Manlove, D. F. (2002) The structure of stable marriage with indifference. *Discrete Applied Mathematics* 122(1-3):167-181.

http://eprints.gla.ac.uk/archive/00000014

The structure of stable marriage with indifference

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Abstract

We consider the stable marriage problem where participants are permitted to express indifference in their preference lists (i.e., each list can be partially ordered). We prove that, in an instance where indifference takes the form of ties, the set of strongly stable matchings forms a distributive lattice. However, we show that this lattice structure may be absent if indifference is in the form of arbitrary partial orders. Also, for a given stable marriage instance with ties, we characterise strongly stable matchings in terms of perfect matchings in bipartite graphs. Finally, we briefly outline an alternative proof of the known result that, in a stable marriage instance with indifference in the form of arbitrary partial orders, the set of super-stable matchings forms a distributive lattice.

Keywords: Stable marriage problem; Partial order; Tie; Strong stability; Super-stability; Distributive lattice

1 Introduction

The classical stable marriage problem (SM) and its many variants have been widely studied in the literature [10, 5, 14]. An instance I of SM involves n men and n women, each of whom ranks all n members of the opposite sex in strict order of preference. A *matching* is a one-one correspondence between the men and women in I. We say that a (man,woman) pair (x, y) is a *blocking pair* for M if x prefers y to $p_M(x)$, and y prefers x to $p_M(y)$, where $p_M(q)$ denotes q's partner in M, for any person q in I. A matching that admits no blocking pair is said to be *stable*. It is known that every instance of SM admits at least one stable matching [3], and in general there may be many [7]. Moreover, the set of stable matchings for a given instance of SM forms a finite distributive lattice under a natural relation of dominance, denoted $\preceq$ (Knuth [10] attributes this result to John Conway). One stable matching M *dominates* another stable matching M' if every man has at least as good a partner in M as he has in M'. The man-oriented Gale/Shapley algorithm [3] finds, in $O(n^2)$ time, the top element of this lattice, called the *man-optimal* stable matching. This is the unique stable matching in which each man has his best possible partner (and each woman her worst) among all stable matchings. Similarly, by considering the woman-oriented version of this algorithm, we may find the bottom element of the lattice, the *woman-optimal* stable matching.

The exploitation of this lattice structure has led to the formulation of efficient algorithms for a number of problems associated with SM, for example finding all stable pairs (i.e., determining, for each man m and woman w, whether m and w are partners in some

*Supported by grant GR/M13329 from the Engineering and Physical Sciences Research Council.

stable matching) [4], generating all stable matchings [4], finding a so-called *egalitarian* stable matching [8] and finding a so-called *minimum regret* stable matching [4][1], and plays a key role in establishing the #P-completeness of the problem of counting stable matchings [7]. Many of these problems arise naturally in practical applications such as large-scale centralised matching schemes. Perhaps the largest and best-known of these schemes is the National Resident Matching Program in the US, which administers the annual assignment of graduating medical students to hospital posts, and employs an extension of the Gale/Shapley algorithm for SM [13].

A natural generalisation of SM arises when each person need not rank all members of the opposite sex in *strict* order. It is possible that some of those involved might be indifferent among certain members of the opposite sex, so that preference lists may be partially ordered. We use SMP to stand for this variant of SM, and in such a setting, it may be convenient to refer to a person's (partially ordered) preference list as his/her *preference poset*. The restriction of SMP in which the indifference takes the form of ties in the preference posets (i.e. each preference poset is a *weak order* [1]) is denoted by SMT. If a person q precedes a person r in a person p's preference poset, then we say that p *strictly prefers* q to r; if q and r are incomparable in p's preference poset, then we say that p is *indifferent* between them. Irving [6] formulates three possible definitions for stability for SMP. A matching M is *weakly stable* if there is no couple (x, y), each of whom strictly prefers the other to his/her partner in M. Also, a matching M is *strongly stable* if there is no couple (x, y) such that x strictly prefers y to his/her partner in M, and y either strictly prefers x to his/her partner in M or is indifferent between them. Finally, a matching M is *super-stable* if there is no couple (x, y), each of whom either strictly prefers the other to his/her partner in M or is indifferent between them. Clearly a super-stable matching is strongly stable, and a strongly stable matching is weakly stable. The definition of a blocking pair for each of these three stability criteria is analogous to that for SM, and henceforth, the particular stability criterion to which the term applies should be clear from the context.

In practical situations, it is arguable that strong stability is the most appropriate stability definition. For, there appears to be no real incentive for a man or woman to form a blocking pair of a matching if each is indifferent between the other and his/her partner in the matching. Thus there is a sense in which the super-stability criterion is too extreme. However, there is also a sense in which the weak stability criterion is too weak: a man m who strictly prefers a woman w to his partner in a matching might offer a bribe to w if she is indifferent between m and her partner in the matching, to try to tempt w into exchanging her partner for m. Clearly the weak stability of a matching cannot exclude blocking pairs that may arise from this practical case; however the strong stability definition does. The latter stability criterion will be the main focus of this paper.

