MOTIVIC DONALDSON–THOMAS THEORY AND THE ROLE OF ORIENTATION DATA

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1. Introduction

The purpose of this expository paper is to introduce the reader, in a gentle way, to orientation data, as it appears in the work of Kontsevich and Soibelman [23]. We do this while keeping things as simple as possible by focusing on a single simple example and also explaining the broad motivations behind motivic Donaldson–Thomas theory in as down to earth a way as possible.

In motivic Donaldson–Thomas theory, the central foundational result states that, given a fixed category $\mathcal{C}$, assumed to be Abelian and also 3-dimensional Calabi-Yau in some appropriate sense (see [23, Sec.3.3]), we may form an *integration map*, which is a ring homomorphism

$$\mathsf{DT}:\mathsf{st}(\mathbf{Ob}(\mathcal{C}))\rightarrow\widetilde{\mathsf{Mot}}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})[[x^{\alpha}|\alpha\in \mathrm{K}(\mathcal{C})]]. \tag{1}$$

Here $\mathsf{st}(\mathbf{Ob}(\mathcal{C}))$ is some version of Joyce's motivic Hall algebra [17], spanned as a group by symbols $[X\rightarrow\mathbf{Ob}(\mathcal{C})]$. These are certain finite type morphisms from Artin stacks into $\mathbf{Ob}(\mathcal{C})$, the stack of objects of $\mathcal{C}$; but one shouldn't be put off at this stage by the presence of stacks, since in this paper, they will be relegated to the background. The coefficients of the target ring are a modification of $\mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})$, the naive Grothendieck ring of $\hat{\mu}$-equivariant complex varieties, where $\hat{\mu}:=\varprojlim\mu_n$, spanned by symbols $[X\rightarrow\operatorname{Spec}\mathbb{C}]$ (we will often omit the structure morphism and just write $[X]$), for X a μ_n-equivariant reduced variety with $\hat{\mu}$-action given by the surjection $\hat{\mu}\rightarrow\mu_n$. The multiplication on this ring is the convolution product defined by Looijenga in [25]. We form $\widetilde{\mathsf{Mot}}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})$ by adding inverses to the motives of all the general linear groups, considered as varieties with trivial $\hat{\mu}$-action, and a formal square root $\mathbb{L}^{1/2}$ to $\mathbb{L}$, the class of the affine line $\mathbb{A}^1_{\mathbb{C}}$, again with the trivial action. On both sides we impose the cut and paste relations[1], i.e. for general Y we identify

$$[X\xrightarrow{f}Y]=[U\xrightarrow{f|_U}Y]+[V\xrightarrow{f|_V}Y], \tag{2}$$

where $U\subset X$ is Zariski open, with complement V. We consider $\widetilde{\mathsf{Mot}}^{\hat{\mu}}$ instead of $\mathsf{Mot}^{\hat{\mu}}$ since, in order to deal with the presence of stacks on the left hand side of the map DT we have to invert the motives corresponding to stabilisers of closed points of these stacks (in fact one operates under the assumption that these stabilisers can always be taken to be subgroups of general linear groups, from which it follows that this localisation can be simply described by the addition of

[1] For technical reasons, there is an extra relation on $\widetilde{\mathsf{Mot}}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})$; if $p:V\rightarrow Y$ is a μ_n-equivariant rank m vector bundle on the μ_n-equivariant scheme Y, we identify $[V]=[Y\times\mathbb{A}^m_{\mathbb{C}}]$, where the μ_n-action on $Y\times\mathbb{A}^m_{\mathbb{C}}$ is the product of the μ_n-action on Y and the trivial action on $\mathbb{A}^m_{\mathbb{C}}$.

a formal inverse to the motive of each general linear group). Finally, we set $\mathrm{K}(\mathcal{C})$ to be some finite rank free Abelian group obtained as a quotient of the Grothendieck group of $\mathcal{C}$, and for $M \in \mathcal{C}$ we write $[M]_{\mathrm{K}}$ for the class of M in $\mathrm{K}(\mathcal{C})$.

In general, a morphism from a finite type scheme Y to $\mathbf{Ob}(\mathcal{C})$ should be thought of as a family of objects of $\mathcal{C}$ parameterised by the scheme Y, and the general principle behind defining any such map DT is that one associates to each closed point $y \in Y$ representing an object M_y a motivic weight $\mathsf{mw}(M_y) \in \mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})$, i.e. $\mathsf{mw}(M_y)$ should be some linear combination $\sum a_i[X_{y,i} \to y]$. These motivic weights are required in fact to form a family, i.e. there should be some linear combination of symbols $\sum a_i[X'_i \to Y]$ such that restriction to each fibre y gives $\mathsf{mw}(M_y)$. Then we 'integrate' across Y, by simply forgetting the maps into Y, or equivalently pushing forward along the structure morphism, i.e. we take $\int \sum a_i[X'_i \to Y] := \sum a_i[X'_i]$. Finally, we assume that all the points M_y satisfied $[M_y]_{\mathrm{K}} = \alpha$, which we may do after decomposition, and define $\mathsf{DT}([Y \to \mathbf{Ob}(\mathcal{C})]) := \sum a_i[X'_i]x^{\alpha}$.

The fact that, no matter what motivic weight mw we choose, the map DT is a group homomorphism, is a direct consequence of the definition, using the cut and paste relation (2). The real goal is to show that DT preserves also a *product*, and so is a ring morphism. On the left hand side, the product is the Hall algebra product, for which, if $Y_1 \to \mathbf{Ob}(\mathcal{C})$ and $Y_2 \to \mathbf{Ob}(\mathcal{C})$ are two families of objects in $\mathbf{Ob}(\mathcal{C})$, we define $[Y_1 \to \mathbf{Ob}(\mathcal{C})] \star [Y_2 \to \mathbf{Ob}(\mathcal{C})]$ to be the family of short exact sequences $0 \to M' \to M \to M'' \to 0$ in $\mathbf{Ob}(\mathcal{C})$, with M' in the family parameterised by Y_1, and M'' in the family parameterised by Y_2. This is considered as a family of objects of $\mathcal{C}$ via the forgetful map that remembers only M. To be a lot more rigorous, using the language of stacks, there are three projections $\pi_i : \mathbf{SES}(\mathcal{C}) \to \mathbf{Ob}(\mathcal{C})$ from the stack of short exact sequences in $\mathcal{C}$ to the stack of objects in $\mathcal{C}$, taking a short exact sequence to its first, second or third term, and one can take the Cartesian product of stacks

$$\begin{tikzcd} Y_3 \arrow[r, "h"] \arrow[d] & \mathbf{SES}(\mathcal{C}) \arrow[d, "\pi_1 \times \pi_3"] \\ Y_1 \times Y_2 \arrow[r, "f_1 \times f_2"] & \mathbf{Ob}(\mathcal{C}) \times \mathbf{Ob}(\mathcal{C}). \end{tikzcd}$$

Then $[Y_1 \to \mathbf{Ob}(\mathcal{C})] \star [Y_2 \to \mathbf{Ob}(\mathcal{C})] := [Y_3 \xrightarrow{\pi_2 \circ h} \mathbf{Ob}(\mathcal{C})]$.

The product on the right hand side of (1) is given by a twisted version of Looijenga's convolution product. To explain what this is, let us first concentrate on the coefficient ring and define the convolution product itself. Given a μ_n-equivariant variety Y, form the mapping torus $[Y \times^{\mu_n} \mathbb{G}_{\mathbb{C}}^{\times} \xrightarrow{(y,z) \mapsto z^n} \mathbb{G}_{\mathbb{C}}^{\times}] \in \mathsf{Mot}^{\mathbb{G}_{\mathbb{C}}^{\times},n}(\mathbb{G}_{\mathbb{C}}^{\times})$, a $\mathbb{G}_{\mathbb{C}}^{\times}$-equivariant variety over $\mathbb{G}_{\mathbb{C}}^{\times}$, with $\mathbb{G}_{\mathbb{C}}^{\times}$ given the weight n action on itself. There is an embedding $\mathsf{Mot}^{\mathbb{G}_{\mathbb{C}}^{\times},n}(\mathbb{G}_{\mathbb{C}}^{\times}) \to \mathsf{Mot}^{\mathbb{G}_{\mathbb{C}}^{\times},n}(\mathbb{A}_{\mathbb{C}}^1)$ induced by the embedding $\mathbb{G}_{\mathbb{C}}^{\times} \to \mathbb{A}_{\mathbb{C}}^1$, and a complement is provided by the embedding $\mathsf{Mot}(\operatorname{Spec}\mathbb{C}) \to \mathsf{Mot}^{\mathbb{G}_{\mathbb{C}}^{\times},n}(\mathbb{A}_{\mathbb{C}}^1)$ taking $[Y]$ to $[Y \times \mathbb{A}_{\mathbb{C}}^1 \xrightarrow{(y,z) \mapsto z} \mathbb{A}_{\mathbb{C}}^1]$. If we denote the image of this second embedding by $\mathcal{I}$, we obtain an isomorphism $\mathsf{Mot}^{\mu_n}(\operatorname{Spec}\mathbb{C}) \cong \mathsf{Mot}^{\mathbb{G}_{\mathbb{C}}^{\times},n}(\mathbb{A}_{\mathbb{C}}^1)/\mathcal{I}$. On $\mathsf{Mot}^{\mathbb{G}_{\mathbb{C}}^{\times},n}(\mathbb{A}_{\mathbb{C}}^1)$ there is a natural associative product

$$[Y_1 \xrightarrow{f_1} \mathbb{A}_{\mathbb{C}}^1] \cdot [Y_2 \xrightarrow{f_2} \mathbb{A}_{\mathbb{C}}^1] = [Y_1 \times Y_2 \xrightarrow{+\circ(f_1 \times f_2)} \mathbb{A}_{\mathbb{C}}^1]$$

for which $\mathcal{I}$ is an ideal, so this product descends to a product on $\mathsf{Mot}^{\mu_n}(\operatorname{Spec}\mathbb{C})$. This defines the product on the coefficient ring on the right hand side of (1). The product on the whole ring of formal power series is given by decreeing that the coefficients commute with the variables x^{α}, and defining[2] $x^{[M]_{\mathrm{K}}} \cdot x^{[N]_{\mathrm{K}}} := \mathbb{L}^{\frac{1}{2}\sum_i (-1)^i \dim(\mathrm{Ext}^i(M,N))} x^{[M]_{\mathrm{K}}+[N]_{\mathrm{K}}}$.

We have described the associative product on each of the Abelian groups of (1). The central foundational result, then, for any candidate for the integration map DT, is that it commutes with these products. The reason this is a desirable feature for an integration map is that there are a plethora of identities in the Hall algebra that we can apply the integration map to in order to obtain product descriptions of motivic generating series. Perhaps the archetypal example is the situation in which we have some stability condition θ on the elements of $\mathcal{C}$, for which every object F admits a unique filtration $0 = F_0 \subset \ldots \subset F_n = F$ such that each subquotient F_{i+1}/F_i is θ-semistable and the slopes $\theta(F_{i+1}/F_i)$ are strictly descending – a Harder–Narasimhan filtration. This translates to the statement in the Hall algebra that the stack of all objects is some ordered product of the stacks of θ-semistable objects of fixed slope. As we perturb the stability condition θ, the terms in this infinite product, the stacks of θ-semistable objects, change, while the product (the stack of all objects in $\mathcal{C}$) stays the same. Applying the integration map to this statement, one obtains an equality of infinite products, that is the famous wall crossing formula (see [18] and [23]).

In this paper we will describe how one builds a map like DT that respects these products, and in particular, how one constructs the motivic weight mw. The idea is to work through a simple example, in order to see the natural candidate for a motivic weight in action. The endpoint is to motivate the introduction of orientation data: we will see how the natural choice for the motivic weight fails to define a map preserving the product of (1), and describe the kind of modification that must be made to fix this defect.

2. Some background: The numerical Donaldson–Thomas count

Let X be a smooth projective 3-fold. Then for a given Hilbert polynomial p we may consider $\mathcal{M}_p$, the moduli space of semistable coherent sheaves $\mathcal{F}$ on X that have Hilbert polynomial p. In order to get a reasonable space we impose some kind of stability condition (Gieseker stability or slope stability), and under suitable conditions (e.g. if there are no strictly semistable objects) this space will be a finite type fine moduli scheme, which we will denote by $\mathcal{M}$ (see for example Huybrechts and Lehn's book [15]). It is an important feature of the scheme $\mathcal{M}$ that it is proper: the Donaldson–Thomas count for $\mathcal{M}$ is the degree of some cycle class of zero-dimensional subschemes of $\mathcal{M}$, and in the proper case this is just given by the count of the points in this class, with multiplicity. In the non-proper case this breaks down somewhat.

We arrive at this zero-dimensional class by next assuming that our 3-fold X was, all along, a Calabi-Yau 3-fold. This implies, in particular, that the expected dimension of $\mathcal{M}$ is zero. More precisely, $\mathcal{M}$ comes equipped with a perfect obstruction theory $L^{\bullet} := [E_1 \to E_0]$ which satisfies the condition $\operatorname{rank}(E_1) = \operatorname{rank}(E_0)$. In [2] Behrend and Fantechi show that from such data one can construct a virtual fundamental class of the correct dimension in $A^*(\mathcal{M})$, i.e. a class $[\mathcal{M}]_{\mathrm{vir}} \in A^0(\mathcal{M})$. Finally, the Donaldson–Thomas count is given by $\deg[\mathcal{M}]_{\mathrm{vir}}$.

[2]For this definition to make sense we must make the obvious additional assumption on $\mathrm{K}_0(\mathcal{C}) \xrightarrow{\pi} \mathrm{K}(\mathcal{C})$.

The justification for taking this virtual fundamental class is the fact that, since our moduli scheme $\mathcal{M}$ 'should' be zero-dimensional, there is an intuition that the correct number (the Donaldson–Thomas count) should be obtained by perturbation. So if $\mathcal{M}$ has a component that is smooth, with an obstruction bundle over it that is just a vector bundle, the contribution from that component should be the Euler class of that vector bundle. Similarly, if $\mathcal{M}$ has a component that has underlying topological space a point, but structure sheaf of length n, then its contribution to the DT invariant should be n, as this component 'should' generically deform to give n points (the inverted commas here are on account of the fact that we remain vague as to where these deformations are taking place). The taking of a virtual fundamental class is a way of using excess intersection theory to make all of this precise.

