F_y[\psi(x)] &= \psi(y)+\delta\psi(y)-\psi(y) \\ &= \int dx\, \delta\psi(x)\, \delta(y-x) \\ &= \int dx\, \delta\psi(x) \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} F_y[\psi(x)] \right)_x \\ &= \int dx\, \delta(y-x)\, \delta\psi(x) \\ &= \int dx \left(F_y[\psi(x)] \frac{\overleftarrow{\delta}}{\delta\psi(x)} \right)_x \delta\psi(x) \end{aligned}$$

we have the same result for the left and right Grassmann functional derivative

$$\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} F_y[\psi(x)] \right)_x = \left(\frac{\overrightarrow{\delta}\,\psi(y)}{\delta\psi(x)} \right)_x \tag{131}$$

$$= \delta(y-x) \tag{132}$$

$$\left(F_y[\psi(x)] \frac{\overleftarrow{\delta}}{\delta\psi(x)} \right)_x = \left(\frac{\psi(y)\overleftarrow{\delta}}{\delta\psi(x)} \right)_x \tag{133}$$

$$= \delta(y-x) \tag{134}$$

so here the left and right functional derivatives are a delta function, just as for c-numbers.

A similar situation applies to the functional $F_{\nabla y}[\psi(x)]$ in Eq.(124) that gives the spatial derivative function $\bigtriangledown_y \psi(y)$. We have the same result for the left and right Grassmann functional derivatives

$$\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} F_{\nabla y}[\psi(x)] \right)_x = \left(\frac{\overrightarrow{\delta}\ \bigtriangledown_y \psi(y)}{\delta\psi(x)} \right)_x$$

$$= \bigtriangledown_y \delta(y-x) \tag{135}$$

$$\left(F_{\nabla y}[\psi(x)] \frac{\overleftarrow{\delta}}{\delta\psi(x)} \right)_x = \left(\frac{\nabla_y \psi(y)\overleftarrow{\delta}}{\delta\psi(x)} \right)_x$$

$$= \nabla_y \delta(y-x) \tag{136}$$

so again the functional derivatives are the spatial derivative of a delta function.

10.4 Grassmann Functional Derivative and Mode Functions

We consider functionals $F[\psi(x),\psi^+(x)]$ of both fields $\psi(x),\psi^+(x)$ and first consider functional derivatives with respect to $\psi(x)$. If a mode expansion for the Grassmann field $\psi(x)$ as in Eq.(3) is performed, then we can obtain an expression for the functional derivative with respect to $\psi(x)$ in terms of mode functions. With

$$\delta\psi(x) = \sum_k \delta g_k \phi_k(x) \tag{137}$$

where δg_1, δg_2, ..., δg_n are Grassmann variables that determmine the change $\delta\psi(x)$ in the Grassmann field $\psi(x)$, we see that

$$\begin{aligned} & F[\psi(x)+\delta\psi(x),\psi^+(x)] - F[\psi(x),\psi^+(x)] \\ \approx & \int dx\, \delta\psi(x) \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} F[\psi(x),\psi^+(x)] \right)_x \\ \approx & \sum_k \delta g_k \int dx\, \phi_k(x) \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} F[\psi(x),\psi^+(x)] \right)_x \end{aligned}$$

Suppose we write $F[\psi(x),\psi^+(x)]$ as a Grassmann function, $F[\psi(x),\psi^+(x)] = f(g_1,\cdots,g_k,\cdots g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+)$. Applying a Taylor series expansion (see Eq. (155) in Ref. [1]) we have

$$\begin{aligned} & f(g_1+\delta g_1,\cdots,g_k+\delta g_k,\cdots g_n+\delta g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+) \\ & -f(g_1,\cdots,g_k,\cdots g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+) \\ = & \sum_k \delta g_k \left\{ \frac{\overrightarrow{\partial}}{\partial g_k} f(g_1,\cdots,g_k,\cdots g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+) \right\} \end{aligned}$$

correct to first order in the δg_k so that we can write

$$\begin{aligned} & F[\psi(x)+\delta\psi(x),\psi^+(x)] - F[\psi(x),\psi^+(x)] \\ = & \sum_k \delta g_k \frac{\overrightarrow{\partial}}{\partial g_k} f(g_1,\cdots,g_k,\cdots g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+) \end{aligned}$$

We therefore have found that

$$\begin{aligned} & \sum_k \delta g_k \int dx\, \phi_k(x) \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} F[\psi(x),\psi^+(x)] \right)_x \\ = & \sum_k \delta g_k \frac{\overrightarrow{\partial}}{\partial g_k} f(g_1,\cdots,g_k,\cdots g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+) \end{aligned}$$

Equating the coefficient of the δg_k and then using the completeness relationship analogous to Eq.(95) gives the key result

$$\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}F[\psi(x),\psi^+(x)]\right)_x = \sum_k \phi_k^*(x)\,\frac{\overrightarrow{\partial}}{\partial g_k}f(g_1,\cdots,g_k,\cdots g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+). \tag{138}$$

This relates the left functional derivative to the mode functions and to the ordinary left Grassmann derivatives of the function $f(g_1,\cdots,g_k,\cdots g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+)$ that was equivalent to the original Grassmann functional $F[\psi(x),\psi^+(x)]$. Again, we see that the result is a function of x. Note that the left functional derivative involves an expansion in terms of the conjugate mode functions $\phi_k^*(x)$ rather than the original modes $\phi_k(x)$. The last result may be put in the form of a useful operational identity

$$\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x = \sum_k \phi_k^*(x)\,\frac{\overrightarrow{\partial}}{\partial g_k} \tag{139}$$

where the left side is understood to operate on an arbitary functional $F[\psi(x),\psi^+(x)]$ and the right side is understood to operate on the equivalent function $f(g_1,\cdots,g_k,\cdots g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+)$.

Similar results can be obtained for the right functional derivative with respect to $\psi(x)$

$$\left(F[\psi(x),\psi^+(x)]\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right)_x = \sum_k \phi_k^*(x)\,f(g_1,\cdots,g_k,\cdots g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+)\frac{\overleftarrow{\partial}}{\partial g_k} \tag{140}$$

and

$$\left(\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right)_x = \sum_k \phi_k^*(x)\,\frac{\overleftarrow{\partial}}{\partial g_k}. \tag{141}$$