On the other hand, an example context in which super-stability is relevant is when there is uncertainty in the preference lists. Suppose that, in a stable marriage instance, we wish to find a stable matching (in the classical sense), but for some or all of the participants we have only partial information regarding preferences. In general, each preference 'list' may be expressable only as a partial order, and the particular linear extension that represents a participant's true preferences is unknown. It is not difficult to see that a matching M in an instance I of SMP is super-stable if and only if M is stable in every instance of SMT

[1]The egalitarian and minimum regret stable matching problems may be defined as follows. Given an instance I of SM and a stable matching M in I, the *cost* of M for a man m is the ranking of $p_M(m)$ in m's preference list. The cost of M for a woman is defined similarly. The egalitarian stable matching problem is to find a stable matching M in I such that the total cost of M summed over all men and women is minimised. The minimum regret stable matching problem is to find a stable matching M in I such that the maximum cost of M taken over all men and women is minimised.

m_1 :	w_1	w_2	w_3	w_1 :	$(m_1$	$m_2)$	m_3
m_2 :	w_1	w_3	w_2	w_2 :	m_1	m_3	m_2
m_3 :	w_1	w_2	w_3	w_3 :	m_2	m_3	m_1

Men's preferences — Women's preferences

Figure 1: An instance of SMT with no man-optimal weakly stable matching.

obtained from I by forming linear extensions of the preference posets in I. Therefore a super-stable matching is one that is stable no matter which linear extensions of the various preference posets represent the true preferences.

Also, in practice, ties seem to be the most natural form of indifference. For, if a man is indifferent between one woman w_1 and another woman w_2, and he is also indifferent between w_2 and a third woman w_3, then it is reasonable to assume that he is indifferent between w_1 and w_3 (i.e., indifference is 'transitive'). Nevertheless, in this paper our standard definition of indifference allows a partially ordered list for full structural generality, since partially ordered preference lists are the natural relaxation of totally ordered lists.

For a given instance I of SMP, the existence of a weakly stable matching is guaranteed: by resolving the indifference in I arbitrarily (i.e. by forming a linear extension of each preference poset), we obtain an instance I' of SM, and it is clear that a stable matching in I' is weakly stable in I. (Thus a weakly stable matching for I may be found in $O(n^2)$ time, using the Gale/Shapley algorithm.) On the other hand, it is straightforward to construct instances of SMT which admit no strongly stable matching and/or no super-stable matching; see [6] for further details. However, Irving [6] presents $O(n^4)$ and $O(n^2)$ algorithms for respectively determining whether a strongly stable matching and/or a super-stable matching exists in a given instance of SMP, and in each case, if such a matching does exist, the appropriate algorithm constructs one.

It is known that, for a given instance of SMP, the set of super-stable matchings forms a finite distributive lattice [15]. However, in the case of weak stability, this structure is absent (under the 'usual' definitions of meet and join – c.f. [5, p.20]) even for SMT: there is an instance I of SMT, containing three men, namely m_1, m_2, m_3, and three women, namely w_1, w_2, w_3, which admits no man-optimal weakly stable matching. This example, due to Roth [13], is reproduced in Figure 1 (in a preference list, persons within parentheses are tied); the matchings $M_1 = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}$ and $M_2 = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$ are the two unique weakly stable matchings in I. Since man m_1 has his first-choice partner in M_1 and his second-choice partner in M_2, whereas man m_2 has his second-choice partner in M_1 and his first-choice partner in M_2, then no man-optimal weakly stable matching in I exists. However, the structure of the set of matchings that are stable with respect to the remaining stability criterion, namely strong stability, has remained open until now.

The main contribution of this paper is to demonstrate that, despite an apparent lack of symmetry in the strong stability definition, strongly stable matchings in an SMT instance nevertheless give rise to a finite distributive lattice. This result, presented in Section 2, is obtained by defining an equivalence relation $\sim$ on the set of strongly stable matchings for a given SMT instance; it is the set of equivalence classes under $\sim$ which forms the distributive lattice under a dominance relation closely related to $\preceq$. Hence there is convincing evidence that the problems mentioned previously for SM [4, 8] are also polynomial-time solvable for SMT under strong stability [2].

[2]For the egalitarian and minimum regret stable matching problems, the cost of a matching in an instance

In Section 3, we give a characterisation of the strongly stable matchings in a given equivalence class in terms of perfect matchings of a suitably defined bipartite graph. One use of this representation is to demonstrate how to generate efficiently the strongly stable matchings in a given equivalence class.

The lattice structure for strong stability in an SMT instance does not carry over to SMP: in Section 4, we construct an example instance of SMP in which the set of strongly stable matchings does not form a lattice.

We also consider super-stable matchings briefly, in Section 5. As previously mentioned, it has been shown that the set of super-stable matchings for an SMP instance I forms a finite distributive lattice [15]. The result is established by noting that there is a set $\mathcal{I}$ of instances of SM such that $I' \in \mathcal{I}$ if and only if I' may be obtained from I by resolving the indifference in I in some way; as previously noted, a matching M is super-stable in I if and only if M is stable in *every* member of $\mathcal{I}$. Thus the set of super-stable matchings in I is equal to $\bigcap_{I' \in \mathcal{I}} S(I')$, where $S(I')$ denotes the set of stable matchings in the instance I' of SM. But each set $S(I')$ forms a distributive lattice, and since the intersection of distributive lattices is also a distributive lattice, the result follows.

