# A new look at the morning commute with household shared-ride: how does school location play a role?

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**Abstract**

Recent studies examined the households' trip-timing decisions where adults have to drive their children to school. These studies focus on the cases where school is near workplace. However, different school locations can affect travels and activity patterns significantly. This study re-looks at the household shared-ride problem where school is near home. It is found that the resulting dynamic commuting equilibrium has very different properties, which have been examined and discussed accordingly. Three management strategies have been proposed to reduce total travel cost, and efficiencies of these strategies are evaluated and compared.

**Keywords:** Morning commute; household shared-ride; school location; schedule coordination; pricing.

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# 1 Introduction

Modelling and managing the peak hour traffic dynamics have attracted substantial attention from both transport scientists and economists. Vickrey (1969) proposed the first bottleneck model to capture the departure time choices of commuters and the corresponding traffic dynamics.[1] Despite the large literature on the morning commute, little attention has been paid to the shared-ride of household members in the morning peak hour, and how intra-household interaction among household members with different schedule preferences could affect the trip-timing choice of a household travel.

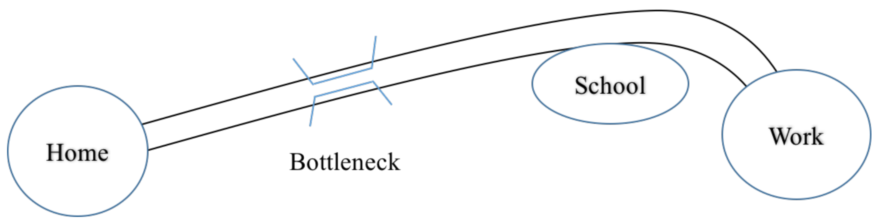
Recently, de Palma et al. (2015) modelled the trip-timing of couples. Particularly, they considered that couples value time at home more when together than when alone. Jia et al. (2016) then explored the equilibrium trip scheduling of households where the adult traveller in the household has to send the child to school before going ahead to the workplace. More recently, Liu et al. (2016) further extended Jia et al. (2016) by looking into a more general situation with both shared-ride travellers (on home-school-work trips) and individual travellers (on home-work trips). In the similar spirit of staggered work hours (as those in Henderson, 1981; Yushimito et al., 2014; Takayama, 2015), Liu et al. (2016) proposed to coordinate the schedules of school and work to reduce traffic congestion and travel cost of road users.

Both Jia et al. (2016) and Liu et al. (2016) examined the household travelling in a particular setting, i.e., household travellers firstly pass through the congested highway bottleneck, which is the major driving part of the trip, and then reach the school, and finally the workplace, as shown in Figure 1a. However, it is quite often that schools are located near homes (for example, it is common in Chinese cities that children go to schools near homes) such that the adult member in a household firstly drives to school with no or very light congestion (generally far from the city center and the congested network), and then pass through the congested highway (the major driving part for the trip) and finally reach the workplace as in Figure 1b.

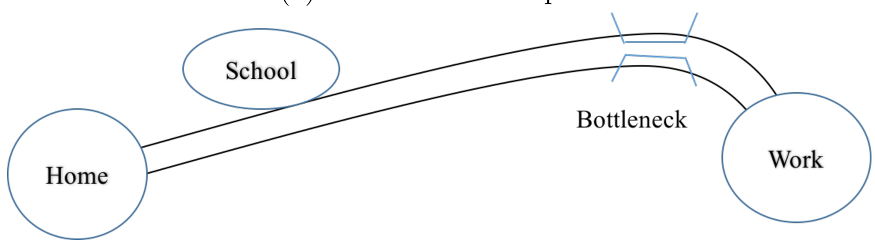
Literature has established evidence that urban planning and land-use pattern interact with transportation and have significant impact on travel behaviour (e.g., Wilson, 1998).

[1] Smith (1984) and Daganzo (1985) established the existence and uniqueness of the user equilibrium solution at a single bottleneck. Thanks to its tractability, the bottleneck model has been extended to study various issues, e.g., stochastic bottleneck capacity and travel demand (Lindsey, 2009; Xiao et al., 2014, 2015); heterogeneous travellers (Arnott et al., 1994; van den Berg and Verhoef, 2011; Liu and Nie, 2011; Liu et al., 2015b; Wang and Xu, 2016); road pricing, tradable credits or parking permits to manage traffic congestion (Arnott et al., 1990; Xiao et al., 2012; Liu et al., 2014b); integrated problem of parking and morning commute (Arnott et al., 1991; Qian et al., 2012; Yang et al., 2013; Liu et al., 2014a); integrated modelling of morning and evening commutes (Zhang et al., 2005, 2008); morning commute with car-pooling (Xiao et al., 2016); activity-based modelling (Li et al., 2014).

This paper considers the school-near-home network structure in Figure 1b, aiming to identify the difference in resulting equilibrium traffic patterns from those in Liu et al. (2016) and elucidate how different school locations (school near home or school near workplace) would shape the morning commute with mixed travels of households and individuals differently.




(a) School near workplace




(b) School near home

Figure 1: Different city networks

After presenting the trip-timing and traffic patterns at the dynamic user equilibrium, we then propose and analyse three strategies to reduce the total social cost, which is the sume of travel time cost and schedule delay cost. The proposed strategies are: schedule coordination of school and work, joint scheme of schedule coordination and pricing, and schedule coordination with differentiated school and work hours. Schedule coordination and differentiated school and work hours can reduce traffic congestion by separating the travelling of different groups of travellers, and pricing can further reduce congestion by preventing traffic intensity from going beyond the highway bottleneck capacity. In particular, two first-best tolling patterns have been identified for the commuting problem with mixed travellers (depending on parameter values). Furthermore, efficiency of the three strategies is evaluated and analysed.

The rest of the paper is organized as follows. Section 2 presents the problem description and the cost formulations for both households and individuals. In Section 3, the dynamic traffic patterns at the departure/arrival equilibrium with mixed travellers are discussed and compared with those in Liu et al. (2016). Section 4 examines the system performance under given numbers of individual and household travellers, and work and school schedules, and then analyses three strategies to reduce the total travel cost. Numerical illustrations and verifications are presented in Section 5, and Section 6 concludes the paper.

# 2 Model Formulation

We start with a thumbnail description of the commuting problem with both household home-school-work shared-ride trips and individual home-work trips, and then formulate the travel costs of individual workers and households respectively.

## 2.1 Thumbnail description

We consider the city network in Figure 1b, where highway bottleneck is between school and workplace. This means that travellers start to experience traffic congestion after passing the school. There are $N_w$ individuals driving to the city centre for work with a desired arrival time of $t_w$. These individuals have to tradeoff between in-vehicle travel time cost related to queue length at the bottleneck and schedule delay cost associated with earliness or lateness for work. As mentioned in Section 1, Vickrey (1969) was among the first to propose the bottleneck model to capture and analyse this trade-off.

Besides the $N_w$ individuals, there are $N_{sw}$ households with home-school-work trips. Following Jia et al. (2016) and Liu et al. (2016), we consider that each household trip consists of a work trip for one adult and a school trip for one child. Therefore, there are $N_{sw}$ vehicles each carrying two travellers: an adult and a child. Throughout the paper, we will frequently refer to the combination of an adult and a child as "household travellers". The desired arrival time for school is $t_s$[2] whilst the desired arrival time for work is $t_w$ as mentioned before.[3] When making departure time choices, household travellers will not only consider the travel cost of the work trip, but also that of the school trip.

Before formulating the travel costs, we assume that the free-flow travel time between home and school is zero and the travel time between school and workplace equals the delay at the highway bottleneck. For simplicity, we also assume that the delay caused by dropping off the child at school is negligible (which is zero then) so that we can focus on how the schedule difference of work and school affects trip-timing of household travels, as well as the commuting traffic equilibrium.[4] It is usually the case in practice that parents arrive early for work if they deliver the child on time for school. Thus in this paper we focus on the situation where the desired work arrival time is later than that for school such that $t_w - t_s \geq 0$ (with

[2]The desired arrival time for school is not necessarily the school start time. It can be the arrival time point under which children can arrive at class comfortably on time. Later, we frequently refer to "later than $t_s$ as "late for school", which, however, does not mean that the child is late for class.

[3]Future research may consider heterogeneous desired arrival times for work associated with different travellers.

[4]The drop-off delay at school can be readily taken into account by adding a constant delay for all households, which will create small variations to the model without affecting the central idea. This is already illustrated in Liu et al. (2016).

zero travel time between school and workplace), which is also an assumption in Jia et al. (2016) and Liu et al. (2016).

## 2.2 Individual work trip

We now formulate the travel cost of an individual. His or her cost consists of in-vehicle travel time cost and schedule delay cost. Specifically, for a traveller departing from home at time $t$, his or her cost is

$$c_w(t) = \alpha \cdot T(t) + \beta \cdot max\{0, t_w - t - T(t)\} + \gamma \cdot max\{0, t + T(t) - t_w\}, \tag{1}$$

where $T(t)$ is the experienced travel time for the traveller, and $\alpha$ is the value of unit travel delay, and $\beta$ and $\gamma$ are the schedule penalties for a unit time of early and late arrivals, respectively. It is assumed that $\gamma > \alpha > \beta$, which is consistent with the empirical evidence. Since $T(t)$ only contains the queuing delay at the highway bottleneck, it is equal to $\frac{q(t)}{s}$, where $q(t)$ is the queue length faced by the traveller departing at time $t$, and $s$ is the constant highway bottleneck capacity. Vehicles can leave the highway at any time and incur no delay until the traffic flow exceeds the constant capacity. Once flow exceeds capacity, deterministic point queue develops.

Given the above standard setting in the literature, at the departure/arrival equilibrium, the departure rates from home (or arrival rates at the bottleneck) of individuals who arrive at the destination before and after desired work arrival time $t_w$ can be determined respectively, which are

$$r_1 = \frac{\alpha}{\alpha - \beta} s; r_2 = \frac{\alpha}{\alpha + \gamma} s. \tag{2}$$

Eq.(2) is a well-established result in the literature and detailed derivations are thus omitted (one can refer to e.g., Arnott et al., 1990, for detailed derivations).

## 2.3 Household shared-ride

For a household trip with one adult's work trip and one child's school trip, the total travel cost is the sum of the costs experienced by the two household members, which is

$$\begin{array}{r} c_{sw}(t) = [\alpha \cdot 0 + \beta \cdot max\{0, t_s - t\} + \gamma \cdot max\{0, t - t_s\}] \\ +[\alpha \cdot T(t) + \beta \cdot max\{0, t_w - t - T(t)\} + \gamma \cdot max\{0, t + T(t) - t_w\}] \end{array}. \tag{3}$$

In Eq.(3), the travel time for school trip is zero as we assume zero free-flow time and zero drop-off delay. Thus, the departure time from home is equal to the arrival time at school.

This is different from that in Liu et al. (2016) where households arrive at school when leaving the highway bottleneck. Specifically, the terms in the first square bracket of Eq.(3) are the travel time cost and schedule delay cost for travelling to school (associated with the child in the household). The second square bracket represents the cost experienced by the adult in the household. Identical value of time, schedule penalties are adopted for both work and school trips. This treatment simplifies the algebra in the paper although still leaving it quite tedious. However, in practice we usually expect the schedule penalties for work and school trips to be different. Future research might take this into account, as well as a more general user heterogeneity among different groups of travellers.

As mentioned, we consider that household travels can be on time for school while early for work i.e., $t_w - t_s \geq 0$ (free-flow time is zero). If we, however, consider a non-zero free-flow time between school and workplace $t_{s-w} > 0$, this assumption should be correspondingly modified to $t_w - t_s \geq t_{s-w}$. In either case, the setting yields two possible situations regarding the earliness/lateness for household travels: i) early for school and early for work; ii) early for school but late for work. The departure rates of households travelling in the two possible situations are respectively:

$$r_1^{'} = \frac{\alpha+\beta}{\alpha-\beta}s; r_2^{'} = \frac{\alpha+\beta}{\alpha+\gamma}s. \tag{4}$$

In Eq.(4), we use superscript " $'$ " to denote the departure rates for household travels. It is worth mentioning that $r_1^{'} > r_1$ and $r_2^{'} > r_2$ because households experience additional earliness for school (corresponding to the $\beta$ in the numerator of $r_1^{'}$ or $r_2^{'}$) and that the inequalities $r_1^{'} > r_1 > s > r_2^{'} > r_2$ always hold.

In Jia et al. (2016) and Liu et al. (2016) where the congested road bottleneck is between home and school, three possible situations for household trips can arise: i) early for school and early for work; ii) late for school but early for work; iii) late for school and late for work. However in the current network structure where major congestion occurs between school and workplace (school is near home), households will never be late for school. In other words, situations ii) and iii) in Liu et al. (2016) never occur. This is generally because the lateness for school is more expensive than the earliness for work, and is also more expensive than the in-vehicle queuing delay. More detailed explanation will be provided in Section 3.1.

# 3 Dynamic User Equilibrium

## 3.1 Traffic patterns at equilibrium

Depending on the values of $N_w$, $N_{sw}$, $\alpha$, $\beta$, $\gamma$, $s$, and $\Delta t = t_w - t_s$, different equilibrium traffic patterns may arise. For instance, if $\Delta t = t_w - t_s$ is extremely large (much larger than $\frac{N_w+N_{sw}}{s}$), households and individuals will travel at very different times and thus their travels will be completely separated; if $\Delta t = t_w - t_s$ approaches zero, one can expect that the two types of users would travel at similar times thus interact with each other through sharing the same network.

We present all possible equilibrium traffic patterns in Figure 2. The time points ($t_1$, $t_2$, $t_3$ and $t_4$) indicated in Figure 2 can be derived based on the equilibrium condition (identical cost for the same type of users) and flow conservation at the bottleneck (cumulative departure should be equal to arrival). Conditions of the occurrence of each traffic pattern can be accordingly determined based on these time points. For example, in Pattern (1), $t_1 \le t_2$ holds and it leads to the condition listed in Table 1 for Pattern (1). We summarize the conditions for the occurrence of each traffic pattern in Table 1.