The equivalent results for left and right functional derivatives with respect to $\psi^+(x)$ are

$$\left(\frac{\overrightarrow{\delta}}{\delta\psi^+(x)}F[\psi(x),\psi^+(x)]\right)_x = \sum_k \phi_k(x)\,\frac{\overrightarrow{\partial}\,f(g_1,\cdots,g_k,\cdots g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+)}{\partial g_k^+} \tag{142}$$

$$\left(\frac{\overrightarrow{\delta}}{\delta\psi^+(x)}\right)_x = \sum_k \phi_k(x)\,\frac{\overrightarrow{\partial}}{\partial g_k^+} \tag{143}$$

$$\left(F[\psi(x),\psi^+(x)]\frac{\overleftarrow{\delta}}{\delta\psi^+(x)}\right)_x = \sum_k \phi_k(x)\,f(g_1,\cdots,g_k,\cdots g_n,g_1^+,\cdots,g_k^+,\cdots g_n^+)\frac{\overleftarrow{\partial}}{\partial g_k^+} \tag{144}$$

$$\left(\frac{\overleftarrow{\delta}}{\delta\psi^+(x)}\right)_x = \sum_k \phi_k(x)\,\frac{\overleftarrow{\partial}}{\partial g_k^+}. \tag{145}$$

The results for left and right functional derivatives can be inverted to give

$$\frac{\overrightarrow{\partial}}{\partial g_k} = \int dx \phi_k(x) \left(\frac{\overrightarrow{\delta}}{\delta \psi(x)} \right)_x \tag{146}$$

$$\frac{\overrightarrow{\partial}}{\partial g_k^+} = \int dx \phi_k^*(x) \left(\frac{\overrightarrow{\delta}}{\delta \psi^+(x)} \right)_x \tag{147}$$

$$\frac{\overleftarrow{\partial}}{\partial g_k} = \int dx \phi_k(x) \left(\frac{\overleftarrow{\delta}}{\delta \psi(x)} \right)_x \tag{148}$$

$$\frac{\overleftarrow{\partial}}{\partial g_k^+} = \int dx \phi_k^*(x) \left(\frac{\overleftarrow{\delta}}{\delta \psi^+(x)} \right)_x . \tag{149}$$

10.5 Basic Rules for Grassmann Functional Derivatives

It is possible to establish useful rules for the functional derivative of the sum of two Grassmann functionals. It is easily shown that

$$\left(\frac{\overrightarrow{\delta}}{\delta \psi(x)} \right)_x \{F[\psi(x)] + G[\psi(x)]\} = \left(\frac{\overrightarrow{\delta}}{\delta \psi(x)} \right)_x F[\,] + \left(\frac{\overrightarrow{\delta}}{\delta \psi(x)} \right)_x G[\,] \tag{150}$$

$$\{F[\,] + G[\,]\} \left(\frac{\overleftarrow{\delta}}{\delta \psi(x)} \right)_x = F[\,] \left(\frac{\overleftarrow{\delta}}{\delta \psi(x)} \right)_x + G[\,] \left(\frac{\overleftarrow{\delta}}{\delta \psi(x)} \right)_x \tag{151}$$

with similar results for functionals of $\psi(x), \psi^+(x)$.

Rules can be established for the functional derivative of the product of two Grassmann functionals that depend on the parity of the functions equivalent to the functionals, and the proofs are quite different to the c-number case. For functionals that are neither even nor odd, results can be obtained by expressing the relevant functional as a sum of even and odd contributions. We will keep the functionals in order to cover the case where the functionals are operators.

Correct to first order in $\delta\psi(x)$, we have from the definitions

$$\begin{aligned}
& \int dx\,\delta\psi(x)\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x \{F[\psi(x)]G[\psi(x)]\} \\
\approx\ & F[\psi(x)+\delta\psi(x)]G[\psi(x)+\delta\psi(x)] - F[\psi(x)]G[\psi(x)] \\
\approx\ & F[\psi(x)+\delta\psi(x)]G[\psi(x)+\delta\psi(x)] - F[\psi(x)]G[\psi(x)+\delta\psi(x)] \\
& +F[\psi(x)]G[\psi(x)+\delta\psi(x)] - F[\psi(x)]G[\psi(x)] \\
\approx\ & \left\{\int dx\,\delta\psi(x)\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x F[\psi(x)]\right\} G[\psi(x)+\delta\psi(x)] \\
& +F[\psi(x)]\left\{\int dx\,\delta\psi(x)\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x G[\psi(x)]\right\} \\
\approx\ & \int dx\,\delta\psi(x)\left\{\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x F[\psi(x)]\right\} G[\psi(x)] \\
& + \int dx\,\delta\psi(x)\,\sigma(F)F[\psi(x)]\left\{\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x G[\psi(x)]\right\}
\end{aligned}$$

where $\sigma(F,G) = +1, -1$ depending on the parity of the function f, g that is equivalent to the functionals F, G. In the first term the factor $G[\psi(x)+\delta\psi(x)]$ is replaced by $G[\psi(x)]$ to discard second order contributions. Hence

$$\begin{aligned}
\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x \{F[\psi(x)]G[\psi(x)]\} &= \left\{\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x F[\psi(x)]\right\} G[\psi(x)] \\
&+ \sigma(F)F[\psi(x)]\left\{\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x G[\psi(x)]\right\} \quad (152)
\end{aligned}$$

A similar derivation covers the right functional derivative. The result is

$$\begin{aligned}
\{F[\psi(x)]G[\psi(x)]\}\left(\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right)_x &= F[\psi(x)]\left\{G[\psi(x)]\left(\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right)_x\right\} \\
&+ \sigma(G)\left\{F[\psi(x)]\left(\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right)_x\right\} G[\psi(x)] \quad (153)
\end{aligned}$$

These two rules are the functional derivative extensions of the previous left and right product rules (see Eq. (154) in Ref. [1]) for Grassmann derivatives. The extension to functionals of $\psi(x), \psi^+(x)$ are obvious. These results are essentially the same as for Grassmann ordinary differentiation.

10.6 Other Rules for Grassmann Functional Derivatives

There are several rules that are needed because of the distinction between left and right functional differentiation. These are analogous to rules applying for

left and right differentiation of Grassmann functions and may be established using the mode based expressions for Grassmann functional derivatives. With $\psi(x)$ a general Grassmann field these include:

(1) Right and left functional derivative relations for even and odd functionals

$$\begin{aligned} \frac{\overrightarrow{\delta}}{\delta\psi(x)}F_E[\psi(x)] &= -F_E[\psi(x)]\frac{\overleftarrow{\delta}}{\delta\psi(x)} \\ \frac{\overrightarrow{\delta}}{\delta\psi(x)}F_O[\psi(x)] &= +F_O[\psi(x)]\frac{\overleftarrow{\delta}}{\delta\psi(x)} \end{aligned} \tag{154}$$

(2) Altering order of functional derivatives

$$\frac{\overrightarrow{\delta}}{\delta\psi(x)}\frac{\overrightarrow{\delta}}{\delta\psi(y)}F[\psi(x)] = -\frac{\overrightarrow{\delta}}{\delta\psi(y)}\frac{\overrightarrow{\delta}}{\delta\psi(x)}F[\psi(x)] \tag{155}$$

An analogous result applies for right functional derivatives.