In this paper, we outline an alternative approach which builds on results from Section 2 and leads to the same conclusion, but which avoids using the above intersection argument, and which, we feel, provides a more intuitive picture of the structure of the set of super-stable matchings in a given SMP instance. [3]

Finally, we present some concluding remarks in Section 6.

2 Strongly stable matchings form a distributive lattice

In this section we show that the set of strongly stable matchings for an instance of SMT gives rise to a finite distributive lattice. Throughout this section, certain results are proved for SMP (rather than SMT) for greater generality, and also with a view to their exploitation in Section 5.

We begin with a definition which will be useful on a number of occasions henceforth.

Definition 2.1. *Let I be an instance of SMP, and let M, M' be two strongly stable matchings in I. Construct an edge-coloured bipartite graph, denoted by $M \oplus M'$, as follows: form a vertex for each person in I, and join any two vertices by a red (resp. blue) edge if the corresponding people are matched in M but not M' (resp. M' but not M).*

It is clear that the connected components of any such graph $M \oplus M'$ are cycles, since everybody is matched in each of M, M'.

Our first result states that if some person p has different partners in two strongly stable matchings M, M', then there is an important structural relationship between M and M'.

Lemma 2.2. *Let I be an instance of SMP, and let M, M' be two strongly stable matchings in I. Suppose that, for any person p in I, $(p, q) \in M$ and $(p, q') \in M'$, where $q \neq q'$ [4], and p strictly prefers q' to q or is indifferent between them. Then there is a cycle in*

of SMP may be defined as follows. For a given person q, assume that $\prec_q$ denotes q's preference poset, where $r \prec_q s$ if and only if q strictly prefers r to s. The cost of a matching M for q is 1 plus the number of predecessors in $\prec_q$ of $p_M(q)$.

[3] We remark that the structural results presented in Sections 2, 3 and 5 may be generalised to the case that each person may declare certain members of the opposite sex as being unacceptable (in other words, he or she would rather be unmatched than be matched with such a person). The details are omitted from this paper.

[4] In this paper, we follow the convention that p is indifferent between q and q' includes the possibility that $q = q'$.

$M \oplus M'$, comprising alternating (man,woman) and (woman,man) pairs involving p, q and q' as follows: for some $r > 1$, there are r people $p_1, \ldots, p_r$ in I, all of the same sex, and r people $q_1, \ldots, q_r$ in I, all of the opposite sex, such that

1. *$p_1 = p$, $q_1 = q$ and $q_2 = q'$.*
2. *$(p_i, q_i) \in M$ $(1 \le i \le r)$ and $(p_i, q_{i+1}) \in M'$ $(1 \le i \le r)$.*
3. Case (i)*: p is indifferent between q' and q implies that (a) p_i is indifferent between q_{i+1} and q_i $(1 \le i \le r)$, and (b) q_i is indifferent between p_i and p_{i-1} $(1 \le i \le r)$.*

   Case (ii)*: p strictly prefers q' to q implies that (a) p_i strictly prefers q_{i+1} to q_i $(1 \le i \le r)$, and (b) q_i strictly prefers p_i to p_{i-1} $(1 \le i \le r)$,*

where $p_0 = p_r$, $p_{r+1} = p_1$ and $q_{r+1} = q_1$.

Proof. Consider $M \oplus M'$ as defined in Definition 2.1. In this graph, the vertices p, q, q' are all in the same connected component G'. We claim that there is a sequence $\langle p_j \rangle_{j \ge 1}$ of people in G', all of the same sex, and a sequence $\langle q_j \rangle_{j \ge 1}$ of people in G', all of the opposite sex, such that, for each $i \ge 1$,

(a). $\{p_i, q_i\}$ is a red edge and $\{p_i, q_{i+1}\}$ is a blue edge.

(b). p_i strictly prefers q_{i+1} to q_i, or is indifferent between them.

We prove the claim by induction on i. The base case $i = 1$ clearly holds with $p_1 = p$, $q_1 = q$ and $q_2 = q'$. For an induction step, suppose that some $k \ge 1$ is given, and assume that the claim is true for $i = k$. We show that the claim holds for $i = k + 1$. Person q_{k+1} is incident to a red edge, $\{p_{k+1}, q_{k+1}\}$ say, such that q_{k+1} strictly prefers p_{k+1} to p_k or is indifferent between them, for otherwise (p_k, q_{k+1}) blocks M. Also person p_{k+1} is incident to a blue edge, $\{p_{k+1}, q_{k+2}\}$ say, such that p_{k+1} strictly prefers q_{k+2} to q_{k+1} or is indifferent between them, for otherwise (p_{k+1}, q_{k+1}) blocks M'. This completes the inductive step.

Since G' is a cycle, $q_{r+1} = q$, $p_{r+1} = p$ and $q_{r+2} = q'$, for some $r > 1$. Also, note that for each $i \ge 2$, q_i strictly prefers p_i to p_{i-1}, or is indifferent between them, for otherwise (p_{i-1}, q_i) blocks M. The remainder of the proof is split into two cases.

