Liu, W., Zhang, F., and Yang, H. (2016) Modeling and managing morning commute with both household and individual travels. *Transportation Research Part B: Methodological*, (doi:10.1016/j.trb.2016.12.002)

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Revised paper submitted to Transportation Research Part B on December 6, 2016

Modeling and Managing Morning Commute with both Household and Individual Travels

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Abstract

This study investigates the morning commute problem with both household and individual travels, where the household travel is a shared ride of household (family) members. In particular, it considers the situation when a proportion of commuters have to drive their children to school first and then go to work (household travel). For household travel, departure time choice is a joint decision based on all household members' preferences. Unlike the standard bottleneck model, the rush-hour dynamic traffic pattern with mixed travelers (household travelers and individual travelers) varies with the numbers of individual travelers and households, as well as the schedule difference between school and work. Given the numbers of individual travelers and households, we show that by appropriately coordinating the schedules of work and school, the traffic congestion at the highway bottleneck can be relieved, and hence the total travel cost can be reduced. This is because, departure/arrival of individual and household travels can be separated by schedule coordination. System performance under schedule coordination is quantified in terms of the relative proportions of the two classes of travelers and is compared with the extreme case when the same desired arrival time applies to both schooling and working. Furthermore, the efficiency of work and school schedule coordination in reducing travel cost is bounded. This efficiency is also compared with that at the system optimum where queuing is fully eliminated and schedule delay cost is minimized (achieved by a joint scheme of first-best pricing and schedule coordination).

Keywords: Morning commute; household travel; individual travel; bottleneck congestion; schedule coordination.

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1. Introduction

Traffic congestion is pervasive in many metropolitan areas and is worsening throughout many countries. In the literature, understanding the dynamic traffic pattern and managing traffic congestion in the morning peak hour have been studied extensively by both transportation scientists and economists. Vickrey (1969) was the first to propose the bottleneck model to capture the traffic dynamics in the rush hour. In Vickrey's model, the congestion is modeled as a deterministic queue behind a bottleneck of fixed capacity. Travelers choose their departure times to minimize individual travel cost including travel delay cost and schedule delay cost. Based on this model, various issues have been studied, e.g., existence and uniqueness of user equilibrium solution at a single bottleneck (Smith, 1984; Daganzo, 1985; Lindsey, 2004); road pricing, tradable credits, and tradable permits to manage traffic congestion (Arnott et al., 1990; Laih, 1994; Xiao et al., 2012; Tian et al., 2013; Nie and Yin, 2013; Wada and Akamatsu, 2013); stochastic bottleneck capacity and travel demand (Arnott et al., 1999; Lindsey, 2009; Xiao et al., 2015); morning commute with heterogeneous travelers (Arnott et al., 1994; van den Berg and Verhoef, 2011; Liu and Nie, 2011; Liu et al., 2014b; Liu et al., 2015a; Liu et al., 2015c); integrated problem of parking and morning commute (Arnott et al., 1991; Zhang et al., 2008; Qian et al., 2011; Qian et al., 2012; Yang et al., 2013; Liu et al., 2014a; Xiao et al., 2016); capacity drop and/or hyper-congestion (Arnott, 2013; Liu et al., 2015b; Liu and Geroliminis, 2016); complementarity formulation or ordinary differential equation formulation (Ramadurai et al., 2010; Wu and Huang, 2015; Wang and Xu, 2016).

However, most of the previous studies focus on analyzing and managing the commuting problem with individual travelers only. Little attention has been paid to household travels. Different from individual trips, a household trip consists of the travels of all household members, i.e., shared ride of household members. Note that ride sharing is encouraged as the same number of travelers can be transported with less vehicles and drivers, and traffic congestion and environmental pollution can be reduced. Recently, de Palma et al. (2015) considered that individuals live as couples and value time at home more when together than when alone. They estimated the trip-time preference for married and unmarried men and women in the Greater Paris region. More recently, Jia et al. (2016) considered the equilibrium trip scheduling for households where each household travel group consists of one adult traveler and one child, and the adult traveler has to send the child to school first and then go to the workplace. For household travel, more than one member in the household will be involved in the departure time choice decision, i.e., all the members' preference of arrival times have to be considered (e.g., for work and for school).

In Jia et al. (2016), all travelers are assumed to be household travelers. This is reasonable as a first step to understand how household travel is different from individual travel in departure time choices. However, this is usually not the case in reality as there will be both household travelers and individual travelers. Therefore, not only the interactions among members within a household can affect departure time decisions and the traffic equilibrium, but also the interactions among household travelers and individual travelers (through sharing the same road network) can re-shape the dynamic traffic pattern in the morning peak. This study, by considering this more realistic case with mixed travelers, will help us to better understand the impact of household travels. Indeed, the model presented in this paper incorporates that in Jia et al. (2016) as a special or extreme case, where the number of individual travelers equals zero (this has been specified in Section 3).

Specifically, we consider that there are two types of travels: individual travel and household travel. An individual travel consists of only one trip, i.e., going to the workplace (given that travelers have a desired arrival time for work). A typical household travel consists of two successive trips, i.e., dropping off the children at the school and then going to the workplace. In this case, there are two desired arrival times: desired arrival time at school and desired arrival time at work. For individual travel, travelers have no cost associated with the school, while the household travel will take into account the costs associated with both school and work. We firstly explore the dynamic equilibrium traffic pattern with both household and individual travels. Then we examine how to coordinate school schedule and work schedule in order to reduce the traffic congestion, and thus reduce the total travel cost. Also, we analyze the efficiency of schedule coordination (for work and school).

It is worth mentioning that the efficiency of schedule coordination depends on relative proportions of the two classes of travelers, and can be bounded. Note that there is a branch of studies looking into staggered work hours (e.g., Henderson 1981; Yushimito et al., 2014; Shirmohammadi et al., 2015; Takayama, 2015). However, all of these studies focus on the coordination of work schedules for individuals. None of them involves household trips or school schedule. Besides, we found that total travel cost can decrease with the proportion of household travelers in the population (which can be counterintuitive as a larger number of households suggest a larger number of travelers in total). We also found that schedule delay cost does not always increase with the difference between the two desired arrival times (for work and school).

The remainder of the paper is organized as follows. Section 2 presents problem description and the cost formulations for both households and individual travelers. The dynamic traffic pattern at departure/arrival equilibrium with mixed travelers is discussed in Section 3. Section 4 analyzes the system performance under given numbers of different travelers and work and school schedules, and then evaluates and bounds the efficiency of coordinating work and school schedules in reducing total travel cost. Numerical illustration and verification are presented in Section 5. Section 6 concludes the paper.

2. Problem Formulation

Consider a bottleneck-constrained highway connecting a residential area and a city center (workplace) as shown in Figure 1. There is a school between home and workplace after the highway bottleneck. In the morning commute, there are two types of travelers: individual travelers and household travelers, which are described in the following.

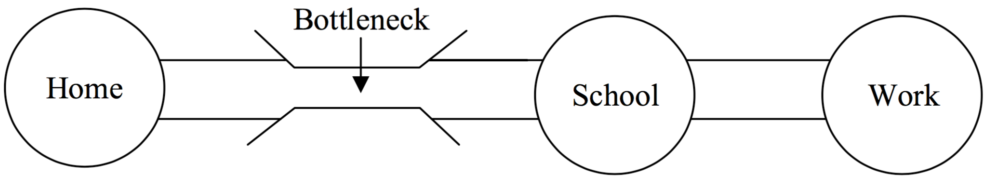


Figure 1. Home-Bottleneck-School-Work Network

Firstly, N_1 individual travelers have to drive to the city center to work, and their desired arrival time at the workplace is t_2^*. For these individual travelers, they will make a trade-off between the travel time cost related to queue length at the highway bottleneck and the schedule delay cost of arriving early or late at work.

Besides the N_1 individual travelers, there are travelers from N_2 households. For simplicity, it is assumed that there is one adult and one child per household (all of them are referred to as household travelers in this paper). While later in the paper we sometimes might use N_2 to refer to household travelers, the total number of household travelers is indeed $2 \cdot N_2$ (including both the adults and the children). The adult members in the households have to drop off their children at the school first and then go to the workplace. The desired arrival time at the workplace is t_2^* (identical to that of the individual travelers) while the desired arrival time at the school is t_1^*.[1] It is assumed that $\Delta t = t_2^* - t_1^* \geq 0$.[2] When making departure

[1] The desired arrival time at school might not be interpreted as the exact school start time. It can be a time point which allows sufficient flexibility (e.g., no need to run) for the child to arrive at class on time. Later in the paper, we refer to "later than desired arrival time for school" as "late for school".

[2] This means that a household on-time for school might still be early for work, which is closer to the practice. From a modeling perspective, the case with $t_1^* > t_2^*$ would be similar to the case with $t_1^* < t_2^*$ if we exchange the "school" and "work".

time decisions, these household travelers will not only consider the travel cost of traveling to work, but also traveling to school.