When the schedule difference $\Delta t = t_w - t_s$ decreases from $+\infty$ to zero, the equilibrium traffic pattern varies along the following path: (1) $\to$ (2) $\to$ (3)-1 or (3)-2 $\to$ (4) $\to$ (5)-1 or (5)-2 $\to$ (6). However with certain combinations of $N_w$, $N_{sw}$, $\alpha$, $\beta$ and $\gamma$, particular conditions in Table 1 can never be met, so that the corresponding traffic patterns would never occur even if $\Delta t$ can be arbitrarily chosen. In particular, Patterns (3)-1 and (3)-2 cannot arise for the same combination of $N_w$ and $N_{sw}$, as they respectively require $N_w \ge \frac{\beta}{\gamma} N_{sw}$ and $N_w < \frac{\beta}{\gamma} N_{sw}$. Similarly, patterns (5)-1 and (5)-2 are for cases with $N_w \ge \frac{\gamma}{\beta} N_{sw}$ and $N_w < \frac{\gamma}{\beta} N_{sw}$ respectively.

Table 1: Travel costs of individuals and households for different cases

| Patterns | Conditions (lower bounds for $\Delta t$) | Conditions (upper bounds for $\Delta t$) |
|---|---|---|
| (1) | $\Delta t \geq \frac{\gamma}{\beta+\gamma}\frac{N_w}{s} + \frac{2\beta}{\alpha+\beta}\frac{N_{sw}}{s}$ | |
| (2) | $\Delta t \geq \frac{\gamma}{\beta+\gamma}\frac{N_w}{s} + (\frac{2\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma})\frac{N_{sw}}{s}$; <br> $\Delta t \geq \frac{2\beta}{\alpha+\beta}\frac{N_{sw}}{s}$ | $\Delta t < \frac{\gamma}{\beta+\gamma}\frac{N_w}{s} + \frac{2\beta}{\alpha+\beta}\frac{N_{sw}}{s}$ |
| (3)-1 | $\Delta t \geq \frac{\gamma}{\beta+\gamma}\frac{\beta}{\alpha}\frac{N_w}{s} + (\frac{2\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma}\frac{\beta}{\alpha})\frac{N_{sw}}{s}$ | $\Delta t < \frac{\gamma}{\beta+\gamma}\frac{N_w}{s} + (\frac{2\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma})\frac{N_{sw}}{s}$ |
| (3)-2 | $\Delta t \geq (\frac{\gamma}{\beta+\gamma} + \frac{\gamma}{\alpha+\beta})\frac{N_w}{s} + (\frac{\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma})\frac{N_{sw}}{s}$ | $\Delta t < \frac{2\beta}{\alpha+\beta}\frac{N_{sw}}{s}$ |
| (4) | $\Delta t \geq \frac{\gamma}{\beta+\gamma}\frac{\beta}{\alpha}\frac{N_w}{s} + (\frac{\gamma}{\beta+\gamma}\frac{\beta}{\alpha} - \frac{\alpha+\gamma}{\alpha+\beta})\frac{N_{sw}}{s}$ <br> $\Delta t \geq (\frac{\beta}{\alpha+\beta}\frac{\beta}{\alpha} - \frac{\beta}{\beta+\gamma})\frac{N_w}{s} + (\frac{\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma})\frac{N_{sw}}{s}$ | $\Delta t < \frac{\gamma}{\beta+\gamma}\frac{\beta}{\alpha}\frac{N_w}{s} + (\frac{2\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma}\frac{\beta}{\alpha})\frac{N_{sw}}{s}$ <br> $\Delta t < (\frac{\gamma}{\beta+\gamma} + \frac{\gamma}{\alpha+\beta})\frac{N_w}{s} + (\frac{\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma})\frac{N_{sw}}{s}$ |
| (5)-1 | $\Delta t \geq -\frac{\beta}{\beta+\gamma}\frac{N_w}{s} + (\frac{\alpha+\gamma}{\alpha+\beta}\frac{\beta}{\alpha} - \frac{\beta}{\beta+\gamma})\frac{N_{sw}}{s}$ | $\Delta t < \frac{\gamma}{\beta+\gamma}\frac{\beta}{\alpha}\frac{N_w}{s} + (\frac{\gamma}{\beta+\gamma}\frac{\beta}{\alpha} - \frac{\alpha+\gamma}{\alpha+\beta})\frac{N_{sw}}{s}$ |
| (5)-2 | $\Delta t \geq \frac{\beta}{\alpha}\frac{N_w}{s} - \frac{\alpha+\gamma}{\alpha+\beta}\frac{N_{sw}}{s}$ | $\Delta t < (\frac{\beta}{\alpha+\beta}\frac{\beta}{\alpha} - \frac{\beta}{\beta+\gamma})\frac{N_w}{s} + (\frac{\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma})\frac{N_{sw}}{s}$ |
| (6) | | $\Delta t < -\frac{\beta}{\beta+\gamma}\frac{N_w}{s} + (\frac{\alpha+\gamma}{\alpha+\beta}\frac{\beta}{\alpha} - \frac{\beta}{\beta+\gamma})\frac{N_{sw}}{s}$ <br> $\Delta t < \frac{\beta}{\alpha}\frac{N_w}{s} - \frac{\alpha+\gamma}{\alpha+\beta}\frac{N_{sw}}{s}$ |

We now qualitatively describe all the possible traffic patterns when the dynamic user equilibrium is achieved. In Figure 2, the red and blue solid lines represent the departures of individuals and households, respectively. The black solid lines represent the arrivals. $r_1$ and $r_2$ are the departure rates of individuals, which are given in Eq.(2). $r_1'$ and $r_2'$ associate with households, which are given in Eq.(4). The light red (dotted lines) extensions of the red departure curves depict the isocost queuing curves for individuals, which are not actual departures.

- Pattern (1): In this case, the schedule gap $\Delta t$ between school and work is relatively large. All households depart from home between $t_1$ and $t_s$, while individuals depart much closer to work start time $t_w$ as they do not need to contend with schooling. The travels of households and individuals are completely separated. However, highway capacity is wasted between $t_2$ and $t_3$ when no one uses it.

- Pattern (2): This pattern is similar to Pattern (1). The difference lies in that, as the schedule gap $\Delta t$ between school and work becomes smaller, the travels of households and individuals are now more connected, i.e., the first individual will join the queue behind the last household. It follows that there is no capacity waste between the arrivals (at work) of households and individuals.

- Pattern (3)-1: Compared to Pattern (2), as the gap $\Delta t$ becomes even smaller, the travels of households and individuals become more connected and households travel in the middle of the peak period as shown in Figure 2c. As the number of individuals is relatively large, i.e., $N_w \geq \frac{\beta}{\gamma} N_{sw}$, some of the individuals are forced to depart earlier than households due to the small gap $\Delta t$. On the other hand, households stick to the time duration close to $t_s$ since they are affected by the school schedule. One can verify that travelling further away from $t_s$ would lead to an increase in travel cost for households.

- Pattern (3)-2: Contrary to Pattern (3)-1, this pattern occurs when the number of households is relatively large, i.e., $N_w < \frac{\beta}{\gamma} N_{sw}$. As the schedule gap $\Delta t$ becomes less than that in Pattern (2), some of the households are forced to be late for work.

- Pattern (4): When $\Delta t$ becomes sufficiently small, no matter $N_w \geq \frac{\beta}{\gamma} N_{sw}$ or $N_w < \frac{\beta}{\gamma} N_{sw}$, households depart in the middle of the peak duration, arriving at school close to the desired arrival time for school $t_s$, and at work between $t_2$ and $t_3$. By doing so households can avoid large schedule delays costs for both school and work. As earliness and lateness are less expensive for individuals (i.e., one unit time of earliness or lateness leads to double penalty for households compared to individual), some of them are forced to arrive relatively early for work and some are relatively late for work (arrive at time points further away from $t_w$).

- Pattern (5)-1: When the number of individuals is relatively large ($N_w \geq \frac{\gamma}{\beta} N_{sw}$) and the schedule gap $\Delta t$ is very small, the morning peak is dominated by individuals as shown in Figure 2f. Similarly, the small number of households will travel close to $t_s$ to reduce their "double schedule delay cost". For other duration along the red solid lines, the traffic pattern is very close to those in the standard bottleneck model with only individuals.

- Pattern (5)-2: Different from Pattern (5)-1, when the number of households is large ($N_w < \frac{\gamma}{\beta} N_{sw}$) and $\Delta t$ is small, all individuals depart earlier than households. Individuals cannot travel close to $t_w$ as households with "double schedule delay cost" have larger willingness-to-pay to travel during this time slot.

- Pattern (6): Similar to Pattern (5)-1, some individuals and all households arrive late for work. The difference is that all individuals now travel earlier than households.

As shown in Figure 2, the last households always depart at the desired arrival time for school $t_s$ in all patterns. They have no incentive to depart later even though doing so can reduce their travel delays and/or earliness for work.

**Proposition 3.1.** *In the school-near-home case shown in Figure 1b, at the commuting equilibrium, there will be no household arriving later than the desired school arrival time* $t_s$.

Proposition 3.1 states that if the child in the household is attending the school near home (major congestion occurs between school and work place), the parent will choose to send the child to school no later than the desired school arrival time $t_s$ even if he or she will encounter very large earliness for work. We now take Pattern (1) in Figure 2a as an example to illustrate the reason.

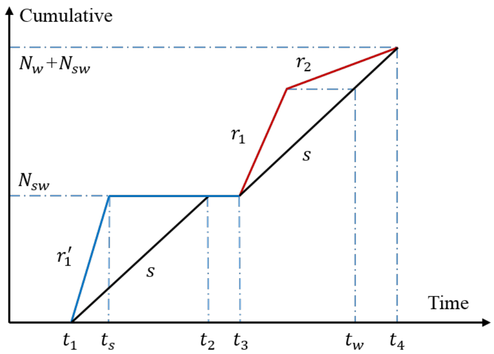
In Figure 2a, households depart from home between $t_1$ and $t_s$. Suppose that the travel cost of a household is $c_0$. We now explain why a household has no incentive to depart from home at a later time $t$, such that $t > t_s$.

If $t_s < t \leq t_2$, travel cost of the household will be $c_0 + (\gamma - \alpha)(t - t_s)$. As $\gamma > \alpha$, i.e., lateness for school is more expensive than travel delay, it is evident that households have no incentive to depart in this period.

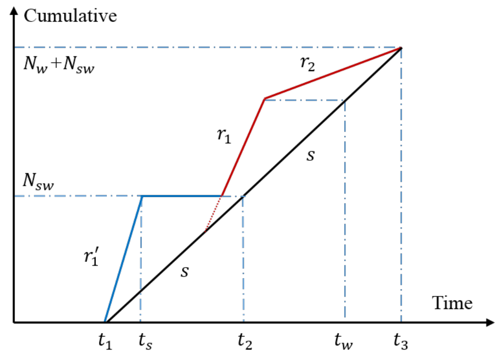
If $t_2 < t \leq t_3$, travel cost of the household will be $c_0 + (\gamma - \alpha)(t_2 - t_s) + (\gamma - \beta)(t - t_2)$. As $\gamma > \beta$, i.e., lateness for school is more expensive that earliness for work, again households have no incentive to depart in this period.

If $t > t_3$, it is obvious that by departing later, the household cannot reduce the cost associated with work trip, while encounter larger late arrival schedule delay cost for school trip.

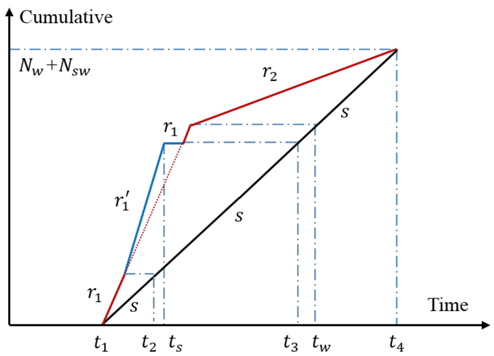
Note that in this paper, we assume that the child members in the households have identical $\alpha$, $\beta$, and $\gamma$ as the adult members, and $\gamma > \alpha > \beta$. However, the validity of Proposition 3.1 does not necessarily rely on these assumptions; instead, it requires that the late arrival penalty for school is larger than the value-of-time for work trip, and also larger than the early arrival penalty for work, which, we conjecture, generally hold in practice.



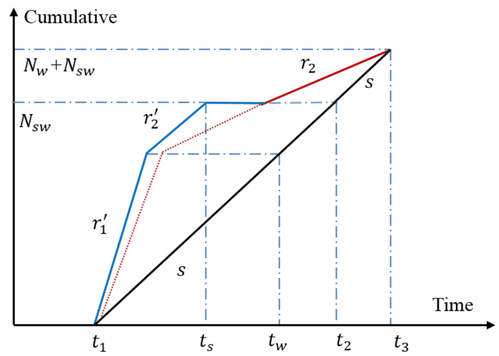

(a) Pattern (1)



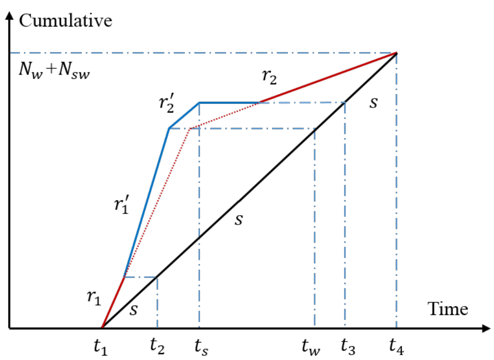

(b) Pattern (2)



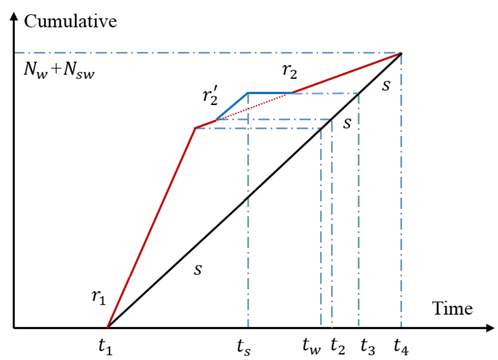

(c) Pattern (3)-1



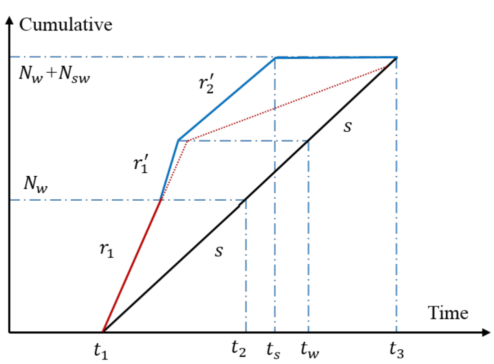

(d) Pattern (3)-2



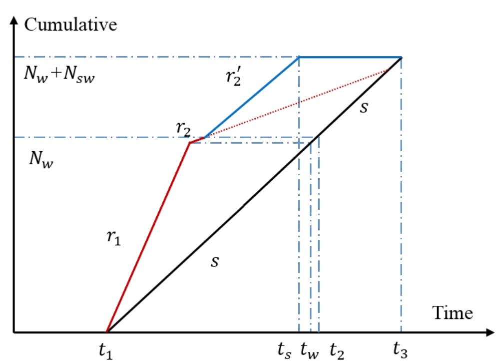

(e) Pattern (4)




(f) Pattern (5)-1




(g) Pattern (5)-2




(h) Pattern (6)

Figure 2: All possible equilibrium traffic patterns

## 3.2 Varying traffic patterns

We now explore how the equilibrium traffic pattern varies with the schedule difference $\Delta t$ and the relative number of the two groups of users $\frac{N_w}{N_{sw}}$. To avoid repetition of algebra, we only present the result with $\alpha^2 - \beta\gamma > 0$. This inequality is consistent with many empirical evidences, e.g., Tseng et al. (2005), where (per hour) the average value of time $\alpha$ is 9.91 (EUR\$), early arrival penalty $\beta$ is 4.66 (EUR\$), and late arrival penalty $\gamma$ is 14.48 (EUR\$). If the converse $\alpha^2 - \beta\gamma \leq 0$ prevails, the traffic pattern will vary in a different way among the possible flow patterns presented in Section 3.1. One of the major differences is that Pattern (6) cannot arise when $\alpha^2 - \beta\gamma > 0$, but it may occur when $\alpha^2 - \beta\gamma \leq 0$.