(3) Mixed functional derivatives

$$\begin{aligned} \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}F[\psi(x)]\right)\frac{\overleftarrow{\delta}}{\delta\psi(y)} &= \frac{\overrightarrow{\delta}}{\delta\psi(x)}\left(F[\psi(x)]\frac{\overleftarrow{\delta}}{\delta\psi(y)}\right) \\ &= \frac{\overrightarrow{\delta}}{\delta\psi(x)}F[\psi(x)]\frac{\overleftarrow{\delta}}{\delta\psi(y)} \end{aligned} \tag{156}$$

10.7 Functional Integration for Grassmann Fields

If the range over variable x for the Grassmann field $\psi(x)$ is divided up into n small intervals $\Delta x_i = x_{i+1} - x_i$ (the ith interval), then we may specify the value ψ_i of the function $\psi(x)$ in the ith interval via the spatial average over the interval

$$\psi_i = \frac{1}{\Delta x_i}\int\limits_{\Delta x_i} dx\,\psi(x). \tag{157}$$

Averaging a Grassmann field over a position interval still results in a linear form involving the Grassmann variables $g_1, \cdots, g_k, \cdots g_n$. As previously, for simplicity we will choose the same number n of intervals as mode functions.

The functional $F[\psi(x)]$ may be regarded as a function $F(\psi_1, \cdots, \psi_i, \cdots, \psi_n)$ of all the n different ψ_i, which in the present case are a set of Grassmann variables. As we will see, these Grassmann variables $\psi_1, \cdots, \psi_i, \cdots, \psi_n$ just involve a linear transformation from the $g_1, \cdots, g_k, \cdots g_n$. Introducing a suitable *weight function* $w(\psi_1, \cdots, \psi_i, \cdots, \psi_n)$ we may then define the *functional integral* via the multiple Grassmann integral

$$\begin{aligned} \int D\psi\, F[\psi(x)] &= \lim_{n\to\infty}\lim_{\epsilon\to 0}\int\cdots\int d\psi_n\cdots d\psi_i\cdots d\psi_1\, w(\psi_1, \cdots, \psi_i, \cdots, \psi_n) \\ &\quad \times F(\psi_1, \cdots, \psi_i, \cdots, \psi_n) \end{aligned} \tag{158}$$

where $\epsilon > \Delta x_i$. As previously, we use left integration and follow the convention in which the symbol $D\psi$ stands for $d\psi_n \cdots d\psi_i \cdots d\psi_1\, w(\psi_1, \cdots, \psi_i, \cdots, \psi_n)$. A total functional integral of a functional of a Grassmann field gives a c-number.

If the functional $F[\psi(x), \psi^+(x)]$ involves pairs of Grassmann fields, then the functional integral will be of the form $\int\int D\psi^+\, D\psi\, F[\psi(x), \psi^+(x)]$, where $D\psi^+ D\psi = d\psi_n^+ \cdots d\psi_i^+ \cdots d\psi_1^+ d\psi_n \cdots d\psi_i \cdots d\psi_1\, w(\psi_1, .., \psi_i .., \psi_n, \psi_1^+, .., \psi_i^+ .., \psi_n^+)$. Similarly to differentiation and integration in ordinary Grassmann calculus, functional integration and differentiation are not inverse processes.

10.8 Functional Integrals and Phase Space Integrals

For a mode expansion such as in Eq.(3) the value ϕ_{ki} of the mode function in the ith interval is also defined via the average

$$\phi_{ki} = \frac{1}{\Delta x_i} \int\limits_{\Delta x_i} dx\, \phi_k(x) \tag{159}$$

Unlike the Grassmann field, this is just a c-number. It is then easy to see that the Grassmann variables $\psi_1, \cdots, \psi_i, \cdots, \psi_n$ are related to the g-number expansion coefficients $g_1, \cdots, g_k, \cdots g_n$ via the linear transformation with c-number coefficients ϕ_{ki}

$$\psi_i = \sum_k \phi_{ki}\, g_k \tag{160}$$

$$= \sum_k g_k\, \phi_{ki}. \tag{161}$$

This shows that the average values in the ith interval of the function ψ_i and the mode function ϕ_{ki} are related via the expansion coefficients g_k. This linear relation enables us to transform the functional Grassmann integral (158) into a Grassmann phase space integral, thereby establishing the link between these integrals.

Using the expression Eq.(5) for the expansion coefficients we then obtain the inverse formula to Eq.(160)

$$g_k = \sum_i \Delta x_i\, \phi_{ki}^*\, \psi_i. \tag{162}$$

The relationship in Eq.(160) shows that the functions $F(\psi_1, \cdots, \psi_i, \cdots, \psi_n)$ and $w(\psi_1, \cdots, \psi_i, \cdots, \psi_n)$ of all the interval values ψ_i can also be regarded as functions of the expansion coefficients g_k which we may write as

$$\begin{aligned} f(g_1, \cdots, g_k, \cdots g_n) &\equiv F(\psi_1(g_1, \cdots, g_k, \cdots g_n), \cdots, \psi_i(g_1, \cdots, g_k, \cdots g_n), \cdots, \psi_n) \\ v(g_1, \cdots, g_k, \cdots g_n) &\equiv w(\psi_1(g_1, \cdots, g_k, \cdots g_n), \cdots, \psi_i(g_1, \cdots, g_k, \cdots g_n), \cdots, \psi_n). \end{aligned} \tag{163}$$

Thus the various values $\psi_1, \cdots, \psi_1, \cdots, \psi_i, \cdots, \psi_n, \cdots, \psi_n$ that the function $\psi(x)$ takes on in the n intervals - and which are integrated over in the functional integration process - are all determined by the choice of the expansion coefficients $g_1, \cdots, g_k, \cdots g_n$. Hence Grassmann integration over all the ψ_i is equivalent to Grassmann integration over all the g_k. This enables us to express the functional integral in Eq.(158) as a Grassmann phase space integral over the expansion coefficients $g_1, \cdots, g_k, \cdots g_n$. However, the derivation of the result differs from the c-number case because the transformation of the product of Grassmann differentials $d\psi_n \cdots d\psi_i \cdots d\psi_1$ into the new product of Grassmann differentials $dg_n \cdots dg_i \cdots dg_2 dg_1$ requires a similar treatment to that explained in Appendix A of Ref. [1] where the transformation between Grassmann integration variables is linear. We cannot just write $d\psi_i = \sum_k \phi_{ki}\, dg_k$ because the differentials are also Grassmann variables. Hence the usual c-number transformation involving the Jacobian does not apply. The required result can be obtained from Appendix A in Paper I [1] (see Eqs.(157), (158), (161) therein) by making the idenifications $g_i \to \psi_i, h_k \to g_k, A_{ik} \to \phi_{ki}$ so

$$d\psi_n \cdots d\psi_i \cdots d\psi_1 = (\mathrm{Det}A)^{-1} dg_n \cdots dg_i \cdots dg_1. \tag{164}$$