2.1. Individual travelers

The individual travel cost (including the travel time cost and the schedule delay cost) by departing at time t is

$$c(t)=\alpha\cdot T(t)+\beta\cdot\max\left\{0,t_2^*-t-T(t)\right\}+\gamma\cdot\max\left\{0,t+T(t)-t_2^*\right\}, \tag{1}$$

where $T(t)$ is the travel time for a departure time t, α is the value of unit travel time, and β and γ are the schedule penalties for a unit time of early arrival and late arrival respectively. It is assumed that $\gamma>\alpha>\beta>0$. Without loss of generality, the fixed component of the travel time (or free-flow travel time) is assumed to be zero, so that $T(t)$ only contains the queuing time at the bottleneck. Therefore, $T(t)=q(t)/s$, where $q(t)$ is the queue length at the bottleneck at time t, and s is the constant service capacity. Traffic can leave the road at any time and incur no delay until the traffic flow exceeds the road's capacity; and once flow exceeds capacity then deterministic (point) queue will develop. Thus,

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t}=\begin{cases}r(t)-s, & r(t)>s \text{ or } q(t)>0\\ 0, & r(t)\le s \text{ and } q(t)=0\end{cases}, \tag{2}$$

where $r(t)$ is the rate of traffic arriving at the bottleneck at time t.

Given the above standard setting in the literature, the departure rates from home (or arrival rates at the bottleneck) for commuters who arrive at the destination before and after desired arrival time t_2^* respectively are given by

$$r_1=\frac{\alpha}{\alpha-\beta}s,\ r_1'=\frac{\alpha}{\alpha+\gamma}s. \tag{3}$$

For detailed derivation of Eq.(3), one may refer to Arnott et al. (1990).

2.2. Household travelers

For household travelers, they have to drive their children to school before going to work. They will try to minimize the travel cost of the household as a whole, which is

$$\begin{aligned}c(t)=&\left[\alpha\cdot T(t)+\beta\cdot\max\left\{0,t_1^*-t-T(t)\right\}+\gamma\cdot\max\left\{0,t+T(t)-t_1^*\right\}\right]\\&+\left[\alpha\cdot T(t)+\beta\cdot\max\left\{0,t_2^*-t-T(t)\right\}+\gamma\cdot\max\left\{0,t+T(t)-t_2^*\right\}\right],\end{aligned} \tag{4}$$

The first part in Eq.(4) is the travel time cost and schedule delay cost for traveling to school (associated with the child in the household), while the second part is for work (associated with the adult in the household). Following the travel cost assumption for individual travelers, the travel cost in Eq.(4) assumes zero free-flow travel time between school and work, and also assumes zero additional travel delay caused by dropping-off children at school (both assumptions can be relaxed with the current framework by adding a constant delay, which is briefly discussed in Appendix A).

Also note that in Eq.(4), the value of time (VOT) and schedule delay penalties for school are considered identical to those for work. This simplification, as an initial step to understand dynamic traffic pattern under mixed travelers, makes the algebra in the paper much less tedious (but indeed still sufficiently complex). While in future study we will incorporate different VOTs and schedule delay penalties, the modeling framework and essential analysis here can still be applied in a similar way.

We focus on the situation where $t_2^* - t_1^* \geq 0$, which means that the household can be on time for school while early for work. Note that, if we consider the free-flow travel time between school and workplace to be a positive constant $t_{sw} > 0$,[3] then the assumption should be modified to $t_2^* - t_1^* \geq t_{sw}$. Given this, there are three possible situations for household travel, i.e., i) early for school and early for work; ii) late for school but early for work; iii) late for school and late for work. For those households early for school and work, the equilibrium departure rate (from home) is

$$r_2 = \frac{\alpha}{\alpha - \beta} s. \tag{5}$$

For those household early for school but late for work, the equilibrium departure rate (from home) is

$$r_2' = \frac{\alpha}{\alpha - \beta + \gamma} s. \tag{6}$$

For those household late for school and work, the equilibrium departure rate (from home) is

$$r_2'' = \frac{\alpha}{\alpha + \gamma} s. \tag{7}$$

Derivation of the three rates given in Eq.(5), Eq.(6) and Eq.(7) is similar to that for the rates in Eq.(3). It is obvious to see that $r_2 > s > r_2' > r_2''$, and $r_2 = r_1$ and $r_2'' = r_1'$, where r_1 and r_1' are given in Eq.(3). Note that $r_2 = r_1$ and $r_2'' = r_1'$ are due to the assumption of identical VOT and schedule delay penalties for school travel and work travel. While there are three possible situations for household, it does not mean that all of them will arise simultaneously under a

[3] We assume $t_{sw} = 0$ throughout the analysis while we discuss the extension to $t_{sw} \neq 0$ in Appendix A.

specific dynamic traffic pattern at the departure/arrival equilibrium given the numbers of individual travelers and households, and the work and school schedule difference (as shown in Section 3, there are four possible specific equilibrium traffic patterns).

3. Commuting Equilibrium with Mixed Travelers

3.1. Equilibrium traffic patterns

With the formulations in Section 2, we can derive the possible dynamic departure/arrival pattern at the user equilibrium. As shown in Figure 2, four possible departure/arrival patterns can appear (later we refer to each pattern by "Case"), depending on $\Delta t = t_2^* - t_1^*$, N_1 and N_2 (t_1^* and t_2^* are desired arrival times at school and at workplace respectively). The conditions for the occurrence of each pattern are summarized in Table 1, and the critical time points in each pattern are summarized in Table 2. Furthermore, three critical departure/arrival patterns (the boundary between two different traffic patterns) are also shown in Figure 2. Note that in Figure 2, the blue lines are departures from home, and the red lines are the arrivals at work (also the arrivals at school as we assume zero free-flow time between school and work place). As one can see from the following discussion, household travelers are generally traveling earlier than individual travelers due to the school schedule.

Table 1. Conditions for the four cases depicted in Figure 2

Equilibrium	Conditions	Figures
Case 1	$t_2^* - t_1^* \geq \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$	Figure 2(a)
Case 2	$t_2^* - t_1^* < \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$ $t_2^* - t_1^* > -\frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$ $t_2^* - t_1^* > \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\beta}{\beta+\gamma}\frac{N_2}{s}$	Figure 2(b)
Case 3	$t_2^* - t_1^* \leq -\frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$ $t_2^* - t_1^* \geq 0$	Figure 2(c)
Case 4	$t_2^* - t_1^* \leq \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\beta}{\beta+\gamma}\frac{N_2}{s}$ $t_2^* - t_1^* \geq 0$	Figure 2(d)

Case 1 in Figure 2(a): the time difference Δt between the two desired arrival times, i.e., t_1^* and t_2^*, is relatively large, which means that the school schedule is much earlier than work schedule. Due to this early school schedule, all household travels are early for work (those arrive between t_1 and t_2). But they can be either early or late for school. Individual travelers will travel around their desired work arrival time t_2^* (to reduce schedule delays), and can be either early or late for work (arrival occurs between t_3 and t_4). The departure and arrival of household and individual travels are completely separated, thus there is no direct flow interaction between the two classes of travelers.

Case 2 in Figure 2(b): Δt is less than that in Case 1, and the departure and arrival of household and individual travels are connected. "Connected" means that the earliest individual travelers meet the latest household travelers at the highway bottleneck (both of them arrive at time t_2), and have to wait upon arrival at the bottleneck until the queue of household travelers disappears. All household travels are early for work ($t_2 < t_2^*$), while might be early or late for school. Individual travelers can be either early or late for work (arrival occurs between t_2 and t_3).

The critical case between Case 1 and Case 2 is depicted in Figure 2(e), where, the first individual traveler arrives at the bottleneck exactly when the last household travelers exit; there is no flow interaction at the highway bottleneck between the two classes of travelers.