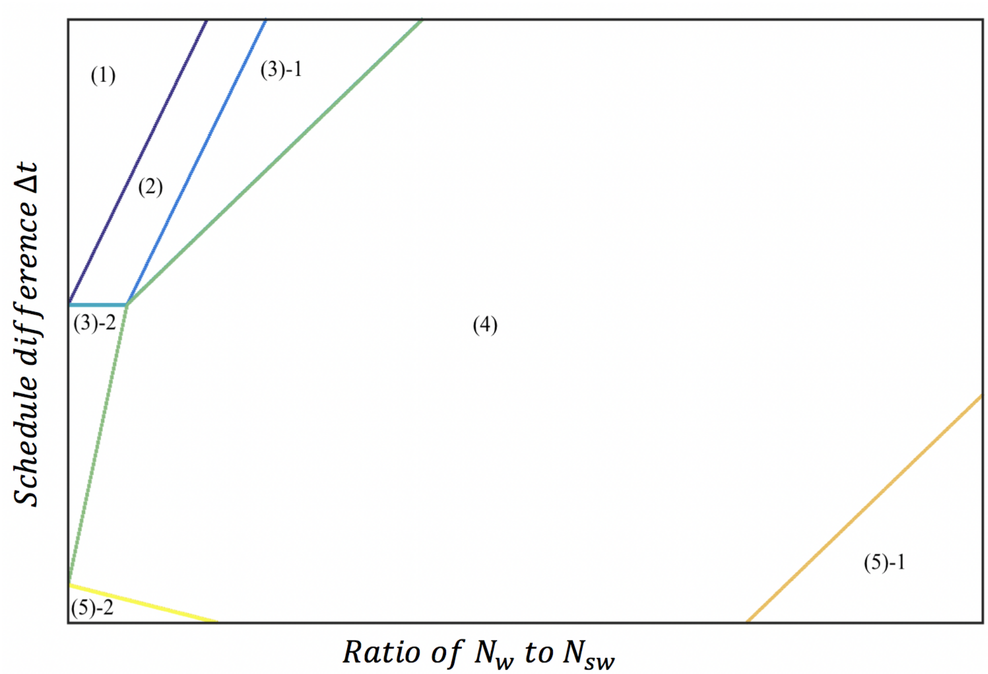



Figure 3: Occurrence of different patterns in the domain of $(\frac{N_w}{N_{sw}}, \Delta t)$

The occurrence of different equilibrium traffic patterns in the domain of $(\frac{N_w}{N_{sw}}, \Delta t)$ is displayed in Figure 3. There are several critical lines in the domain that divide it into seven regions. Each region corresponds to a particular traffic pattern. In Figure 3, $x$-axis represents $\frac{N_w}{N_{sw}}$ and $y$-axis represents $\Delta t$. Line 1 separating (1) and (2) corresponds to $y = (\frac{\gamma}{\beta+\gamma}x + \frac{2\beta}{\alpha+\beta})\frac{N_{sw}}{s}$; Line 2 separating (2) and (3)-1 corresponds to $y = (\frac{\gamma}{\beta+\gamma}x + (\frac{2\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma}))\frac{N_{sw}}{s}$; Line 3 separating (2) and (3)-2 corresponds to $y = \frac{2\beta}{\alpha+\beta}\frac{N_{sw}}{s}$; Line 4 separating (3)-1 and (4) corresponds to $y = (\frac{\gamma}{\beta+\gamma}\frac{\beta}{\alpha}x + (\frac{2\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma}\frac{\beta}{\alpha}))\frac{N_{sw}}{s}$; Line 5 separating (3)-2 and (4) corresponds to $y = ((\frac{\gamma}{\beta+\gamma} + \frac{\gamma}{\alpha+\beta})x + (\frac{\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma}))\frac{N_{sw}}{s}$; Line 6 separating (4) and (5)-1 corresponds to $y = (\frac{\gamma}{\beta+\gamma}\frac{\beta}{\alpha}x + (\frac{\gamma}{\beta+\gamma}\frac{\beta}{\alpha} - \frac{\alpha+\gamma}{\alpha+\beta}))\frac{N_{sw}}{s}$; and Line 7 separating (4) and (5)-2 corresponds to

$y = ((\frac{\beta}{\alpha+\beta}\frac{\beta}{\alpha} - \frac{\beta}{\beta+\gamma})x + (\frac{\beta}{\alpha+\beta} - \frac{\beta}{\beta+\gamma}))\frac{N_{sw}}{s}$. These lines are established from the conditions in Table 1. Later, these lines are numerically illustrated in Figure 8.

For given $N_w$, $N_{sw}$, $\alpha$, $\beta$, and $\gamma$, how the traffic pattern varies with $\Delta t$ is summarized in Table 2. For ease of presentation, we define the following critical values:

$$\lambda_1 = \frac{\alpha^2\beta + \alpha^2\gamma + \alpha\gamma^2 - \beta^2\gamma}{(\alpha+\beta)\beta\gamma}; \lambda_2 = \frac{\alpha(\gamma - \alpha)}{\alpha^2 + \alpha\beta - \beta^2 - \beta\gamma}; \lambda_3 = \frac{\beta}{\gamma}; \tag{5}$$

where $\lambda_2$ is only valid when $\alpha^2 + \alpha\beta - \beta^2 - \beta\gamma$ is not equal to zero. These critical values are derived based on conditions for the occurrence of Pattern (5)-1 and Pattern (5)-2 listed in Table 1. As can be seen in Table 2, these critical values are related to the occurrence of Pattern (5)-1 and Pattern (5)-2. Furthermore, since $\alpha^2 - \beta\gamma > 0$ and $\alpha > \beta$, it can be verified that $\alpha^2 + \alpha\beta - \beta^2 - \beta\gamma \neq 0$ and the following inequality holds.

$$\lambda_1 > \lambda_2 > \lambda_3. \tag{6}$$

It can be readily verified that when $\Delta t$ decreases from $+\infty$ to zero, the traffic pattern will change from one pattern to another according to the order in Table 2, based on the equilibrium conditions in Table 1. Table 2 will be utilized when we explore how individual or household travel cost and total travel cost vary with $\Delta t$.

Table 2: Varying equilibrium traffic pattern against $\Delta t$

| Range of $\frac{N_w}{N_{sw}}$ | Varying traffic pattern when $\Delta t$ decreases (from $+\infty$ to zero) |
|---|---|
| $[\lambda_1, \infty)$ | (1) $\to$ (2) $\to$ (3)-1 $\to$ (4) $\to$ (5)-1 |
| $[\lambda_2, \lambda_1)$ | (1) $\to$ (2) $\to$ (3)-1 $\to$ (4) |
| $[\lambda_3, \lambda_2)$ | (1) $\to$ (2) $\to$ (3)-1 $\to$ (4) $\to$ (5)-2 |
| $[0, \lambda_3)$ | (1) $\to$ (2) $\to$ (3)-2 $\to$ (4) $\to$ (5)-2 |

# 4 Managing System Performance

## 4.1 User travel costs

In Section 3, we have described all the possible commuting traffic patterns in detail. Now we examine the users' travel costs under different patterns. Based on Section 3 and with some manipulations,[5] we can obtain the equilibrium travel cost of individuals $c_w$ and households $c_{sw}$, both of which are summarized in Table 3.

Table 3: Individual and household travel costs under different equilibrium traffic patterns

| Pattern | Individual cost $c_w$ & Household cost $c_{sw}$ (User Equilibrium) |
|---|---|
| (1) | $c_w = \frac{\beta\gamma}{\beta+\gamma}\frac{N_w}{s}$ <br> $c_{sw} = \beta\Delta t + 2\beta\frac{\alpha-\beta}{\alpha+\beta}\frac{N_{sw}}{s}$ |
| (2) | $c_w = -\gamma\Delta t + \gamma\frac{N_w}{s} + \frac{2\beta\gamma}{\alpha+\beta}\frac{N_{sw}}{s}$ <br> $c_{sw} = \beta\Delta t + 2\beta\frac{\alpha-\beta}{\alpha+\beta}\frac{N_{sw}}{s}$ |
| (3)-1 | $c_w = \frac{\beta\gamma}{\beta+\gamma}\frac{N_w}{s} + \frac{\beta\gamma}{\beta+\gamma}\frac{N_{sw}}{s}$ <br> $c_{sw} = \frac{\beta\gamma}{\beta+\gamma}\frac{N_w}{s} + (\frac{\beta\gamma}{\beta+\gamma} + \frac{\beta(\alpha-\beta)}{\alpha+\beta})\frac{N_{sw}}{s}$ |
| (3)-2 | $c_w = -\frac{\gamma(\alpha+\beta)}{\alpha+2\beta+\gamma}\Delta t + \gamma\frac{N_w}{s} + \frac{2\beta\gamma}{\alpha+2\beta+\gamma}\frac{N_{sw}}{s}$ <br> $c_{sw} = -\frac{\beta(\gamma-\alpha)}{\alpha+2\beta+\gamma}\Delta t + \frac{2\beta(\alpha+\gamma)}{\alpha+2\beta+\gamma}\frac{N_{sw}}{s}$ |
| (4) | $c_w = \frac{\beta\gamma}{\beta+\gamma}\frac{N_w}{s} + \frac{\beta\gamma}{\beta+\gamma}\frac{N_{sw}}{s}$ <br> $c_{sw} = -\frac{\alpha\beta(\beta+\gamma)}{\alpha^2+\alpha\beta-\beta^2+\alpha\gamma}\Delta t + \frac{(\alpha+\gamma)(\alpha+\beta)}{\alpha^2+\alpha\beta-\beta^2+\alpha\gamma}\frac{\beta\gamma}{\beta+\gamma}\frac{N_w}{s} + \frac{(\alpha+\gamma)(\alpha+\beta)}{\alpha^2+\alpha\beta-\beta^2+\alpha\gamma}(\frac{\beta\gamma}{\beta+\gamma} + \frac{\beta(\alpha-\beta)}{\alpha+\beta})\frac{N_{sw}}{s}$ |
| (5)-1 | $c_w = \frac{\beta\gamma}{\beta+\gamma}\frac{N_w}{s} + \frac{\beta\gamma}{\beta+\gamma}\frac{N_{sw}}{s}$ <br> $c_{sw} = \frac{\beta\gamma}{\beta+\gamma}\frac{N_w}{s} + (\frac{\beta\gamma}{\beta+\gamma} + \frac{\beta(\alpha+\gamma)}{\alpha+\beta})\frac{N_{sw}}{s}$ |
| (5)-2 | $c_w = \frac{\beta(\alpha+\beta)}{\alpha+2\beta+\gamma}\Delta t + \beta(1 - \frac{\beta}{\alpha}\frac{\alpha+\beta}{\alpha+2\beta+\gamma})\frac{N_w}{s} + \frac{\beta(\alpha+\gamma)}{\alpha+2\beta+\gamma}\frac{N_{sw}}{s}$ <br> $c_{sw} = -\frac{\beta(\gamma-\alpha)}{\alpha+2\beta+\gamma}\Delta t + (\alpha+\gamma)\frac{\beta}{\alpha}\frac{\alpha+\beta}{\alpha+2\beta+\gamma}\frac{N_w}{s} + \frac{2\beta(\alpha+\gamma)}{\alpha+2\beta+\gamma}\frac{N_{sw}}{s}$ |
| (6) | $c_w = \frac{\alpha\beta}{\alpha+\beta+\gamma}\Delta t + \frac{\beta(\alpha+\gamma)}{\alpha+\beta+\gamma}\frac{N_w}{s} + \frac{\beta(\alpha+\gamma)}{\alpha+\beta+\gamma}\frac{\alpha}{\alpha+\beta}\frac{N_{sw}}{s}$ <br> $c_{sw} = \frac{\alpha\beta}{\alpha+\beta+\gamma}\Delta t + \frac{\beta(\alpha+\gamma)}{\alpha+\beta+\gamma}\frac{N_w}{s} + \frac{\beta(\alpha+\gamma)}{\alpha+\beta+\gamma}(1 + \frac{\alpha+\gamma}{\alpha+\beta})\frac{N_{sw}}{s}$ |

**Proposition 4.1.** *Equilibrium travel cost of a user group (household or individual) is more sensitive to the number of users in that group than the number in the other group, i.e., wherever* $\frac{\partial c_i}{\partial N_i}$ *and* $\frac{\partial c_i}{\partial N_j}$ *exist, we have*

$$\frac{\partial c_i}{\partial N_i} \geq \frac{\partial c_i}{\partial N_j}, \tag{7}$$

*where* $i \neq j$, *and* $i, j \in \{w, sw\}$.