The perfect obstruction theory $L^\bullet$ constructed by Richard Thomas in [30] has the extra property that it is symmetric, in the sense of [3, Def.1.10], that is there is an isomorphism $\theta : L^\bullet \to (L^\bullet)^\vee[1]$ in the derived category of coherent sheaves on $\mathcal{M}$ satisfying $\theta^\vee[1] = \theta$. So, returning to the situation in which a component $\mathcal{M}_1$ of $\mathcal{M}$ is smooth, the obstruction bundle is automatically a vector bundle, and isomorphic to the cotangent bundle, and in this case we can say exactly what we think the contribution to the Donaldson–Thomas count of the component should be: $(-1)^{\dim(\mathcal{M}_1)}\chi(\mathcal{M}_1)$. This is the first indication that in fact the contribution of every component should be (and actually is, in the case in which the perfect obstruction theory with which we calculate our Donladson–Thomas count is symmetric) a weighted Euler characteristic, with the weighting of smooth points given by the parity of the dimension, and the weight of isolated points given by the length of their structure sheaves.

The goal, then, is to associate to an arbitrary finite-type scheme Y a constructible function ν_Y, with image lying in the integers, such that, in the event that Y is compact and is equipped with a symmetric perfect obstruction theory, there is an equality

$$\deg[Y]_{\mathrm{vir}} = \sum_{n\in\mathbb{Z}} n \cdot \chi(\nu_Y^{-1}(n)), \tag{3}$$

where the class on the left hand side is the virtual fundamental class constructed from the symmetric perfect obstruction theory. For schemes defined over $\mathbb{C}$, this is achieved by Kai Behrend in [1], and this function ν_Y is Behrend's microlocal function for Y. Note that in the case in which Y is a noncompact scheme with a symmetric perfect obstruction theory, the machinery of [2] still gives us a virtual fundamental class $[Y]_{\mathrm{vir}}$, for which (3) does not make sense, since $\deg[Y]_{\mathrm{vir}}$ will be undefined. In this case, however, we can take the right hand side as our definition of the Donaldson–Thomas count.

Recall the moduli space $\mathcal{M}$ we started with. For gauge-theoretic reasons (see [18, Sec.5.1]), a complex analytic neighborhood of an arbitrary sheaf $\mathcal{F}$, considered as a point in $\mathcal{M}$, is given by the following setup. Let $\mathbb{C}^t$ be some affine space, and let f be the germ of an analytic function defined and equal to zero at the origin. Then a complex analytic neighborhood of $\mathcal{F}$ is isomorphic to a neighborhood of the origin in the critical locus of f. This becomes an important observation given the following fact regarding the microlocal function $\nu_{\mathcal{M}}$: if a scheme Y is given by the critical locus of some function f on some smooth d-dimensional scheme, at least analytic locally around some point $y \in Y$, then

$$\nu_Y(y) = (-1)^d(1 - \chi(\mathsf{mf}(f, y))), \tag{4}$$

where $\mathsf{mf}(f, y)$ is the Milnor fibre of f at the point y (see [1, Sec.1.2] for a discussion of this point, and more generally of the definition of ν_Y, or [27, Cor.2.4] for a proof of (4)).

3. Motivic vanishing cycles and Milnor fibres

If X is a finite type scheme, we may equip X with the trivial $\hat{\mu}$-action, and define $\mathsf{Mot}^{\hat{\mu}}(X)$, as a group, to be generated by $\hat{\mu}$-equivariant maps $[Y \to X]$, where the $\hat{\mu}$-action on the finite type reduced scheme Y is induced from some μ_n-action, for $n \in \mathbb{N}$. We impose the further technical assumption on these generators that each closed point of Y lies in a μ_n-equivariant open affine subscheme of Y. If we make $\mathsf{Mot}(\operatorname{Spec}\mathbb{C})$ into a ring via the product structure $[Y_1] \times [Y_2] = [Y_1 \times Y_2]$ then $\mathsf{Mot}^{\hat{\mu}}(X)$ is a $\mathsf{Mot}(\operatorname{Spec}\mathbb{C})$-module via the action $[Y] \cdot [Z \xrightarrow{g} X] := [Y \times Z \xrightarrow{g \circ \pi_Z} X]$.

Let $h : X_1 \to X_2$ be a finite type morphism of schemes. Then we obtain a morphism $h_* : \mathsf{Mot}^{\hat{\mu}}(X_1) \to \mathsf{Mot}^{\hat{\mu}}(X_2)$ by sending $[f : Y \to X_1]$ to $[h \circ f : Y \to X_2]$. The pullback morphism $h^* : \mathsf{Mot}^{\hat{\mu}}(X_2) \to \mathsf{Mot}^{\hat{\mu}}(X_1)$ is defined by sending $[f : Y \to X_1]$ to $[Y \times_{X_1} X_2 \to X_2]$.

The motive

$$\sum_{n \in \mathbb{Z}} n[\nu_X^{-1}(n)] \in \mathsf{Mot}(\operatorname{Spec}\mathbb{C}) \tag{5}$$

is in some sense a motivic refinement of the Donaldson–Thomas count, but it is a somewhat unnatural halfway point. For we have replaced the measure χ with a motivic measure, without replacing the weight by a motivic weight. The natural refinement of our weight, from a number to a motive, is given by taking the motivic vanishing cycle, instead of Behrend's constructible function. So we next recall some of the definitions and formulae regarding motivic vanishing cycles and nearby fibres – the proper background for this material is to be found in [25] and [12].

Let f be a regular function from a smooth complex finite type scheme X to $\mathbb{C}$. Let

$$\begin{array}{c} Y \\ \big\downarrow{\scriptstyle h} \\ X \end{array}$$

be an embedded resolution of the function f. Then the motivic nearby cycle $[\psi_f]$, as defined by Denef and Loeser in [12], in terms of arc spaces, has an explicit formula in terms of this embedded resolution, which we will now describe. The level set $(fh)^{-1}(0)$ consists of a set of divisors, indexed by a set forever denoted J, with each divisor D_i meeting every other one transversally. We use the symbol a_i to denote the order of vanishing of fh on D_i. Given $I \subset J$, a nonempty subset, let D_I^0 be the complement in the intersection of all the divisors in I of the union of the divisors that are not in I. So the D_I^0 form a stratification of $(fh)^{-1}(0)$, with deeper strata coming from larger subsets $I \subset J$.

Let $I \subset J$ be a subset. The function fh defines a section of $\mathcal{O}_Y(-\sum_{i \in I} a_i D_i)$, and so a regular map, f_I, linear along the fibres, from the total space of $\mathcal{O}_Y(\sum_{i \in I} a_i D_i)$ to $\mathbb{C}$. The restriction of this bundle to D_I^0 is just $\otimes_{i \in I} N_{D_i|Y}^{\otimes a_i}$, so that the restriction $f_I|_{D_I^0}$ becomes a $\mathbb{G}_{\mathbb{C}}^{\times}$-equivariant function, where $\mathbb{G}_{\mathbb{C}}^{\times}$ acts by rescaling each copy of $N_{D_i|Y}$, and acts on the target $\mathbb{C}$ with weight $\sum_{i\in I} a_i$. We define $\tilde{D}_I = f_I^{-1}|_{D_I^0}(1)$, and via the natural projection we obtain étale covers

$$\begin{tikzcd} \tilde{D}_I \arrow[d, "p_I"] \\ D_I^0. \end{tikzcd}$$

The scheme $\tilde{D}_I$ over D_I^0 carries the obvious action under the group of a_Ith roots of unity, and so we obtain an element of $\mathsf{Mot}^{\hat{\mu}}(X)$ by pushforward from D_I^0 to X along h. Finally, the formula is

$$[\psi_f] = \sum_{\emptyset\neq I\subset J} (1-\mathbb{L})^{|I|-1}[\tilde{D}_I \xrightarrow{h\circ p_I} X] \in \mathsf{Mot}^{\hat{\mu}}(X). \tag{6}$$

Let T be a constructible subset of X. Restriction to T defines a map from $\hat{\mu}$-equivariant motives over X to $\hat{\mu}$-equivariant motives over T. Pushforward from T to a point gives us an absolute $\hat{\mu}$-equivariant motive. We let $\int_T$ denote the composition of these two maps. Explicitly, $\int_T [Y \xrightarrow{g} X] := [g^{-1}(T)]$.

Let f be as above, and let $p \in X$ be a point in $f^{-1}(0)$. Then the *motivic Milnor fibre* of f at p is defined to be

$$\mathsf{MF}(f,p) := \int_p [\psi_f] \in \mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C}).$$

If X is affine space, and f is a function vanishing at the origin, then we define

$$\mathsf{MF}(f) := \mathsf{MF}(f,0).$$

Finally, define the *motivic vanishing cycle*:

$$[\phi_f] := [\psi_f] - [f^{-1}(0) \to X] \in \mathsf{Mot}^{\hat{\mu}}(X). \tag{7}$$

In the equation (7), $[f^{-1}(0)]$ carries the trivial $\hat{\mu}$-action.

We close this section with a fundamental theorem regarding motivic vanishing cycles.

Theorem 3.1. *(Motivic Thom-Sebastiani)*[11] *Let V and V' be vector bundles on smooth schemes X and X' respectively. Let π and π' be the projections from $X \times X'$ to X and X' respectively. Let f and f' be algebraic functions on the vector bundles V and V' respectively. Denote by $f \oplus f'$ the sum of the pullbacks of f and f' to the vector bundle $\pi^*(V) \oplus \pi'^*(V')$. Then there is an equality*

$$[-\phi_{f\oplus f'}] = \pi^*([-\phi_f]) \cdot \pi'^*([-\phi_{f'}]) \in \mathsf{Mot}^{\hat{\mu}}(X \times X'). \tag{8}$$

The product structure on the right hand side of (8) is defined as in the introduction via the natural product structure on $\mathsf{Mot}^{\mathbb{G}_{\mathbb{C}}^{\times},n}(X \times X' \times \mathbb{A}^1_{\mathbb{C}})$:

$$[Y_1 \xrightarrow{g_1} X \times X' \times \mathbb{A}^1_{\mathbb{C}}] \cdot [Y_2 \xrightarrow{g_2} X \times X' \times \mathbb{A}^1_{\mathbb{C}}] := [Y_1 \times_{X\times X'} Y_2 \xrightarrow{+\circ g_1 \times g_2} X \times X' \times \mathbb{A}^1_{\mathbb{C}}].$$

Given a moduli space $\mathcal{M}$, in order to refine the weight with which we integrate from $\nu_{\mathcal{M}}$ to ϕ_f, we should first find a way to express $\mathcal{M}$, at least Zariski locally, as the critical locus of a regular function f on a smooth ambient variety Y. Recently there has been significant progress in this direction, through the work of Ben–Bassat, Brav, Bussi, Dupont, Joyce, Szendrői and Meinhardt, see the papers [6], [7], [4] and [5]. The first of these is most relevant for the geometric applications above, for it guarantees that moduli spaces $\mathcal{M}$ of coherent sheaves on X a Calabi–Yau 3-fold are Zariski locally modelled on critical loci of regular functions. The second and fourth provide a detailed description of the way in which one cooks up a motivic weight on $\mathcal{M}$ from a sheaf theoretic version of orientation data in the sense of Kontsevich and Soibelman.

There are, broadly speaking, two stages to the motivation of orientation data. The first is most clearly stated in the context of the work of Joyce et al., and comes to the fact that the bare knowledge that a scheme can be expressed Zariski locally as the critical locus of a regular function f on a smooth ambient variety Y is not enough to fix a motivic weight, since there may be more than one such pair (Y, f), and different choices will give rise to different motivic weights. A very simple example is given by letting $Y_1 = Y_2 = \mathbb{C} \times \mathbb{C}^* = \mathrm{Spec}(\mathbb{C}[x, y^{\pm 1}])$, letting $f_1 = x^n$, and letting $f_2 = yx^n$. One can express essentially the same problem in the language of the current paper by saying that the knowledge that a family of objects come from a cyclic Calabi–Yau category is not enough to fix the motivic weight from their (non-minimal) potentials, as this weight will not be invariant under changing the quasi–isomorphism class of $\mathcal{C}$. In the framework of this paper the problem of fixing a motivic weight, will be less prominent, as in the motivic context there *is* a canonical choice, given by taking the *minimal* potential – see Section 8. This solution is not available in the sheaf theoretic context, as it relies on passing to a constructible decomposition – see Section (6). The second stage of motivation for the introduction of orientation data is most easily expressed in the categorical framework, and is expressed by saying that even though one has a canonical choice for the motivic weight given by taking minimal potentials, this is the wrong one for obtaining a ring homomorphism, so that we must modify this canonical choice in a coherent way – this modification is the orientation data. It is this second point that we will focus on, with the aid of the main example of this paper.

4. A basic example

Let $B = \mathbb{C}[x]/\langle x^3 \rangle$. We will study $B\text{-}\mathsf{mod}$, the moduli space of finite-dimensional modules over B. We will let the class of a B-module M in $\mathrm{K}(B\text{-}\mathsf{mod}) \cong \mathbb{Z}$ be the dimension. In fact B is a special example of a 'Jacobi algebra', or a 'superpotential' algebra. Let Q be the quiver with one vertex and one loop. Then $\mathbb{C}Q \cong \mathbb{C}\langle a \rangle$, where $\mathbb{C}Q$ denotes the free path algebra of the quiver Q. Let W be the cyclic word in this quiver given by $W = a^4$. Then, in forming the Jacobi algebra that this data defines, we are meant to form the 'noncommutative differentials' of W by differentiating it with respect to each of the arrows in Q (see [14, Sec.1.3] or [9] for an explanation of what this means). Here, this noncommutative generalization of differential calculus reduces to familiar calculus, since $\mathbb{C}Q$ is commutative. So the only noncommutative differential we need to think about is

$$\frac{\partial}{\partial a} W = 4a^3.$$

The statement that B is a Jacobi algebra amounts to saying that

$$B \cong \mathbb{C}Q/\langle \frac{\partial}{\partial a} W \rangle.$$

This puts us in a special situation, noted by Ginzburg and E. Segal in [14, Sec.2] and [28] and exploited by Szendrői in his study of the noncommutative conifold in [29], in which we have a way of coherently embedding the representation spaces of B-modules as subschemes of smooth schemes. The word 'coherently' doesn't yet have a precise meaning here, but has to do with the problem of comparing the motivic weight associated to extensions of modules to the motivic weights of those modules themselves, which in turn will be the central difficulty when it comes to checking that putative integration maps from families of B-modules to motives (as in (1)) preserve associative products. This in turn is the central problem motivating the introduction of orientation data.