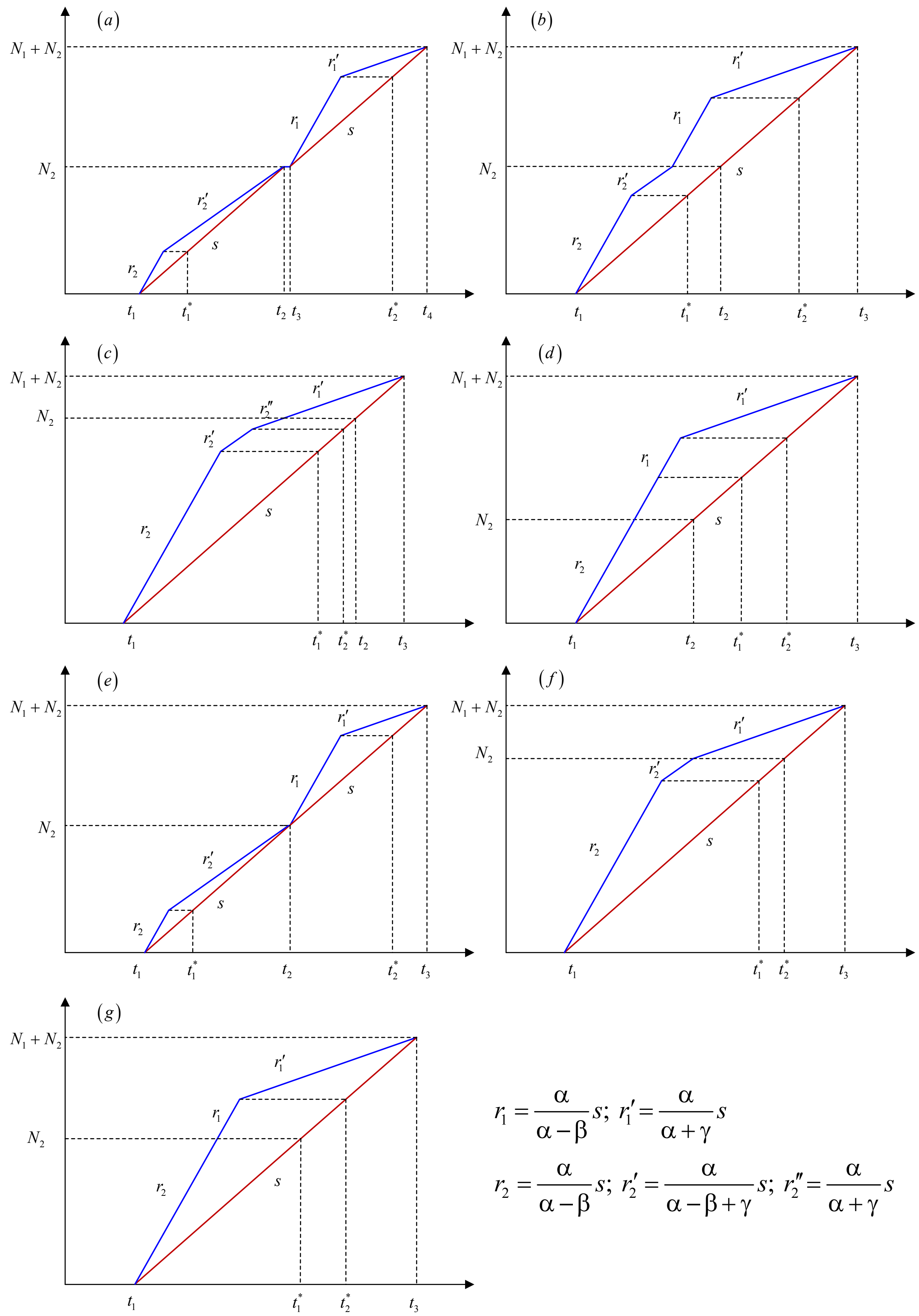
Case 3 in Figure 2(c): Δt is relatively small, and the number of household N_2 is relatively large, thus some household travelers have to be late for work (they are also late for school). All the arrivals earlier than t_2^* are household travels. However, some households are late for both school and work (arrive after t_2^*). Figure 2(c) is only illustrative where these households late for school and work arrive at destination between t_2^* and t_2 (those arrive after t_2 are individual travelers). Indeed the departure and arrival of these household (late for both school and work) can be mixed with the individual travelers. In this "mixed" period, some individual travelers can be earlier than household travelers. However, even the departure order of these travelers can be different (any combination is possible as long as the departure rate is $r_2'' = r_1'$), the traffic pattern will still be the same.

The critical case between Case 2 and Case 3 is depicted in Figure 2(f), where the first individual traveler meets the last household traveler, and both of them are on time for work. There is no household travel which is late for school and work.

Case 4 in Figure 2(d): Δt is also relatively small. However, different from Case 3, the number of individual traveler N_1 is relatively large, thus all households are pushed to arrive early for both school and work. Individual travelers can be either early or late for work. The departure and arrival of household are indeed mixed with the early arrival individual travelers. Similar to Case 3 and for illustration purpose, Figure 2(d) only depicts the situation that all household travelers arrive earlier than t_2. However, while the departure order of these people can be different (i.e., mixed with individual travelers of early arrival), the traffic pattern will still be the same because $r_2 = r_1$. Also note that the traffic pattern is identical to the situation where all the travelers are individual travelers.

The critical case between Case 2 and Case 4 is depicted in Figure 2(g), where, all the household travelers are early for work and school, and the last household traveler is just on time for school. Similar to Case 4, the traffic pattern is identical to the situation where all the travelers are individual travelers.

By assuming that all travelers are household travelers, Jia et al. (2016) characterizes two situations: small school-work schedule difference and large school-work schedule difference. The "small school-work schedule difference" case corresponds to Figure 2(c) by letting $N_1 = 0$, while the "large school-work schedule difference" corresponds to Figure 2(a) (imagine that the current N_1 individual travelers all become household travelers). Later, readers can find in Figure 3 that the model in Jia et al. (2016) corresponds to the y-axis (thus $N_1 = 0$) in the two-dimension domain of (N_1, N_2).



$$r_1 = \frac{\alpha}{\alpha-\beta}s;\ r_1' = \frac{\alpha}{\alpha+\gamma}s$$

$$r_2 = \frac{\alpha}{\alpha-\beta}s;\ r_2' = \frac{\alpha}{\alpha-\beta+\gamma}s;\ r_2'' = \frac{\alpha}{\alpha+\gamma}s$$

(a) Case 1; (b) Case 2; (c) Case 3; (d) Case 4; (e) Critical case between Case 1 and Case 2; (f) Critical case between Case 2 and Case 3; (g) Critical case between Case 2 and Case 4

Figure 2. Possible commuting patterns with both household and individual travels

3.2. Properties of the equilibrium traffic patterns

For given Δt, Figure 3 displays the occurrence of each case (departure/arrival pattern) in the domain of (N_1, N_2). As can be seen, the domain of (N_1, N_2) can be divided into four regions by the three lines (Line 1, Line 2 and Line 3), i.e., region (1), (2), (3) and (4), which correspond to Cases 1, 2, 3, 4 respectively. Note that Line 1 is $N_2 = \frac{\beta+\gamma}{2\beta}\cdot\Delta t\cdot s - \frac{\gamma}{2\beta}\cdot N_1$ and corresponds to the critical case between Case 1 and Case 2 shown in Figure 2(e); Line 2 is $N_2 = \frac{\beta+\gamma}{2\beta}\cdot\Delta t\cdot s + \frac{\gamma}{\beta}N_1$ and corresponds to the critical case between Case 2 and Case 3 shown in Figure 2(f); and Line 3 is $N_2 = -\frac{\beta+\gamma}{\beta}\cdot\Delta t\cdot s + \frac{\gamma}{\beta}N_1$ and corresponds to the critical case between Case 2 and Case 4 shown in Figure 2(g). The four lines indeed correspond to the conditions in Table 1. As mentioned, the model in Jia et al. (2016) corresponds to the y-axis ($N_1 = 0$) in the two-dimension domain of (N_1, N_2).

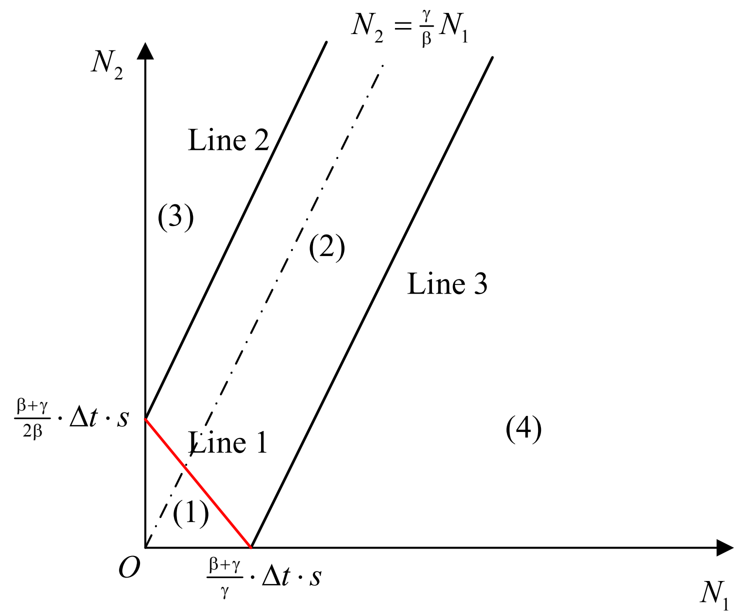


Figure 3. Domain of (N_1, N_2) for cases in Figure 2 with given Δt

From Figure 3, we can see that, when $N_2 \geq \frac{\gamma}{\beta}N_1$, only Case 1, Case 2 and Case 3 can arise; when $N_2 < \frac{\gamma}{\beta}N_1$, only Case 1, Case 2 and Case 4 can arise. We summarize these results in more details as follows.