Proposition 4.1 is verified using Table 3 (details relayed to Appendix A). This proposition

[5] The equilibrium travel costs for households and individuals are derived from the time points in Figure 2.

dictates that intra-group externality is more significant than inter-group externality. Similar results have been found by Liu et al. (2016) and Lindsey (2004), although a different school location is considered here and the combined trip-timing decision of household members' shared-ride brings further complexity to the model.

It can be inferred from Table 3 that how household and individual travel cost change with $\Delta t$ depends on which traffic pattern prevails. Table 4 summarizes $\frac{dc_i}{\Delta t}$ for $i \in \{w, sw\}$ for all traffic patterns.

Table 4: Derivatives of costs with respect to schedule difference $\Delta t$

| Pattern | $\frac{dc_w}{d\Delta t}$ | $\frac{dc_{sw}}{d\Delta t}$ |
|---|---|---|
| (1) | $0$ | $\beta > 0$ |
| (2) | $-\gamma < 0$ | $\beta > 0$ |
| (3)-1 | $0$ | $0$ |
| (3)-2 | $-\frac{\gamma(\alpha+\beta)}{\alpha+2\beta+\gamma} < 0$ | $-\frac{\beta(\gamma-\alpha)}{\alpha+2\beta+\gamma} < 0$ |
| (4) | $0$ | $-\frac{\alpha\beta(\beta+\gamma)}{\alpha^2+\alpha\beta-\beta^2+\alpha\gamma} < 0$ |
| (5)-1 | $0$ | $0$ |
| (5)-2 | $\frac{\beta(\alpha+\beta)}{\alpha+2\beta+\gamma} > 0$ | $-\frac{\beta(\gamma-\alpha)}{\alpha+2\beta+\gamma} < 0$ |
| (6) | $\frac{\alpha\beta}{\alpha+\beta+\gamma} > 0$ | $\frac{\alpha\beta}{\alpha+\beta+\gamma} > 0$ |

The results in Table 4 are briefly explained in the following and meanwhile readers can refer to the corresponding traffic patterns in Figure 2 for a better understanding of Table 4.

In Pattern (1), travels of households and individuals are completely separated. A marginal increase in $\Delta t$ does not affect the individuals as they still travel around $t_w$, i.e., $\frac{dc_w}{d\Delta t} = 0$. However, the larger schedule difference $\Delta t$ will enlarge the earliness for work experienced by households as their departure is constrained by the school schedule, and thus $\frac{dc_{sw}}{d\Delta t} > 0$.

In Pattern (2), similar to Pattern (1), a larger $\Delta t$ will force the households to depart further away from $t_w$ (depart earlier, so that $\frac{dc_{sw}}{d\Delta t} > 0$). Thus, the travels of the two types of users will be more separated, and the individual work trips will be less affected by the households. As a result, on average the individual traveller will experience less queuing, and smaller schedule delay cost ($\frac{dc_w}{d\Delta t} < 0$).

In Pattern (3)-1 and Pattern (5)-1, given a marginal increase in $\Delta t$, households will not change their relative departure/arrival time with respect to $t_s$. Therefore, the cost associated with the school trip will remain constant. The relative scale of queuing and schedule delay cost associated with the work trip will change with $\Delta t$, but their sum will not. It follows then $\frac{dc_w}{d\Delta t} = 0$ and $\frac{dc_{sw}}{d\Delta t} = 0$.

In Pattern (3)-2 and Pattern (4), a marginal increase in $\Delta t$ would reduce the number of late-for-work households. This is because the households are constrained by the school trip, and they depart relatively earlier with respect to $t_w$ if $\Delta t$ is larger. As less late-for-work households implies less congestion delays (see Figure 2d and Figure 2e), we then have $\frac{dc_{sw}}{d\Delta t} < 0$. In Pattern (3)-2, as households depart further away from $t_w$, individuals can be less affected and thus can reduce their late arrival penalties, and $\frac{dc_w}{d\Delta t} < 0$. However, for Pattern (4), similar to Pattern (3)-1 and Pattern (5)-1, the change in $\Delta t$ will change the relative scale of queuing delay cost and schedule delay cost for the work trip, but it does not change the sum of queuing delay cost and schedule delay cost, and we have $\frac{dc_w}{d\Delta t} = 0$.

In Pattern (5)-2 and Pattern (6), a larger $\Delta t$ will lead households to depart further away from $t_w$ (depart earlier). As all individuals travel earlier than households, individuals will be forced to travel even earlier. As a result, we have $\frac{dc_w}{d\Delta t} > 0$. In Pattern (5)-2, similar to Pattern (3)-2 and Pattern (4), there will be less households late for work, and their queuing delays will be reduced. Thus, $\frac{dc_{sw}}{d\Delta t} < 0$. However in Pattern (6) a marginal increase in $\Delta t$ will not reduce the number of late-for-work households, but will enlarge the (average) queuing delay for households, and $\frac{dc_{sw}}{d\Delta t} > 0$.

## 4.2 Schedule coordination

Now we explore how to coordinate the schedules of school and work (by appropriately choosing $\Delta t$) to reduce travel cost. Based on Table 3, we can obtain the total travel cost for all travellers

$$TC = N_w \cdot c_w + N_{sw} \cdot c_{sw}, \tag{8}$$

where $c_w$ and $c_{sw}$ are the cost of individuals and households respectively. The optimal $\Delta t$ that minimizes total travel cost can then be determined, which is denoted by $(\Delta t)^*$.[6] We present $(\Delta t)^*$ in Proposition 4.2.

**Proposition 4.2.** *For given $N_w$ and $N_{sw}$, i) if $N_w \geq \frac{\beta}{\gamma} N_{sw}$*

$$(\Delta t)^* = \frac{\gamma}{\beta + \gamma}\frac{N_w}{s} + \frac{2\beta}{\alpha + \beta}\frac{N_{sw}}{s}; \tag{9}$$

[6] This can be verified by looking at the first-order derivative of $TC$ with respect to $\Delta t$, $\frac{dTC}{d\Delta t}$, and by comparing different local optima.

*ii) and if* $N_w < \frac{\beta}{\gamma} N_{sw}$

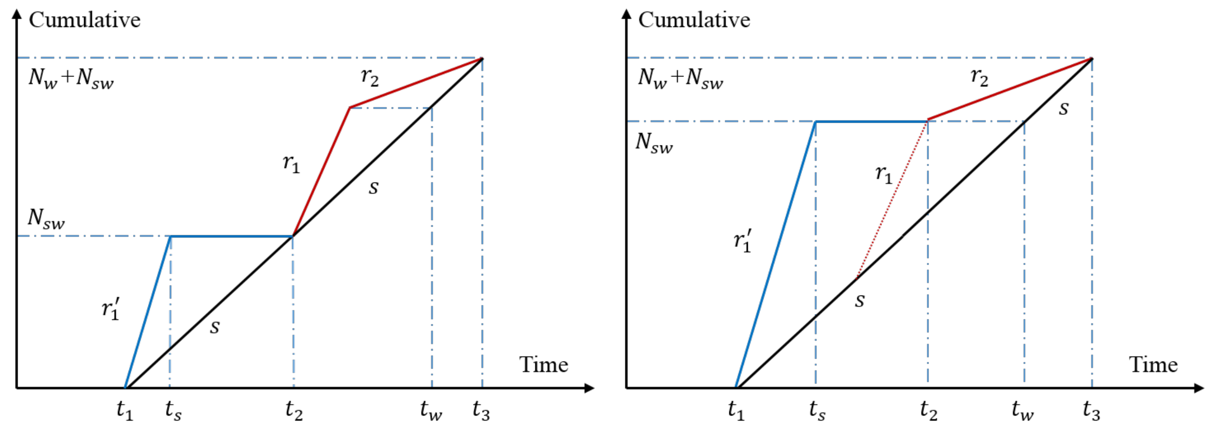
$$(\Delta t)^* = \frac{2\beta}{\alpha + \beta} \frac{N_{sw}}{s}. \tag{10}$$

Proposition 4.2, with referring to Table 1, implies that when $N_w \geq \frac{\beta}{\gamma} N_{sw}$, the optimal $\Delta t$ leads to the boundary equilibrium traffic pattern between patterns (2) and (1). When $N_w < \frac{\beta}{\gamma} N_{sw}$, the optimal $\Delta t$ leads to the boundary between patterns (3)-2 and (2). The corresponding optimal traffic patterns are depicted in Figure 4, where the blue and red lines are the departures from home associated with households and individuals respectively.

In the boundary traffic pattern between patterns (2) and (1), the departures/arrivals of households and individuals are separated, thus the congestion is reduced. In this case, the first individual arrives at work just after the last households, and thus there is no capacity waste of the highway bottleneck (no gap in the arrivals at work between households' and individuals').

In the boundary traffic pattern between patterns 3-(2) and (2), the last household and first individual are exactly on time for work. By comparing Figure 4a and Figure 4b, it is intuitive that the congestion delay is larger under the boundary pattern between 3-(2) and (2) than the boundary pattern between (2) and (1). We could increase $\Delta t$ to further reduce congestion , but it would increase the schedule delay cost for households, as the gap between school and work schedules becomes larger. When $N_w < \frac{\beta}{\gamma} N_{sw}$ (the number of households is relatively large), increasing $\Delta t$ can lead to a very large increase in total travel cost because a relatively large number of households would encounter larger schedule delay cost. Thus, there is no incentive to increase $\Delta t$ from the system's perspective, even if doing so can reduce congestion.

However, when the network structure of Liu et al. (2016) is in effect, the boundary case between patterns (2) and (3)-2 in Figure 4b is always socially unfavourable compared to the case in Figure 4a. This is because, under the framework of Liu et al. (2016), households might arrive at school later than $t_s$ and their arrivals at work can be closer to $t_w$. Consequently, the increase in schedule delay cost caused by the separation of two user groups is less significant.




(a) Boundary case between patterns (1) & (2) (b) Boundary case between patterns (2) & (3)-2

Figure 4: Two critical traffic patterns

From Proposition 4.2 the total travel cost under the optimal $\Delta t$ can be derived, which is

$$TC^* = \begin{cases} (\frac{\beta\gamma}{\beta+\gamma}\frac{N_w}{s}) \cdot N_w + (\frac{\beta\gamma}{\beta+\gamma}\frac{N_w}{s} + \frac{2\alpha\beta}{\alpha+\beta}\frac{N_{sw}}{s}) \cdot N_{sw} & N_w \geq \frac{\beta}{\gamma}N_{sw} \\ (\gamma\frac{N_w}{s}) \cdot N_w + (\frac{2\alpha\beta}{\alpha+\beta}\frac{N_{sw}}{s}) \cdot N_{sw} & N_w < \frac{\beta}{\gamma}N_{sw} \end{cases}. \tag{11}$$

$TC^*$ in Eq.(11) is the minimum cost that can be achieved by adjusting the schedule difference between school and work. It is a second-best optimum. After introducing the joint scheme of schedule coordination and pricing in next subsection, we will evaluate the efficiency of pure schedule coordination against the optimal joint scheme that leads to the first-best situation. Before doing so, we now compare the school-near-home case with the school-near-workplace case when the schedule difference is optimized. According to Liu et al. (2016), when school is near workplace, it is optimal to set the school and work schedule difference as

$$(\Delta t)^{**} = \frac{\gamma}{\beta+\gamma}\frac{N_w}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_{sw}}{s}. \tag{12}$$

The resulting minimum total travel cost under the $(\Delta t)^{**}$ is

$$TC^{**} = (\frac{\beta\gamma}{\beta+\gamma}\frac{N_w}{s}) \cdot N_w + (\frac{\beta\gamma}{\beta+\gamma}\frac{N_w}{s} + \frac{2\beta\gamma}{\beta+\gamma}\frac{N_{sw}}{s}) \cdot N_{sw}. \tag{13}$$

With respectively optimized schedule difference, the relative efficiency of different school locations is evaluated by this ratio:

$$\eta = \frac{TC^*}{TC^{**}}. \tag{14}$$

We have the following results regarding the ratio $\eta$.

**Proposition 4.3.** *The relative efficiency of school-near-home against school-near-workplace, i.e., $\eta$, satisfies*

$$\eta = \begin{cases} \frac{\lambda^2+\lambda+\frac{2\alpha(\beta+\gamma)}{\gamma(\alpha+\beta)}}{\lambda^2+\lambda+2} < 1 & \lambda \geq \frac{\beta}{\gamma} \\ \frac{\frac{\beta+\gamma}{\beta}\lambda^2+\frac{2\alpha(\beta+\gamma)}{\gamma(\alpha+\beta)}}{\lambda^2+\lambda+2} < 1 & 0 \leq \lambda < \frac{\beta}{\gamma} \end{cases}, \tag{15}$$

*where $\lambda = \frac{N_w}{N_{sw}}$ and $N_{sw} \neq 0$.*

*Proof.* Based on Eqs.(11), (13), and (14), it is easy to establish that $\eta$ can be determined as a function of $\lambda = \frac{N_w}{N_{sw}}$ where $N_{sw} > 0$, which is given in Eq.(15). We now show that $\eta < 1$.

Firstly, as $\gamma > \alpha > \beta > 0$ holds, we have

$$\alpha(\beta+\gamma) < \gamma(\alpha+\beta) \iff \frac{2\alpha(\beta+\gamma)}{\gamma(\alpha+\beta)} < 2.$$

Secondly, when $0 \leq \lambda < \frac{\beta}{\gamma}$,

$$\frac{\gamma}{\beta}\lambda^2 < \lambda \iff \frac{\beta+\gamma}{\beta}\lambda^2 < \lambda^2 + \lambda$$

Based on the above, one can readily verify with Eq.(15) that $\eta < 1$ holds. □

The implication of Proposition 4.3 is twofold. Firstly, the relative efficiency $\eta$ is determined by the relative size of the user groups $\lambda = \frac{N_w}{N_{sw}}$. Secondly, $\eta$ is always smaller than 1 as long as $\gamma > \alpha > \beta > 0$, meaning that with household shared-ride and respectively optimized schedule difference, *ceteris paribus*, the school-near-home plan considered in this paper is a more efficient setting than the school-near-workplace plan as in Jia et al. (2016) and Liu et al. (2016). This is mainly because, in the school-near-home case, the arrival rate at school $r_1'$ as shown in Figure 4 is not constrained by the road bottleneck capacity $s$, and thus larger than that in Liu et al. (2016), which is equal to $s$. The arrivals at school are then more concentrated around the desired school arrival time and the total schedule delay cost associated with the school trips can be smaller.