How this works out in our case is as follows. Define

$$\mathrm{Rep}_n(B) := \mathrm{Hom}_{\mathrm{alg}}(B, \mathrm{Mat}_{n\times n}(\mathbb{C})),$$

the set of homomorphisms of unital algebras. This is a scheme, the points of which correspond to representations of B. In general the more natural object to study is perhaps the stack formed under the conjugation action of $\mathrm{GL}_n(\mathbb{C})$, but for the time being we will really just be looking at the above scheme. Similarly, we define

$$\mathrm{Rep}_n(\mathbb{C}Q) := \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}Q, \mathrm{Mat}_{n\times n}(\mathbb{C})).$$

Then since a representation of B is just a representation of $\mathbb{C}Q$ satisfying some relations, $\mathrm{Rep}_n(B)$ is defined as a Zariski closed subscheme of this *smooth* scheme. There is a map

$$\mathrm{ev}_a : \mathrm{Rep}_n(\mathbb{C}Q) \to \mathrm{Mat}_{n\times n}(\mathbb{C})$$

that sends

$$\theta \mapsto \theta(a).$$

In fact this is clearly an isomorphism. It turns out (and this is a general fact about Jacobi algebras) that

$$\mathrm{Rep}_n(B) = \mathrm{crit}(\mathbf{tr}((\mathrm{ev}_a)^4)), \tag{9}$$

where the object on the right hand side of (9) is the scheme-theoretic critical locus. For a general Jacobi algebra we replace $(\mathrm{ev}_a)^4$ with a function of evaluation maps built from W, and the corresponding statement remains true.

The goal of this subject is to define motivic Donaldson–Thomas counts, that soup up the old one, which was just the Euler characteristic weighted by a microlocal function ν. Recall that the microlocal function of a scheme at a point x, at which the scheme is locally described as $\mathrm{crit}(f)$ for some f on a d-dimensional ambient smooth scheme, is just $(-1)^d(1-\chi(\mathsf{mf}(f,x)))$. Consider just

$$\mathrm{Rep}_1(B) \cong \mathrm{Spec}(B).$$

The fact that we have an explicit presentation of our space as a critical locus enables us to go ahead and refine the microlocal function $\nu_{\mathrm{Rep}_1(B)}$ to a motivic weight, which is given by minus the (absolute) motivic vanishing cycle of the function x^4. Here and elsewhere we will adopt the shorthand that where a function $f(x_1,\dots,x_n)$ appears without reference to a space that it is a function on, that space will always be assumed to be affine n-space, and the motivic vanishing/nearby cycle of it is the motivic vanishing/nearby cycle of the function on affine n-space. We define[3]

$$\mathsf{DT}(\mathrm{Rep}_1(B)) := \int_{\mathrm{Rep}_1(B)} [-\phi_{x^4}]\,\mathbf{x}\,.$$

In order to establish uniform notation with what follows we rewrite this as

$$\mathsf{DT}(\mathrm{Rep}_1(B)) := \int_{\mathrm{Rep}_1(B)} [-\phi_{\mathbf{tr}(T^4)}]\,\mathbf{x}, \tag{10}$$

where $\mathbf{tr}(T^4)$ is considered as a function on $\mathbb{C}$ by identifying $\mathbb{C}$ with the ring of 1×1 matrices. Since $\mathrm{Rep}_1(B)$ is just a point, in this case we have

$$\mathsf{DT}(\mathrm{Rep}_1(B)) = (1-\mathsf{MF}(x^4))\,\mathbf{x}\,.$$

The unique closed point of the space $\mathrm{Rep}_1(B)$ is given by a 1×1 matrix, the zero matrix. Call this representation M. Considered as a module for the quiver algebra $\mathbb{C}Q/\langle a^3\rangle$, this is the one-dimensional simple module killed by all the arrows of Q. In this example it is easy enough to explain what we mean by 'preservation of the ring structure'. Define

$$\mathrm{Rep}_1(B)\star\mathrm{Rep}_1(B)$$

to be the stack of flags $M\subset N$ with $N/M\cong M$. This stack is defined and its properties studied by Joyce in [16, Sec.10]. The stabilizer at any point is given by $\mathrm{Hom}(M,M)\cong\mathbb{C}$, and in fact this stack can be described explicitly as a group quotient of the space $\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})$ of strictly upper-triangular 2 by 2 matrices by the trivial action of the additive group $\mathbb{C}\cong\mathrm{Hom}(M,M)$. So we write the motive of this stack as

$$[\mathrm{Rep}_1(B)\star\mathrm{Rep}_1(B)] = [\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})]/\mathbb{L}. \tag{11}$$

Now what we want is the identity, in $\widetilde{\mathsf{Mot}}^{\hat{\mu}}(\mathrm{Spec}\,\mathbb{C})[[\mathbf{x}]]$:

$$\mathsf{DT}[\mathrm{Rep}_1(B)\star\mathrm{Rep}_1(B)] = \mathsf{DT}[\mathrm{Rep}_1(B)]\cdot\mathsf{DT}[\mathrm{Rep}_1(B)] = (1-\mathsf{MF}(x^4))^2\,\mathbf{x}^2, \tag{12}$$

where on the right hand side we use Looijenga's product on the ring of motives. From the motivic Thom-Sebastiani Theorem 3.1, we deduce that

$$\mathsf{DT}[\mathrm{Rep}_1(B)]\cdot\mathsf{DT}[\mathrm{Rep}_1(B)] = (1-\mathsf{MF}(x^4+y^4))\,\mathbf{x}^2\,.$$

Proposition 4.1. *Denote the representation ring of $\mathbb{Z}_4$ by $\mathbb{Z}[\alpha]/\alpha^4$, where α is the 1-dimensional representation sending $1\in\mathbb{Z}_4$ to multiplication by i. There is an equality of motives*

$$\mathsf{MF}(x^4+y^4) = [C_1]-4\mathbb{L},$$

where C_1 is a genus 3 curve with the representation $2(\alpha+\alpha^2+\alpha^3)$ on its middle cohomology.

We defer the proof of this proposition to the start of Appendix B.

[3]The attentive reader will wonder what has happened to the sign $(-1)^d$ of (4). The answer runs as follows. In order to fix, once and for all, the contribution of a module M to the numerical DT count of moduli spaces it occurs as a closed point of, we always pull back the microlocal function, and the motivic weight, from the *stack* of finite-dimensional B-modules. This stack is in fact zero-dimensional, so we can safely forget about signs.

By using Proposition 4.1 and the motivic Thom-Sebastiani theorem we can calculate the right hand side of equation (12). What, then, of the left hand side? Well, first we should define it! This we do as follows: the coarse moduli space $\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})$ of our stack $\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})/\mathbb{A}^1$ is a subscheme of $\mathrm{Rep}_2(B)$. Let

$$\iota : \mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C}) \hookrightarrow \mathrm{Rep}_2(B)$$

be the inclusion. Then recall that we want a motivic refinement of the weighted Euler characteristic

$$\sum_{n\in\mathbb{Z}} n \cdot \chi(\iota^*(\nu_{\mathrm{Rep}_2}(B))^{-1}(n)).$$

It is clear enough what this should be. The space $\mathrm{Rep}_2(B)$ occurs again as a critical locus of a function on a smooth space, the function $\mathbf{tr}(T^4)$ on the space of 2×2 matrices, and so a refinement of the pullback of the microlocal function is already at hand, we can just pull back the motivic vanishing cycle of the function $\mathbf{tr}(T^4)$ along the inclusion ι, i.e. take $\int_{\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})}[-\phi_{\mathbf{tr}(T^4)}]$. In terms of the weight mw from the introduction, the general idea here is to set $\mathsf{mw}(M') = \int_x \phi_{\mathbf{tr}(W)}$, where M' is any n-dimensional B-module, and x is a closed point of $\mathrm{Rep}_n(B)$ representing it. The content of the word 'coherently' in the statement that a Jacobi algebra presentation enables us to coherently express different representation spaces as critical loci will amount to the claim that this naive pulling back actually gives a good answer, one that gives the equality (12). Let us unpick this particular case.

We follow, then, the natural suggestion for defining the left hand side of (12), that is we write

$$\mathsf{DT}[\mathrm{Rep}_1(B) \star \mathrm{Rep}_1(B)] := \int_{\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})} [-\phi_{\mathbf{tr}(T^4)}]\mathbb{L}^{-1}\,\mathbf{x}^2\,. \tag{13}$$

The $\mathbb{L}^{-1}$ term here comes from the $\mathbb{L}$ in the denominator of (11). Working out what the right hand side of (13) is will occupy the next section.

5. Verifying preservation of ring structure: an example

To start with, we should work out an embedded resolution of

$$\mathbf{tr}(T^4) : \mathrm{Mat}_{2\times 2}(\mathbb{C}) \to \mathbb{C}.$$

The function $\mathbf{tr}(T^4)$ has its worst singularity at 0, and is homogeneous, so a good start would be to blow up at the zero matrix. Write $X = \mathrm{Mat}_{2\times 2}(\mathbb{C})$ and let

$$\begin{array}{c} \tilde{X} \\ \big\downarrow{\scriptstyle h} \\ X \end{array}$$

be the blowup at the zero matrix. The strict transform of $(\mathbf{tr}(T^4))^{-1}(0)$ in $\tilde{X}$, intersected with the exceptional $\mathbb{P}^3$, is the projective surface cut out by the homogeneous equation $\mathbf{tr}(T^4)$. Call this projective variety $V(\mathbf{tr}(T^4))$.

Let

$$\begin{tikzcd} Y \arrow[d, "h_p"] \\ \mathbb{P}^3 \end{tikzcd}$$

be an embedded resolution of the singular projective variety $V(\mathbf{tr}(T^4))$. Then we have a diagram

$$\begin{tikzcd} \tilde{X}_1 \arrow[r, "h_1"] \arrow[d, "\pi_1"] & \tilde{X} \arrow[r, "h"] \arrow[d, "\pi"] & X \\ Y \arrow[r, "h_p"] & \mathbb{P}^3 \end{tikzcd}$$

with the leftmost square a pullback (in fact this is a pullback of a vector bundle, since $\tilde{X}$ is the total space of the tautological bundle for $\mathbb{P}^3$, and π is the projection). It is not hard to see that $h' := h \circ h_1$ is an embedded resolution for $\mathbf{tr}(T^4)$. It follows from the fact that $\mathbf{tr}(T^4) \circ h$ vanishes to order 4 on $\mathbb{P}^3$ that there is an equality of divisors

$$(\mathbf{tr}(T^4) \circ h')^*(0) = (h_p \circ \pi_1)^*(V(\mathbf{tr}(T^4))) + 4Y \tag{14}$$

where Y is considered as a divisor on $\tilde{X}_1$, the zero section of the vector bundle $\tilde{X}_1 \to Y$.

So we just need to work out an embedded resolution of $V(\mathbf{tr}(T^4))$. Note that $\mathrm{PSL}(2, \mathbb{C})$ acts on X by conjugation, $\mathbf{tr}(T^4)$ is invariant under this action, the action lifts to $\tilde{X}$, and $V(\mathbf{tr}(T^4))$ is also invariant under the action. There are exactly three orbits of the $\mathrm{PSL}(2, \mathbb{C})$-action in $V(\mathbf{tr}(T^4))$. Define

(1) S_1 to be the orbit consisting of matrices whose eigenvalues differ by a factor of $e^{i\pi/4}$,
(2) S_2 to be the orbit consisting of matrices whose eigenvalues differ by a factor of $e^{3i\pi/4}$,
(3) S_3 to be the orbit of nilpotent matrices.

Proposition 5.1. *In the ring* $\mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})$ *there are equalities*

$$[S_1] = [S_2] = [\mathbb{P}^1 \times \mathbb{C}], \tag{15}$$

where all of these motives carry the trivial $\hat{\mu}$*-action.*

Proof. Fix two nonzero numbers a and b differing by a factor of $e^{i\pi/4}$. Then to pick a matrix with these two numbers as eigenvalues is the same as to pick two distinct vectors (up to rescaling) to be the respective eigenvalues. So pick the eigenvector for a first, this gives us a $\mathbb{P}^1$ of choices, then pick the eigenvector for b, giving a $\mathbb{C}$ of choices, one can in fact see that S_1 is a line bundle over $\mathbb{P}^1$. The motive of any line bundle is the same as the motive of the trivial line bundle – any ordered open cover underlying a trivialization induces a stratification on which each restriction of the line bundle is trivial. □

Proposition 5.2. *There is an isomorphism* $S_3 \cong \mathbb{P}^1$.