Proposition 3-1. *For given* N_1 *and* N_2*, as* Δt *increases from* 0^+ *to* $+\infty$*:*

i) if $N_2 \geq \frac{\gamma}{\beta} N_1$*, the equilibrium traffic pattern will vary according to the path: Case 3* $\rightarrow$ *Case 2* $\rightarrow$ *Case 1;*

ii) if $N_2 < \frac{\gamma}{\beta} N_1$*, the equilibrium traffic pattern will vary according to the path: Case 4* $\rightarrow$ *Case 2* $\rightarrow$ *Case 1.*

Proposition 3-1 can be verified according to the conditions in Table 1. When $N_2 \geq \frac{\gamma}{\beta} N_1$, the following condition $\frac{2\beta}{\beta+\gamma}\frac{N_2}{s} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} \geq 0$ holds. As $\Delta t \leq \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s}$, conditions for Case 3 (in Table 2) hold. As Δt becomes larger but is still less than $\frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$, conditions for Case 2 (in Table 2) hold. One can further verify that when Δt becomes even larger and is greater than the critical value $\frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$, conditions for Case 1 (in Table 2) hold. We can verify the results for the situation with $N_2 < \frac{\gamma}{\beta} N_1$ similarly.

Proposition 3-1 summarizes how dynamic traffic pattern varies with Δt. From Proposition 3-1, we see that, as Δt increases, the departure/arrival pattern changes in the way that the departure/arrival of the two classes of travelers becomes more separated in the situation with either $N_2 \geq \frac{\gamma}{\beta} N_1$ or $N_2 < \frac{\gamma}{\beta} N_1$. It is possible that given N_1 and N_2, by appropriately choosing Δt, we can separate travels of the two classes of travelers, thus can reduce the traffic congestion temporally (one may refer to the Critical case between Case 1 and Case 2), and in the end reduce the total travel cost of travelers. This is also the motivation to discuss the coordination of school and work schedules in Section 4.

Proposition 3-2. *For given* N_1 *and* Δt*, as* N_2 *increases from* 0 *to* $+\infty$*:*

i) if $N_1 < \frac{\beta+\gamma}{\gamma} \cdot \Delta t \cdot s$*, the equilibrium traffic pattern will vary according to the path: Case 1* $\rightarrow$ *Case 2* $\rightarrow$ *Case 3;*

ii) if $N_1 \geq \frac{\beta+\gamma}{\gamma} \cdot \Delta t \cdot s$*, the equilibrium traffic pattern will vary according to the path: Case 4* $\rightarrow$ *Case 2* $\rightarrow$ *Case 3.*

The reasoning to verify Proposition 3-2 is similar to that for Proposition 3-1. Proposition 3-2 summarizes how dynamic traffic pattern varies with N_2, given Δt and N_1. More specifically, given Δt, when the number of individual travelers is relatively small, i.e., $N_1 < \frac{\beta+\gamma}{\gamma} \cdot \Delta t \cdot s$, Case 1 can arise; otherwise, i.e., $N_1 \geq \frac{\beta+\gamma}{\gamma} \cdot \Delta t \cdot s$, Case 1 would not occur. This means that when there are too many individual travelers, even if there are very few households, the departure/arrival of the two classes of travelers cannot be separated. Proposition 3-2 also indicates that increasing the number of households might force the individual travelers to experience larger queuing delays (Case 1 $\rightarrow$ Case 2 $\rightarrow$ Case 3), and might force all individual travelers to be late for work (Case 2 $\rightarrow$ Case 3).

Proposition 3-3. *For given N_2 and Δt, as N_1 increases from 0 to $+\infty$:*

i) if $N_2 < \frac{\beta+\gamma}{2\beta} \cdot \Delta t \cdot s$, the equilibrium traffic pattern will vary according to the path: Case 1 $\rightarrow$ Case 2 $\rightarrow$ Case 4;

ii) if $N_2 \geq \frac{\beta+\gamma}{2\beta} \cdot \Delta t \cdot s$, the equilibrium traffic pattern will vary according to the path: Case 3 $\rightarrow$ Case 2 $\rightarrow$ Case 4.

Similarly, we can verify Proposition 3-3. Proposition 3-3 summarizes how dynamic traffic pattern will vary with N_1, given Δt and N_2. When N_2 is large, i.e., $N_2 \geq \frac{\beta+\gamma}{2\beta} \cdot \Delta t \cdot s$, Case 1 cannot arise, i.e., the departure/arrival of the two classes of travelers cannot be separated. Proposition 3-3 also indicates that increasing the number of individual travelers might force the household travelers to experience larger queuing delays (e.g., Case 1 $\rightarrow$ Case 2 $\rightarrow$ Case 4), and might force all the household travelers to be early for work (e.g., Case 3 $\rightarrow$ Case 2), or to be early for school (Case 2 $\rightarrow$ Case 4).

Table 2. Time points for each pattern depicted in Figure 2

Time	**Figure 2(a)**	**Figure 2(b)**
t_1	$t_1^* - \frac{\gamma-\beta}{\beta+\gamma}\frac{N_2}{s}$	$\frac{1}{3}t_1^* + \frac{2}{3}t_2^* - \frac{2\gamma}{3(\beta+\gamma)}\frac{N_1}{s} - \frac{\beta+3\gamma}{3(\beta+\gamma)}\frac{N_2}{s}$
t_2	$t_1^* + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$	$\frac{1}{3}t_1^* + \frac{2}{3}t_2^* - \frac{2\gamma}{3(\beta+\gamma)}\frac{N_1}{s} + \frac{2\beta}{3(\beta+\gamma)}\frac{N_2}{s}$
t_3	$t_2^* - \frac{\gamma}{\beta+\gamma}\frac{N_1}{s}$	$\frac{1}{3}t_1^* + \frac{2}{3}t_2^* + \frac{3\beta+\gamma}{3(\beta+\gamma)}\frac{N_1}{s} + \frac{2\beta}{3(\beta+\gamma)}\frac{N_2}{s}$
t_4	$t_2^* + \frac{\beta}{\beta+\gamma}\frac{N_1}{s}$	N/A
Time	**Figure 2(c)**	**Figure 2(d)**
t_1	$\frac{1}{2}\left(t_1^* + t_2^*\right) - \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\gamma}{\beta+\gamma}\frac{N_2}{s}$	$t_2^* - \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\gamma}{\beta+\gamma}\frac{N_2}{s}$
t_2	$\frac{1}{2}\left(t_1^* + t_2^*\right) - \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{\beta}{\beta+\gamma}\frac{N_2}{s}$	$t_2^* - \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{\beta}{\beta+\gamma}\frac{N_2}{s}$
t_3	$\frac{1}{2}\left(t_1^* + t_2^*\right) + \frac{\beta}{\beta+\gamma}\frac{N_1}{s} + \frac{\beta}{\beta+\gamma}\frac{N_2}{s}$	$t_2^* + \frac{\beta}{\beta+\gamma}\frac{N_1}{s} + \frac{\beta}{\beta+\gamma}\frac{N_2}{s}$

From Figure 2 and Table 2, it is straightforward to see that, the peak start time t_1 depends on N_1 and N_2, and Δt. As $\Delta t = t_2^* - t_1^*$, to have comparable results when looking at how t_1

change with Δt, we fix the mid-point between t_1^* and t_2^*, i.e., $0.5\cdot\left(t_1^*+t_2^*\right)$ is fixed. (This assumption is only valid for Lemma 3-1 and Proposition 3-4.)

Lemma 3-1. *For given N_1 and N_2, when we fix $0.5\cdot\left(t_1^*+t_2^*\right)$: i) t_1 is non-decreasing with Δt when $\Delta t < \frac{\gamma}{\beta+\gamma}\frac{N_1}{s}+\frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$; ii) t_1 is decreasing with Δt when $\Delta t > \frac{\gamma}{\beta+\gamma}\frac{N_1}{s}+\frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$.*

Proof. See Appendix B.

Proposition 3-4. *For given N_1 and N_2, when we change Δt, but fix $0.5\cdot\left(t_1^*+t_2^*\right)$, the peak start time t_1 reaches the maximum when $\Delta t = \frac{\gamma}{\beta+\gamma}\frac{N_1}{s}+\frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$.*

Proposition 3-4 is straightforward based on Lemma 3-1. It indicates that for given N_1 and N_2, when we try to coordinate the school and work schedules by changing Δt, the latest peak start time occurs at $\Delta t = \frac{\gamma}{\beta+\gamma}\frac{N_1}{s}+\frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$, where the traffic pattern is the critical case between Case 1 and Case 2. Note that a later peak start time generally indicates lower travel cost for travelers (e.g., less schedule delay cost and identical queueing delay cost for the traveler departing at the peak start time). Later in Section 4, we will show that by letting $\Delta t = \frac{\gamma}{\beta+\gamma}\frac{N_1}{s}+\frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$, we can reduce travel cost compared with an arbitrary Δt.