## 4.3 Schedule coordination and pricing

Subsection 4.2 discussed how to appropriately adjust $\Delta t$ to reduce total travel cost and achieve the second-best situation. The first-best situation can be achieved by simultaneously implementing pricing to avoid queueing and coordinating schedules of work and school to avoid additional schedule penalties. We begin with describing the first-best situation, to which we refer as the System Optimum (SO).

Queues should be eliminated at the SO to minimize the travel delay cost such that the departure rate should be exactly equal to the highway bottleneck capacity $s$. To minimize schedule delay cost, arrivals at work should occur between $t_1 = t_w - \frac{\gamma}{\beta+\gamma}\frac{N_w+N_{sw}}{s}$ and $t_4 = t_w + \frac{\beta}{\beta+\gamma}\frac{N_w+N_{sw}}{s}$, such that the number of early arrival travellers is $\frac{\gamma}{\beta}$ times the number of late arrival travellers. Similarly, arrivals at school should occur between $t_2 = t_s - \frac{\gamma}{\beta+\gamma}\frac{N_{sw}}{s}$ and $t_3 = t_s + \frac{\beta}{\beta+\gamma}\frac{N_{sw}}{s}$. These results are standard in the bottleneck model literature so the detailed derivations are omitted. By appropriately setting $t_s$ and $t_w$, we can minimize the schedule delay cost associated with both school and work trips. Specifically, to achieve this, we let $[t_2,\, t_3]$ be within $[t_1,\, t_4]$ as shown in Figure 5.

Setting the above as the target traffic pattern, we can derive the optimal time-varying toll given an appropriate $\Delta t$. To achieve SO where $t_2 \geq t_1$, we should set $\Delta t \leq \frac{\gamma}{\beta+\gamma}\frac{N_w}{s}$ if $N_w \geq \frac{\beta}{\gamma}N_{sw}$, and $\Delta t \leq \frac{\beta}{\beta+\gamma}\frac{N_{sw}}{s}$ if $N_w < \frac{\beta}{\gamma}N_{sw}$. In other words, $\Delta t$ should be upper bounded, because if the gap between $t_s$ and $t_w$ is too large, the inequality $t_2 \geq t_1$ will be violated, and there would be very significant schedule delay costs for household travellers (one may imagine the case with an extremely large $\Delta t$).

We now briefly depict how to derive the first-best time-varying toll. Similar to Eq.(1) and Eq.(3), the travel cost after taking into account pricing can be written as follows. For individual home-work trips:

$$c_w(t) = \alpha \cdot T(t) + \beta \cdot max\{0, t_w - t - T(t)\} + \gamma \cdot max\{0, t + T(t) - t_w\} + \tau(t), \tag{16}$$

where $\tau(t)$ is the toll for the traveller departing at time $t$; and for household home-school-work trips, we have

$$\begin{array}{r} c_{sw}(t) = \alpha \cdot 0 + \beta \cdot max\{0, t_s - t\} + \gamma \cdot max\{0, t - t_s\} \\ +\alpha \cdot T(t) + \beta \cdot max\{0, t_w - t - T(t)\} + \gamma \cdot max\{0, t + T(t) - t_w\} + \tau(t) \end{array}. \tag{17}$$

Note that here the toll is charged per vehicle rather than per passenger. Given $t_s$ and $t_w$ and $t_1 = t_w - \frac{\gamma}{\beta+\gamma}\frac{N_w+N_{sw}}{s}$, $t_2 = t_s - \frac{\gamma}{\beta+\gamma}\frac{N_{sw}}{s}$, $t_3 = t_s + \frac{\beta}{\beta+\gamma}\frac{N_{sw}}{s}$, $t_4 = t_w + \frac{\beta}{\beta+\gamma}\frac{N_w+N_{sw}}{s}$, we can derive the time-varying tolls by setting $T(t) = 0$ and taking the first-order derivatives of Eq.(16) and Eq.(17) with respect to $t$. Depending on the relative magnitude of $t_3 = t_s + \frac{\beta}{\beta+\gamma}\frac{N_{sw}}{s}$ with respect to $t_w$, two toll pattens can arise.

When $\frac{\beta}{\beta+\gamma}\frac{N_{sw}}{s} \leq \Delta t \leq \frac{\gamma}{\beta+\gamma}\frac{N_w}{s}$ ($t_3 < t_w$ holds and $\tau(t_1) = 0$ is assumed), the first-best toll is

$$\tau(t)=\begin{cases}0 & t\in(-\infty,t_1)\\ \beta(t-t_w)+\frac{\beta\gamma}{\beta+\gamma}\frac{N_w+N_{sw}}{s} & t\in[t_1,t_2)\\ 2\beta(t-t_s)-\beta\Delta t+\frac{\beta\gamma}{\beta+\gamma}\frac{N_w+2N_{sw}}{s} & t\in[t_2,t_s)\\ (\beta-\gamma)(t-t_s)-\beta\Delta t+\frac{\beta\gamma}{\beta+\gamma}\frac{N_w+2N_{sw}}{s} & t\in[t_s,t_3)\\ \beta(t-t_w)+\frac{\beta\gamma}{\beta+\gamma}\frac{N_w+N_{sw}}{s} & t\in[t_3,t_w)\\ -\gamma(t-t_w)+\frac{\beta\gamma}{\beta+\gamma}\frac{N_w+N_{sw}}{s} & t\in[t_w,t_4)\\ 0 & t\in[t_4,+\infty)\end{cases}, \tag{18}$$

The corresponding toll pattern is displayed in Figure 5a, where $t_1$, $t_2$, $t_3$ and $t_4$ are defined in the above. Households arrive between $t_2$ and $t_3$ (the blue solid lines represent the tolls experienced by households). The premier condition $\frac{\beta}{\beta+\gamma}\frac{N_{sw}}{s}\le\Delta t\le\frac{\gamma}{\beta+\gamma}\frac{N_w}{s}$ implies $\frac{N_{sw}}{s}\le\frac{\gamma}{\beta+\gamma}\frac{N_w+N_{sw}}{s}$, meaning that all households can complete their trips within a time interval less than or equal to $\frac{\gamma}{\beta+\gamma}$ times of the total peak duration $[t_1,t_4]$. However, if there are too many households such that $\frac{N_{sw}}{s}>\frac{\gamma}{\beta}\frac{N_w}{s}$, the toll pattern in Figure 5a is no longer feasible. In fact, when $0\le\Delta t\le\frac{\beta}{\beta+\gamma}\frac{N_{sw}}{s}$ $(t_3\ge t_w)$, the first-best toll is

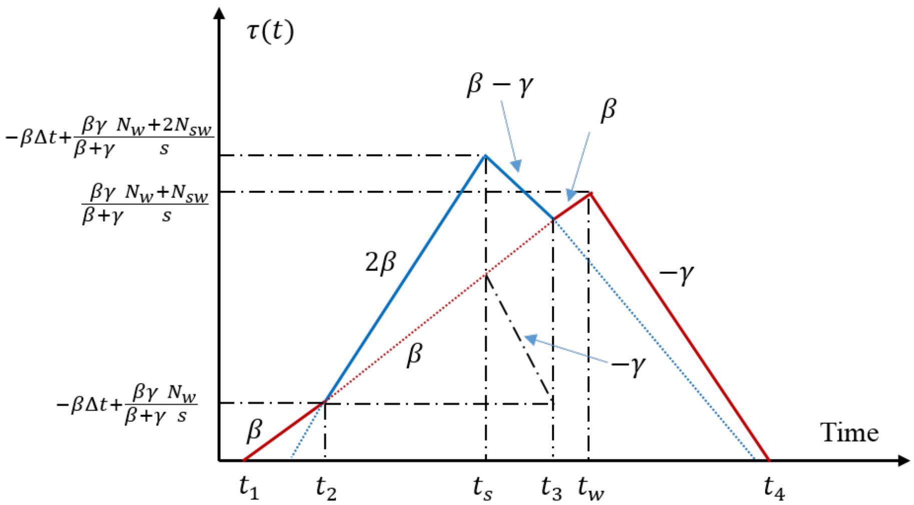
$$\tau(t)=\begin{cases}0 & t\in(-\infty,t_1)\\ \beta(t-t_w)+\frac{\beta\gamma}{\beta+\gamma}\frac{N_w+N_{sw}}{s} & t\in[t_1,t_2)\\ 2\beta(t-t_s)-\beta\Delta t+\frac{\beta\gamma}{\beta+\gamma}\frac{N_w+2N_{sw}}{s} & t\in[t_2,t_s)\\ (\beta-\gamma)(t-t_s)-\beta\Delta t+\frac{\beta\gamma}{\beta+\gamma}\frac{N_w+2N_{sw}}{s} & t\in[t_s,t_w)\\ -2\gamma(t-t_w)-\gamma\Delta t+\frac{\beta\gamma}{\beta+\gamma}\frac{N_w+2N_{sw}}{s} & t\in[t_w,t_3)\\ -\gamma(t-t_w)+\frac{\beta\gamma}{\beta+\gamma}\frac{N_w+N_{sw}}{s} & t\in[t_3,t_4)\\ 0 & t\in[t_4,+\infty)\end{cases}, \tag{19}$$

where $t_1$, $t_2$, $t_3$ and $t_4$ still follow the same formulas as in Eq.(18). The toll pattern determined by Eq.(19) is displayed in Figure 5b. It is different from the former case in the sense that there will be some households late for both school and work between $t_w$ and $t_3$. During this time interval, the toll decreases over time at the rate of $-2\gamma$ as shown in Figure 5b.

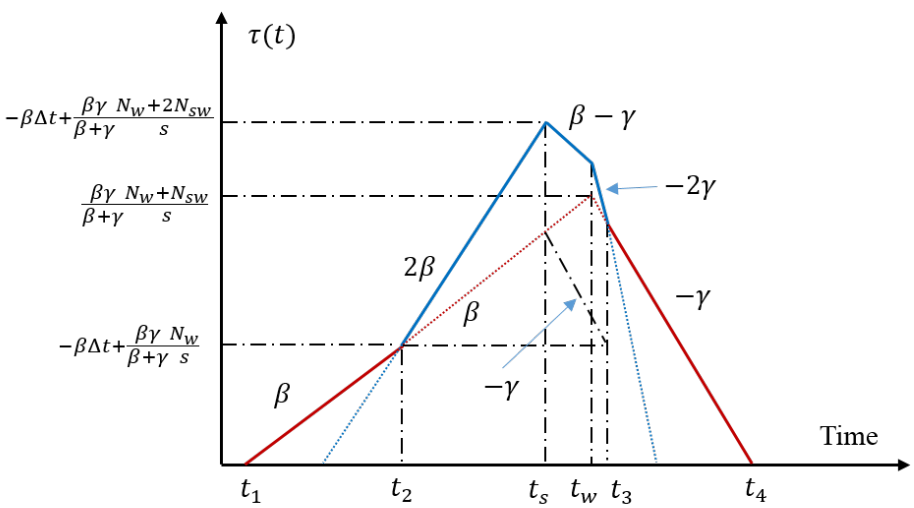
In a special case where $\Delta t=0$, which is always feasible, the toll segment between $t_s$ and $t_w$ (decreasing at a rate of $(\beta-\gamma)$) in Figure 5b will disappear.[7] The resulting toll pattern over time is similar to the optimal toll for the case with heterogeneous individual travellers (a group of $N_w$ with schedule penalties of $\beta$ and $\gamma$, and a group of $N_{sw}$ with schedule penalties

[7] This special case of first-best toll is described in Liu et al. (2016) for efficiency comparison purpose.

of $2\beta$ and $2\gamma$), which has been well studied in the literature.




(a) Toll pattern (1): $\frac{\beta}{\beta+\gamma}\frac{N_{sw}}{s} \leq \Delta t \leq \frac{\gamma}{\beta+\gamma}\frac{N_w}{s}$




(b) Toll pattern (2): $0 \leq \Delta t \leq \frac{\beta}{\beta+\gamma}\frac{N_{sw}}{s}$

Figure 5: First-best time-varying toll under different $\Delta t$

It is evident from Figure 5 that the maximum toll for households ($-\beta\Delta t + \frac{\beta\gamma}{\beta+\gamma}\frac{N_w+2N_{sw}}{s}$) is always imposed on the households that arrive on time for school (departing at time $t_s$). We summarize the relevant findings in the following propositions.

**Proposition 4.4.** *Under the first-best toll given in Eq.(18) or Eq.(19), the maximum toll for a household occurs at time $t_s$. The value of this maximum toll varies with $\Delta t$, reaching the upper bound*

$$\tau_{up} = \frac{\beta\gamma}{\beta+\gamma}\frac{N_w + 2N_{sw}}{s}, \tag{20}$$

*at $\Delta t = 0$; and the lower bound*

$$\tau_{low} = \begin{cases} \frac{\beta\gamma}{\beta+\gamma}\frac{N_w+N_{sw}}{s} & N_w \geq \frac{\beta}{\gamma}N_{sw} \\ \frac{\beta\gamma}{\beta+\gamma}\frac{N_w+(2-\frac{\beta}{\gamma})N_{sw}}{s} & N_w < \frac{\beta}{\gamma}N_{sw} \end{cases}. \tag{21}$$

*at $\Delta t = \frac{\gamma}{\beta+\gamma}\frac{N_w}{s}$ when $N_w \geq \frac{\beta}{\gamma}N_{sw}$, and at $\Delta t = \frac{\beta}{\beta+\gamma}\frac{N_{sw}}{s}$ when $N_w < \frac{\beta}{\gamma}N_{sw}$.*

As tolls for individuals are quite standard in the literature, we omit detailed discussions. While the tolls experienced by households or individuals vary with $\Delta t$, the total toll revenue of all travellers does not change with $\Delta t$, under a given toll scheme defined in either Eq.(18) or Eq.(19) (this can be readily verified).