Case (i): p is indifferent between q' and q. Then a similar induction to the above (swapping the colours red and blue, and interpreting the indices of each p_i, q_j appropriately) establishes that, for each $i \ge 1$, p_i strictly prefers q_i to q_{i+1} or is indifferent between them, and q_i strictly prefers p_{i-1} to p_i or is indifferent between them. Hence $p_1, \ldots, p_r$ and $q_1, \ldots, q_r$ satisfy the required properties.

Case (ii): p strictly prefers q' to q. Then the fact that $p_1, \ldots, p_r$ and $q_1, \ldots, q_r$ satisfy the required properties may be established by considering the argument from the start of the proof up to Case (i), and removing all occurrences of the phrase "or is indifferent between them". □

The concepts of strict preference and indifference may be defined between matchings, as well as between men and women, as follows. Let M, M' be any matchings in I, and let q be any person in I. We say that q *strictly prefers* M to M' if q strictly prefers $p_M(q)$ to $p_{M'}(q)$. Also, we say that q is *indifferent* between M and M' if q is indifferent between $p_M(q)$ and $p_{M'}(q)$.

Lemma 2.2 leads to the following theorem, which plays an important role in establishing the lattice structure.

Theorem 2.3. *Let I be an instance of SMP, and let M, M' be two strongly stable matchings in I. Suppose that m and w are partners in M but not in M'. Then* either

1. *one of m, w strictly prefers M to M', and the other strictly prefers M' to M,* or
2. *both m and w are indifferent between M and M'.*

Proof. By Case (i) of Lemma 2.2, m is indifferent between M and M' if and only if w is indifferent between M and M'. Now suppose that m strictly prefers M to M'. If w strictly prefers M to M', then (m, w) blocks M'. Hence, since w is not indifferent between M and M', w strictly prefers M' to M. Finally, suppose that m strictly prefers M' to M. Then by Case (ii) of Lemma 2.2, w strictly prefers M to M'. □

We proceed with the definition of the equivalence relation that will be central to our construction of the lattice structure involving strongly stable matchings.

Definition 2.4. *Let $\mathcal{M}$ be the set of strongly stable matchings for a given SMT instance I. Define an equivalence relation $\sim$ on $\mathcal{M}$ as follows. For two strongly stable matchings $M, M' \in \mathcal{M}$, $M \sim M'$ if and only if each man is indifferent between M and M' . Denote by $\mathcal{C}$ the set of equivalence classes of $\mathcal{M}$ under $\sim$, and denote by $[M]$ the equivalence class containing M, for $M \in \mathcal{M}$.*

Note that $\sim$ is an equivalence relation, for indifference is transitive in an instance of SMT. However, the transitivity of indifference cannot be guaranteed in an arbitrary SMP instance (we see the consequences of this in Section 4). Definition 2.4 leads to the following proposition, whose proof is immediate from the strong stability definition:

Proposition 2.5. *Let I be an instance of SMT, and let M, M' be two strongly stable matchings in I. Then $M \sim M'$ implies that each woman in I is indifferent between M and M', where $\sim$ is as defined in Definition 2.4.*

Recall from Section 1 the dominance partial order defined on the set of stable matchings in an SM instance, where one stable matching M dominates another, M', if every man has at least as good a partner in M as he has in M'. We now give a formal definition of this partial order defined on strongly stable matchings.

Definition 2.6. *Let I be an instance of SMP and let M, M' be two strongly stable matchings. Then M* dominates *M', written $M \preceq M'$, if each man either strictly prefers M to M', or is indifferent between them.*

We may extend $\preceq$ to a partial order $\preceq^*$ defined on equivalence classes as follows.

Definition 2.7. *Let I be an instance of SMT and let $\mathcal{C}$ be as defined in Definition 2.4. Define a partial order $\preceq^*$ on $\mathcal{C}$ as follows: for any two equivalence classes $[M], [M'] \in \mathcal{C}$, $[M] \preceq^* [M']$ if and only if $M \preceq M'$, where $\preceq$ is as defined in Definition 2.6.*

Note that the definition of $(\mathcal{C}, \preceq^*)$ is independent of the particular choices of representatives of the equivalence classes $[M]$ and $[M']$: if $M \preceq M'$, then $P \preceq P'$ for any $P \in [M]$ and $P' \in [M']$. Also, observe that $(\mathcal{C}, \preceq^*)$ is a partial order, for, given any two strongly stable matchings M, M', $M \sim M'$ if and only if $M \preceq M'$ and $M' \preceq M$. We aim to show that $(\mathcal{C}, \preceq^*)$ is a finite distributive lattice.

In order to establish the existence of a lattice structure, we require to define the meet and the join of two equivalence classes. To this end, we make the following definition: given two strongly stable matchings M, M' in an SMP instance I, let $U_{in}(M, M')$ be the set of men in I who are indifferent between M and M' (note that possibly $U_{in}(M, M') = \emptyset$.) Also, by Case (i) of Lemma 2.2, there is a set of women $W_{in}(M, M')$ such that $|W_{in}(M, M')| = |U_{in}(M, M')|$ and each woman in $W_{in}(M, M')$ is indifferent between M and M'. Clearly, for every $m \in U_{in}(M, M')$, $\{p_M(m), p_{M'}(m)\} \subseteq W_{in}(M, M')$, and similarly, for every $w \in W_{in}(M, M')$, $\{p_M(w), p_{M'}(w)\} \subseteq U_{in}(M, M')$.