Proof. Give $\mathbb{P}^3$ coordinates $(X : Y : Z : W)$ by writing matrices as

$$\begin{pmatrix} X & Z \\ W & Y \end{pmatrix}.$$

Then the nilpotent matrices are precisely those satisfying **trace** $=$ **det** $= 0$. So they are the ones satisfying

$$X = -Y,$$
$$XY = WZ,$$

giving a $\mathbb{P}^1$ inside $\mathbb{P}^3$. □

The singular locus of $V(\mathbf{tr}(T^4))$ is precisely S_3. Since S_3 is a $\mathrm{PSL}(2,\mathbb{C})$-orbit, the singularity is the same all along this $\mathbb{P}^1$. We restrict to an affine patch U by setting $W \neq 0$. On this patch we use the coordinates

$$(x,y,z) \mapsto \begin{pmatrix} x & z \\ 1 & y \end{pmatrix}.$$

There is an isomorphism $U \cap S_3 \cong \mathbb{C}$, and $U \cap S_3$ can be parameterised as follows

$$\mathbb{C} \to U \cap S_3 \tag{16}$$

$$t \mapsto \begin{pmatrix} t & -t^2 \\ 1 & -t \end{pmatrix}. \tag{17}$$

We can extend this to a coordinate system (t,a,b) for U, given by

$$(t,a,b) \mapsto \begin{pmatrix} t+a & b-t^2 \\ 1 & -t \end{pmatrix}. \tag{18}$$

In these coordinates the local defining equation for $\mathbf{tr}(T^4)$ becomes

$$\mathbf{tr}(T^4) = a^4 + 4a^3t + 4a^2b + 2a^2t^2 + 4abt + 2b^2,$$

or, after rearranging,

$$\mathbf{tr}(T^4) = -a^4 + 2(at+b+a^2)^2.$$

After replacing b with $b' = b + at + a^2$ we get that the local defining equation for $\mathbf{tr}(T^4)$ is

$$\mathbf{tr}(T^4) = -a^4 + 2b'^2,$$

and so we have a $\mathbb{P}^1$ of A_3 singularities along S_3. If we blow up S_3 we replace this with an exceptional divisor (the projectivization of the normal bundle of S_3), on which there is another $\mathbb{P}^1$ of singularities, this time of type D_4. Blowing up this new $\mathbb{P}^1$ gives our embedded resolution

$$\begin{tikzcd} Y \arrow[d, "h_p"] \\ \mathbb{P}^3. \end{tikzcd}$$

Let J be the set of divisors in $(\mathbf{tr}(T^4) \circ h')^{-1}(0)$. We wish to calculate the absolute equivariant motive

$$\int_{\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})} [\psi_{\mathbf{tr}(T^4)}] = \sum_{\emptyset \neq I \subset J} (1-\mathbb{L})^{|I|-1} \int_{h'^{-1}(\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C}))} [\tilde{D}_I]. \tag{19}$$

We abuse notation a little, and leave out the maps $\tilde{D}_I \to \tilde{X}_1$, since we are only interested in the absolute motive anyway. Consider the decomposition

$$\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C}) = \{0\} \amalg H,$$

where $\{0\}$ is the zero matrix, and $H \cong \mathbb{C}^*$ is the complement. This decomposition induces a decomposition of the sum (19): if we define

$$M_{\mathrm{nt}} = \int_H [\psi_{\mathbf{tr}(T^4)}] = \sum_{\emptyset \neq I \subset J} (1-\mathbb{L})^{|I|-1} \int_{h'^{-1}(H)} [\tilde{D}_I] \tag{20}$$

$$M_t = \int_{\{0\}} [\psi_{\mathbf{tr}(T^4)}] = \sum_{\emptyset \neq I \subset J} (1-\mathbb{L})^{|I|-1} \int_{h'^{-1}(\{0\})} [\tilde{D}_I], \tag{21}$$

then

$$\int_{\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})} [\psi_{\mathbf{tr}(T^4)}] = M_{\mathrm{nt}} + M_t.$$

Since H is just the complement to the zero section in the fibre $\pi^{-1}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$, and $V(\mathbf{tr}(T^4))$ has an A_3 singularity at this matrix, i.e. the singularity defined by the singular curve $x^4 + y^2$, the following proposition follows from equation (14).

Proposition 5.3. *There are equalities of absolute motives*

$$M_{\mathrm{nt}} = (\mathbb{L} - 1)\,\mathsf{MF}(x^4 + y^2) \tag{22}$$

$$= (\mathbb{L} - 1)([C_2] - 2\mathbb{L}) \tag{23}$$

where C_2 is a torus with the representation $\alpha + \alpha^3$ on its middle cohomology.

Proof. Only the second equality needs proving. This is implied by Proposition B.1. □

Proposition 5.4. *There is an equality of absolute motives*

$$M_t = (1 - \mathbb{L})\,\mathsf{MF}(x^4 + y^2) + \mathbb{L}\,\mathsf{MF}(x^4 + y^4).$$

Proof. One of the terms in the sum (21) comes from setting $I = \{Y_s\}$, the proper transform of the copy of $\mathbb{P}^3 \subset \tilde{X}$ we obtained by blowing up at the zero matrix. Now fh' vanishes to order 4 on Y_s, and so $\tilde{D}_I$ is a 4-sheeted étale cover over the complement of $V(\mathbf{tr}(T^4))$ in $\mathbb{P}^3$. It follows from Proposition B.2 that

$$\int_{h'^{-1}(\{0\})} [\tilde{D}_{\{Y_s\}}] = [\tilde{D}_{\{Y_s\}}] = \mathbb{L}\,\mathsf{MF}(x^4 + y^4) + (\mathbb{L} - 1)\mathbb{L}\,\mathsf{MF}(x^4 + y^2) + 2\mathbb{L}(\mathbb{L}^2 - 1).$$

The subvariety of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ cut out by $\mathbf{tr}(T^4)$ has two components, the cones over the divisors $S_1 \cup S_3$ and $S_2 \cup S_3$, and we denote the strict transform of these divisors in the embedded resolution $\tilde{X}_1$ by F_1 and F_2, respectively. These divisors occur with multiplicity 1. Since we only blow up along S_3, there is an isomorphism $\tilde{D}_{\{F_i, Y_s\}} \cong S_i$ for $i = 1, 2$. So these two subsets of J each contribute

$$(1 - \mathbb{L}) \int_{h'^{-1}(\{0\})} [D_{\{Y_s, F_i\}}] = (1 - \mathbb{L})[\mathbb{P}^1 \times \mathbb{C}]$$

to M_t, by Proposition 5.1. All the other contributions to (21) come from the modifications made to the singular locus of $V(\mathbf{tr}(T^4))$, i.e. from subsets $I \subset J$ that contain Y and at least one divisor occurring as the cone over an exceptional divisor of

$$\begin{tikzcd} Y \arrow[d, "h_P"] \\ \mathbb{P}^3. \end{tikzcd}$$

At the first blowup, along the $\mathbb{P}^1$ of A_3-singularities S_3, we introduce a $\mathbb{P}^1$-bundle, along with a $\mathbb{P}^1$ of new singularities. Since we are working in the motivic ring, we can assume that the bundle in question is trivial. The same is true for the second blowup. The result is the equation

$$\sum_{\emptyset \neq J \subset D | J \not\subseteq \{Y_s, F_1, F_2\}} (1-\mathbb{L})^{|J|-1} \int_{h'^{-1}(\{0\})} [\tilde{D_J}] = (1-\mathbb{L})[\mathbb{P}^1]\,\mathsf{MF}(x^2+y^4).$$

Putting all this together gives the result. □

It turns out, then, that we have exactly what we want:

Proposition 5.5. *There is an equality of $\hat{\mu}$-equivariant motives*

$$\mathbb{L}^{-1} \int_{\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})} [-\phi_{\mathbf{tr}(T^4)}] = 1 - \mathsf{MF}(x^4+y^4), \tag{24}$$

and so there is an equality in $\widetilde{\mathsf{Mot}}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})[[\mathbf{x}]]$

$$\mathsf{DT}[\mathrm{Rep}_1(B) \star \mathrm{Rep}_1(B)]\,\mathbf{x}^2 = \mathsf{DT}[\mathrm{Rep}_1(B)] \cdot \mathsf{DT}[\mathrm{Rep}_1(B)]\,\mathbf{x}^2, \tag{25}$$

where these 'DT counts' are as defined in (10) and (13).

Remark 5.1. We have shown this equality directly, but also it turns out to be a comparatively simple application of the Kontsevich–Soibelman integral identity (see Section 4.4 of [23] for a discussion of this identity, and see [31] for a proof). This motivic identity implies that this motivic refinement of the Donaldson–Thomas count preserves ring structure for more general moduli spaces of objects in the Abelian category of B-modules, and more general Jacobi algebras.

6. Towards motivic Donaldson–Thomas counts

The above calculations show that a 'naive' motivic refinement of the Donaldson–Thomas count preserves ring structure, at least in our basic example. It will turn out that the key ingredient for achieving this was the *extra* data provided by a realisation of our algebra as a superpotential algebra, which in turn enables us to realise the representation spaces of finite dimensional modules for our Jacobi algebra B as critical loci in such a way that the integration map defined via the associated motivic weight $-\phi_{\mathbf{tr}(W)}$ preserves the ring structure. The question is: can we do without this extra data?

Question 6.1. ***If we are handed a 'Calabi-Yau 3-dimensional category', whatever that may turn out to be, can we construct a motivic integration map from the Hall algebra of stack functions, preserving the product?***

There is a notion of quasi-equivalence of Calabi-Yau categories (see, for example, [19]), that in particular induces quasi-isomorphisms of homomorphism spaces and quasi-isomorphisms of endomorphism spaces as cyclic A_∞-algebras. Again, we needn't worry at the moment about what that means precisely, but already an implication for a satisfactory theory of motivic Donaldson–Thomas counts follows from the fact that quasi-equivalences of Calabi-Yau categories induce isomorphisms of derived categories:

Requirement 6.2. ***The motivic Donaldson–Thomas count associated to a stack function should be invariant under pullback along quasi-equivalences of Calabi-Yau 3-dimensional categories.***

Consider again our archetypal Donaldson–Thomas setup in Section (2): producing numbers 'counting' sheaves $\mathcal{F}$ in fine moduli spaces $\mathcal{M}$. Recall that if $\mathcal{F}$ is a coherent sheaf on our Calabi-Yau 3-fold X, the constructible function $\nu_{\mathcal{M}}(\mathcal{F})$ depends solely on the scheme structure of the moduli space $\mathcal{M}$, where we use the common abuse of notation whereby $\mathcal{F}$ also denotes the point of $\mathcal{M}$ representing it. The fact that the scheme structure of $\mathcal{M}$ tells us what kind of contribution $\mathcal{F}$ should make to the Donaldson–Thomas count is explained by the fact that we assumed that $\mathcal{M}$ is a fine moduli space, and so carries information about infinitesimal deformations of $\mathcal{F}$.

The idea is that the contribution of an object $\mathcal{F}$ need not be calculated from the local structure around $\mathcal{F}$ in some moduli scheme $\mathcal{M}$. In the example above we used a particular way of realising our moduli spaces as critical loci in order to give a motivic refinement of the Donaldson–Thomas count, but of course this application of extra data means that we have not provided an affirmative answer to Question 6.1.

The contribution of an object $\mathcal{F}$, sitting inside a fine moduli space $\mathcal{M}$, to the ordinary Donaldson–Thomas count is a function of the Euler characteristic of the Milnor fibre of a function

$$f : \mathbb{C}^t \to \mathbb{C},$$

for some t, satisfying the condition that $\mathrm{crit}(f)$ looks (locally) like a formal neighborhood of the point x representing $\mathcal{F}$ in $\mathcal{M}$. The crucial observation is that some version of a critical locus description around $\mathcal{F}$ can be read off straight from the formal deformation theory of that object, which can be expressed purely in terms of category theory. So we try to refine the Donaldson–Thomas contribution to a motive by building such an f directly from the category, and as a preliminary step we should find somewhere for f to *live*, e.g. as a function on a vector bundle on a stack of objects. It turns out that a reasonable candidate for f, at x, is a function defined on $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$. Now our aim was to write down motivic Donaldson–Thomas counts for arbitrary families, at which point we are confronted by the fact that the dimension of $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$ is liable to jump as we vary $\mathcal{F}$, so we cannot hope that our f will be a function on a vector bundle. The appropriate sheaf (which we will call $\mathcal{EXT}^1$ here) will, rather, be a *constructible* vector bundle.

7. Some remarks on constructible vector bundles

Let X be a locally Noetherian scheme. By a constructible decomposition of X we will hereafter mean a decomposition of X into locally closed subschemes such that there is a cover of X by open affine schemes U_i for which the restriction of the decomposition of X to each U_i is a finite constructible decomposition. A constructible vector bundle V on X is given by a constructible decomposition of X, and a vector bundle on each component of the decomposition. There is, in principle, no reason why one must impose any kind of finite-dimensionality of V in the definition, but we will see shortly that doing so makes the category of such constructible vector bundles much better behaved. We identify a constructible vector bundle with the one obtained, by restrictions, on a subordinate constructible decomposition. A morphism between two constructible vector bundles V_1 and V_2 is given by taking a constructible decomposition subordinate to the two decompositions defining V_1 and V_2, and giving a morphism, for each X_i in the decomposition, from $V_1|_{X_i}$ to $V_2|_{X_i}$. We identify a morphism f with the morphism obtained by restricting f to a constructible decomposition subordinate to the one defined by f. Every constructible vector bundle $\mathcal{V}$ on a scheme X defines a constructible function $\dim_{\mathcal{V}} : x \mapsto \dim(\mathcal{V}_x)$. We will only work with locally finite constructible vector bundles $\mathcal{V}$, meaning that X can be covered by affine open subschemes on which this function is bounded. Constructible vector bundles are to a large extent all trivial:

Proposition 7.1. *Let $\mathcal{V}$ be a locally finite dimensional constructible vector bundle. Then*

$$\mathcal{V} \cong \coprod_{n \in \mathbb{N}} \mathcal{O}^{\oplus n}_{\dim_{\mathcal{V}}^{-1}(n)}.$$

Proposition 7.2. *Let X be a locally Noetherian scheme, and let $\mathcal{V}_{\text{fin}}$ be the full subcategory of the category of constructible vector bundles on X consisting of locally finite-dimensional vector bundles. Then $\mathcal{V}_{\text{fin}}$ is a semisimple Abelian category.*

We define the (ordinary) category of constructible differential graded vector bundles on X as the category with objects given by pairs of a constructible decomposition of X, and on each subscheme of the decomposition a differential graded vector bundle. Morphisms are given by morphisms of such objects that preserve degree and commute with the differential, and we make the obvious identifications of objects and morphisms under subordinate decompositions.

Corollary 7.3. *(Formality) Let $V^\bullet$ be a constructible differential graded vector bundle on a locally Noetherian scheme X such that each fibre of $V^\bullet$ is finite-dimensional in each degree, and on each of the subschemes X_i of X defined by the constructible decomposition associated to $V^\bullet$, the homology $\mathrm{H}^i(V^\bullet)$, considered as a constructible vector bundle on X_i, is nonzero for only finitely many i. Then there is a quasi-isomorphism from a constructible differential graded vector bundle with zero differential to $V^\bullet$.*

Proof. We can define the ith homology of $V^\bullet$, in the category of constructible vector bundles, since the category $\mathcal{V}_{\text{fin}}$ of Proposition 7.2 is Abelian. Then the formality follows from the fact that $\mathcal{V}_{\text{fin}}$ is semisimple, and our local finiteness assumption on the homology. □

Remark 7.1. Kernels in the category $\mathcal{V}_{\text{fin}}$ above are maybe a little surprising. For instance, the homomorphism of $\mathbb{C}[x]$-modules

$$\mathbb{C}[x] \xrightarrow{\cdot x} \mathbb{C}[x]$$

is of course an injection of coherent sheaves on the scheme $\mathbb{C}$. Considered as a morphism of constructible vector bundles, however, one readily verifies that the kernel consists of a rank 1 vector bundle over the origin. The same example shows that the homology of a differential graded vector bundle, considered as a constructible differential graded vector bundle, can be very different from the homology of the vector bundle considered as a complex of coherent sheaves.