**Proposition 4.5.** *Under the first-best toll in either Eq.(18) or Eq.(19), the total toll revenue is constantly equal to $\frac{1}{2s}\frac{\beta\gamma}{\beta+\gamma}(N_w^2 + N_w N_{sw} + 2N_{sw}^2)$ regardless of $\Delta t$.*

The total travel cost (tolls excluded) at the System Optimum is

$$TC_{SO} = 0.5 \cdot \frac{\beta\gamma}{\beta+\gamma}\frac{N_w+N_{sw}}{s}(N_w + N_{sw}) + 0.5 \cdot \frac{\beta\gamma}{\beta+\gamma}\frac{N_{sw}}{s}N_{sw}. \tag{22}$$

The relative efficiency of the second-best situation achieved through pure schedule coordination as in Subsection 4.2 with respect to the System Optimum (first-best situation) can be evaluated by examining the ratio $\theta$:

$$\theta = \frac{TC^*}{TC_{SO}} \tag{23}$$

With some manipulations, we have the following proposition.

**Proposition 4.6.** *The relative efficiency of pure schedule coordination (second-best solution) with respect to the System Optimum (first-best solution) satisfies*

$$\theta = \begin{cases} \frac{\lambda^2+\lambda+\frac{2\alpha(\beta+\gamma)}{\gamma(\alpha+\beta)}}{0.5\lambda^2+\lambda+1} < 2 & \lambda \geq \frac{\beta}{\gamma} \\ \frac{\frac{\beta+\gamma}{\beta}\lambda^2+\frac{2\alpha(\beta+\gamma)}{\gamma(\alpha+\beta)}}{0.5\lambda^2+\lambda+1} \leq \frac{2\alpha(\beta+\gamma)}{\gamma(\alpha+\beta)} < 2 & 0 \leq \lambda < \frac{\beta}{\gamma} \end{cases}, \tag{24}$$

*where $\lambda = \frac{N_w}{N_{sw}}$ and $N_{sw} \neq 0$.*

It is obvious that $\theta > 1$ holds always. Proposition 4.6 further indicates that, firstly, the relative efficiency $\theta$ depends on the relative size of user groups $\lambda = \frac{N_w}{N_{sw}}$; secondly, the relative efficiency can always be bounded by 2 regardless of the value of time $\alpha$ and schedule penalties $\beta$ and $\gamma$. This means that the minimum total travel cost achieved under the optimal schedule coordination (second-best) will always be less than twice the minimum total travel cost at the System Optimum. Note that if $\Delta t$ is not appropriately chosen, the travel cost at the dynamic user equilibrium (without tolling) can be much more than twice of the cost at System Optimum. This is different from the standard results with identical travellers (identical value of time, schedule penalties and desired arrival time for work) in, e.g., Arnott et al. (1990), as household shared-ride is incorporated.

## 4.4 Differentiated school and work schedules

This subsection will briefly show that if children are attending different schools near the residential area, it is beneficial for the transportation system to encourage differentiated school schedules at different schools. The rational is that differentiated schedules can separate the travels of households and avoid temporally-concentrated demand. We illustrate the idea with two groups of schools, of which the desired arrival times are respectively $t_{s,1}$ and $t_{s,2}$, as shown in Figure 6.

To avoid tedious repetition, we do not elaborate on all the possible situations with two different school schedules. The focus is given to the two critical traffic patterns in Figure 4, which are the (second-best) optima under uniform school schedule. Based on this benchmark, the travel cost of households can be further reduced by separating the departure/arrival of the households tied with the two different school schedules. Assuming $xN_{sw}$ ($x \in [0,1]$) households attend the school with $t_{s,1}$, and $(1-x)N_{sw}$ attend the other with $t_{s,2}$, the proposed school schedules are

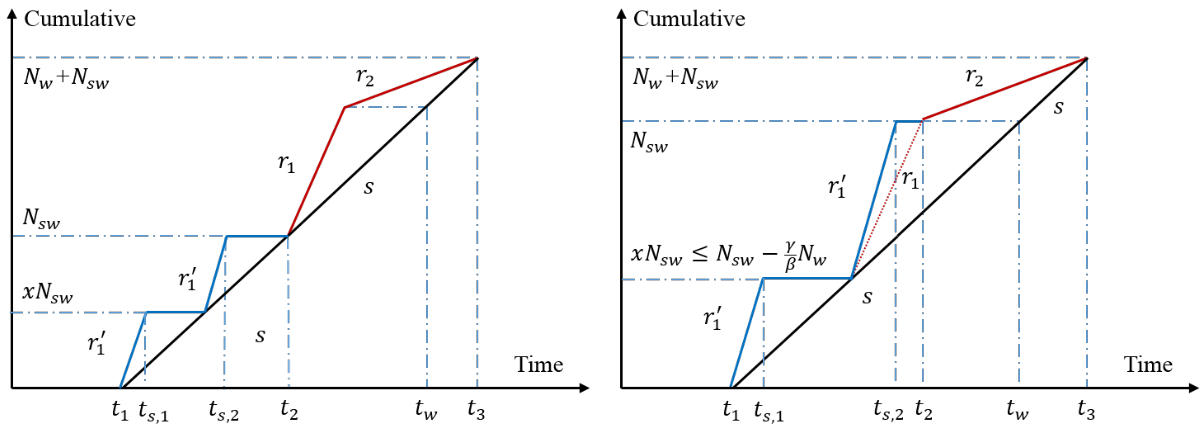
$$\begin{array}{l} t_{s,1} = t_w - \frac{\gamma}{\beta+\gamma}\frac{N_w}{s} - \frac{2\beta}{\alpha+\beta}\frac{N_{sw}}{s} - \frac{\alpha-\beta}{\alpha+\beta}\frac{(1-x)N_{sw}}{s} \\ t_{s,2} = t_w - \frac{\gamma}{\beta+\gamma}\frac{N_w}{s} - \frac{2\beta}{\alpha+\beta}\frac{(1-x)N_{sw}}{s} \end{array}, \tag{25}$$

if $N_w \geq \frac{\beta}{\gamma}N_{sw}$ and $x \in [0,1]$; and

$$\begin{array}{l} t_{s,1} = t_w - \frac{2\beta}{\alpha+\beta}\frac{N_{sw}}{s} - \frac{\alpha-\beta}{\alpha+\beta}\frac{(1-x)N_{sw}}{s} \\ t_{s,2} = t_w - \frac{2\beta}{\alpha+\beta}\frac{(1-x)N_{sw}}{s} \end{array}, \tag{26}$$

if $N_w < \frac{\beta}{\gamma}N_{sw}$ and $x \in [0, 1 - \frac{\gamma}{\beta}\frac{N_w}{N_{sw}}]$.

Figure 6a and Figure 6b display the equilibrium traffic pattern with $N_w \geq \frac{\beta}{\gamma}N_{sw}$ and $N_w < \frac{\beta}{\gamma}N_{sw}$ respectively. When $N_w < \frac{\beta}{\gamma}N_{sw}$, $x \leq 1 - \frac{\gamma}{\beta}\frac{N_w}{N_{sw}}$ should hold so the equilibrium pattern in Figure 6b is valid (the red dotted line, representing the isocost queuing curve, is to the right of the blue solid line which represents the departures of households with desired school arrival time $t_{s,2}$).




(a) Boundary case between patterns (1) & (2) (b) Boundary case between patterns (2) & (3)-2

Figure 6: Differentiated school schedules for two critical traffic patterns

The travel cost reduced by the proposed differentiated school schedules with respect to the second-best optima with uniform school schedule is:

$$\Delta TC_s = x(1-x)(\beta\frac{\alpha-\beta}{\alpha+\beta}\frac{N_{sw}}{s} + \alpha\frac{2\beta}{\alpha+\beta}\frac{N_{sw}}{s})N_{sw} \tag{27}$$

The above $\Delta TC$ reaches the maximum at $x = 0.5$ if $N_w \geq \frac{\beta}{\gamma}N_{sw}$; at $x = 0.5$ if $N_w < \frac{\beta}{\gamma}N_{sw}$ and $1 - \frac{\gamma}{\beta}\frac{N_w}{N_{sw}} \geq 0.5$; at $x = 1 - \frac{\gamma}{\beta}\frac{N_w}{N_{sw}}$ if $N_w < \frac{\beta}{\gamma}N_{sw}$ and $1 - \frac{\gamma}{\beta}\frac{N_w}{N_{sw}} < 0.5$. This indicates that as households distribute more evenly between the two schools, a larger efficiency can be achieved by implementing differentiated school schedules.

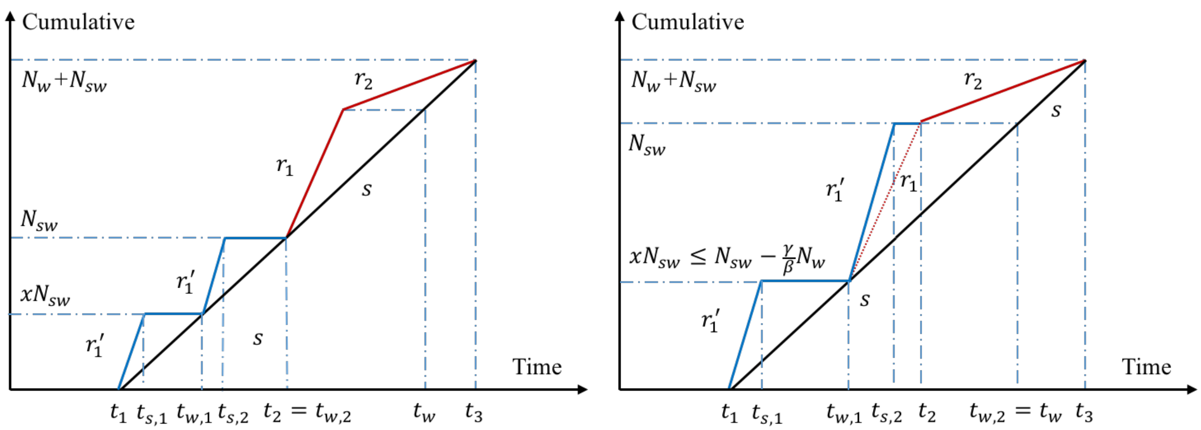
We now examine how to appropriately differentiate work schedules to be compatible with the differentiated school schedules. Suppose that the differentiated school schedules are still $t_{s,1}$ and $t_{s,2}$ as defined in the above, and households can also have differentiated work schedules. For $xN_{sw}$ ($x \in [0,1]$) households with $t_{s,1}$, and $(1-x)N_{sw}$ with $t_{s,2}$, the proposed work schedules are respectively

$$\begin{array}{l} t_{w,1} = t_w - \frac{\gamma}{\beta+\gamma}\frac{N_w}{s} - (1-x)\frac{N_{sw}}{s} \\ t_{w,2} = t_w - \frac{\gamma}{\beta+\gamma}\frac{N_w}{s} \end{array}, \tag{28}$$

if $N_w \geq \frac{\beta}{\gamma} N_{sw}$ and $x \in [0, 1]$, as shown in Figure 7a; and the work schedules are respectively

$$\begin{aligned} t_{w,1} &= t_w - (1-x)\frac{N_{sw}}{s} \\ t_{w,2} &= t_w \end{aligned}, \tag{29}$$

if $N_w < \frac{\beta}{\gamma} N_{sw}$ and $x \in [0, 1 - \frac{\gamma}{\beta}\frac{N_w}{N_{sw}}]$, as shown in Figure 7b. Note that differentiated work schedules can be simultaneously implemented for individuals, which are not included here to again avoid tedious repetition given that staggered work-hour scheme has been well studied in the literature in for example Henderson (1981).




(a) Boundary case between patterns (1) & (2) (b) Boundary case between patterns (2) & (3)-2

Figure 7: Differentiated school and work schedules for two critical traffic patterns

When the desired arrival times (or work start times) are set as compatible with the school schedules, the total travel cost of all travellers can be further reduced by (compared to the cases with only differentiated school schedules in Figure 6):

$$\Delta TC_w = [\frac{\gamma}{\beta+\gamma}\frac{N_w}{s} + x(1-x)\frac{N_{sw}}{s}]N_{sw} \tag{30}$$

if $N_w \geq \frac{\beta}{\gamma} N_{sw}$ and $x \in [0, 1]$; and

$$\Delta TC_w = x(1-x)\frac{N_{sw}}{s}N_{sw} \tag{31}$$

if $N_w < \frac{\beta}{\gamma} N_{sw}$ and $x \in [0, 1 - \frac{\gamma}{\beta}\frac{N_w}{N_{sw}}]$. Similarly, one can verify that $\Delta TC_w$ reaches the maximum at $x = 0.5$ if $N_w \geq \frac{\beta}{\gamma} N_{sw}$; at $x = 0.5$ if $N_w < \frac{\beta}{\gamma} N_{sw}$ but $1 - \frac{\gamma}{\beta}\frac{N_w}{N_{sw}} \leq 0.5$; and at $x = 1 - \frac{\gamma}{\beta}\frac{N_w}{N_{sw}}$ if $N_w < \frac{\beta}{\gamma} N_{sw}$ and $1 - \frac{\gamma}{\beta}\frac{N_w}{N_{sw}} < 0.5$. These results together with those from Eq.(27) suggest that the more evenly do the households distribute between the two schools, the more largely the cost can be reduced through differentiated and compatible school and work schedules.

# 5 Numerical Experiments

This section presents numerical results to illustrate and verify the analysis in the paper. We start with introducing the numerical settings. Following Liu et al. (2015a), the value of time $\alpha$ is 9.91 (EUR\$/hour), early arrival penalty $\beta$ is 4.66 (EUR\$/hour), and late arrival penalty $\gamma$ is 14.48 (EUR\$/hour). The capacity of the highway bottleneck is $s = 30$ (veh/min).