The following result will be of use in our formulation of a meet operation between two equivalence classes.

Lemma 2.8. *Let I be an instance of SMP, and let M, M' be two strongly stable matchings in I. Let M^* be a set of (man,woman) pairs defined as follows: for each man $m \in U_{in}(M, M')$, m has in M^* the same partner as in M, and for each man $m \notin U_{in}(M, M')$, m has in M^* the better of his partners in M and M'. Then M^* is a strongly stable matching.*

Proof. Firstly, we show that M^* is a matching. Suppose that men m and m' both have some woman w as their partner in M^*. Without loss of generality, suppose that $(m, w) \in M$ and $(m', w) \in M'$; then m strictly prefers M to M' or is indifferent between them, and m' strictly prefers M' to M. Theorem 2.3 applied to the pair (m, w) implies that w strictly prefers M' to M or is indifferent between them. But Theorem 2.3 applied to the pair (m', w) implies that w strictly prefers M to M', a contradiction. Hence M^* is indeed a matching.

Now suppose that M^* is blocked by some pair (m, w). Suppose firstly that m strictly prefers w to $p_{M^*}(m)$. Then w strictly prefers m to $p_{M^*}(w)$ or is indifferent between them. Also, m strictly prefers w to both $p_M(m)$ and $p_{M'}(m)$. If $p_{M^*}(w) = p_M(w)$ then (m, w) blocks M, and if $p_{M^*}(w) = p_{M'}(w)$ then (m, w) blocks M'. But $p_{M^*}(w) \in \{p_M(w), p_{M'}(w)\}$, so we reach a contradiction.

Hence m is indifferent between w and $p_{M^*}(m)$. Thus w strictly prefers m to $p_{M^*}(w)$. Also, m strictly prefers w to $p_M(m)$ or is indifferent between them, and m strictly prefers w to $p_{M'}(m)$ or is indifferent between them. If $p_{M^*}(w) = p_M(w)$ then (m, w) blocks M, and if $p_{M^*}(w) = p_{M'}(w)$ then (m, w) blocks M'. Again $p_{M^*}(w) \in \{p_M(w), p_{M'}(w)\}$, so we have a contradiction. Hence M^* is strongly stable. □

We denote by $M \wedge M'$ the set of (man,woman) pairs in which each man $m \in U_{in}(M, M')$ receives the same partner as in M, and each man $m \notin U_{in}(M, M')$ receives the better of his partners in M and M'; by Lemma 2.8, $M \wedge M'$ is a strongly stable matching. Note that it is not the case that, in general, $M \wedge M' = M' \wedge M$.

We now present a result along the same lines as Lemma 2.8, which will be of use in our definition of a join operation between two equivalence classes.

Lemma 2.9. *Let I be an instance of SMP, and let M, M' be two strongly stable matchings in I. Let M^* be a set of (man,woman) pairs defined as follows: for each man $m \in U_{in}(M, M')$, m has in M^* the same partner as in M, and for each man $m \notin U_{in}(M, M')$, m has in M^* the poorer of his partners in M and M'. Then M^* is a strongly stable matching.*

Proof. Clearly each woman in $W_{in}(M, M')$ has the same partner in M^* as she has in M. If each man $m \notin U_{in}(M, M')$ is given the poorer of his partners in M and M', then by Theorem 2.3, each woman $w \notin W_{in}(M, M')$ receives the better of her partners in M and M'. The remainder of the proof is essentially the same as the proof of Lemma 2.8, with the roles of the men and women reversed. □

We denote by $M \vee M'$ the set of (man,woman) pairs in which each man $m \in U_{in}(M, M')$ receives the same partner as in M, and each man $m \notin U_{in}(M, M')$ receives the poorer of his partners in M and M'; by Lemma 2.9, $M \vee M'$ is a strongly stable matching.

The operations $\wedge$ and $\vee$ on strongly stable matchings have a number of properties, as indicated by the following lemma.

Lemma 2.10. *Let I be an instance of SMT, and let $\mathcal{M}$ be the set of strongly stable matchings in I. Then, for any $X, Y, Z \in \mathcal{M}$,*

1. $X \wedge X = X$
2. $X \vee X = X$
3. $X \wedge Y \sim Y \wedge X$
4. $X \vee Y \sim Y \vee X$
5. $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$
6. $X \vee (Y \vee Z) = (X \vee Y) \vee Z$
7. $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$
8. $X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$

where $\sim$ is the equivalence relation on $\mathcal{M}$ as defined in Definition 2.4.

Proof. (1)-(4) are obvious from the definitions.

To show (5), let m be any man, let $R = X \wedge (Y \wedge Z)$ and let $S = (X \wedge Y) \wedge Z$. If m is not indifferent between any pair of X, Y, Z, then it is clear that $p_R(m) = p_S(m)$. Suppose that m is indifferent between X, Y and Z. Then $p_R(m) = p_X(m) = p_S(m)$.