Proposition 3-5. *For given t_1^* and t_2^*, peak start time t_1 is non-increasing with N_1, and is decreasing with N_2.*

Similar to Proposition 3-4, Proposition 3-5 can be obtained with Table 2 and Proposition 3-1. This result is expected as more travelers generally would indicate earlier peak start time. However, when the traffic pattern belongs to Case 1 (when the departure/arrival of household and individual travelers are completely separated), a marginal increase in individual travelers will not affect the peak start time (no direct interaction exists between households and individual travelers, and the peak start is fully determined by household travelers).

4. User Cost and System Performance

In Section 3, we analyzed all the possible commuting traffic patterns in detail. Now we turn to the users' travel cost and total travel cost.

4.1. Users' travel cost

Based on Section 3 and with some manipulations, we can obtain the travel cost of individual travelers c_1 and travel cost of households c_2, which are summarized in Table 3. Note that c_1 is based on Eq.(1) and c_2 is based on Eq.(4) for each case.

Table 3. Travel costs of individual travelers and households for different cases

Case	Individual cost c_1	Household cost c_2
1	$\frac{\beta\gamma}{\beta+\gamma}\frac{N_1}{s}$	$\beta\Delta t+\frac{2\beta(\gamma-\beta)}{\beta+\gamma}\frac{N_2}{s}$
2	$-\frac{1}{3}\gamma\Delta t+\frac{(3\beta+\gamma)\gamma}{3(\beta+\gamma)}\frac{N_1}{s}+\frac{2\beta\gamma}{3(\beta+\gamma)}\frac{N_2}{s}$	$-\frac{1}{3}\beta\Delta t+\frac{4\beta\gamma}{3(\beta+\gamma)}\frac{N_1}{s}+\frac{2\beta(\beta+3\gamma)}{3(\beta+\gamma)}\frac{N_2}{s}$
3	$-\frac{1}{2}\gamma\Delta t+\frac{\beta\gamma}{\beta+\gamma}\frac{N_1}{s}+\frac{\beta\gamma}{\beta+\gamma}\frac{N_2}{s}$	$\frac{2\beta\gamma}{\beta+\gamma}\frac{N_1}{s}+\frac{2\beta\gamma}{\beta+\gamma}\frac{N_2}{s}$
4	$\frac{\beta\gamma}{\beta+\gamma}\frac{N_1}{s}+\frac{\beta\gamma}{\beta+\gamma}\frac{N_2}{s}$	$-\beta\Delta t+\frac{2\beta\gamma}{\beta+\gamma}\frac{N_1}{s}+\frac{2\beta\gamma}{\beta+\gamma}\frac{N_2}{s}$

Proposition 4-1. *For given N_1 and N_2 and where $dc_1/d\Delta t$ and $dc_2/d\Delta t$ exist, we have:*

i) If $\Delta t<\frac{\gamma}{\beta+\gamma}\frac{N_1}{s}+\frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$,

$$\frac{dc_1}{d\Delta t}\le 0;\ \frac{dc_2}{d\Delta t}\le 0. \tag{8}$$

ii) Otherwise,

$$\frac{dc_1}{d\Delta t}=0;\ \frac{dc_2}{d\Delta t}>0. \tag{9}$$

Proof. When $\Delta t<\frac{\gamma}{\beta+\gamma}\frac{N_1}{s}+\frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$, the arrivals of household travelers and individual travelers are connected (Cases 2, 3 and 4), and otherwise the arrivals of the two classes of travelers are separated (Case 1). Based on Table 3, we see that Eq.(8) and Eq.(9) hold. ■

Note that c_1 and c_2 are continuous while not differentiable at the boundary cases. Part (i) of Proposition 4-1 indicates that when departure/arrival of two classes of travelers is close to each other (the arrivals are connected), increasing Δt is beneficial. This is because, separating departure/arrival of two classes of travelers can reduce congestion, and the reduced congestion cost overweighs the schedule delay increase led by a larger Δt (if any). The rational of Part (ii) of Proposition 4-1 is that when the arrivals of two classes of travelers are already fully disconnected, increasing Δt will only increase the schedule delay cost of household travelers, as well as their travel cost.

Furthermore, with Table 3, we can find the first-order derivatives of c_1 and c_2 with respect to both N_1 and N_2, which are summarized in Table 4. As we can see, in most cases we have $\frac{\partial c_1}{\partial N_1}>0$, $\frac{\partial c_1}{\partial N_2}>0$, $\frac{\partial c_2}{\partial N_1}>0$ and $\frac{\partial c_2}{\partial N_2}>0$. This is because, more travelers lead to more severe congestion and larger schedule delay cost. However, in Case 1, when the two classes of travelers have no interaction, $\frac{\partial c_1}{\partial N_2}=0$ and $\frac{\partial c_2}{\partial N_1}=0$. We also notice the following result.

Proposition 4-2. Wherever $\frac{\partial c_1}{\partial N_1}$, $\frac{\partial c_1}{\partial N_2}$, $\frac{\partial c_2}{\partial N_1}$ and $\frac{\partial c_2}{\partial N_2}$ exist, we have

$$\frac{\partial c_1}{\partial N_1} \geq \frac{\partial c_1}{\partial N_2}; \frac{\partial c_2}{\partial N_2} \geq \frac{\partial c_2}{\partial N_1}. \qquad (10)$$

Proposition 4-2 indicates that the equilibrium travel cost of a user group tends to be more sensitive to the number of users in that group than the number in the other group. This result is similar to that in Lindsey (2004) even if the combined preferences of two members in a household bring further complexity (note that Lindsey's paper only considers individual travelers).

Table 4. Travel costs of individual traveler and household for different cases

Case	$\frac{\partial c_1}{\partial N_1}$	$\frac{\partial c_1}{\partial N_2}$	$\frac{\partial c_2}{\partial N_1}$	$\frac{\partial c_2}{\partial N_2}$
1	$\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s}$	0	0	$\frac{2\beta(\gamma-\beta)}{\beta+\gamma}\frac{1}{s}$
2	$\frac{(3\beta+\gamma)\gamma}{3(\beta+\gamma)}\frac{1}{s}$	$\frac{2\beta\gamma}{3(\beta+\gamma)}\frac{1}{s}$	$\frac{4\beta\gamma}{3(\beta+\gamma)}\frac{1}{s}$	$\frac{2\beta(\beta+3\gamma)}{3(\beta+\gamma)}\frac{1}{s}$
3	$\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s}$	$\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s}$	$\frac{2\beta\gamma}{\beta+\gamma}\frac{1}{s}$	$\frac{2\beta\gamma}{\beta+\gamma}\frac{1}{s}$
4	$\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s}$	$\frac{\beta\gamma}{\beta+\gamma}\frac{1}{s}$	$\frac{2\beta\gamma}{\beta+\gamma}\frac{1}{s}$	$\frac{2\beta\gamma}{\beta+\gamma}\frac{1}{s}$

Proposition 4-3. *For given Δt, suppose $N_1+N_2=N$ is fixed, where the following derivatives exist, we have*

$$\frac{dc_1}{dx} \leq 0; \frac{dc_2}{dx} \geq 0, \qquad (11)$$

where $x = N_2/N$.

Proof. See Appendix C.

Proposition 4-3 indicates that, given the total number of users N and the schedule difference Δt, increasing the proportion of households (i.e., increase of x, which means there will be less individual travelers) is beneficial to individual travelers (their travel costs decrease or at least do not increase) and is unfavorable to households (their travel costs increase or at least do not decrease). Furthermore, from Eq.(32), we know that within Case 3 and Case 4, where one class of travelers is dominating the population (Case 3: proportion of households is relatively large; Case 4: proportion of individual travelers is relatively large), the variation of the proportions of two classes of travelers will not change the travel costs (per household or per individual) of both households and individual travelers (however, total travel cost of all travelers will change).

4.2. Total cost and schedule coordination

Given the travel costs of individual travelers and households presented in Table 3, we can easily obtain the total travel cost of all the users in all cases, i.e.,

$$TC = c_1 \cdot N_1 + c_2 \cdot N_2. \tag{12}$$

Suppose the system authority is able to coordinate schedules of school and work, we have the following result.

Proposition 4-4. *Given N_1 and N_2, it is optimal to set Δt such that the departure/arrival pattern is the critical case between Case 1 and Case 2. Therefore, we have*

$$\left(\Delta t\right)^* = \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}. \tag{13}$$

Proof. See Appendix D.