Now with $A_{ik} = \phi_{ki}$ we have using the completeness relationship in Eq.(95)

$$\begin{aligned}
(AA^\dagger)_{ij} &= \sum_k \phi_{ki}\phi^*_{kj} \\
&= \sum_k \frac{1}{\Delta x_i} \int\limits_{\Delta x_i} dx\, \phi_k(x) \frac{1}{\Delta x_j} \int\limits_{\Delta x_j} dy\, \phi^*_k(y) \\
&= \frac{1}{\Delta x_i} \int\limits_{\Delta x_i} dx\, \frac{1}{\Delta x_j} \int\limits_{\Delta x_j} dy\, \delta(x-y) \\
&= \delta_{ij} \frac{1}{(\Delta x_i)^2} \int\limits_{\Delta x_i} dx \\
&= \delta_{ij}\, (\Delta x_i)^{-1}.
\end{aligned} \tag{165}$$

Thus we have

$$|(\mathrm{Det}A)| = \prod_i (\Delta x_i)^{-1/2} \tag{166}$$

Hence we have using the result in Eq.(165) in Ref. [1] for transforming phase space integrals

$$\begin{aligned}
\int D\psi\, F[\psi(x)] &= \lim_{n\to\infty} \lim_{\epsilon\to 0} \int \cdots \int dg_n \cdots dg_k \cdots dg_1 \prod_i (\Delta x_i)^{1/2} \\
&\quad \times v(g_1, \cdots, g_k, \cdots g_n)\, f(g_1, \cdots, g_k, \cdots g_n)
\end{aligned} \tag{167}$$

This key result expresses the original Grassmann functional integral as a Grassmann phase space integral over the g-number expansion coefficients g_k for the

Grassmann field $\psi(x)$ in terms of the mode functions $\phi_k(x)$. Note that this result is different to the previous c-number case, where the factor is $\prod_i(\Delta x_i)^{-1/2}$instead of $\prod_i(\Delta x_i)^{1/2}$. This is because the Grassmann differentials transform via $(\mathrm{Det}A)^{-1}$ instead of $(\mathrm{Det}A)^{+1}$.

The general result can be simplified with a special choice of the weight function

$$w(\psi_1, \cdots, \psi_i, \cdots, \psi_n) = \prod_i (\Delta x_i)^{-1/2} \tag{168}$$

and we then get a simple expression for the Grassmann functional integral

$$\int D\psi\, F[\psi(x)] = \lim_{n\to\infty} \lim_{\epsilon\to 0} \int \cdots \int dg_1 \cdots dg_k \cdots dg_n\ f(g_1, \cdots, g_k, \cdots g_n). \tag{169}$$

In this form of the Grassmann functional integral the original Grassmann functional $F[\psi(x)]$ has been replaced by the equivalent function $f(g_1, \cdots, g_k, \cdots g_n)$ of the g-number expansion coefficients g_k, and the functional integration is now replaced by a Grassmann phase space integration over the expansion coefficients.

For two Grassmann fields $\psi(x), \psi^+(x)$ a straightforward extension of the last result gives

$$\begin{aligned} & \int D\psi^+\, D\psi\, F[\psi(x), \psi^+(x)] \\ = & \lim_{n\to\infty} \lim_{\epsilon\to 0} \int \cdots \int dg_n^+ \cdots dg_k^+ \cdots dg_1^+ dg_n \cdots dg_k \cdots dg_1 \\ \times & \ f(g_1, \cdots, g_k, \cdots g_n, g_1^+, \cdots, g_k^+, \cdots g_n^+) \end{aligned} \tag{170}$$

where the weight function is now

$$w(\psi_1, \cdots, \psi_i, \cdots, \psi_n, \psi_1^+, \cdots, \psi_i^+, \cdots, \psi_n^+) = \prod_i (\Delta x_i)^{-1} \tag{171}$$

and $\psi^+(x)$ is given via (4).

10.9 Functional Integration by Parts

A useful integration by parts rule can often be established from Eq.(152). Consider the Grassmann functional $H[\psi(x)] = F[\psi(x)]G[\psi(x)]$. Then

$$\begin{aligned} & F[\psi(x)] \left\{ \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} \right)_x G[\psi(x)] \right\} \\ = & \ \sigma(F) \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} \right)_x \{F[\psi(x)]G[\psi(x)]\} - \sigma(F) \left\{ \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} \right)_x F[\psi(x)] \right\} G[\psi(x)] \end{aligned}$$

Then

$$\int D\psi\, F[\,]\left\{\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x G[\,]\right\} = \sigma(F)\int D\psi\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x H[\,] - \sigma(F)\int D\psi\left\{\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x F[\,]\right\} G[\,]$$

If we now introduce mode expansions and use Eq.(138) for the functional derivative of $H[\psi(x)]$ and Eq.(169) for the first of the two functional integrals on the right hand side of the last equation then

$$\begin{aligned}
&\int D\psi\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x H[\psi(x)] \\
&= \lim_{n\to\infty}\lim_{\epsilon\to 0}\int dg_1\cdots dg_k\cdots dg_n\sum_k \phi_k^*(x)\frac{\overrightarrow{\partial}}{\partial g_k}h(g_1,\cdots,g_k,\cdots) \\
&= \lim_{n\to\infty}\lim_{\epsilon\to 0}\sum_k \phi_k^*(x)\int\cdots\int dg_1\cdots dg_{k-1}dg_{k+1}\cdots dg_n \\
&\times\ (-1)^{n-k}\{\int dg_k\,\frac{\overrightarrow{\partial}}{\partial g_k}h(g_1,\cdots,g_k,\cdots)\}
\end{aligned}$$

so that the functional integral of this term reduces to the Grassmann integral of a Grassmann derivative. This is zero, since differentiation removes the g_k dependence. Hence the Grassmann functional integral involving the functional derivative of $H[\psi(x)]$ vanishes and we have the integration by parts result

$$\int D\psi\, F[\psi(x)]\left\{\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x G[\psi(x)]\right\} = -\sigma(F)\int D\psi\left\{\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x F[\psi(x)]\right\} G[\psi(x)] \tag{172}$$

A similar result involving right functional differentiation can be established.

10.10 Differentiating a Functional Integral

Functionals can be defined via functional integration processes and it is useful to find rules for their functional derivatives. This leads to a rule for differentiating a functional integral.