8. Formal deformation theory

Since we are working in the ring of motives, we may treat the constructible vector bundle $\mathcal{EXT}^1$ (once it is properly defined) as though it were a vector bundle. In the original setup, in which we were working out Donaldson–Thomas counts associated to fine moduli spaces, this constructible vector bundle played an important role: it is naturally identified with the Zariski tangent space of our scheme $\mathcal{M}$ (see [15] for example).

Given an object $\mathcal{F}$ in a Calabi-Yau 3-dimensional category $\mathcal{C}$, we obtain an A_∞-algebra $A = \mathrm{Hom}^\bullet(\mathcal{F},\mathcal{F})$. Such an algebra is like a differential graded algebra, in that it has two operations $m_1 : A \to A[1]$ and $m_2 : A \otimes A \to A$, but it also has countably many higher operations

$$m_n : A^{\otimes n} \to A[2-n]$$

which are required to satisfy some compatibility conditions (see Keller's introduction [20] or Lefèvre-Hasegawa's thesis [24] for a longer exposition). Given such a set of m_n we define a set of b_n making the following diagram commute

$$\begin{tikzcd} A^{\otimes n} \arrow[r, "m_n"] \arrow[d, "S^{\otimes n}"] & A[2-n] \arrow[d, "S^n"] \\ A[1]^{\otimes n} \arrow[r, "b_n"] & A[2], \end{tikzcd}$$

where S is the degree -1 map sending $a \in A$ to a in $A[1]$. Clearly these b_n contain the same information as the m_n, so we may just as well describe an A_∞-algebra using them. Here begins the constant tension in this subject between the m_n, which naturally extend our notions of ordinary algebras and differential graded algebras, but have increasingly awkward sign rules, and the b_n, which do not.

We can describe the formal deformation theory of $\mathcal{F}$, using the functor

$$\begin{aligned} Def_{\mathcal{F}} : \text{Artinian nonunital algebras} \to & Sets \\ \mathbf{m} \mapsto & \{\gamma \in \mathbf{m} \otimes \mathrm{Hom}^1(\mathcal{F},\mathcal{F}) | MC(\gamma) = 0\} \end{aligned}$$

where $MC : \mathrm{Hom}^1(\mathcal{F},\mathcal{F}) \to \mathrm{Hom}^2(\mathcal{F},\mathcal{F})$ is given by the formal sum of the degree n functions

$$MC_n(a) = b_n(\gamma, \dots, \gamma),$$

and b_n are the higher multiplications of $\mathrm{Hom}^1(\mathcal{F},\mathcal{F})$. We have shifted from the usual maps $m_n : A^{\otimes n} \to A[2-n]$ to maps $b_n : A[1]^{\otimes n} \to A[2]$ just to make the signs trivial here.

The fact that $\mathcal{C}$ is supposed to be a Calabi-Yau 3-dimensional category over some ground field k enables us to make some extra assumptions on our A_∞-Yoneda algebra $\mathrm{End}^\bullet(\mathcal{F})$, namely we assume that it has a cyclic structure. What exactly this means is spelt out in detail elsewhere, for example in Kajiura's paper [19], but for the present purposes it is sufficient to note that this extra structure implies that we have a nondegenerate antisymmetric pairing

$$\langle \bullet, \bullet \rangle : \mathrm{Hom}^1 \otimes \mathrm{Hom}^2 \to k$$

and that if we define

$$W_n(x) = \frac{1}{n}\langle b_{n-1}(x, \dots, x), x\rangle$$

and let W be the formal sum of these degree n functions, we have that $dW = MC$. This makes sense once one views $\mathrm{End}^2(\mathcal{F})$ as the vector dual of $\mathrm{End}^1(\mathcal{F})$ via the pairing $\langle \bullet, \bullet \rangle$ and identifies each fibre of the cotangent space of $\mathrm{End}^1(\mathcal{F})$ with the vector dual of $\mathrm{End}^1(\mathcal{F})$ in the natural way.

It follows, then, that we are given a formal critical locus description for $\mathcal{F}$ without any reference to a moduli space, directly from the structure of a 3-dimensional Calabi-Yau category.

The W we have here, though, is in some sense not yet intrinsic to the category – it changes as we vary the representative we take of the quasi-equivalence class of the category $\mathcal{C}$, varying by quasi-isomorphisms the representative we take of the A_∞-algebra $\mathrm{End}^\bullet(\mathcal{F})$. Help is at hand though: it turns out (see Theorem 5 and Corollary 2 of [23], as well as [22], [19]) that we can always find a (noncanonical!) minimal cyclic model for our category, at least around a neighbourhood of our object $\mathcal{F}$, after constructible decomposition of the space of objects in the category. So there is a quasi-isomorphism (at least after we replace $\mathcal{C}$ by the full subcategory whose objects are a constructible neighborhood of $\mathcal{F}$):

$$\mathcal{C} \xrightarrow{\sim} \mathcal{C}' \tag{26}$$

to a Calabi-Yau A_∞-category $\mathcal{C}'$ where the morphism spaces have zero differential, and so we have the identification $\mathrm{End}^1_{\mathcal{C}'}(\mathcal{F}) \cong \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{F})$. This is good, since the graded vector space of Exts between two objects, as opposed to the differential graded vector space of Homs, is a true invariant under quasi-isomorphisms of A_∞-categories. In addition, since we have taken this minimal model in the category of *cyclic* A_∞-categories, this new $\mathrm{End}^1_{\mathcal{C}'}(\mathcal{F})$ comes also with its potential function, denoted $W_{\min}$. Finally, the really good news is that this $W_{\min}$ doesn't depend on the choice of minimal model (up to some changes that have no effect on motivic Milnor fibres). So $W_{\min}$, considered as a formal function on the constructible vector bundle $\mathcal{EXT}^1$, presents itself as a likely candidate for our intrinsic critical locus description of the category.

9. An example in the general framework

Let us see how some of this theory works in our specific example. First we fix some data. We will start by defining A, an A_∞-algebra. Such an algebra has an underlying graded vector space, which in our case is just going to be

$$A = \mathbb{C} \oplus \mathbb{C}[-1] \oplus \mathbb{C}[-2] \oplus \mathbb{C}[-3].$$

Such an algebra comes also with a countable collection of operations

$$m_n : A^{\otimes n} \to A[2-n],$$

for $n \geq 1$, satisfying some compatibilities (see e.g. Keller's [20, Sec.3]). For example, in the case where $m_n = 0$ for all $n \geq 3$ the algebra can be thought of (and indeed really is) just a differential graded algebra, with m_2 equal to the multiplication and m_1 giving the differential; in this case, the compatibility conditions say exactly that our algebra satisfies the conditions required of a differential graded algebra. The A we are going to consider is slightly different. We first set $m_1 = 0$, i.e. the differential is zero – this puts us in the 'minimal' situation of (26). Next, we set the thing to be unital. So there is some $1 \in A^0 = \mathbb{C}$ which functions just like the identity under m_2, and such that $m_i(\ldots, 1, \ldots) = 0$ for all $i \geq 3$. Let us extend this unit to a basis

$$\{1 \in A^0, a \in A^1, a^* \in A^2, w \in A^3\}$$

so that we have a graded basis for the whole of A. Next, set

$$\begin{aligned} m_2(a, a^*) = m_2(a^*, a) &= w \\ m_2(a, a) &= 0. \end{aligned}$$

For degree reasons, this and the unital property determine m_2 entirely. We define $m_i = 0$ unless $i \in \{2, 3\}$. We let $m_3(a, a, a,) = a^*$, and set m_3 to be zero on all other 3-tuples of basis elements.

This algebra was not plucked from nowhere: it is the A_∞ Koszul dual (as in [26]) of the Ginzburg differential graded algebra $\Gamma(Q, W)$ associated to the quiver with potential we considered in Section 4. This is a differential graded algebra with cohomology concentrated in negative degrees, with zeroeth cohomology isomorphic to our algebra B as defined in Section 4 – one should consult Ginzburg's paper [14, Sec.5] for a full definition of this algebra. So the Abelian category of B-modules sits inside the derived category of $\Gamma(Q, W)$-modules as the heart of the natural t-structure, and A is the Yoneda algebra $\mathrm{Ext}^\bullet_{\Gamma(Q,W)\text{-}\mathsf{mod}_\infty}(M, M)$ of the 1-dimensional simple module M of Section 4. Note that this algebra is very different from the Yoneda algebra $\mathrm{Ext}^\bullet_{B\text{-}\mathsf{mod}_\infty}(M, M)$, which is concentrated in infinitely many degrees.

Under Koszul duality, the B-module M gets sent to the free (right) A-module. But it is maybe worth forgetting that for now, and just taking some category of modules over A to be our Calabi-Yau category, and seeing what the programme sketched above, involving $W_{\min}$, does in this case.

As in Section 4 we will be interested in some very simple spaces of modules over A (indeed the same spaces, under Koszul duality). First we need to write down our version of the superpotential coming from the structure of our category. To this end we introduce the symmetric pairing

$$\langle \bullet, \bullet \rangle : A \otimes A \to \mathbb{C}[-3]$$

given by letting $\langle a, a^* \rangle = \langle 1, w \rangle = 1$. This gives us our W: if we let x be a coordinate on $\mathrm{Ext}^1(M, M) \cong A^1$, then

$$W = x^4.$$

(Recall that W is actually defined in terms of the b_n, maps from $A[1]^{\otimes n}$ to $A[2]$, but up to sign this makes no difference to our W.) The only modules we will be interested in are A and extensions of A by itself. Denote by N the free left A-module. We denote by N_α the cone of a morphism $\alpha : N[-1] \to N$. Such a module is really just the extension determined by $\alpha \in \mathrm{Ext}^1(N, N)$, but souped up to an object in an A_∞-category. Such an extension has, as underlying A-module, $N_1 \oplus N_2$, where we have labelled the two copies of N merely for convenience. N_α has a differential determined by α:

$$d\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = m_2\left(\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right) = \begin{pmatrix} m_2(\alpha, a_2) \\ 0 \end{pmatrix}. \tag{27}$$

By a slight abuse of notation we denote the 2 by 2 matrix appearing in (27) simply by α. By a slightly larger abuse of notation we have used the same m_i as appear in the definition of A to denote the natural extension to matrix calculus. What we are really interested in is $\mathrm{End}^\bullet(N_\alpha)$.

Proposition 9.1. *The A_∞-algebra $\mathrm{End}^\bullet(N_\alpha)$ has a model whose underlying graded vector space is*

$$\begin{aligned} H :=& \mathrm{End}^\bullet(N_1, N_1) \oplus \mathrm{End}^\bullet(N_1, N_2) \oplus \mathrm{End}^\bullet(N_2, N_1) \oplus \mathrm{End}^\bullet(N_2, N_2) \\ =& A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22} \\ =& M_{2\times 2}(A) \end{aligned} \tag{28}$$

where the subscripts do not change the mathematical object denoted by the terms they are subscripts to, and are just added for notational convenience. This algebra carries natural higher products coming from A, which we denote by $m_{2\times 2,n}$, or the shifted version by $b_{2\times 2,n}$, and twist by setting

$$b_{\alpha,i}(a_1, \ldots, a_i) = \sum_{n \geq i} b_{2\times 2,n}(\alpha, \ldots, \alpha, a_1, \alpha \ldots \alpha, a_2, \alpha, \ldots, \alpha, a_i, \alpha, \ldots, \alpha). \tag{29}$$

See [20, Sec.7] for an explanation of where this model is coming from, or Kontsevich's original paper [21]. Note that the sum in (29) is actually finite: any term in which α appears in consecutive places is automatically zero, from the definition of $b_{2\times 2,n}$. So, for example

$$b_{\alpha,1}(a) = b_{2\times 2,2}(a, \alpha) + b_{2\times 2,2}(\alpha, a) + b_{2\times 2,3}(\alpha, a, \alpha). \tag{30}$$

Let δ_N be the scheme consisting of a single closed point, which we make into a parameter space of A-modules by decreeing that the module over the point is just N. In the language of stack functions, this is just the map $\mathrm{Spec}\,\mathbb{C} \to \mathbf{Ob}(\mathcal{C})$ sending the point to N. The stack function/parameter space $\delta_N \star \delta_N$ is, as in Section 4, just $\mathrm{Ext}^1(N_2, N_1)/\mathbb{A}^1$, where the point $\alpha \in \mathrm{Ext}^1(N_2, N_1)$ parameterises the module N_α.

Definition 9.2. *We define a graded vector bundle $\mathcal{END}$ over the vector space $\mathrm{Ext}^1(N_2, N_1)$, given by the trivial bundle with fibre H as defined in (28). This differential graded vector bundle has operations*

$$m_{\mathcal{END},i} : \mathcal{END}^{\otimes i} \to \mathcal{END}$$

as defined fibrewise in (29).

While $\mathcal{END}$ is a useful object, it isn't quite right for our purposes, since it isn't minimal. In particular, if we build the function W using it, as it is, it has quadratic terms, since $m_{\mathcal{END}^1,1} \neq 0$ (as in (30)). Consider the decomposition

$$\mathrm{Ext}^1(N_2, N_1) = E_t \amalg E_{\mathrm{nt}}$$

where $E_t = 0$ and $E_{\mathrm{nt}} \cong \mathbb{C}^*$ is the complement of E_t. Consider first the part E_t. Here $\alpha = 0$, and so $\mathcal{END}^\bullet|_{E_t}$ *is* minimal, and there is nothing for us to do.