Now suppose that m is indifferent between two of X, Y, Z only, namely A, B, where $A \in \{X, Y\}$ and $B \in \{Y, Z\}$, without loss of generality. Let C be such that $\{A, B, C\} = \{X, Y, Z\}$. If m strictly prefers C to each of A, B, then $p_R(m) = p_C(m) = p_S(m)$. Otherwise, m strictly prefers each of A, B to C, and $p_R(m) = p_A(m) = p_S(m)$.

The proof that (6) holds is similar.

To show (7), let m be any man, let $R = X \wedge (Y \vee Z)$ and let $S = (X \wedge Y) \vee (X \wedge Z)$. If m is not indifferent between any pair of X, Y, Z, then the proof that $p_R(m) = p_S(m)$ follows by the corresponding proof in Theorem 1.3.2 of [5].

Now suppose that m is indifferent between Y and Z only. If m strictly prefers X to Y, then $p_R(m) = p_X(m) = p_S(m)$. Otherwise, m strictly prefers Y to X, and $p_R(m) = p_Y(m) = p_S(m)$.

If m is indifferent between X, Y and Z, or m is indifferent between X and Y only, or m is indifferent between X and Z only, then $p_R(m) = p_X(m) = p_S(m)$.

The proof that (8) holds is similar. ☐

We are now in a position to present our main result of this section.

Theorem 2.11. *Let I be an instance of SMT, and let $\mathcal{M}$ be the set of strongly stable matchings in I. Let $\mathcal{C}$ be the set of equivalence classes of $\mathcal{M}$ under $\sim$, and let $\preceq^*$ be the dominance partial order on $\mathcal{C}$, where $\sim$ and $\preceq^*$ are as defined in Definitions 2.4 and 2.7 respectively. Then $(\mathcal{C}, \preceq^*)$ forms a finite distributive lattice, with $[M \wedge M']$ representing the meet of $[M]$ and $[M']$, and $[M \vee M']$ the join, for two equivalence classes $[M], [M'] \in \mathcal{C}$.*

Proof. By Lemmas 2.8 and 2.9, $[M \wedge M'] \in \mathcal{C}$ and $[M \vee M'] \in \mathcal{C}$. Also, by Lemma 2.8, $M \wedge M' \preceq M$ and $M \wedge M' \preceq M'$, so that $[M \wedge M'] \preceq^* [M]$ and $[M \wedge M'] \preceq^* [M']$. Similarly, by Lemma 2.9, $M \preceq M \vee M'$ and $M' \preceq M \vee M'$, so that $[M] \preceq^* [M \vee M']$ and $[M'] \preceq^* [M \vee M']$,

Now suppose that $[M^*] \preceq^* [M]$ and $[M^*] \preceq^* [M']$, for any strongly stable matching M^*. Then $M^* \preceq M$ and $M^* \preceq M'$, so that each man has at least as good a partner in

M^* as he has in each of M and M'. Hence $M^* \preceq M \wedge M'$, so that $[M^*] \preceq^* [M \wedge M']$, and hence $[M \wedge M']$ is the greatest lower bound of $[M]$ and $[M']$. By a similar argument, $[M \vee M']$ is the least upper bound of $[M]$ and $[M']$. Hence by Lemma 2.10, $(\mathcal{C}, \preceq^*)$ is a finite distributive lattice. □

3 Characterising strongly stable matchings in an equivalence class

In this section we give a representation of the strongly stable matchings in an equivalence class in terms of perfect matchings of a bipartite graph. We begin with a definition which will feature in the construction of this graph.

Definition 3.1. *Let I be an instance of SMT. For any persons p, q in I, not both of the same sex, let $T(p, q)$ denote the set of people tied with q in p's preference list.*

Note that in the above definition, $T(p, q) \neq \emptyset$, since $q \in T(p, q)$. We now define the bipartite graph that forms the basis of our equivalence class representation.

Definition 3.2. *Let I be an instance of SMT, and let M be a strongly stable matching in I. Define the* equivalence graph*, $H_M = (V, E)$, of M as follows: let $V = U \cup W$, where U, W are the sets of men and women in I respectively, and*

$$E = \{(m, w) : w \in T(m, p_M(m)) \wedge m \in T(w, p_M(w))\}.$$

For a given equivalence class C, the perfect matchings in the corresponding equivalence graph are exactly the strongly stable matchings in C, as the following result indicates.

Theorem 3.3. *Let I be an instance of SMT, and let M be a strongly stable matching in I. Let M' be any matching in I. Then M' is strongly stable with $M' \sim M$ if and only if M' is a perfect matching in H_M, where $\sim$ is as defined in Definition 2.4 and $H_M = (V, E)$ is the equivalence graph of M.*

Proof. Suppose that M' is strongly stable and $M' \sim M$. Let (m, w) be any (man,woman) pair in M'. Then m is indifferent between w and $p_M(m)$, so that $w \in T(m, p_M(m))$. Also, by Proposition 2.5, w is indifferent between m and $p_M(w)$, so that $m \in T(w, p_M(w))$. Hence $(m, w) \in E$, so that M' is a perfect matching in H_M.