Suppose we have a functional $G[\chi(x)]$ determined from another functional $F[\psi(x)]$ via a functional integral that involves a left transfer functional $A_{GF}[\chi(x),\psi(x)]$

$$G[\chi(x)] = \int D\psi\, A_{GF}[\chi(x),\psi(x)]\, F[\psi(x)]. \tag{173}$$

Applying the definition of the left Grassmann functional derivatives of $G[\chi(x)]$

and $A_{GF}[\chi(x)+\delta\chi(x),\psi(x)]$ with respect to $\chi(x)$ we have

$$\begin{aligned} & G[\chi(x)+\delta\chi(x)] \\ = & \int D\psi\, A_{GF}[\chi(x)+\delta\chi(x),\psi(x)]\, F[\psi(x)] \\ = & \int D\psi\, \{A_{GF}[\chi(x),\psi(x)]\}\, F[\psi(x)] \\ & + \int D\psi \left\{ \int dx\, \delta\chi(x) \left\{ \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} \right)_x A_{GF}[\chi(x),\psi(x)] \right\} \right\} F[\psi(x)] \\ = & G[\chi(x)] + \int dx\, \delta\chi(x) \int D\psi \left\{ \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} \right)_x A_{GF}[\chi(x),\psi(x)] \right\} F[\psi(x)] \end{aligned}$$

since (for reasonably well-behaved quantities) the functional integration over $D\psi$ and the ordinary integration over dx can be carried out in either order, given that both just involve processes that are limits of sumnations. Hence from the definition of the functional derivative we have

$$\left(\frac{\overrightarrow{\delta}}{\delta\chi(x)} \right)_x G[\chi(x)] = \int D\psi \left\{ \left(\frac{\overrightarrow{\delta}}{\delta\psi(x)} \right)_x A_{GF}[\chi(x),\psi(x)] \right\} F[\psi(x)] \tag{174}$$

which is the required rule for left differentiating a functional defined via a functional of another function. Clearly the rule is to just differentiate the transfer functional under the functional integration sign, a rule similar to that applying in ordinary calculus.

A similar rule can be obtained for a functional $H[\chi(x)]$ determined from another functional $F[\psi(x)]$ via a functional integral that involves a right transfer functional $A_{GF}[\chi(x),\psi(x)]$ in the form

$$H[\chi(x)] = \int D\psi\, F[\psi(x)]\, A_{GF}[\chi(x),\psi(x)]. \tag{175}$$

We find that

$$H[\chi(x)] \left(\frac{\overleftarrow{\delta}}{\delta\chi(x)} \right)_x = \int D\psi\, F[\psi(x)] \left\{ A_{GF}[\chi(x),\psi(x)] \left(\frac{\overleftarrow{\delta}}{\delta\psi(x)} \right)_x \right\} \tag{176}$$

which is the required rule for right differentiating a functional defined via a functional of another function. The proof is left as an exercise. Clearly the rule is to just differentiate the transfer functional under the functional integration sign, a rule similar to that applying in ordinary calculus (except that right and left differentiation are different).

As a particular case, consider the Fourier like Grassmann transfer functional

$$A_{GF}[\chi(x),\psi(x)] = \exp\left\{ i \int dx\, \chi(x)\, \psi(x) \right\} \tag{177}$$

This Grassmann transfer functional is equivalent to a even Grassmann function of the expansion coefficients g_k for $\psi(x)$ and h_k for $\chi(x)$. In this case

$$\begin{aligned}
A_{GF}[\chi(x)+\delta\chi(x),\psi(x)] &= \exp\left\{i\int dx\,(\chi(x)+\delta\chi(x))\,\psi(x)\right\} \\
&= \exp\left\{i\int dx\,\chi(x)\,\psi(x)\right\}\exp\left\{i\int dx\,\delta\chi(x)\,\psi(x)\right\} \\
&\approx \exp\left\{i\int dx\,\chi(x)\,\psi(x)\right\}(1+i\int dx\,\delta\chi(x)\,\psi(x)) \\
&= A_{GF}[\chi(x),\psi(x)]+A_{GF}[\chi(x),\psi(x)]\,i\int dx\,\delta\chi(x)\,\psi(x) \\
&= A_{GF}[\chi(x),\psi(x)]+\int dx\,\delta\chi(x)\,iA_{GF}[\chi(x),\psi(x)]\,\psi(x)
\end{aligned}$$

where we have used the Baker-Haussdorff theorem (see Eq. (130) in Ref. [1]) with $A=\int dx\,\chi(x)\,\psi(x)$ and $B=\int dx\,\delta\chi(x)\,\psi(x)$ together with the commutator result based on Grassmann fields anti-commuting

$$\begin{aligned}
[A,B] &= \int dx\,\chi(x)\,\psi(x)\int dy\,\delta\chi(y)\,\psi(y)-\int dy\,\delta\chi(y)\,\psi(y)\int dx\,\chi(x)\,\psi(x) \\
&= \int dx\int dy\,\chi(x)\,\psi(x)\delta\chi(y)\,\psi(y)-(-1)^{2+2}\int dx\int dy\,\chi(x)\,\psi(x)\delta\chi(y)\psi(y) \\
&= 0
\end{aligned}$$

to establish the third line of the derivation. The last line follows from $A_{GF}[\chi(x),\psi(x)]$ being equivalent to an even Grassmann function and therefore commuting with the Grassmann field $\delta\chi(x)$. Hence for the left functional derivative

$$\left(\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)_x A_{GF}[\chi(x),\psi(x)] = A_{GF}[\chi(x),\psi(x)]\times i\psi(x) \tag{178}$$

and

$$G[\chi(x)] = \int D\psi\,A_{GF}[\chi(x),\psi(x)]\,F[\psi(x)] \tag{179}$$

$$\left(\frac{\overrightarrow{\delta}}{\delta\chi(x)}\right)_x G[\chi(x)] = \int D\psi\,\{A_{GF}[\chi(x),\psi(x)]\times(i\psi(x))\}\,F[\psi(x)] \tag{180}$$

A similar result follows for the right transfer functional case. We have

$$H[\chi(x)] = \int D\psi\,F[\psi(x)]\,A_{GF}[\chi(x),\psi(x)] \tag{181}$$

$$H[\chi(x)]\left(\frac{\overleftarrow{\delta}}{\delta\chi(x)}\right)_x = \int D\psi\,F[\psi(x)]\,\{A_{GF}[\chi(x),\psi(x)]\times(-i\psi(x))\} \tag{182}$$

11 Appendix D - Functional P Distribution

11.1 Characteristic and P Distribution Functional Relation

The characteristic functional - distribution functional relationship'can be established in terms of functional integrals.

In the characteristic functional definition (26) the Grassmann fields h, h^+ are expanded in terms of mode functions

$$h(x) = \sum_i h_i \phi_i(x) \qquad h^+(x) = \sum_i h_i^+ \phi_i^*(x) \tag{183}$$

and it is easy to see using orthogonality of the modes that

$$\int dx\, \hat{\Psi}(x) h^+(x) = \sum_i \hat{c}_i h_i^+ \qquad \int dx\, h(x) \hat{\Psi}^\dagger(x) = \sum_i h_i \hat{c}_i^\dagger \tag{184}$$

Hence from Eqs.(26) and (**??**) we see that

$$\chi[h(x), h^+(x)] \equiv \chi(h, h^+) \tag{185}$$

so the characteristic functional of the fields $h(x), h^+(x)$ is entirely equivalent to the original fermionic characteristic function of the g-number expansion coefficients h_i, h_i^+.