Given $N_{sw} = 2000$, by varying $\frac{N_w}{N_{sw}}$ and $\Delta t$, we obtain the contour map of individual cost $c_w$, household cost $c_{sw}$, and total cost $TC$ in Figure 8. The occurrence of different traffic patterns in the domain of $(\frac{N_w}{N_{sw}}, \Delta t)$ is shown in Figure 8a which is a numerical verification of Figure 3. Figure 8b shows that in most cases the individual travel cost remains constant with fixed $\frac{N_w}{N_{sw}}$ and varying $\Delta t$, indicating that in various cases the individual travel cost is not affected by $\Delta t$. However, as $\Delta t$ increases and the traffic pattern shifts from Pattern(3)-1 to (2) to (1) (varying within Pattern (2)), the individual cost decreases sharply. Another observation from Figure 8b is that individual cost increases with $\frac{N_w}{N_{sw}}$ for given $\Delta t$, which is straightforward because when there are more individuals, $\frac{\partial c_w}{\partial N_w} > 0$ and $\frac{\partial c_{sw}}{\partial N_w} \geq 0$.

Different from the individual travel cost, the household travel cost ($c_{sw}$) is more sensitive to $\Delta t$ when $\frac{N_w}{N_{sw}}$ is fixed. In Figure 8c, $c_{sw}$ firstly decreases and then increases with $\Delta t$ for given $\frac{N_w}{N_{sw}}$. For given $\frac{N_w}{N_{sw}}$, the household cost achieves the minimum within the domain for Pattern (3)-1 or Pattern (3)-2 mostly rather than the boundary pattern between Pattern (1) and Pattern (2). This is because, a larger $\Delta t$ (comparing Pattern (2) with Pattern (3)-1 or (3)-2) leads to larger earliness (for work) for the households as their departures are constrained by the school trip. When the number of households is significant compared to individuals ($N_w < \frac{\beta}{\gamma} N_{sw}$), to avoid costly large earliness (for work) for a large number of users, we should set the $\Delta t$ to realize the boundary pattern between Pattern (3)-2 and Pattern (1) as discussed in Subsection 4.2. The total cost contours in Figure 8d can also explain why the optimal schedule coordination often occurs at the boundary case between Pattern (1) and Pattern (2).

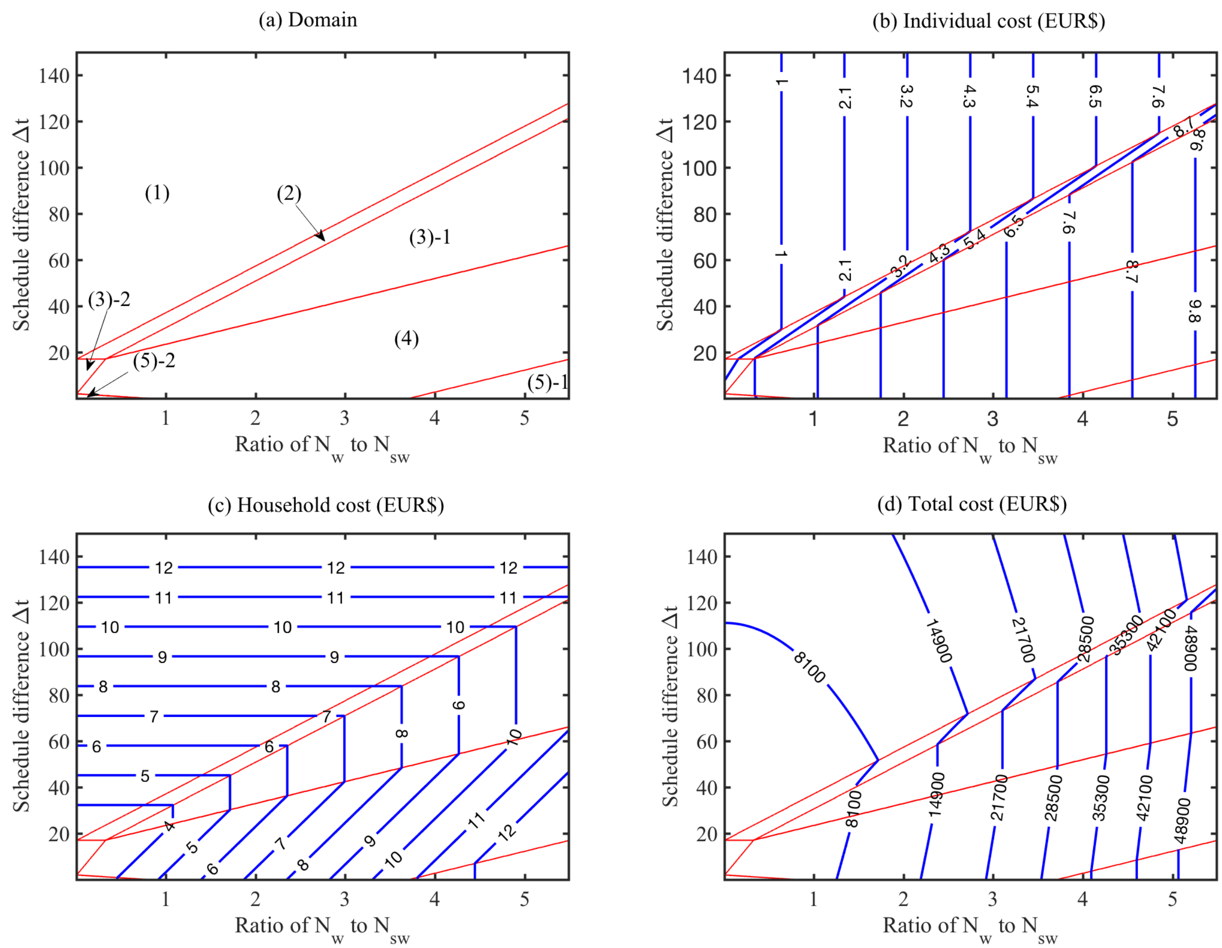



Figure 8: Costs contours in the domain of $(\frac{N_w}{N_{sw}}, \Delta t)$

Figure 9 shows two situations with different ratios of $\frac{N_w}{N_{sw}}$. It displays how individual cost, household cost, and total cost vary with schedule difference $\Delta t$. Specifically, Figure 9a and 9b correspond to $N_w = 5500$ and $N_{sw} = 2000$, and thus $\frac{N_w}{N_{sw}} = 2.75 > \frac{\beta}{\gamma} = 0.32$; and Figure 9c and 9d correspond to $N_w = 300$ and $N_{sw} = 2000$, and thus $\frac{N_w}{N_{sw}} = 0.15 < \frac{\beta}{\gamma} = 0.32$.

Given $\alpha$, $\beta$, and $\gamma$, we obtain $\lambda_1 = 3.71$, $\lambda_2 = 0.82$, and $\lambda_3 = 0.32$, which are defined in Eq.(5). We have derived in Table 2 that increasing $\Delta t$ from zero will lead the traffic pattern vary through (4) $\rightarrow$ (3)-1 $\rightarrow$ (2) $\rightarrow$ (1) when $\frac{N_w}{N_{sw}} = 2.75$, or in line with (5)-2 $\rightarrow$ (4) $\rightarrow$ (3)-2 $\rightarrow$ (2) $\rightarrow$ (1) when $\frac{N_w}{N_{sw}} = 0.15$. Both paths are now verified by Figure 9.

In Figure 9a where $\frac{N_w}{N_{sw}} > \frac{\beta}{\gamma}$, the decrease in individual travel cost over $\Delta t$ is sharper than the increase of household cost in Pattern (2). Since there is a relatively large number of individuals, the total cost reaches the minimum at the boundary between Pattern (2) and Pattern (1) which coincides with the minimum of individual travel cost over $\Delta t$ as shown in Figure 9b.

Similarly in Figure 9c where $\frac{N_w}{N_{sw}} < \frac{\beta}{\gamma}$, the decrease in individual travel cost over $\Delta t$

is sharper than the increase of household cost under Pattern (2). However, the number of households is relatively large with respect to individuals. Thus, the total increase of household cost overweights the total decrease of individual cost. Therefore, the total cost reaches the minimum at the boundary between Pattern (3)-2 and Pattern (2), coinciding with the minimum of household travel cost.

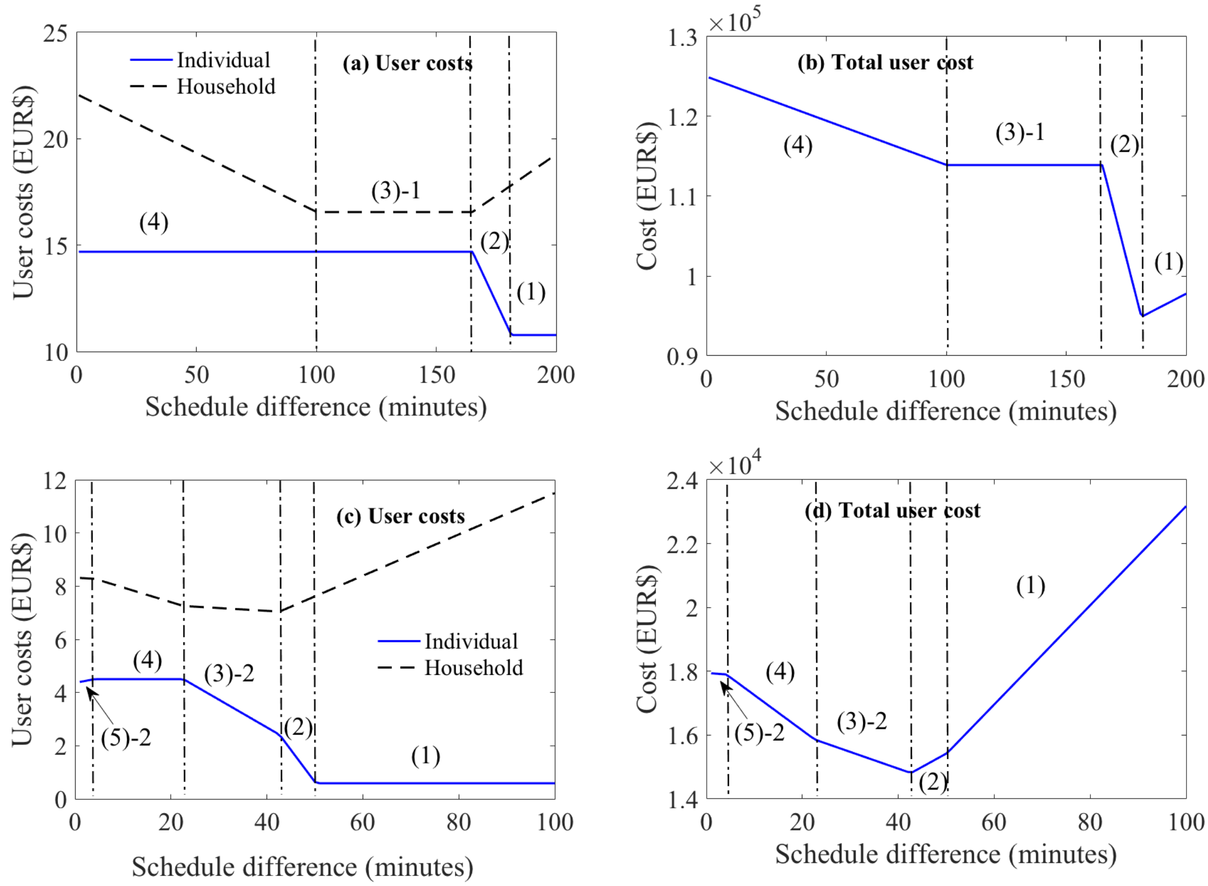



Figure 9: Individual cost, and household cost, and total cost vary with $\Delta t$: (a)-(b): $N_w = 5500$, $N_{sw} = 2000$; (c)-(d): $N_w = 300$, $N_{sw} = 2000$

Figure 10 further shows how the derivatives of user costs with respect to number of users vary with $\Delta t$ at which $N_w = 5500$, $N_{sw} = 2000$, and $\frac{N_w}{N_{sw}} = 2.75$. Similarly, the traffic pattern varies through (4) $\rightarrow$ (3)-1 $\rightarrow$ (2) $\rightarrow$ (1) as $\Delta t$ increases. In Figure 10, 'w-w', 'w-sw', 'sw-w', and 'sw-sw' represent $\frac{\partial c_w}{\partial N_w}$, $\frac{\partial c_w}{\partial N_{sw}}$, $\frac{\partial c_{sw}}{\partial N_w}$, and $\frac{\partial c_{sw}}{\partial N_{sw}}$, respectively. It is evident that $\frac{\partial c_w}{\partial N_w} \geq \frac{\partial c_w}{\partial N_{sw}}$ and $\frac{\partial c_{sw}}{\partial N_w} < \frac{\partial c_{sw}}{\partial N_{sw}}$ always hold, which is a numerical illustration of Proposition 4.1, indicating that the equilibrium travel cost of a user group (household or individual in this paper) is more sensitive to the number of users in that group than the number of users in the other group.

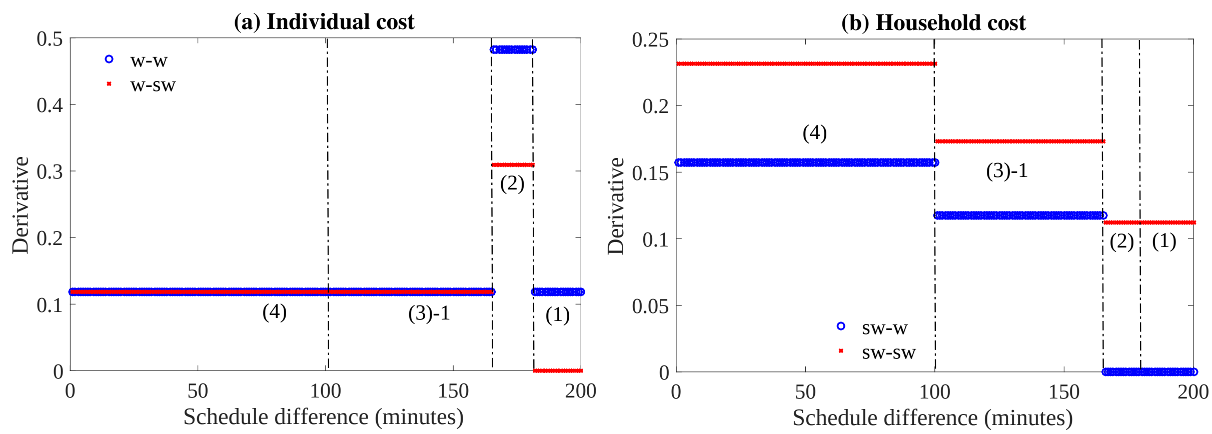



Figure 10: Derivatives of user costs with respect to numbers of users

# 6 Conclusion and Discussion

This study re-examines the morning commute problem with both individuals' home-work trips and households' home-school-work shared-ride trips in the context of a different school location from the literature. We consider that the congested road bottleneck is between the school and workplace, which is quite common in reality as children often go to school near home, rather than between school and home. In this case, the adult member in a household will firstly drive to the school with zero or very light congestion, and then pass through the congested highway bottleneck where the major delay occurs, and finally reach the workplace.