Now take the part E_{nt}. The vector bundle $\mathcal{END}^0|_{E_{\mathrm{nt}}}$ is spanned by sections

$$1_{ij} \in \mathrm{Ext}^0(N_i, N_j) \cong A_{ij},$$

where as before the subscripts are being used to distinguish the two copies of N, not to pick out degrees, and our differential acts on these as follows:

$$\begin{aligned} d(1_{11}) =& a_{21}\alpha, \\ d(1_{12}) =& -a_{11}\alpha + a_{22}\alpha, \\ d(1_{21}) =& 0, \\ d(1_{22}) =& -a_{21}\alpha, \end{aligned}$$

where α denotes a coordinate on $\mathrm{Ext}^1(N_2, N_1)$ and the vector bundle $\mathcal{END}^1|_{E_{\mathrm{nt}}}$ is spanned by sections

$$a_{ij} \in \mathrm{Ext}^1(N_i, N_j),$$

which in turn are acted on as follows

$$d(a_{11}) = 0, \tag{31}$$
$$d(a_{12}) = \alpha^2 a_{21}^*, \tag{32}$$
$$d(a_{21}) = 0, \tag{33}$$
$$d(a_{22}) = 0. \tag{34}$$

So the section a_{11} gives us an embedding of $\mathcal{EXT}^1|_{E_{\mathrm{nt}}}$ into $\mathcal{END}^1|_{E_{\mathrm{nt}}}$. In fact we can almost realise $\mathcal{EXT}^\bullet|_{E_{\mathrm{nt}}}$ as a sub A_∞-vector bundle of $\mathcal{END}|_{E_{\mathrm{nt}}}$, by writing

$$\mathcal{EXT}^\bullet|_{E_{\mathrm{nt}}} = \{1_{11} + 1_{22}, 1_{21}, a_{11} + a_{22}, a_{11}^* + a_{22}^*, w_{11} + w_{22}, w_{12}\}. \tag{35}$$

The identity (35) isn't quite right though, since this sub-bundle isn't closed under the operations $m_{\mathcal{END}^\bullet,i}$. The fix involves tweaking the inclusion $i : \mathcal{EXT}^\bullet|_{E_{\mathrm{nt}}} \to \mathcal{END}^\bullet|_{E_{\mathrm{nt}}}$ - we are working with A_∞-morphisms - with 'higher' parts that can be modified to counteract the failure of our sub-bundle to be closed under the A_∞-operations $m_{\mathcal{END},i}$; this is the process of taking a minimal model. None of this technicality matters to us at the moment, since the thing we really care about, $m_{\mathcal{EXT}^\bullet,i}$, is unchanged by these modifications, and so we can read off our function $W_{E_{\mathrm{nt}},\min}$ – it is just the function x^4 (after rescaling) on the 1-dimensional vector bundle $\mathcal{EXT}^1|_{E_{\mathrm{nt}}}$.

We are working with the idea that our motivic refinement, which we will denote "DT" for now, looks something like

$$\begin{aligned} \text{"}\mathsf{DT}\text{"} : \text{stack functions for } A\text{-}\mathsf{mod} &\to \mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C}) \\ S &\mapsto \int_S (1 - \mathsf{MF}(W_{\min})). \end{aligned} \tag{36}$$

There will in general be some twists by powers of $\mathbb{L}^{1/2}$, a formal square root of the motive of the affine line, but we have conveniently picked our example so that these powers are all trivial, in the end. Let us work out what this map does in our example. It turns out we have already done most of the work. Firstly one can easily check that

$$\text{“}\mathsf{DT}\text{”}([E_t]\mathbb{L}^{-1}) = (1 - \mathsf{MF}(\mathbf{tr}(T^4)))\mathbb{L}^{-1}$$

and so

$$\text{“}\mathsf{DT}\text{”}([E_t]\cdot\mathbb{L}^{-1}) = \mathbb{L}^{-1} - (1-\mathbb{L})\mathbb{L}^{-1}\,\mathsf{MF}(x^4+y^2) - \mathsf{MF}(x^4+y^4)$$

by Proposition 5.4. Secondly, we have that

$$\text{“}\mathsf{DT}\text{”}([E_{\mathrm{nt}}]\mathbb{L}^{-1}) = (\mathbb{L}-1)\,\mathbb{L}^{-1} - (\mathbb{L}-1)\,\mathsf{MF}(x^4)\mathbb{L}^{-1}.$$

In order for the map “DT” to preserve the ring structure, then, we need

$$(\mathsf{MF}(x^4) - \mathsf{MF}(x^4+y^2)) = 0. \tag{37}$$

While equalities in the ring of motives can perhaps be a little elusive, there are certain realisations from the ring of motives to more manageable rings that make inequalities easier to identify. For example, from the functoriality of the weight filtration of the mixed Hodge structure of a scheme X, it follows that if a finite group G acts on X we may form an equivariant version $\chi_{eq,q}$ of the Serre polynomial for X. Using Propositions 4.1 and B.1 one can show

$$\chi_{eq,q}(\mathsf{MF}(x^4) - \mathsf{MF}(x^4+y^2)) = (\alpha+\alpha^2+\alpha^3 - 2(\alpha+\alpha^2+\alpha^3)\sqrt{q} - q)$$

from which we deduce that our map “DT” does *not* preserve the ring structure, as it stands. At a first approximation, this is because we have left out powers of $\mathbb{L}^{1/2}$, a formal square root of the element $\mathbb{L} \in \mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})$. The correct integration map looks more like

$$\begin{aligned} \mathsf{DT}_{\mathbb{L}^{1/2}} : \text{stack functions for } A\text{-}\mathsf{mod} &\to \mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})[\mathbb{L}^{-1/2}] \\ S &\mapsto \int_S (1-\mathsf{MF}(W_{\min}))\,\mathbb{L}^{\sum_{i\le 1}(-1)^i \dim(\operatorname{Ext}^i(\bullet,\bullet))/2}. \end{aligned} \tag{38}$$

Over E_t this makes no difference, but over E_{nt} an extra $\mathbb{L}^{1/2}$ factor appears, as the nontrivial self-extension of N has 2-dimensional endomorphism ring, but only 1 nontrivial self-extension. Then, in order to demonstrate that $\mathsf{DT}_{\mathbb{L}^{1/2}}$ preserves the ring structure, we end up instead having to prove

$$\mathsf{MF}(x^4)\,\mathbb{L}^{1/2} = \mathsf{MF}(x^4+y^2),$$

where the right hand side contains this formal square root of $\mathbb{L}$, while the left hand side doesn't. At least without making some kind of identification of $\mathbb{L}^{1/2}$ with something truly belonging to $\mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})$, this doesn't improve the situation much (see Appendix A for more on why such a move doesn't work).

Let us compare the case where things looked better, Section 4, with what has happened here. The following basic observation makes this easier.

Proposition 9.3. *Let* $\alpha \in \operatorname{Ext}^1(M_2, M_1)$. *Define*

$$W_\alpha(a) = \sum_{n\ge 2} \frac{1}{n} W_{\alpha,n}(a),$$

a function on 2×2 *matrices with entries in* $Ext^1(M,M)$, *by*

$$W_{\alpha,n}(a) = \langle b_{\alpha,n-1}(a,\ldots,a), a\rangle.$$

Write $W := W_0$. *Then* $W_\alpha(a) = W(\alpha + a)$.

There is a smooth function $+ : \mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})\times\mathrm{Mat}_{2\times 2}(\mathbb{C}) \to \mathrm{Mat}_{2\times 2}(\mathbb{C})$ given by matrix addition, and the proposition states that $+^*(W) = W_-$, the function on $\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})\times M_{2\times 2}(\mathbb{C})$ that restricts to W_α over $\alpha \in \mathrm{Mat}_{\mathrm{sut},2\times 2}$. It follows by the properties of the transformation of the motivic vanishing cycle under pullback that $\int_{\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})\times\{0\}}[-\phi_{\mathbf{tr}(W_-)}] = \int_{\mathrm{Mat}_{\mathrm{sut},2\times 2}(\mathbb{C})}[-\phi_{\mathbf{tr}(W)}]$. So as well as integrating motivic weights across the same 1-dimensional subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ both times, we have actually been integrating against the same motivic weight $[-\phi_{\mathbf{tr}(T^4)}]$ both times as well, *almost*. The almost here comes from the fact that along E_{nt} we have modified the function W_α, breaking it into a quadratic part and a part with cubic and higher terms – this is what we do when we restrict to the minimal superpotential $W_{\min}$. What is this quadratic part? As noted in [23, Sec.6.3], to a first approximation it is just $W_{\alpha,2}$ on the constructible vector space

$$V = \mathcal{HOM}^1/\,\mathrm{Ker}(b_{\mathcal{HOM}^\bullet,1}). \tag{39}$$

On E_t this is trivial, so we concentrate on E_{nt}. Here, V is spanned by a_{12} (see (31)), and the quadratic function induced by W_- equals $\alpha^2 y^2$, where y is the coordinate on the vector space $\langle a_{12}\rangle \subset H$, as defined in (28). After rescaling, this is just the function y^2. If we had modified "DT" so that instead of integrating $(1 - \mathsf{MF}(x^4))$ along E_{nt} we integrated $(1 - \mathsf{MF}(x^4)) \cdot (1 - \mathsf{MF}(y^2)) = (1 - \mathsf{MF}(x^4 + y^2))$ we would have arrived at the right answer, by the motivic Thom–Sebastiani Theorem 3.1.

10. The role of orientation data in fixing preservation of ring structure

So let us recall the situation we have arrived at. Firstly, our goal was to associate motivic Donaldson–Thomas counts to arbitrary stack functions of a Calabi–Yau 3-dimensional category $\mathcal{C}$. In the example of the Abelian category of modules over a superpotential algebra (in our case, $\mathbb{C}[x]/\langle x^3\rangle$), we have a good idea of how to do this, that seems to work, with the product preserved on account of an application of the Kontsevich–Soibelman integral identity, followed by the motivic Thom-Sebastiani Theorem. If we just start from the data of a 3-Calabi-Yau category $\mathcal{C}$, we have some proxy for the critical locus description, the minimal superpotential $W_{\min}$ considered as a function on the constructible vector bundle $\mathcal{EXT}^1$, the problem is that we don't know how to apply the integral identity. More precisely, in the case of two stack functions from single points both parameterising the object N, we *do* have something to apply the integral identity to – the induced potential on the differential graded vector bundle $\mathcal{END}^\bullet$ over $\mathrm{Ext}^1(N,N)$, defined as in Definition 9.2 – but away from the origin, quadratic terms show up, that are removed when we only consider the minimal superpotential $W_{\min}$.

The same story occurs if we replace the two stack functions we were multiplying before, which were both ν_N, with arbitrary ν_{E_i}, for $E_1, E_2 \in \mathcal{C}$. Let us denote the version of the vector bundle V from (39) that we get after making these replacements by V_{E_1,E_2}, so V_{E_1,E_2} is a vector bundle on $\mathrm{Ext}^1(E_2,E_1)$. The key, then, is to get some control over the constructible vector bundle V_{E_1,E_2}, and its associated quadratic form, which we will denote Q_{E_1,E_2}, so that we know how to correct our map "DT" in order to get something that preserves products. It turns out that (up to a notion of equivalence that induces isomorphisms of motivic Milnor fibers in the ring

$\mathsf{Mot}^{\hat{\mu}}(\mathrm{Ext}^1(E_2,E_1)))$ the pair of the vector bundle $(V_{E_1,E_2},Q_{E_1,E_2})$ is intrinsic to the category $\mathcal{C}$, i.e. if we had picked a different minimal model for the category consisting just of the two objects E_1,E_2, and so obtained a new pair of a vector bundle with nondegenerate quadratic form, $(V'_{E_1,E_2},Q'_{E_1,E_2})$, the modification to the motivic Milnor fibre obtained by replacing the motivic weight

$$(1-\mathsf{MF}(W_{\min}))\,\mathbb{L}^{\sum_{i\leq 1}(-1)^i\dim(\mathrm{Ext}^i(\bullet,\bullet))/2}$$

by

$$(1-\mathsf{MF}(W_{\min}))\,\mathbb{L}^{\sum_{i\leq 1}(-1)^i\dim(\mathrm{Ext}^i(\bullet,\bullet))/2}(1-\mathsf{MF}(Q'_{E_1,E_2}))\,\mathbb{L}^{-\dim(V'_{E_1,E_2})/2}$$

would be the equal to

$$(1-\mathsf{MF}(W_{\min}))\,\mathbb{L}^{\sum_{i\leq 1}(-1)^i\dim(\mathrm{Ext}^i(\bullet,\bullet))/2}(1-\mathsf{MF}(Q_{E_1,E_2}))\,\mathbb{L}^{-\dim(V_{E_1,E_2})/2}.$$

This shouldn't come as a great shock: the failure of our naive "DT" map to preserve the product is again intrinsic to $\mathcal{C}$, by construction. So the dream is not dead at this point: if we can come up with a way to coherently counteract the error term introduced by ignoring the contribution from $(V_{E_1,E_2},Q_{E_1,E_2})$ we will have come up with a fix that is invariant under quasi-equivalences of Calabi-Yau categories.

This then, defines the role of orientation data in the theory of motivic Donaldson–Thomas theory:

Condition 10.1. Orientation data provides a way of replacing $(\mathcal{EXT}^1,W_{\min})$ with a pair $(\mathcal{EXT}^1\oplus\mathcal{V},W_{\min}\oplus Q)$ in such a way that the map Φ defined by integrating with respect to the weight which, over an element $M\in\mathcal{C}$ is $(1-\mathsf{MF}(W_{\min}\oplus Q))\mathbb{L}^{-\dim(V)/2+\sum_{i\leq 1}(-1)^i\dim(\mathrm{Ext}^i(M,M))/2}$ provides an integration map preserving associative products.

Coming back to, and generalising, our main example, the following theorem is proved in [10].