The distribution function $P(g, g^+) \equiv P[\psi(x), \psi^+(x)]$ is related to the characteristic function $\chi(h, h^+)$ as in Eq. (34) in Ref. [1]. Using

$$\int dx\, \psi(x) h^+(x) = \sum_i g_i h_i^+ \qquad \int dx\, h(x) \psi^+(x) = \sum_i h_i g_i^+ \tag{186}$$

we see that

$$\begin{aligned} & \chi[h(x), h^+(x)] \\ = & \int \prod_i dg_i^+ dg_i \, \exp i \left\{ \int dx\, \psi(x) h^+(x) \right\} P[\psi(x), \psi^+(x)] \, \exp i \left\{ \int dx\, h(x) \psi^+(x) \right\} \\ = & \int D\psi^+ D\psi \, \exp i \left\{ \int dx\, \psi(x) h^+(x) \right\} P[\psi(x), \psi^+(x)] \, \exp i \left\{ \int dx\, h(x) \psi^+(x) \right\} \end{aligned} \tag{187}$$

giving the characteristic functional as a functional integral involving the P distribution functional.

11.2 Correspondence Rules

For the P distribution functional these are

$$\hat{\rho} \;\Rightarrow\; \hat{\Psi}_\alpha(x)\,\hat{\rho} \qquad P[\psi(x)] \Rightarrow \psi_\alpha(x)\,P \tag{188}$$

$$\hat{\rho} \;\Rightarrow\; \hat{\rho}\,\hat{\Psi}_\alpha(x) \qquad P[\psi(x)] \Rightarrow P\left(+\frac{\overleftarrow{\delta}}{\delta\psi^+_\alpha(x)} - \psi_\alpha(x)\right) \tag{189}$$

$$\hat{\rho} \;\Rightarrow\; \hat{\Psi}^\dagger_\alpha(x)\,\hat{\rho} \qquad P[\psi(x)] \Rightarrow \left(+\frac{\overrightarrow{\delta}}{\delta\psi_\alpha(x)} - \psi^+_\alpha(x)\right) P \tag{190}$$

$$\hat{\rho} \;\Rightarrow\; \hat{\rho}\,\hat{\Psi}^\dagger_\alpha(x) \qquad P[\psi(x)] \Rightarrow P\,\psi^+_\alpha(x). \tag{191}$$

and can be derived from those for the B distribution functional together with the relationship

$$P[\psi(x)] = B[\psi(x)]\,\exp(+\int dx\,\psi(x)\psi^+(x)) \tag{192}$$

As will be seen there is a mixture of Grassmann functional derivatives and Grassman fields.

12 Appendix E - Ito Stochastic Field Equations

In the notation where $\widetilde{g}_p \to \widetilde{g}^A_{\alpha i}$ the Ito stochastic equations (50) for the stochastic phase variables become

$$\widetilde{g}^A_{\alpha i}(t+\delta t) - \widetilde{g}^A_{\alpha i}(t) = C^A_{\alpha i}(\widetilde{g}(t))\delta t + \sum_a B^{\alpha i\, A}_a(\widetilde{g}(t)) \int_t^{t+\delta t} dt_1 \Gamma_a(t_1) \tag{193}$$

so multiplying by the mode functions $\xi^A_{\alpha i}(r)$ and summing over i leads to

$$\widetilde{\psi}_{\alpha A}(r, t+\delta t) - \widetilde{\psi}_{\alpha A}(r,t) = C_{\alpha A}[\widetilde{\psi}(r), r]\delta t + \sum_a B^{\alpha A}_a[\widetilde{\psi}(r), r] \int_t^{t+\delta t} dt_1 \Gamma_a(t_1) \tag{194}$$

using the expansion in (49) for the stochastic fields and where

$$\begin{aligned} C_{\alpha A}[\widetilde{\psi}(r), r] &= \sum_i C^A_{\alpha i}(\widetilde{g})\, \xi^A_{\alpha\, i}(r) \\ B^{\alpha A}_a[\widetilde{\psi}(r), r] &= \sum_i B^{\alpha i\, A}_a(\widetilde{g})\, \xi^A_{\alpha i}(r) \end{aligned} \tag{195}$$

Hence we have

$$\frac{\partial}{\partial t}\widetilde{\psi}_{\alpha A}(r) = C_{\alpha A}[\widetilde{\psi}(r), r] + \sum_a B^{\alpha A}_a[\widetilde{\psi}(r), r]\, \Gamma_a(t_+) \tag{196}$$

as required. The right side is the sum of a classical field term and a noise field term. The classical and noise field terms involve Grassmann functionals $C_{\alpha A}[\widetilde{\psi}(r), r]$ and $B^{\alpha A}_a[\widetilde{\psi}(r), r]$ defined by

$$C_{\alpha A}[\widetilde{\psi}(r), r] = -\sum_i A^A_{\alpha i}(\widetilde{g})\, \xi^A_{\alpha\, i}(r) \tag{197}$$

$$B^{\alpha A}_a[\widetilde{\psi}(r), r] = \sum_i B^{\alpha i\, A}_a(\widetilde{g})\, \xi^A_{\alpha i}(r) \tag{198}$$

Note that the classical field term is also stochastic because it involves a functional of the stochastic fields $\widetilde{\psi}_{\alpha A}(r)$. The noise field term is stochastic, not only for the same reason but also because it involves the Gaussian-Markoff nose terms.

13 Appendix F - Fermi Gas Functional Fokker-Planck Equation

In this Appendix we denote the distribution functional $B[\psi_u(\mathbf{r}), \psi_u^+(\mathbf{r}), \psi_d(\mathbf{r}), \psi_d^+(\mathbf{r})]$ as $B[\boldsymbol{\psi}(\mathbf{r})]$ for short. The Hamiltonian is given in Eq.(71). The correspondence rules are given in Eqs.(38) with a simple extension to give rules for the spatial derivatives of field operators. We will assume the modes are restricted to a cut-off K.