Proposition 4-4 provides the optimal schedule difference between school and work. This proposition implies that by appropriately coordinating the school and work schedules, the total travel cost can be reduced. Under the optimal schedule coordination given in Eq.(13), the departure/arrival of the households and individual travelers will be completely separated. Intuitively, when departure/arrival of the households and individual travelers are fully separated (at $\Delta t = \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$), a further increase in Δt will not be beneficial as it only leads to capacity waste between the arrivals of individual travelers and households ($\frac{dTC}{d\Delta t} > 0$). More specifically, increasing Δt (traffic pattern belongs to Case 1), the queuing delay will not decrease while schedule delay cost of households will increase (due to a larger school and work schedule gap Δt). We further display how the total travel cost, schedule delay cost and queuing delay cost will vary with Δt given the numbers of individual travelers and households in the following Figure 4. Note that in some intervals in Case 3 and Case 4, the departure/arrival of households and individual travelers can be mixed, the queuing delay cost and schedule delay cost cannot be determined uniquely (the total travel cost is identical).

Therefore, we take the illustrative situations depicted in Figure 2(c) and Figure 2(d) to compute the queuing delay cost and schedule delay cost for Figure 4.

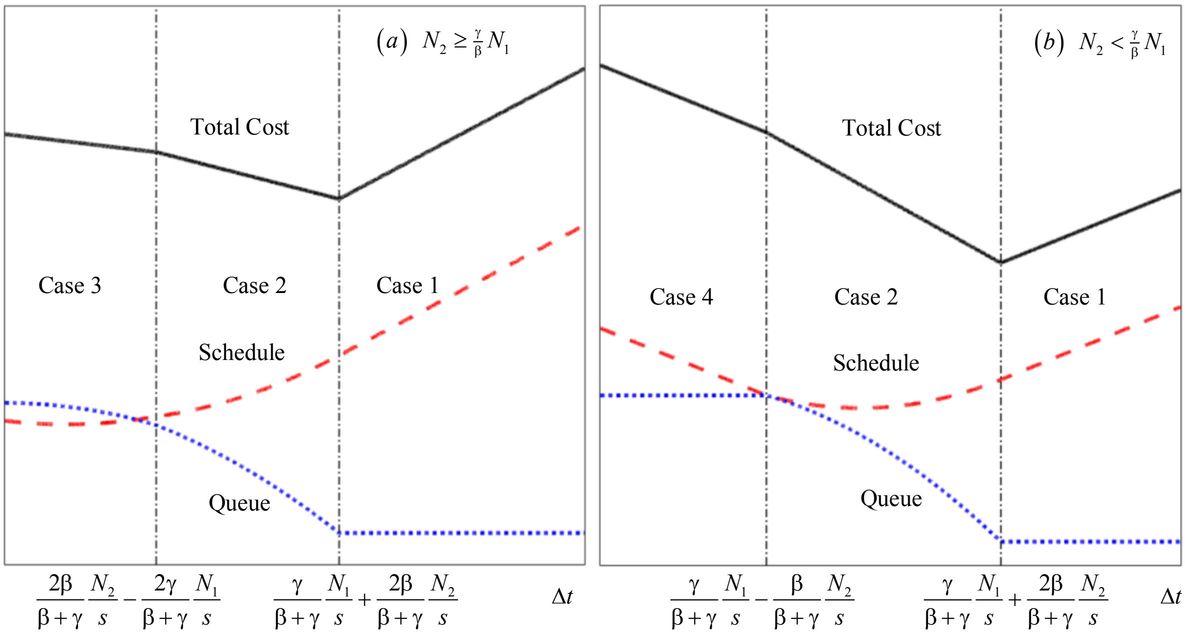


Figure 4. Costs vary with Δt : total travel cost, schedule delay cost and queuing delay cost: (a) The case with $N_2 \geq \frac{\gamma}{\beta} N_1$; (b) The case with $N_2 < \frac{\gamma}{\beta} N_1$

As can be seen in Figure 4, when Δt increases, there is a trade-off between the total queuing delay cost and total schedule delay cost. The queuing delay cost is non-increasing over Δt. This is expected as a larger Δt further separates the departure/arrival of households and individual travelers, and traffic congestion is temporally relieved.

However, the schedule delay cost might not always increase with Δt. For example, for Case 4 in Figure 4(b), the schedule delay cost is decreasing. This is because, the number of individual travelers is relatively large and all households are forced to arrive earlier for school and work (one may refer to the traffic pattern in Figure 2(d)). By increasing Δt (suppose t_2^* is fixed, then it is equivalent to decreasing t_1^*), without changing the traffic pattern, the households will have less earliness for school. Also, for Case 2 (in Figure 4(b)), the schedule delay cost is decreasing with Δt in the beginning. This is because, a larger Δt will allow more individual travelers to arrive at the destination during the Δt (between t_1^* and t_2^*), and there will be less late arrival individual travelers (note that late arrival penalty is larger than the value of time and early arrival penalty). In Case 1, the schedule delay cost always increases with Δt, simply because a larger work and school schedule gap leads to larger schedule delays (associated with work) for households.

4.3. Efficiency of schedule coordination

Following Proposition 4-4, the total travel cost of travelers under the optimal Δt in Eq.(13) can be given as follows:

$$TC^* = \left(\frac{\beta\gamma}{\beta+\gamma} \frac{N_1}{s} \right) \cdot N_1 + \left(\frac{\beta\gamma}{\beta+\gamma} \frac{N_1}{s} + \frac{2\beta\gamma}{\beta+\gamma} \frac{N_2}{s} \right) \cdot N_2. \tag{14}$$

We consider $\Delta t = 0$ as the **benchmark case** for efficiency analysis in which the total travel cost is

$$TC^0 = \left(\frac{\beta\gamma}{\beta+\gamma} \frac{N_1}{s} + \frac{\beta\gamma}{\beta+\gamma} \frac{N_2}{s} \right) \cdot N_1 + \left(\frac{2\beta\gamma}{\beta+\gamma} \frac{N_1}{s} + \frac{2\beta\gamma}{\beta+\gamma} \frac{N_2}{s} \right) \cdot N_2. \tag{15}$$

Note that, in this benchmark case, as the work schedule coincides with the school schedule, all travelers have the same desired arrival time, and the commuting traffic pattern would be identical to that under standard bottleneck model (e.g., Arnott et al., 1990), which is similar to that in Figure 2(d).

We then can define the percentage of cost reduction as

$$\theta = \frac{TC^0 - TC^*}{TC^0}, \tag{16}$$

where $TC^0 - TC^*$ is the cost reduction under optimal schedule coordination, compared with the benchmark case with $\Delta t = 0$, and TC^0 is the travel cost at the benchmark case. The percentage θ can be regarded as relative efficiency of the schedule coordination. Regarding this relative efficiency, we have the following proposition.

Proposition 4-5. *The percentage of cost reduction by setting Δt according to Eq.(13) is*

$$\theta = \frac{2N_1N_2}{(N_1+N_2)(N_1+2N_2)}. \tag{17}$$

With Eq.(14) and Eq.(15), after some manipulations, the relative efficiency defined in Eq.(16) can be rewritten as that in Eq.(17). From Eq.(17), it is straightforward to see that school and work schedule coordination have relatively small efficiency as either $N_1 \to 0$ or $N_2 \to 0$. This is because, the travel cost reduction of schedule coordination comes from the separation between the departure/arrival of the two classes of travelers. As either N_1 or N_2 is relatively small, the impact of schedule coordination is limited by the small amount of traffic, i.e., $\min\{N_1, N_2\}$.

Proposition 4-5 also indicates that the relative efficiency of the schedule coordination can be determined in the way only related to the numbers of individual travelers and households. Furthermore, as we can define $x = N_2/N$ (in Proposition 4-3), thus $1-x = N_1/N$, we have

$$\theta = \frac{2x(1-x)}{1+x}. \tag{18}$$

This means that the efficiency depends on the relative proportions of the two classes of travelers instead of the magnitude of the numbers of travelers. We further bound this efficiency as shown in the next proposition.

Proposition 4-6. *The percentage reduction in Eq.(17) satisfies*

$$\theta \le 6-4\sqrt{2}. \tag{19}$$

Note that $6-4\sqrt{2} \approx 34.31\%$. It is easy to verify that when $x = \sqrt{2}-1$ (around 41.42%), Eq.(18) will reach the maximum value $6-4\sqrt{2}$.