Theorem 10.1. *Let B' be a Jacobi algebra defined by a quiver with potential (Q,W). Then there are $2^{|Q_0|}$ nonisomorphic choices of orientation data on the Abelian category B'-$\mathsf{mod}_{\mathrm{nilp}}$ of nilpotent finite dimensional B'-modules, where $|Q_0|$ is the number of vertices of Q.*

Appendix A. Why setting $\mathbb{L}^{1/2}=(1-\mathsf{MF}(x^2))$ isn't enough to make $\mathsf{DT}_{\mathbb{L}^{1/2}}$ an algebra homomorphism

Let us continue to assume that $k=\mathbb{C}$. There is a final move one could make, in order to try to tweak the map $\mathsf{DT}_{\mathbb{L}^{1/2}}$ of (38) to produce a map preserving the product, without considering the extra structure of orientation data. Recall that, when we modify with the appropriate $\mathbb{L}^{1/2}$ powers in $\mathsf{DT}_{\mathbb{L}^{1/2}}$, we should be integrating across E_{nt} with weight $\mathbb{L}^{1/2}(1-\mathsf{MF}(x^4))$, rather than the weight $(1-\mathsf{MF}(x^4))$. Furthermore, as long as the ground field k contains a square root for -1, we already have a square root for $\mathbb{L}$ in the ring $\mathsf{Mot}^{\hat{\mu}}(\mathrm{Spec}(k))$ given by $1-\mathsf{MF}(x^2)$ (this is a neat exercise in the use of the motivic Thom-Sebastiani theorem, using the fact that x^2+y^2 can be rewritten as $x'y'$ for new variables x' and y', and the explicit formula for the motivic nearby cycle). So we may view the target ring of (38) as a rather unnatural place to work, and instead push forward along the natural ring homomorphism $\pi:\mathsf{Mot}^{\hat{\mu}}(\mathrm{Spec}(\mathbb{C}))[\mathbb{L}^{-1/2}]\xrightarrow{\mathbb{L}^{1/2}\mapsto(1-\mathsf{MF}(x^2))}$

$\mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})[\mathbb{L}^{-1}]$. In this case, after we remember to include the $\mathbb{L}^{1/2}$ factor in the motivic weight for the nontrivial selfextension of N (as defined in Section 9), we have (in the image of π) that its motivic weight was chosen to be $(1-\mathsf{MF}(x^4+y^2))$, where we use the motivic Thom Sebastiani theorem here, and the map $\pi\circ\mathsf{DT}_{\mathbb{L}^{1/2}}$ *does* preserve the Hall algebra product in the special example being considered, i.e.

$$\pi\circ\mathsf{DT}_{\mathbb{L}^{1/2}}(\delta_N\star\delta_N)=\pi\circ\mathsf{DT}_{\mathbb{L}^{1/2}}(\delta_N)^2.$$

This is a crucial point for this paper. We are supposed to be motivating the introduction of orientation data, with our example showing how the integration map $\mathsf{DT}_{\mathbb{L}^{1/2}}$ fails to preserve the product if we ignore it, but on the other hand, it seems it should be easier, and perhaps more natural, to take the lesson from the example to be simply that we should instead direct our efforts towards proving the claim that $\pi\circ\mathsf{DT}_{\mathbb{L}^{1/2}}$ is a ring homomorphism. There are two reasons to reject this approach. The first, presented in the following theorem, is that the claim is false. The second, discussed in Remark A.1, is that working with the integration map obtained by composing with π, even once one uses orientation data modifications in order to be able to prove that this map an algebra homomorphism, yields a substantially weaker theorem – see Remark A.1.

Theorem A.1. *There exists a cyclic three dimensional Calabi-Yau category $\mathcal{C}$, such that the map*

$$\pi\circ\mathsf{DT}_{\mathbb{L}^{1/2}}:\mathsf{st}(\mathbf{Ob}(\mathcal{C}))\to\widetilde{\mathsf{Mot}}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})[[\mathbf{x}]]$$

obtained by integrating with respect to the motivic weight $(1-\mathsf{MF}(W_{\min}))\,\mathbb{L}^{\sum_{i\leq 1}(-1)^i\dim(\operatorname{Ext}^i(\bullet,\bullet))/2}$ *and composing with π is not an algebra homomorphism.*

The idea is as follows. Instead of letting $\mathcal{C}$ be the category B-mod of finitely generated B-modules, we let $\mathcal{C}$ be a family, over the base $\mathbb{C}^*$, of copies of B-mod, except that over the point $z\in\mathbb{C}^*$ we scale the Calabi-Yau pairing $\langle\bullet,\bullet\rangle$ by z. To be more precise, the objects of $\mathcal{C}$ are injective morphisms of sets $\tau:S\to\mathbb{C}^*$, where S is a finite set of finite-dimensional B-modules, and $\operatorname{Hom}_{\mathcal{C}}(\tau_1,\tau_2):=\bigoplus_{\tau_1(\eta_1)=\tau_2(\eta_2),\eta_i\in S_i}\operatorname{Hom}_{\mathsf{mod}\text{-}B}(\eta_1,\eta_2)$. Equivalently, we may define $\mathcal{C}$ in the same way, but instead of considering S to be a finite set of finite-dimensional B-modules, we take it to be a finite set of perfect differential graded modules over the A_∞-algebra A used above. What this word 'perfect' means here needn't concern us, it is sufficient to mention that N and self-extensions of N are perfect. Let N_α be the nontrivial self-extension of N. Then there is a family of objects $\mathcal{X}_{N_\alpha}$ of $\mathcal{C}$ lying over $\mathbb{C}^*$, with the object over z defined by the map of sets $\tau:\{N_\alpha\}\to\{z\}\subset\mathbb{C}^*$. There is a natural construction of orientation data for the category $\mathcal{C}$, and for the family $\mathcal{X}_{N_\alpha}$ it is given in terms of motivic vanishing cycles by considering a trivial 1-dimensional vector bundle V on $\mathbb{C}^*$, with coordinate x, and multiplying the motivic weight by $\mathbb{L}^{-1/2}(1-\mathsf{MF}(zx^2))$, where z is the coordinate on the base $\mathbb{C}^*$. Now the unmodified integration map $\mathsf{DT}_{\mathbb{L}^{1/2}}$ has a $\mathbb{L}^{1/2}$ factor in the motivic weight above a point of $\mathcal{X}_{N_\alpha}$, whereas the modified integration map, taking account of orientation data, replaces this with (minus) the motivic vanishing cycle of zx^2.

Let $\iota_z:\operatorname{Spec}(\mathbb{C})\to\mathbb{C}^*$ be the inclusion of a point. Projecting the integration map along π, and then considering the restriction to the fibre ι_z, there is no change in the motivic weight contribution of the orientation data, i.e. after fixing z we have $\iota_z^*(1-\mathsf{MF}(Q_{\mathsf{OD}}))=(1-\mathsf{MF}(zx^2))=$

$\pi(\mathbb{L}^{1/2})$, and so there is a fibrewise equality $\iota_z^*((1-\mathsf{MF}(W_{\min}))(\pi(\mathbb{L}^{1/2}))) = \iota_z^*((1-\mathsf{MF}(W_{\min}))(1-\mathsf{MF}(Q_{\mathsf{DD}})))$. But integrating across the entire family, varying z, the motive *does* change - this should come as no surprise, since the motivic vanishing cycle of zx^2 on $\mathbb{C}^*\times\mathbb{C}$ is zero, and not $(\mathbb{L}-1)(1-\mathsf{MF}(x^2))$. This is easy enough to see: the nearby fibre over a point of $\mathbb{C}^*\times\{0\}$, the critical locus of zx^2, is just two points, and going around the torus swaps these two points, so that the integrated nearby fibre is just a copy of $\mathbb{C}^*$, which is the same as the zero fibre (we use here the extra relation on $\mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})$, which specifies that any μ_n-action on an affine space can be taken to be trivial). This is enough to suggest that there is at least a difference between the putative integration map $\mathsf{DT}_{\mathbb{L}^{1/2}}$ and the more advanced version, incorporating orientation data. It then becomes reasonable to suspect that in this case the map $\mathsf{DT}_{\mathbb{L}^{1/2}}$ may prove to be defective, and the following (sketch) proof demonstrates this in a case containing the family $\mathcal{X}_{N_\alpha}$.

Sketch proof of Theorem A.1. Let $\mathcal{C}$ be as above. Let $\mathcal{X}_N$ be the family of objects of $\mathcal{C}$ over $\mathbb{C}^*$, with the object over z the map of sets $\{N\}\to\{z\}\subset\mathbb{C}^*$. The minimal potential for the object lying over the point z is just zx^4, where x is the coordinate on $\operatorname{Ext}^1_{\mathsf{mod}\text{-}A}(N,N)$. It follows that $\mathsf{DT}_{\mathbb{L}^{1/2}}(\mathcal{X}_N)$ is the motivic vanishing cycle of the function zx^4 on $\mathbb{C}^*\times\mathbb{C}$. The nearby fibre is just a torus, since above any point of $\mathbb{C}^*$ it is 4 points, and the monodromy action cyclically permutes these points. So $\mathsf{DT}_{\mathbb{L}^{1/2}}(\mathcal{X}_N)^2=(0\cdot\mathbf{x})^2=0$. The theorem will then follow from the observation that $\pi\circ\mathsf{DT}_{\mathbb{L}^{1/2}}(\mathcal{X}_N\star\mathcal{X}_N)\neq 0$.

The family $\mathcal{X}_N\star\mathcal{X}_N$, as a family of A modules, can be broken up, constructibly, into three components

$$\mathcal{X}_N\star\mathcal{X}_N=\mathcal{Y}\amalg\mathcal{X}_{E_t}\amalg\mathcal{X}_{E_{nt}}. \tag{40}$$

The family $\mathcal{Y}$ is parameterised by the scheme $(\mathbb{C}^*)^2_{z\neq w}$, the space of pairs of disjoint ordered points (z,w) of $\mathbb{C}^*$. Each point represents a module $N\oplus N$, and the minimal potential is a function on the 2-dimensional vector bundle $\operatorname{Ext}^1_{\mathsf{mod}\text{-}A}(N)\oplus\operatorname{Ext}^1_{\mathsf{mod}\text{-}A}(N)$ with coordinates x and y, with $W_{\min}=zx^4+wy^4$. We denote by $\overline{W}_{\min}$ the natural extension of this function on the trivial rank 2 vector bundle over $(\mathbb{C}^*)^2$. Then by the motivic Thom Sebastiani theorem, and the fact that $\int_{\mathbb{C}^*\times\mathbb{C}}\phi_{xy^4}=0$, we deduce that $\int_{(\mathbb{C}^*)^2_{z\neq w}}\phi_{W_{\min}}=\int_{(\mathbb{C}^*)^2}\phi_{\overline{W}_{\min}}-\int_{(\mathbb{C}^*)^2_{z=w}}\phi_{\overline{W}_{\min}}=-\int_{\mathbb{C}^*\times\mathbb{C}^2}\phi_{z(x^4+y^4)}$.[4]

The second factor in (40) should be thought of as a copy of E_t over each point of $\mathbb{C}^*$. That is, up to division by $\mathbb{L}$, it is a family parameterised by the scheme $\mathbb{C}^*$, with the fibre over z equal to the map of sets sending $N\oplus N$ to z. Precisely,

$$\mathsf{DT}_{\mathbb{L}^{1/2}}(\mathcal{X}_{E_t})=\mathbb{L}^{-1}\int_{\mathbb{C}^*\times\begin{pmatrix}0&0\\0&0\end{pmatrix}}-\phi_{z\,\mathbf{tr}(T^4)}\cdot\mathbf{x}^2\,.$$

[4]In general one has to be a bit careful with equalities of the form of the first equality here, since in general one shouldn't expect $\int_X\phi_f=\int_U\phi_{f|_U}+\int_V\phi_{f|_V}$

An embedded resolution for the function $z\,\mathbf{tr}(T^4)$ is obtained by taking the fibre product of our old embedded resolution for $\mathbf{tr}(T^4)$ with the extra factor $\mathbb{C}^*$. We deduce from formula (6)

$$\mathsf{DT}_{\mathbb{L}^{1/2}}(\mathcal{X}_{E_t}) = \mathbb{L}^{-1}((1-\mathbb{L})\int -\phi_{z(x^4+y^2)} + \mathbb{L}\int -\phi_{z(x^4+y^4)}) \cdot \mathbf{x}^2\,.$$

Finally,

$$\mathsf{DT}_{\mathbb{L}^{1/2}}(\mathcal{X}_{E_{nt}}) = -\,\mathbb{L}^{-1}\,\mathbb{L}^{1/2}(\mathbb{L}-1)\phi_{zx^4}\cdot \mathbf{x}^2 = 0$$

since $W_{\min} = zx^4$ on E_{nt}, and $\int_{\mathbb{C}^*\times\mathbb{C}} \phi_{zx^4} = 0$. So to prove the theorem, it is enough to show that $\int_{\mathbb{C}^*\times\mathbb{C}^2} \phi_{z(x^4+y^2)} \neq 0$. Now we leave it to the reader to verify that this is given by the naive motivic vanishing cycle

$$\int_{\mathbb{C}^*\times\mathbb{C}^2} -\phi_{z(x^4+y^2)} = [(x,y,z)\in\mathbb{C}^2\times\mathbb{C}^* | z(x^4+y^2)=0] - [(x,y,z)\in\mathbb{C}^2\times\mathbb{C}^* | z(x^4+y^2)=1]$$

and that this final quantity is equal to

$$\int_{\mathbb{C}^*\times\mathbb{C}^2} -\phi_{z(x^4+y^2)} = \mathbb{L}^2 - \mathbb{L},$$

so we may deduce that

$$\pi\circ\mathsf{DT}_{\mathbb{L}^{1/2}}(\mathcal{X}_N \star \mathcal{X}_N) = -(\mathbb{L}-1)^2\cdot\mathbf{x}^2\,.$$

□

Remark A.1. In fact there are separate reasons for not composing the integration map with π, anyway. There are substantive statements that can be deduced from the fact that $\mathsf{DT}_{\mathsf{OD}}$, the version of $\mathsf{DT}_{\mathbb{L}^{1/2}}$ modified by a suitable contribution from orientation data, is a ring homomorphism, which cannot be proved from the same claim regarding $\pi\circ\mathsf{DT}_{\mathsf{OD}}$. For an example, we consider a result from the interface between cluster theory and Donaldson–Thomas theory. In Efimov's work on quantum cluster algebras and positivity [13], which also contains relevant background to what follows, it is proved that the quantum cluster coefficients arising in quantum cluster mutation of a skew-symmetrizable quantum cluster algebra are given by applying a weight polynomial, in a variable $q^{1/2}$, to an element in $\mathbb{Z}[\mathbb{L}^{1/2}]\cap\mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})$. This intersection is just $\mathbb{Z}[\mathbb{L}]$, since we do not make the identification $\mathbb{L}^{1/2} = 1-\mathsf{MF}(x^2)$. From this one immediately deduces the vanishing of odd powers of $q^{1/2}$, as $\chi_q(\mathbb{L}) = q$. However note that applying the weight polynomial to elements in $\pi(\mathbb{Z}[\mathbb{L}^{1/2}])\cap\pi(\mathsf{Mot}^{\hat{\mu}}(\operatorname{Spec}\mathbb{C})) = \pi(\mathbb{Z}[\mathbb{L}^{1/2}])$, we can no longer deduce this, and we are handed the problem (see, for instance, [8]) of having to prove a difficult-looking theorem regarding vanishing of odd (critical) cohomology. For a more precise reference for how keeping the formal square root of $\mathbb{L}^{1/2}$ distinct from $1-\mathsf{MF}(x^2)$ buys us this vanishing result see [13, Thm.5.3].