13.1 Kinetic Energy Terms

If $\widehat{\rho} \to \widehat{T}\widehat{\rho}$ then

$$B[\boldsymbol{\psi}(\mathbf{r})] \to \frac{\hbar^2}{2m} \sum_{\alpha} \sum_{\mu} \int d\mathbf{s} \left\{ \left(\partial_\mu \frac{\overrightarrow{\delta}}{\delta \psi_{\alpha 1}(\mathbf{s})} \right) (\partial_\mu \psi_{\alpha 1}(\mathbf{s})) \right\} B[\boldsymbol{\psi}(\mathbf{s})]$$

and if $\widehat{\rho} \to \widehat{\rho}\widehat{T}$ then

$$B[\boldsymbol{\psi}(\mathbf{r})] \to \frac{\hbar^2}{2m} \sum_{\alpha} \sum_{\mu} \int d\mathbf{s} B[\boldsymbol{\psi}(\mathbf{s})] \left\{ (\partial_\mu \psi_{\alpha 2}(\mathbf{s})) \left(\partial_\mu \frac{\overleftarrow{\delta}}{\delta \psi_{\alpha 2}(\mathbf{s})} \right) \right\}$$

so for $\widehat{\rho} \to -i/\hbar[\widehat{T}, \widehat{\rho}]$ then

$$\begin{aligned} & B[\boldsymbol{\psi}(\mathbf{r})] \\ & \quad \to -\frac{i}{\hbar} \left\{ \frac{\hbar^2}{2m} \sum_{\alpha} \sum_{\mu} \int d\mathbf{s} \left\{ \left(\partial_\mu \frac{\overrightarrow{\delta}}{\delta \psi_{\alpha 1}(\mathbf{s})} \right) (\partial_\mu \psi_{\alpha 1}(\mathbf{s})) \right\} B[\boldsymbol{\psi}(\mathbf{s})] \right\} \\ & \qquad -\frac{i}{\hbar} \left\{ -\frac{\hbar^2}{2m} \sum_{\alpha} \sum_{\mu} \int d\mathbf{s} B[\boldsymbol{\psi}(\mathbf{s})] \left\{ (\partial_\mu \psi_{\alpha 2}(\mathbf{s})) \left(\partial_\mu \frac{\overleftarrow{\delta}}{\delta \psi_{\alpha 2}(\mathbf{s})} \right) \right\} \right\} . \end{aligned} \tag{199}$$

Using the mode expansions in Eqs.(139) and (145) for the functional deriva-

tives we have for the first term

$$\int d\mathbf{s}\left\{\left(\partial_\mu \frac{\overrightarrow{\delta}}{\delta\psi_{\alpha 1}(\mathbf{s})}\right)\left(\partial_\mu\psi_{\alpha 1}(\mathbf{s})\right)\right\}B[\boldsymbol{\psi}(\mathbf{s})]$$

$$=\sum_i\sum_j\int d\mathbf{s}\left\{\left(\partial_\mu\xi^2_{\alpha i}(\mathbf{s})\frac{\overrightarrow{\partial}}{\partial g^1_{\alpha i}}\right)\left(\partial_\mu\xi^1_{\alpha j}(\mathbf{s})\,g^1_{\alpha j}\right)\right\}B(\mathbf{g})$$

$$=-\sum_i\sum_j\int d\mathbf{s}\left\{\left(\xi^2_{\alpha i}(\mathbf{s})\frac{\overrightarrow{\partial}}{\partial g^1_{\alpha i}}\right)\left(\partial^2_\mu\xi^1_{\alpha j}(\mathbf{s})\,g^1_{\alpha j}\right)\right\}B(\mathbf{g})$$

$$=-\int d\mathbf{s}\left\{\left(\frac{\overrightarrow{\delta}}{\delta\psi_{\alpha 1}(\mathbf{s})}\right)\left(\partial^2_\mu\psi_{\alpha 1}(\mathbf{s})\right)\right\}B[\boldsymbol{\psi}(\mathbf{s})]$$

$$=-\int d\mathbf{s}B[\boldsymbol{\psi}(\mathbf{s})]\left\{\partial^2_\mu\psi_{\alpha 1}(\mathbf{s})\left(\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha 1}(\mathbf{s})}\right)\right\} \tag{200}$$

where we have used spatial integration by parts and then Eq.(154) for reversing the functional derivative, noting that $\left(\partial^2_\mu\psi_{\alpha 1}(\mathbf{s})\right)B[\boldsymbol{\psi}(\mathbf{s})]$ is an odd Grassmann function. Similarly

$$\int d\mathbf{s}B[\boldsymbol{\psi}(\mathbf{s})]\left\{\left(\partial_\mu\psi_{\alpha 2}(\mathbf{s})\right)\left(\partial_\mu\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha 2}(\mathbf{s})}\right)\right\}$$

$$=-\int d\mathbf{s}B[\boldsymbol{\psi}(\mathbf{s})]\left\{\left(\partial^2_\mu\psi_{\alpha 2}(\mathbf{s})\right)\left(\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha 2}(\mathbf{s})}\right)\right\} \tag{201}$$

though here no reversal of the functional derivative is needed. Combining these results for $\widehat{\rho}\to -i/\hbar[\widehat{T},\widehat{\rho}]$ gives the kinetic energy term in the functional Fokker-Planck equation as

$$\left(\frac{\partial}{\partial t}B[\boldsymbol{\psi}(\mathbf{s})]\right)_K$$

$$=\frac{-i}{\hbar}\left\{-\sum_\alpha\int d\mathbf{s}\left\{\left(\sum_\mu\frac{\hbar^2}{2m}\partial^2_\mu\psi_{\alpha 1}(\mathbf{s})\,B[\boldsymbol{\psi}(\mathbf{s})]\right)\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha 1}(\mathbf{s})}\right\}\right\}$$

$$\frac{-i}{\hbar}\left\{+\sum_\alpha\int d\mathbf{s}\left\{\left(\sum_\mu\frac{\hbar^2}{2m}\partial^2_\mu\psi_{\alpha 2}(\mathbf{s})\,B[\boldsymbol{\psi}(\mathbf{s})]\right)\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha 2}(\mathbf{s})}\right\}\right\} \tag{202}$$

13.2 Potential Energy Terms

If $\widehat{\rho} \to \widehat{V}\widehat{\rho}$ then

$$\begin{aligned} B[\boldsymbol{\psi}(\mathbf{r})] &\to \sum_{\alpha} \int d\mathbf{s} \left\{ \left(\frac{\overrightarrow{\delta}}{\delta \psi_{\alpha 1}(\mathbf{s})} \right) V_{\alpha}(\mathbf{s}) \left(\psi_{\alpha 1}(\mathbf{s}) \right) \right\} B[\boldsymbol{\psi}(\mathbf{s})] \\ &= \sum_{\alpha} \int d\mathbf{s} \left\{ V_{\alpha}(\mathbf{s}) \left(\psi_{\alpha 1}(\mathbf{s}) \right) \right\} B[\boldsymbol{\psi}(\mathbf{s})] \left(\frac{\overleftarrow{\delta}}{\delta \psi_{\alpha 1}(\mathbf{s})} \right) \end{aligned} \tag{203}$$

where the functional derivative in the first term has been reversed, and we have used the fact that $\{V_{\alpha}(\mathbf{s}) \left(\psi_{\alpha 1}(\mathbf{s}) \right)\} B[\boldsymbol{\psi}(\mathbf{s})]$ is an odd Grassman function.