We now further look at the system optimum where queueing is fully eliminated and schedule delay is minimized. This can be achieved by letting $\Delta t = 0$ and implementing a first-best time-varying toll given as follows (joint scheme of schedule coordination and pricing).

$$\tau(t) = \begin{cases} 0 & t < t_1 \\ \frac{\beta\gamma}{\beta+\gamma}\frac{N_1+N_2}{s} - \beta\left(t_1^* - t\right) & t_1 \le \Delta t < t_2 \\ \frac{\beta\gamma}{\beta+\gamma}\frac{N_1+2N_2}{s} - 2\beta\left(t_1^* - t\right) & t_2 \le \Delta t < t_1^* \\ \frac{\beta\gamma}{\beta+\gamma}\frac{N_1+2N_2}{s} - 2\gamma\left(t - t_1^*\right) & t_1^* \le \Delta t < t_3 \\ \frac{\beta\gamma}{\beta+\gamma}\frac{N_1+N_2}{s} - \gamma\left(t - t_1^*\right) & t_3 \le \Delta t \le t_4 \\ 0 & t > t_4 \end{cases} \tag{20}$$

where $t_1 = t_1^* - \frac{\gamma}{\beta+\gamma}\frac{N_1+N_2}{s}$, $t_2 = t_1^* - \frac{\gamma}{\beta+\gamma}\frac{N_2}{s}$, $t_3 = t_1^* + \frac{\beta}{\beta+\gamma}\frac{N_2}{s}$ and $t_4 = t_1^* + \frac{\beta}{\beta+\gamma}\frac{N_1+N_2}{s}$. The optimal toll is depicted in Figure 5. It is worth mentioning that this toll is similar to the first-best time-varying toll for the single bottleneck model with two classes of travelers (with identical t^*): N_1 travelers with schedule penalties β and γ, and N_2 travelers with schedule penalties 2β and 2γ.

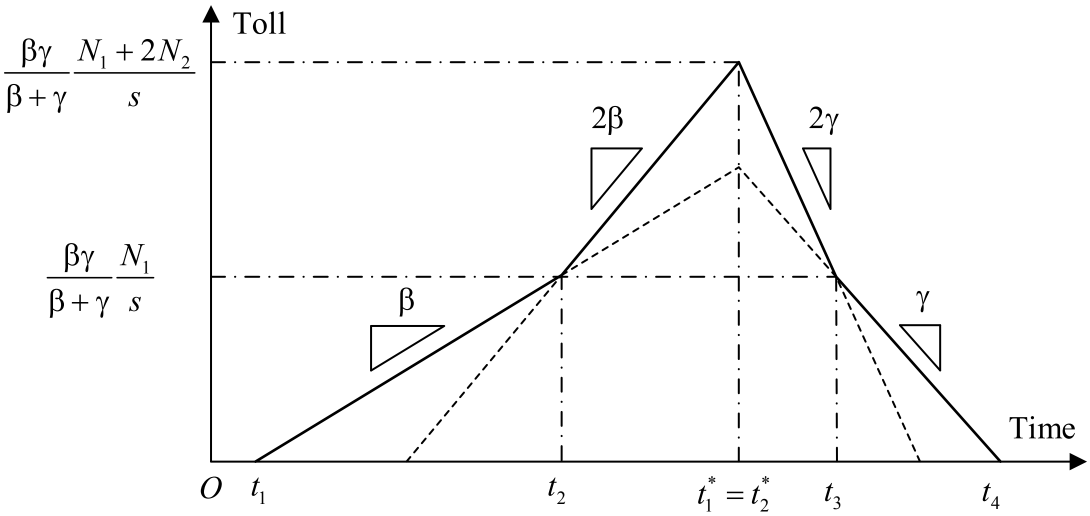


Figure 5. The optimal time-varying toll considering schedule coordination

The total social cost under the above discussed system optimum is

$$TC^{SO} = \left(\frac{1}{2}\cdot\frac{\beta\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{\beta\gamma}{\beta+\gamma}\frac{N_2}{s}\right)\cdot N_1 + \left(\frac{\beta\gamma}{\beta+\gamma}\frac{N_2}{s}\right)\cdot N_2. \tag{21}$$

This cost can be regarded as the lower bound of the total travel cost of travelers. Similarly, the relative efficiency of the optimal joint scheme of schedule coordination and time-varying pricing can be defined as

$$\theta^{SO} = \frac{TC^0 - TC^{SO}}{TC^0}. \tag{22}$$

The relative efficiency is further simplified in Eq.(23).

Proposition 4-7. *The efficiency of the optimal joint scheme of schedule coordination and time-varying pricing is*

$$\theta^{SO} = \frac{1}{2} + \frac{1}{2}\frac{N_1 N_2}{(N_1+N_2)(N_1+2N_2)}. \tag{23}$$

It is obvious that θ^{SO} is greater than 50%. However, $\theta^{SO} \to 50\%$ if either N_1 or N_2 approaches zero. Suppose $N_2 \to 0$, the situation approaches the case with identical individual travelers, and the first-best pricing described in Figure 5 would lead to $\theta^{SO} = 50\%$, which is well noticed in the literature. Moreover, the departure/arrival order of the travelers will make no difference as they are identical. However, when $N_2 > 0$, an appropriate departure/arrival order of the two classes of travelers (the one described in Figure 5) is needed to minimize the schedule delay cost. This additional gain arising from appropriate departure/arrival order results in an efficiency larger than 50%.

Similarly, the efficiency in Eq.(23) can be rewritten as

$$\theta^{SO} = \frac{1}{2} + \frac{1}{2}\frac{x(1-x)}{1+x}. \tag{24}$$

where $x = N_2/N$ (in Proposition 4-3), thus $1-x = N_1/N$. The efficiency then only depends on the relative proportions of the two classes of travelers. We further bound the efficiency as the following.

Proposition 4-8. *The efficiency of the optimal joint scheme of schedule coordination and time-varying pricing satisfies*

$$\theta^{SO} \le 2-\sqrt{2}. \tag{25}$$

Note that $2-\sqrt{2} \approx 58.58\%$. Combining Proposition 4-7 and Proposition 4-8, we notice that the efficiency given in Eq.(22) satisfying $50\% \le \theta^{SO} \le 58.58\%$. Also, it is easy to verify that when $x = \sqrt{2}-1$ (around 41.42%), Eq.(25) will reach the maximum value $2-\sqrt{2}$.

Without pricing, while we can reduce traffic congestion and travel cost by coordination of work and school schedules, we lose certain level of efficiency. We define the loss of efficiency as follows:

$$l = \frac{TC^0 - TC^*}{TC^0 - TC^{SO}}, \tag{26}$$

where $TC^0 - TC^*$ is the amount of travel cost that can be further reduced by the joint optimum scheme compared with pure schedule coordination, and $TC^0 - TC^{SO}$ is the maximum cost reduction that can be achieved by implementing the joint scheme. We have the following proposition regarding Eq.(26).

Proposition 4-9. *The efficiency loss of the schedule coordination without pricing compared with the optimal joint scheme of schedule coordination and time-varying pricing is*

$$l = \frac{2N_1N_2}{(N_1)^2 + 4N_1N_2 + 2(N_2)^2}. \tag{27}$$

Similarly, the efficiency loss in Eq.(27) can be written as

$$l = \frac{2x(1-x)}{1+2x-x^2}. \tag{28}$$

where $x = N_2/N$ (in Proposition 4-3), thus $1-x = N_1/N$. The efficiency loss can be bounded as follows.

Proposition 4-10. *The efficiency loss in Eq.(27) satisfies*

$$l \le 1 - \frac{\sqrt{2}}{2}. \tag{29}$$

Note that $1-\frac{\sqrt{2}}{2} \approx 29.29\%$. It is easy to verify again that when $x=\sqrt{2}-1$, Eq.(25) will reach the maximum value $1-\frac{\sqrt{2}}{2}$.

In summary, schedule coordination can help reduce travel cost (against benchmark case) by up to 34.31%, while the joint scheme of coordination and pricing can reduce total cost by at least 50% and at most 58.58%. Schedule coordination loses efficiency compared with the joint scheme. However, this efficiency loss can be upper bounded.

5. Numerical Analysis

In this section, numerical experiments are conducted to verify and illustrate the essential ideas in the paper. Following Liu et al. (2015b), we take the value of time, schedule delay penalties as follows: $\alpha = 9.91$ (EUR\$/hour), $\beta = 4.66$ (EUR\$/hour), $\gamma = 14.48$ (EUR\$/hour). The highway bottleneck capacity is $s = 30$(veh/min).

5.1. Cost contours in the domain of (N_1, N_2)

Given Δt (here we use 30 minutes), Figure 6 displays the total travel cost contours, individual travel cost contours, and the household travel cost contours in the domain of (N_1, N_2). In Figure 6, the red solid line represents Line 1 in Figure 3, and the red dashed line represents Line 2 in Figure 3, and the red dash-dotted line represent Line 3 in Figure 3. These three lines divide the domain of (N_1, N_2) into four regions, which correspond to Case 1, Case 2, Case 3 and Case 4 (as shown in Figure 3).