Conversely suppose that M' is a perfect matching in H_M. Suppose that some unmatched (man,woman) pair (m, w) blocks M'. Now m is indifferent between $p_M(m)$ and $p_{M'}(m)$. Also, w is indifferent between $p_M(w)$ and $p_{M'}(w)$. Hence (m, w) blocks M, a contradiction. Thus M' is strongly stable in I. Clearly $M' \sim M$. □

A consequence of the representation of Theorem 3.3 is that strongly stable matchings may be efficiently generated in a given equivalence class. More specifically, we now show that, having found a strongly stable matching M in an SMT instance I of size n (which may be achieved in $O(n^4)$ time [6]), we may generate all of the remaining strongly stable matchings in $[M]$ with a delay of $O(n^2)$ time per matching.

Corollary 3.4. *Let I be an instance of SMT, and let $\mathcal{M}$ be the set of strongly stable matchings in I. Let $\mathcal{C}$ be the set of equivalence classes of $\mathcal{M}$ under $\sim$, where $\sim$ is as defined in Definition 2.4. Then for any $[M] \in \mathcal{C}$, the strongly stable matchings in $[M]$ may be generated efficiently.*

$$
\begin{array}{llcll}
m_1: & w_1 \;\; w_2 \;\; \ldots & \qquad & w_1: & m_2 \;\; m_5 \;\; m_1 \;\; \ldots \\
m_2: & w_2 \;\; w_1 \;\; \ldots & & w_2: & m_1 \;\; m_2 \;\; \ldots \\
m_3: & w_4 \;\; w_3 \;\; \ldots & & w_3: & m_3 \;\; m_6 \;\; m_4 \;\; \ldots \\
m_4: & w_3 \;\; w_4 \;\; \ldots & & w_4: & m_4 \;\; m_3 \;\; \ldots \\
m_5: & \overbrace{w_5 \;\; w_1 \;\; \ldots}^{w_6} & & w_5: & (m_5 \;\; m_6) \;\; \ldots \\
m_6: & \overbrace{w_5 \;\; w_3 \;\; \ldots}^{w_6} & & w_6: & (m_5 \;\; m_6) \;\; \ldots
\end{array}
$$

Figure 2: An instance of SMP with no man-optimal strongly stable matching.

Proof. Let H_M be the equivalence graph of M. By Theorem 3.3, $M' \in [M]$ if and only if M' is a perfect matching in H_M. The set of perfect matchings in a bipartite graph G may be generated efficiently [2]: once an initial perfect matching in G has been found (in this case, such a matching is M), all of the remaining perfect matchings in G may be generated with a delay of $O(n^2)$ time per matching. The whole enumeration process takes $O(n^3)$ space. □

The question as to whether this enumeration may be achieved by constructing an initial strongly stable matching in $O(n^2)$ time, and then by generating subsequent strongly stable matchings with a delay of $O(n)$ time per strongly stable matching, and furthermore by using $O(n^2)$ space overall, which are all optimal bounds (and achievable for stable matchings in an instance of SM [4]), remains open.

4 Strongly stable matchings in an SMP instance

In this section, we construct an instance I of SMP with the property that the strongly stable matchings in I do not form a lattice (with respect to the meet and join definitions of Section 2). The instance I contains six men, namely $m_1, m_2, \ldots, m_6$, and six women, namely $w_1, w_2, \ldots, w_6$. The preference posets for each person in I are shown in Figure 2 (the symbol '...' in a person's preference poset denotes all remaining people of the opposite sex listed in arbitrary strict order at the point where the symbol appears). Note that each of men m_5, m_6 is indifferent between woman w_6 and woman w_i, for $1 \leq i \leq 5$. Given that the indifference in the men's preference posets does not take the form of ties, the instance I of SMP is not an instance of SMT.

Now consider the matchings $M_1 = \{(m_i, w_i) : 1 \leq i \leq 6\}$ and $M_2 = \{(m_1, w_2), (m_2, w_1), (m_3, w_4), (m_4, w_3), (m_5, w_6), (m_6, w_5)\}$. It may be verified that each of M_1, M_2 is strongly stable in I. However, each of men m_1, m_2 has his first-choice partner in M_1 and his second-choice partner in M_2, whereas each of men m_3, m_4 has his second-choice partner in M_1 and his first-choice partner in M_2. Now let M be any strongly stable matching in I. It may be verified that $(m_1, w_1) \in M$ if and only if $(m_2, w_2) \in M$, and also $(m_3, w_4) \in M$ if and only if $(m_4, w_3) \in M$. Suppose, without loss of generality, that $(m_1, w_1) \in M$ and $(m_4, w_3) \in M$. Then each of m_5, m_6 must be matched in M with a partner strictly better than w_1, w_3, respectively. Since the only such woman is w_5 in each case, no such M exists.

Hence no man-optimal strongly stable matching (or equivalence class) in I exists, which rules out the possibility of there being a lattice structure for the strongly stable matchings in I, given the the meet and join definitions of Section 2.

Note that in the example I above, indifference occurs on both the men's side and the women's side. In general, this is a necessary condition for the lattice structure to be absent. For, if indifference occurs on one side only, then the strong stability criterion is equivalent to the super-stability criterion, and super-stable matchings in an arbitrary SMP instance do form a lattice [15].