If $\widehat{\rho} \to \widehat{\rho}\widehat{V}$ then

$$B[\boldsymbol{\psi}(\mathbf{r})] \to \sum_{\alpha} \int d\mathbf{s} B[\boldsymbol{\psi}(\ \mathbf{s})] \left\{ \left(\psi_{\alpha 2}(\mathbf{s}) \right) V_{\alpha}(\mathbf{s}) \left(\frac{\overleftarrow{\delta}}{\delta \psi_{\alpha 2}(\mathbf{s})} \right) \right\}$$

so for $\widehat{\rho} \to -i/\hbar[\widehat{V}, \widehat{\rho}]$ then the potential energy term in the functional Fokker-Planck equation becomes

$$\begin{aligned} & \left(\frac{\partial}{\partial t} B[\boldsymbol{\psi}(\mathbf{s})] \right)_V \\ = \ & -i/\hbar \left\{ + \sum_{\alpha} \int d\mathbf{s} \left(V_{\alpha}(\mathbf{s}) \, \psi_{\alpha 1}(\mathbf{s}) B[\boldsymbol{\psi}(\mathbf{s})] \right) \left(\frac{\overleftarrow{\delta}}{\delta \psi_{\alpha 1}(\mathbf{s})} \right) \right\} \\ & -i/\hbar \left\{ - \sum_{\alpha} \int d\mathbf{s} (V_{\alpha}(\mathbf{s}) \, \psi_{\alpha 2}(\mathbf{s}) B[\boldsymbol{\psi}(\mathbf{s})]) \left(\frac{\overleftarrow{\delta}}{\delta \psi_{\alpha 2}(\mathbf{s})} \right) \right\} \end{aligned} \tag{204}$$

13.3 Interaction Energy Terms

If $\widehat{\rho} \to \widehat{U}\widehat{\rho}$ then

$$\begin{aligned} & B[\boldsymbol{\psi}(\mathbf{s})] \\ \to \ & \frac{g}{2} \sum_{\alpha} \int d\mathbf{s} \left\{ \left(+ \frac{\overrightarrow{\delta}}{\delta \psi_{\alpha 1}(\mathbf{s})} \right) \left(+ \frac{\overrightarrow{\delta}}{\delta \psi_{-\alpha 1}(\mathbf{s})} \right) \psi_{-\alpha 1}(\mathbf{s}) \psi_{\alpha 1}(\mathbf{s}) \right\} B[\boldsymbol{\psi}(\mathbf{s})] \\ = \ & -\frac{g}{2} \sum_{\alpha} \int d\mathbf{s} \left\{ \left(+ \frac{\overrightarrow{\delta}}{\delta \psi_{\alpha 1}(\mathbf{s})} \right) \left\{ \psi_{-\alpha 1}(\mathbf{s}) \psi_{\alpha 1}(\mathbf{s}) B[\ \boldsymbol{\psi}(\mathbf{s})] \left(+ \frac{\overleftarrow{\delta}}{\delta \psi_{-\alpha 1}(\mathbf{s})} \right) \right\} \right\} \\ = \ & -\frac{g}{2} \sum_{\alpha} \int d\mathbf{s} \left\{ \left\{ \psi_{-\alpha 1}(\mathbf{s}) \psi_{\alpha 1}(\mathbf{s}) B[\boldsymbol{\psi}(\ \mathbf{s})] \left(+ \frac{\overleftarrow{\delta}}{\delta \psi_{-\alpha 1}(\mathbf{s})} \right) \right\} \left(+ \frac{\overleftarrow{\delta}}{\delta \psi_{\alpha 1}(\mathbf{s})} \right) \right\} \end{aligned} \tag{205}$$

where the second line is obtained using Eq.(154) noting that $\psi_{-\alpha 1}(\mathbf{s}) \psi_{\alpha 1}(\mathbf{s}) B[\boldsymbol{\psi}(\mathbf{s})]$ is an even Grassmann function, and the third line is obtained from the same

equation, but noting that $\left\{\psi_{-\alpha 1}(\mathbf{s})\psi_{\alpha 1}(\mathbf{s})B[\boldsymbol{\psi}(\mathbf{s})]\left(+\frac{\overleftarrow{\delta}}{\delta\psi_{-\alpha 1}(\mathbf{s})}\right)\right\}$ is an odd Grassmann function.

If $\widehat{\rho}\rightarrow\widehat{\rho}\widehat{U}$ then

$$\begin{aligned} & B[\boldsymbol{\psi}(\mathbf{s})] \\ \rightarrow\ & \frac{g}{2}\sum_{\alpha}\int d\mathbf{s}B[\ \boldsymbol{\psi}(\mathbf{s})]\left\{\psi_{\alpha 2}(\mathbf{s})\psi_{-\alpha 2}(\mathbf{s})\left(+\frac{\overleftarrow{\delta}}{\delta\psi_{-\alpha 2}(\mathbf{s})}\right)\left(+\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha 2}(\mathbf{s})}\right)\right\} \\ =\ & -\frac{g}{2}\sum_{\alpha}\int d\mathbf{s}B[\boldsymbol{\psi}(\mathbf{s})]\left\{\psi_{-\alpha 2}(\mathbf{s})\psi_{\alpha 2}(\mathbf{s})\left(+\frac{\overleftarrow{\delta}}{\delta\psi_{-\alpha 2}(\mathbf{s})}\right)\left(+\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha 2}(\mathbf{s})}\right)\right\} \end{aligned} \tag{206}$$

where we have reversed the order of the fields.

Hence if $\widehat{\rho}\rightarrow -i/\hbar[\widehat{U},\widehat{\rho}]$ then the interaction energy term in the functional Fokker-Planck equation becomes

$$\begin{aligned} & \left(\frac{\partial}{\partial t}B[\boldsymbol{\psi}(\mathbf{s})]\right)_{U} \\ =\ & \frac{i}{\hbar}\frac{g}{2}\sum_{\alpha}\int d\mathbf{s}\left\{\psi_{-\alpha 1}(\mathbf{s})\psi_{\alpha 1}(\mathbf{s})B[\boldsymbol{\psi}(\mathbf{s})]\left(+\frac{\overleftarrow{\delta}}{\delta\psi_{-\alpha 1}(\mathbf{s})}\right)\left(+\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha 1}(\mathbf{s})}\right)\right\} \\ & -\frac{i}{\hbar}\frac{g}{2}\sum_{\alpha}\int d\mathbf{s}\left\{\psi_{-\alpha 2}(\mathbf{s})\psi_{\alpha 2}(\mathbf{s})B[\boldsymbol{\psi}(\mathbf{s})]\left(+\frac{\overleftarrow{\delta}}{\delta\psi_{-\alpha 2}(\mathbf{s})}\right)\left(+\frac{\overleftarrow{\delta}}{\delta\psi_{\alpha 2}(\mathbf{s})}\right)\right\}. \end{aligned} \tag{207}$$