Specifically, we examine the dynamic user equilibrium with mixed travellers (individual and household) under the "school near home" network, and compare it with that in Liu et al. (2016) under "school near workplace" network (the major differences of this paper with Liu et al. (2016) are summarized in Appendix B). We find that the dynamic traffic equilibrium is significantly affected by school locations. In the "school near home" network, households always arrive at school no later than the desired school arrival time. This is because, lateness for school is more expensive than earliness for work, and is more expensive than in-vehicle queuing delay.

Three strategies aiming at reducing total social cost have been proposed and analysed: "school-work schedule coordination", "school-work schedule coordination and pricing", and "differentiated school and work schedules". Two first-best toll patterns arising under different parameter values have been derived and analysed for the joint scheme of schedule coordination and pricing. Efficiencies of the three strategies are examined and compared.

The modelling framework of this paper is general for analysing household members' shared-ride. The analysis and results apply to not only school-work trips but also to households with two workers that carpool when one workplace is relatively closer to home.

As a first step to understand the impacts of different school locations on the morning commuting equilibrium with household home-school-work trips, we have assumed deterministic travel demand and highway capacity. This indeed allows tractability for analytical insights. However, it is important to account for stochasticity in both demand and supply in the model (e.g., the stochastic nature of traffic flow dynamics). The highway capacity and travel demand fluctuations can lead to variability in queue length behind the highway bottleneck and variability of travel time and trip cost, and thus can affect the departure time choices of both households and individual workers. This means that stochasticity can create variations in the dynamic traffic patterns studied in the paper. To model the stochasticity, similar approaches as those in Lindsey (2009); Xiao et al. (2015) can be adopted. While the central ideas (e.g., school hours push households to depart earlier, and schedule coordination separate travels of households and individuals) can be extended, the exact dynamic traffic patterns and the optimal decisions (e.g., schedule coordination, pricing) need careful reexamination with demand and supply stochasticity. The efficiency gains from the proposed management strategies should be re-evaluated accordingly; whilst the current deterministic analysis can serve as comparable measures.

This study can also be extended in several other directions. Firstly, the ride-sharing of non-family members, as well as the trip-timing and coordination of travellers, can be analysed. In this case, travellers have to tradeoff between the inconvenience caused by ride-sharing and the reduced monetary cost through ride-sharing, which further complicates the joint trip-timing choice of a shared-ride and the morning commuting dynamics. Secondly, future study requires a general queuing network with schools and workplaces distributed over different places. An intuitive example would be the two-tandem bottleneck network considered in Kuwahara (1990), in which we can model a joint equilibrium of departure time choice, work choice, and school choice of households. Thirdly, a multi-modal transportation system can be incorporated so that household members can either share ride through driving or take public transport, where public transport service is responsive to traffic conditions, such as those considered in Zhang et al. (2014, 2016). Fourthly, in this study, identical value of time and schedule penalties are adopted for both work and school trips. However in practice, the schedule penalties and values of time for work and school trips are usually different. Future research will take this into account, as well as more general user heterogeneity among different groups of travellers (in either group of households or individuals).

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# Appendix

## Appendix A. Proof of Proposition 4.1

*Proof.* To prove Proposition 4.1, we just need to calculate $\frac{\partial c_w}{\partial N_w}$, $\frac{\partial c_w}{\partial N_{sw}}$, $\frac{\partial c_{sw}}{\partial N_w}$ and $\frac{\partial c_{sw}}{\partial N_{sw}}$ under different equilibrium traffic patterns, and then compare them accordingly. Based on Table 3, the mentioned derivatives (when exist) can be readily obtained, and it follows that:

| Pattern | $\frac{\partial c_w}{\partial N_w} \geq \frac{\partial c_w}{\partial N_{sw}}$ | $\frac{\partial c_{sw}}{\partial N_w} \leq \frac{\partial c_{sw}}{\partial N_{sw}}$ |
|---|---|---|
| (1) | $\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s} > 0$ | $0 < \frac{2\beta(\alpha-\beta)}{\alpha+\beta}\frac{1}{s}$ |
| (2) | $\gamma\frac{1}{s} > \frac{2\beta\gamma}{\alpha+\gamma}\frac{1}{s}$ | $0 < \frac{2\beta(\alpha-\beta)}{\alpha+\beta}\frac{1}{s}$ |
| (3)-1 | $\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s} = \frac{\beta\gamma}{\beta+\gamma}\frac{1}{s}$ | $\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s} < (\frac{\beta\gamma}{\beta+\gamma} + \frac{\beta(\alpha-\beta)}{\alpha+\beta})\frac{1}{s}$ |
| (3)-2 | $\gamma\frac{1}{s} > \frac{2\beta\gamma}{\alpha+2\beta+\gamma}\frac{1}{s}$ | $0 < \frac{2\beta(\alpha+\gamma)}{\alpha+2\beta+\gamma}\frac{1}{s}$ |
| (4) | $\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s} = \frac{\beta\gamma}{\beta+\gamma}\frac{1}{s}$ | $\frac{(\alpha+\gamma)(\alpha+\beta)}{\alpha^2+\alpha\beta-\beta^2+\beta\gamma}\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s} < \frac{(\alpha+\gamma)(\alpha+\beta)}{\alpha^2+\alpha\beta-\beta^2+\beta\gamma}(\frac{\beta\gamma}{\beta+\gamma} + \frac{\beta(\alpha-\beta)}{\alpha+\beta})\frac{1}{s}$ |
| (5)-1 | $\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s} = \frac{\beta\gamma}{\beta+\gamma}\frac{1}{s}$ | $\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s} < (\frac{\beta\gamma}{\beta+\gamma} + \frac{\beta(\alpha+\gamma)}{\alpha+\beta})\frac{1}{s}$ |
| (5)-2 | $\beta(1 - \frac{\beta}{\alpha}\frac{\alpha+\beta}{\alpha+2\beta+\gamma})\frac{1}{s} > \frac{\beta(\alpha+\gamma)}{\alpha+2\beta+\gamma}\frac{1}{s}$ | $\frac{\beta}{\alpha}\frac{(\alpha+\beta)(\alpha+\gamma)}{\alpha+2\beta+\gamma}\frac{1}{s} < \frac{2\beta(\alpha+\gamma)}{\alpha+2\beta+\gamma}\frac{1}{s}$ |
| (6) | $\frac{\beta(\alpha+\gamma)}{\alpha+\beta+\gamma}\frac{1}{s} > \frac{\beta(\alpha+\gamma)}{\alpha+\beta+\gamma}\frac{\alpha}{\alpha+\beta}\frac{1}{s}$ | $\frac{\beta(\alpha+\gamma)}{\alpha+\beta+\gamma}\frac{1}{s} < \frac{\beta(\alpha+\gamma)}{\alpha+\beta+\gamma}(1 + \frac{\alpha+\gamma}{\alpha+\beta})\frac{1}{s}$ |

The proposition is then proved. ☐

## Appendix B. The major differences between this paper and Liu et al. (2016)

| Aspects | This paper | Liu et al. (2016) |
|---|---|---|
| School location | near home | near workplace |
| Number of possible flow patterns | 8 | 4 |
| School arrival later than $t_s$ | Never | Possible |
| Flow patterns under the optimized schedule coordination | Depending on $N_w$ and $N_{sw}$ | Unique |
| Total cost under optimized schedule coordination | Smaller | Larger |
| First-best toll patterns | Identified | Special case given |
| Differentiated school and/or work schedules considered | Yes | No |

# References


Arnott, R., De Palma, A., and Lindsey, R. (1990). Economics of a bottleneck. *Journal of Urban Economics*, 27(1):111–130.

Arnott, R., De Palma, A., and Lindsey, R. (1991). A temporal and spatial equilibrium analysis of commuter parking. *Journal of Public Economics*, 45(3):301–335.

Arnott, R., De Palma, A., and Lindsey, R. (1994). The welfare effects of congestion tolls with heterogeneous commuters. *Journal of Transport Economics and Policy*, pages 139–161.

Daganzo, C. F. (1985). The uniqueness of a time-dependent equilibrium distribution of arrivals at a single bottleneck. *Transportation Science*, 19(1):29–37.

de Palma, A., Lindsey, R., and Picard, N. (2015). Trip-timing decisions and congestion with household scheduling preferences. *Economics of Transportation*, 4(1):118–131.

Henderson, J. V. (1981). The economics of staggered work hours. *Journal of Urban Economics*, 9(3):349–364.

Jia, Z., Wang, D. Z., and Cai, X. (2016). Traffic managements for household travels in congested morning commute. *Transportation Research Part E: Logistics and Transportation Review*, 91:173–189.

Kuwahara, M. (1990). Equilibrium queueing patterns at a two-tandem bottleneck during the morning peak. *Transportation Science*, 24(3):217–229.

Li, Z.-C., Lam, W. H., and Wong, S. (2014). Bottleneck model revisited: An activity-based perspective. *Transportation Research Part B: Methodological*, 68:262–287.

Lindsey, R. (2004). Existence, uniqueness, and trip cost function properties of user equilibrium in the bottleneck model with multiple user classes. *Transportation Science*, 38(3):293–314.

Lindsey, R. (2009). Cost recovery from congestion tolls with random capacity and demand. *Journal of Urban Economics*, 66(1):16–24.

Liu, W., Yang, H., and Yin, Y. (2014a). Expirable parking reservations for managing morning commute with parking space constraints. *Transportation Research Part C: Emerging Technologies*, 44:185–201.

Liu, W., Yang, H., Yin, Y., and Zhang, F. (2014b). A novel permit scheme for managing parking competition and bottleneck congestion. *Transportation Research Part C: Emerging Technologies*, 44:265–281.

Liu, W., Yin, Y., and Yang, H. (2015a). Effectiveness of variable speed limits considering commuters long-term response. *Transportation Research Part B: Methodological*, 81:498–519.

Liu, W., Zhang, F., and Yang, H. (2016). Modeling and managing morning commute with both household and individual travels. *Transportation Research Part B: Methodological*, in press. DOI:10.1016/j.trb.2016.12.002.

Liu, Y. and Nie, Y. M. (2011). Morning commute problem considering route choice, user heterogeneity and alternative system optima. *Transportation Research Part B: Methodological*, 45(4):619–642.

Liu, Y., Nie, Y. M., and Hall, J. (2015b). A semi-analytical approach for solving the bottleneck model with general user heterogeneity. *Transportation Research Part B: Methodological*, 71:56–70.

Qian, Z. S., Xiao, F. E., and Zhang, H. (2012). Managing morning commute traffic with parking. *Transportation Research Part B: Methodological*, 46(7):894–916.

Smith, M. J. (1984). The existence of a time-dependent equilibrium distribution of arrivals at a single bottleneck. *Transportation Science*, 18(4):385–394.

Takayama, Y. (2015). Bottleneck congestion and distribution of work start times: The economics of staggered work hours revisited. *Transportation Research Part B: Methodological*, 81:830–847.

Tseng, Y., Ubbels, B., and Verhoef, E. (2005). Value of time, schedule delay and reliability–estimation based on choice behaviour of dutch commuters facing congestion. In *45th ERSA Congress, VU University Amsterdam*.

van den Berg, V. and Verhoef, E. T. (2011). Congestion tolling in the bottleneck model with heterogeneous values of time. *Transportation Research Part B: Methodological*, 45(1):60–78.

Vickrey, W. S. (1969). Congestion theory and transport investment. *The American Economic Review*, 59(2):251–260.

Wang, D. Z. and Xu, L. (2016). Equilibrium trip scheduling in single bottleneck traffic flows considering multi-class travellers and uncertainty–a complementarity formulation. *Transportmetrica A: Transport Science*, 12(4):297–312.

Wilson, A. G. (1998). Land-use/transport interaction models: past and future. *Journal of Transport Economics and Policy*, 32(1):3–26.

Xiao, F., Shen, W., and Zhang, H. M. (2012). The morning commute under flat toll and tactical waiting. *Transportation Research Part B: Methodological*, 46(10):1346–1359.

Xiao, L.-L., Huang, H.-J., and Liu, R. (2015). Congestion behavior and tolls in a bottleneck model with stochastic capacity. *Transportation Science*, 49(1):46–65.

Xiao, L.-L., Liu, R., and Huang, H.-J. (2014). Stochastic bottleneck capacity, merging traffic and morning commute. *Transportation Research Part E: Logistics and Transportation Review*, 64:48–70.

Xiao, L.-L., Liu, T.-L., and Huang, H.-J. (2016). On the morning commute problem with carpooling behavior under parking space constraint. *Transportation Research Part B: Methodological*, 91:383–407.

Yang, H., Liu, W., Wang, X., and Zhang, X. (2013). On the morning commute problem with bottleneck congestion and parking space constraints. *Transportation Research Part B: Methodological*, 58:106–118.

Yushimito, W. F., Ban, X., and Holguín-Veras, J. (2014). A two-stage optimization model for staggered work hours. *Journal of Intelligent Transportation Systems*, 18(4):410–425.

Zhang, F., Lindsey, R., and Yang, H. (2016). The downs–thomson paradox with imperfect mode substitutes and alternative transit administration regimes. *Transportation Research Part B: Methodological*, 86:104–127.

Zhang, F., Yang, H., and Liu, W. (2014). The downs–thomson paradox with responsive transit service. *Transportation Research Part A: Policy and Practice*, 70:244–263.

Zhang, X., Huang, H.-J., and Zhang, H. (2008). Integrated daily commuting patterns and optimal road tolls and parking fees in a linear city. *Transportation Research Part B: Methodological*, 42(1):38–56.

Zhang, X., Yang, H., Huang, H.-J., and Zhang, H. M. (2005). Integrated scheduling of daily work activities and morning–evening commutes with bottleneck congestion. *Transportation Research Part A: Policy and Practice*, 39(1):41–60.