5 Super-stability

In this section, we briefly outline an alternative strategy from the one used by Spieker [15] to show that the set of super-stable matchings for an SMP instance forms a finite distributive lattice. The basis of our method is the following lemma, which demonstrates that if a person has different partners in two super-stable matchings, then he/she cannot be indifferent between them. This useful property of super-stable matchings is not stated explicitly in [15].

Lemma 5.1. *Let I be an instance of SMP, and let M, M' be two super-stable matchings in I. Suppose that, for any person p in I, $(p, q) \in M$ and $(p, q') \in M'$, where p is indifferent between q and q'. Then $q = q'$.*

Proof. Suppose $q \neq q'$. Since M and M' are strongly stable, we may invoke Case (i) of Lemma 2.2. We then find that (p, q') blocks M, a contradiction. □

Let $\mathcal{M}$ be the set of super-stable matchings for a given SMP instance I. Consider $\preceq$, the dominance partial order of Definition 2.6, now defined on $\mathcal{M}$. The insight into the structure of super-stable matchings in an SMP instance provided by Lemma 5.1 allows us to follow an approach along the lines of that employed in Section 2, in order to show that $(\mathcal{M}, \preceq)$ forms a finite distributive lattice. We begin with the analogue of Theorem 2.3 for super-stability.

Theorem 5.2. *Let I be an instance of SMP, and let M, M' be two super-stable matchings in I. Suppose that m and w are partners in M but not in M'. Then one of m, w strictly prefers M to M', and the other strictly prefers M' to M.*

Proof. By Lemma 5.1, neither m nor w is indifferent between M and M'. Hence, as M, M' are both strongly stable, the result follows by Theorem 2.3. □

It is a straightforward matter to modify the proofs of Lemmas 2.8 and 2.9 to establish the super-stability analogues; we omit the details. Thus we have:

Lemma 5.3. *Let I be an instance of SMP, and let M, M' be two super-stable matchings in I. Let M^* be a set of (man,woman) pairs defined by giving each man m the better of his partners in M and M'. Then M^* is a super-stable matching.*

Lemma 5.4. *Let I be an instance of SMP, and let M, M' be two super-stable matchings in I. Let M^* be a set of (man,woman) pairs defined by giving each man m the poorer of his partners in M and M'. Then M^* is a super-stable matching.*

We denote by $M \wedge M'$ (resp. $M \vee M'$) the set of (man,woman) pairs in which each man receives the better (resp. poorer) of his partners in M and M'. We are now in a position to state our main result of this section.

Theorem 5.5. *Let I be an instance of SMP, and let $\mathcal{M}$ be the set of super-stable matchings in I. Then $(\mathcal{M}, \preceq)$ forms a finite distributive lattice, with $M \wedge M'$ representing the meet of M and M', and $M \vee M'$ the join, for two super-stable matchings $M, M' \in \mathcal{M}$, where $\preceq$ is the dominance partial order of Definition 2.6, now defined on $\mathcal{M}$.*

Proof. By Lemmas 5.3 and 5.4, $M \wedge M' \in \mathcal{M}$ and $M \vee M' \in \mathcal{M}$. The proof that $M \wedge M'$ is the greatest lower bound of M and M', and $M \vee M'$ is the least upper bound of M and M', is along the same lines as the corresponding part of the proof of Theorem 2.11. Clearly, each of the meet and join operations is idempotent, commutative and associative. It also follows by Theorem 1.3.2 of [5] that the meet and join operations distribute over each other. Hence $(\mathcal{M}, \preceq)$ is a finite distributive lattice. □

6 Conclusions and open problems

As mentioned previously, it remains open as to whether the lattice structure for SMT under strong stability can be exploited so as to yield efficient algorithms for the problems of finding an egalitarian strongly stable matching, a minimum regret strongly stable matching and all strongly stable pairs (i.e. determining, for each man m and woman w, whether m and w are partners in some strongly stable matching) for a given SMT instance. Although we have suggested a strategy for listing strongly stable matchings in a given equivalence class under $\sim$, the general problem of listing *all* strongly stable matchings in an SMT instance also remains open. Each of these four problems is also open for SMP under super-stability, but the existence of a lattice structure, and Lemma 5.1 in particular, provides strong evidence that all four problems are efficiently solvable. On the other hand, the situation for SMP under strong stability is likely to be different, in view of Section 4. Indeed, the problem of deciding whether a given instance of SMP admits a strongly stable matching is NP-complete [11].

In the case of weak stability however, the efficient solvability of three of the aforementioned problems is known to be unlikely. The problems of finding an egalitarian weakly stable matching and a minimum regret weakly stable matching in an SMT instance I are both NP-hard, and are not approximable within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, where n is the number of people in I, unless P=NP [9, 12]. In addition, the problem of determining, for a given SMT instance and a given man m and woman w, whether m and w are partners in some weakly stable matching, is also NP-hard [12]. Each of these results holds in the restricted case that every tie is of length two, there is at most one tie per preference list, and the ties occur in the preference lists of one sex only (these conditions holding simultaneously). Finally, the question as to whether all weakly stable matchings in a given SMT instance may be generated efficiently (in the sense described in Section 3) remains open, though we conjecture that the answer will be in the negative.

Acknowledgements

I would like to thank Rob Irving and the anonymous referees for valuable comments which helped simplify the proof of Lemma 2.2 and improve the presentation of the paper.

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