Appendix B. Deferred motivic calculations

Recall Proposition 4.1, which stated the equality of $\hat{\mu}$-equivariant motives

$$\mathsf{MF}(x^4+y^4) = [C_1] - 4\mathbb{L}, \tag{41}$$

where C_1 is a genus 3 complex curve, with the action $2(\alpha+\alpha^2+\alpha^3)$ on its middle cohomology.

Proof. One can show this as follows: first, note that if $X=\mathbb{C}^2$, the blowup at the origin

$$\begin{tikzcd} \tilde{X} \arrow[d, "h"] \\ X \end{tikzcd}$$

provides an embedded resolution of $f = x^4+y^4$. As ever, let J denote the set of divisors in $(fh)^{-1}(0)$, as in the formula (6). There are then 5 elements in J, which we denote E, D_1, D_2, D_3, D_4, where E is the exceptional $\mathbb{P}^1$. The preimage $h^{-1}(0)$ is E, which intersects all of the divisors of J nontrivially. So there are 5 terms in the sum

$$\sum_{\emptyset\neq I\subset J}(1-\mathbb{L})^{|I|-1}\int_{\{0\}}[\tilde{D_J}] \tag{42}$$

coming from the 4 sets $\{E, D_i\}$ as well as from the singleton set $\{E\}$. All divisors of $(fh)^{-1}(0)$ apart from the exceptional $\mathbb{P}^1$ have multiplicity 1, so it follows that the étale cover corresponding to each of the points $E\cap D_i$ is just the 1-sheeted cover. So each of these points contributes $(1-\mathbb{L})$ to (42) . There remains the étale cover over the complement to the projective variety $V(x^4+y^4)$ in E, which is denoted, as in the formula (6) by $\tilde{D}_{\{E\}}$. This cover is 4-sheeted, since fh vanishes to order 4 along E. One can complete in the obvious way the resulting 4-sheeted étale cover to a branched cover

$$\begin{tikzcd} C_1 \arrow[d] \\ \mathbb{P}^1 \end{tikzcd}$$

of $\mathbb{P}^1$. Since this branched cover is simply ramified at each branch point of $\mathbb{P}^1$, i.e. there is only one point in the fibre of each branch point, it follows that the cover is connected, and C_1 is a genus 3 curve. One can work out the equivariant Euler characteristic of C_1 by taking a good cover, in the analytic topology, of $\mathbb{P}^1$, such that any open set in the cover contains at most one of the branchpoints. This calculation yields

$$\begin{aligned}\chi_{eq}(C_1) =&(1+\alpha+\alpha^2+\alpha^3)\chi(\mathbb{P}^1-\{4 \text{ points}\})+4\\ =&2-2(\alpha+\alpha^2+\alpha^3).\end{aligned}$$

Since we know that $\mathbb{Z}_4$ acts trivially on the top and bottom cohomology, we deduce that C_1 has the cohomology stated in the proposition. Putting everything together we have

$$\begin{aligned}\mathsf{MF}(x^4+y^4) =&([C_1]-4)+4(1-\mathbb{L})\\ =&[C_1]-4\mathbb{L}.\end{aligned}$$

□

In similar fashion we can explicitly describe $\mathsf{MF}(x^4+y^2)$:

Proposition B.1. *There is an equality of $\hat{\mu}$-equivariant motives*

$$\mathsf{MF}(x^4+y^2)=[C_2]-2\mathbb{L} \tag{43}$$

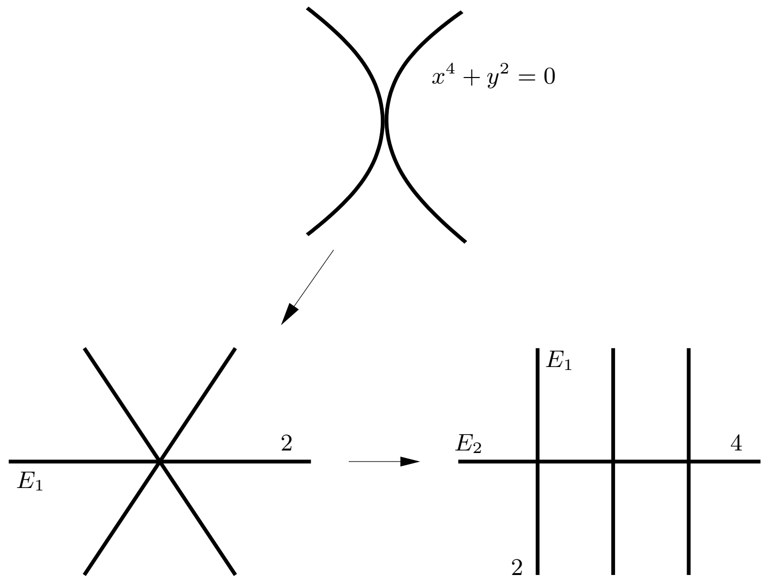

FIGURE 1. Resolved $x^4 + y^2$

where C_2 is a genus 1 curve with the action $\alpha + \alpha^3$ on its middle cohomology.

Proof. The motivic Milnor fibre of $x^4 + y^2$ is obtained by performing a couple of blowups as in our resolution of S_3, the $\mathbb{P}^1$ of A_3 singularities in the projective variety $V(\mathbf{tr}(T^4))$. After the first blowup we introduce an exceptional $\mathbb{P}^1$, which the two components of the strict transform of the divisor given by the original vanishing locus of $x^4 + y^2$ meet in a single point, as in the leftmost part of Figure 1. Blowing up this point gives us the rightmost arrangement of divisors of Figure 1. The new exceptional $\mathbb{P}^1$ we label E_2, and the strict transform of the first exceptional $\mathbb{P}^1$ we label E_1. Let

$$\begin{array}{c} \tilde{Z} \\ \Big\downarrow{\scriptstyle s} \\ \mathbb{C}^2 \end{array}$$

be the map of schemes obtained by performing these two blowups. Then the numbers next to the exceptional divisors in Figure 1 indicate the order of vanishing of the function $(x^4 + y^2)s$ on those divisors.

The preimage $s^{-1}(0)$ is equal to the union $E_1 \cup E_2$. The complement to E_2 in E_1 is a copy of $\mathbb{C}$, from which it follows that our 2-sheeted étale cover of it, $\tilde{D}_{\{E_2\}}$, must be the trivial $\mathbb{Z}_2$-torsor. The (resolved) completion of the 4-sheeted étale cover of E_1, which we denote C_2, is again connected, since two of its branching points are simply ramified. So we can use the same trick as for Proposition 4.1 to work out its cohomology using equivariant Euler characteristics. This gives that

$$\chi(C_2) = (1 + \alpha + \alpha^2 + \alpha^3)\chi(\mathbb{P}^1 - \{3 \text{ points}\}) + 2 + (1 + \alpha^2) = 2 - (\alpha + \alpha^3),$$

implying that C_2 is a torus with the action of $\mathbb{Z}_4$ on its middle cohomology given by the sum $\alpha+\alpha^3$. Putting all the pieces together,

$$\begin{aligned}\mathsf{MF}(x^4+y^2) =&[C_2]-(2+(1+\alpha^2))+(1+\alpha^2)\mathbb{L}+(1-\mathbb{L})(2+(1+\alpha^2))\\ =&[C_2]-2\mathbb{L}.\end{aligned}$$

□

Next we tidy up the unfinished business of calculating $[\tilde{D}_{\{Y\}}]$ from Proposition 5.4.

Proposition B.2. *There is an equality of absolute equivariant motives*

$$[\tilde{D}_{\{Y\}}] =\mathbb{L}[C_1]+\mathbb{L}(\mathbb{L}-1)[C_2]-2\mathbb{L}(\mathbb{L}+1) \tag{44}$$

$$=\mathbb{L}\,\mathsf{MF}(x^4+y^4)+(\mathbb{L}-1)\mathbb{L}\,\mathsf{MF}(x^4+y^2)+2\mathbb{L}(\mathbb{L}^2-1). \tag{45}$$

Proof. We stratify the cover $\tilde{D}_{\{Y\}}$ by stratifying the base $D(\mathbf{tr}(T^4))$, the complement in $\mathbb{P}^3$ to $V(\mathbf{tr}(T^4))$. Denote matrices of $D(\mathbf{tr}(T^4))$ by

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

Note that there is a $\mathbb{C}^*$-action on $D(\mathbf{tr}(T^4))$ given by

$$t\cdot\begin{pmatrix} a & b \\ c & d\end{pmatrix}\mapsto\begin{pmatrix} a & tb \\ t^{-1}c & d\end{pmatrix}.$$

(1) First consider the subscheme $P_1\subset D(\mathbf{tr}(T^4))$ of matrices with nonzero trace, and $c\neq 0$. P_1 is acted on freely by $\mathbb{C}^*$ with the above action. So we may take the quotient, and multiply the motive we get by $(\mathbb{L}-1)$. So we fix the trace to be equal to 1, thereby fixing an element in the line of matrices determined by an arbitrary matrix with nonzero trace, and set $c=1$, thereby passing to the quotient by the $\mathbb{C}^*$-action. Once we have fixed the trace, the complement $D(\mathbf{tr}(T^4))$ is determined entirely by the determinant, it is given by those matrices with determinant not equal to $\theta_1=1+\sqrt{1/2}$ or $\theta_2=1-\sqrt{1/2}$. There is an isomorphism

$$\begin{aligned}\mathbb{C}\times(\mathbb{C}-\{\theta_1,\theta_2\})\to&P_1/\mathbb{C}^*\\ (x,y)\mapsto&\begin{pmatrix} x & x(1-x)-y \\ 1 & 1-x\end{pmatrix}.\end{aligned}$$

Now

$$p^4+q^4=(p+q)^4-4pq(p+q)+2(pq)^2$$

from which it follows that the local defining function for $\mathbf{tr}(T^4)$ on $P_1/\mathbb{C}^*$ is $1-4y+2y^2$. The function $2y^2-4y+1$ defines a 4-sheeted étale cover in the usual way, and this is just the étale cover occurring in the calculation of the motivic Milnor fibre of $\mathsf{MF}(x^4+y^2)$, since we form a homogeneous quartic from $2y^2-4y+1$ by introducing the variable z and taking $2y^2z^2-4yz^3+z^4$, which vanishes to order 2 at infinity. This is just the cover obtained by removing the branchpoints from the equivariant curve C_2 of Proposition B.1. We conclude that there is an equality of absolute equivariant motives

$$\int_{P_1} [D_{\{Y\}}] = \mathbb{L}(\mathbb{L}-1)([C_2] - (3+\alpha^2)). \tag{46}$$

(2) Next let $P_2 \subset D(\mathbf{tr}(T^4))$ be the subscheme of matrices with nonzero trace, $c = 0$, and $b \neq 0$. Again we take representatives with trace equal to 1, and again we use the free $\mathbb{C}^*$-action to assume that $b = 1$. Then there is an isomorphism

$$\begin{aligned} \mathbb{C} - \{\text{roots of } p(z) = z^4 + (1-z)^4\} &\to P_2/\mathbb{C}^* \\ x &\mapsto \begin{pmatrix} x & 1 \\ 0 & 1-x \end{pmatrix}. \end{aligned}$$

The local defining function for $\mathbf{tr}(T^4)$ becomes $x^4 + (1-x)^4$. This polynomial has 4 separate roots, so the 4-sheeted étale cover it defines over $\mathbb{C}$ is the curve C_1, minus the branchpoints, and also minus the 4 points lying over infinity. So

$$\int_{P_2} [D_{\{Y\}}] = (\mathbb{L}-1)([C_1] - 4 - (1+\alpha+\alpha^2+\alpha^3)). \tag{47}$$

(3) Let $P_3 \subset \mathbb{P}^3$ be the subscheme consisting of matrices with trace equal to zero, $a \neq 0$, and $c \neq 0$. Then we can assume $a = 1$, after taking an appropriate scalar multiple. Furthermore we again have a free $\mathbb{C}^*$-action, and so we take the quotient again, and assume $c = 1$. There is an isomorphism

$$\begin{aligned} \mathbb{C}^* &\to P_3/\mathbb{C}^* \\ x &\mapsto \begin{pmatrix} 1 & x-1 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$

The local defining equation for $\mathbf{tr}(T^4)$ becomes $2x^2$. The resulting 4-sheeted cover of $\mathbb{C}^*$ has 2 components, each a torus, and we conclude that

$$\int_{P_3} [D_{\{Y\}}] = (\mathbb{L}-1)(1+\alpha^2)(\mathbb{L}-1). \tag{48}$$

(4) Let $P_4 \subset \mathbb{P}^3$ be the subscheme consisting of matrices with zero trace, $a \neq 0$, $c = 0$, $b \neq 0$. We again may assume $a = 1$. P_4 is just a single free $\mathbb{C}^*$-orbit, and so we conclude that

$$\int_{P_4} [D_{\{Y\}}] = (\mathbb{L}-1)(1+\alpha+\alpha^2+\alpha^3). \tag{49}$$

(5) Let $P_5 \subset \mathbb{P}^3$ be the subscheme of diagonal matrices. Then $P_5 \cong \mathbb{P}^1$, and $V(\mathbf{tr}(T^4)) \cap P_5$ consists of four points. It follows that the étale cover, restricted to P_5 is just the étale cover occurring in the calculation of the motivic Milnor fibre of $x^4 + y^4$, and so

$$\int_{P_5} [D_{\{Y\}}] = [C_1] - 4. \tag{50}$$

(6) Let $P_6 \subset \mathbb{P}^3$ be the subscheme consisting of off-diagonal matrices. Both entries b and c must be nonzero for the matrix to be in $D(\mathbf{tr}(T^4))$. So we may assume $c = 1$. On this orbit $\mathbb{C}^*$ again doesn't act freely, so we will ignore it. There is an isomorphism

$$\begin{aligned} \mathbb{C}^* \to & P_6 \\ x \mapsto & \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

The local defining equation for $\mathbf{tr}(T^4)$ is $2x^2$. So the resulting 4-sheeted étale cover of $\mathbb{C}^*$ is given by a cover by 2 tori, and we have the equality

$$\int_{P_6} [D_{\{Y\}}] = (\mathbb{L} - 1)(1 + \alpha^2). \tag{51}$$

Putting all this together gives equation (44). In light of Propositions 4.1 and B.1 we also deduce equation (45).

□

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