As can be seen in Figure 6(a), the total travel cost increases with N_1 and N_2. This is because, a larger number of travelers indicates a larger travel cost for either household or individual traveler (one can verify from Figure 6(b) and Figure 6(c)). This total travel cost increase is nonlinear as the contours become denser when N_1 or N_2 is larger (for Figure 6(a)). We also notice that the individual travel cost in Figure 6(b) and the household travel cost in Figure 6(c) linearly increase (or at least do not decrease) with N_1 or N_2 in each region (there are four regions in total, therefore, household travel cost and individual travel cost are both piece-wise linear over N_1 and N_2). These results are consistent with those in Table 4.

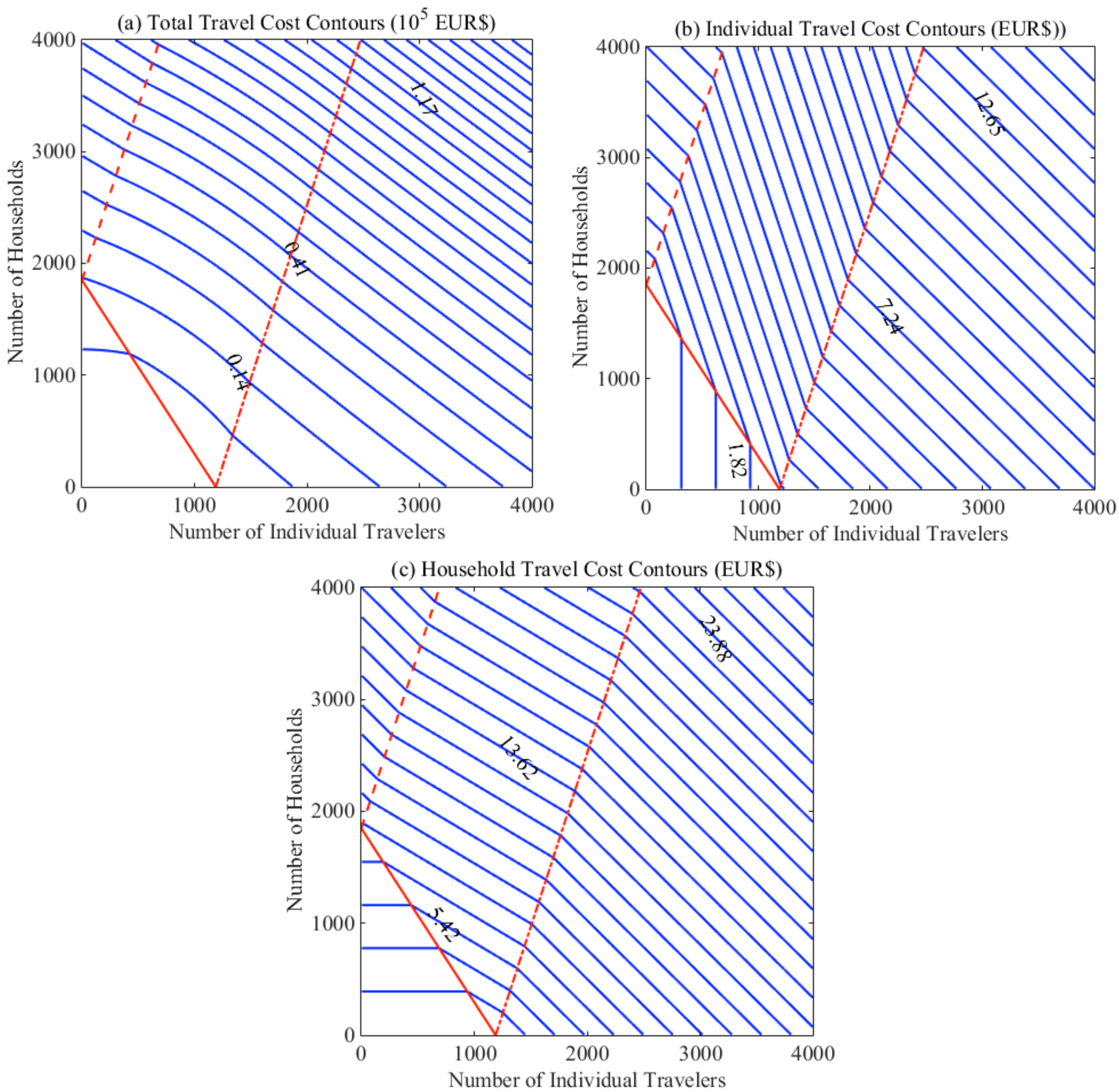
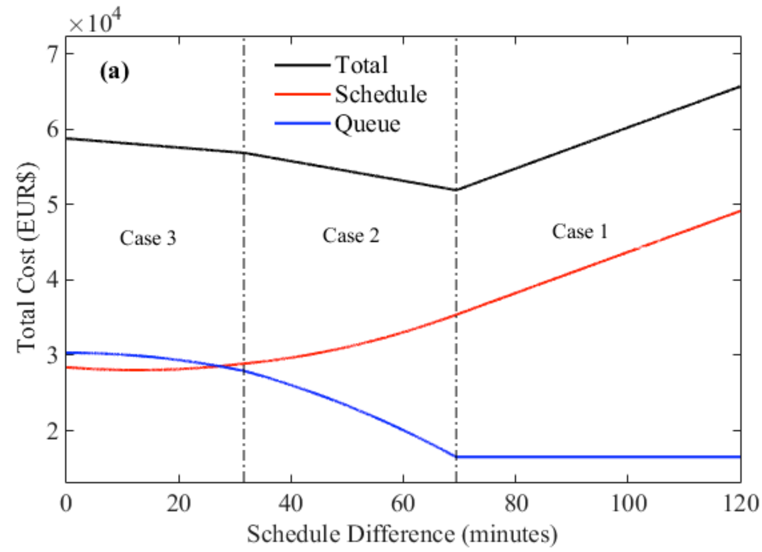


Figure 6. Costs Contours in the domain of (N_1, N_2)

5.2. Total cost, schedule delay cost, and queuing delay cost

Given the travel demand in each user class, Figure 7 and Figure 8 show how total travel cost, total travel delay cost, total schedule delay cost, as well as these total costs for individual travelers or households, vary with Δt. Figure 7 shows the case with $N_2 \geq \frac{\gamma}{\beta} N_1$ ($N = 4000$, $N_1 = 500$ and $N_2 = 3500$), and Figure 8 shows the case with $N_2 < \frac{\gamma}{\beta} N_1$ ($N = 4000$, $N_1 = 2000$ and $N_2 = 2000$). As mentioned in our discussion about Figure 4, for Case 3 and Case 4, we take the illustrative situations depicted in Figure 3(c) and Figure 3(d) to compute the queuing delay cost and schedule delay cost.



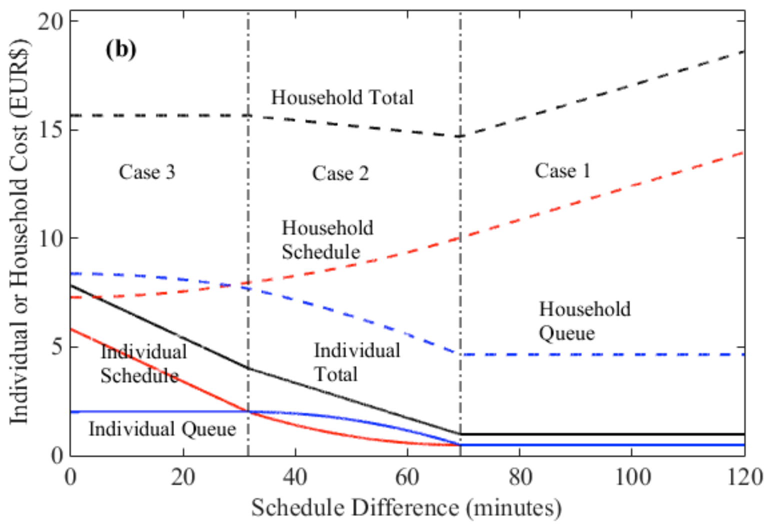


Figure 7. Costs vary with the schedule difference Δt : the case with $N_2 \geq \frac{\gamma}{\beta} N_1$

As stated in Proposition 3-1, for the case with $N_2 \geq \frac{\gamma}{\beta} N_1$, when Δt increases, the traffic pattern shifts from Case 3 to Case 2, and then to Case 1. This is also displayed in Figure 7. Indeed Figure 7(a) is the numerical verification of Figure 4(a). However, with Figure 7(b), we further see that total schedule delay cost of individual travelers decreases or at least does not increase with Δt. This is because, as Δt becomes larger, the impact of households on individual travelers is lessening (i.e., less individual travelers are pushed to arrive late, the critical case (Case 1) is that Δt becomes large enough, departure/arrival of the two classes of travelers are completely separated), thus individual travelers have smaller schedule delay costs. Also we notice that the queueing delay costs for households and individual travelers are both non-increasing with Δt , which means both of them benefit from the schedule coordination in terms of travel delays.

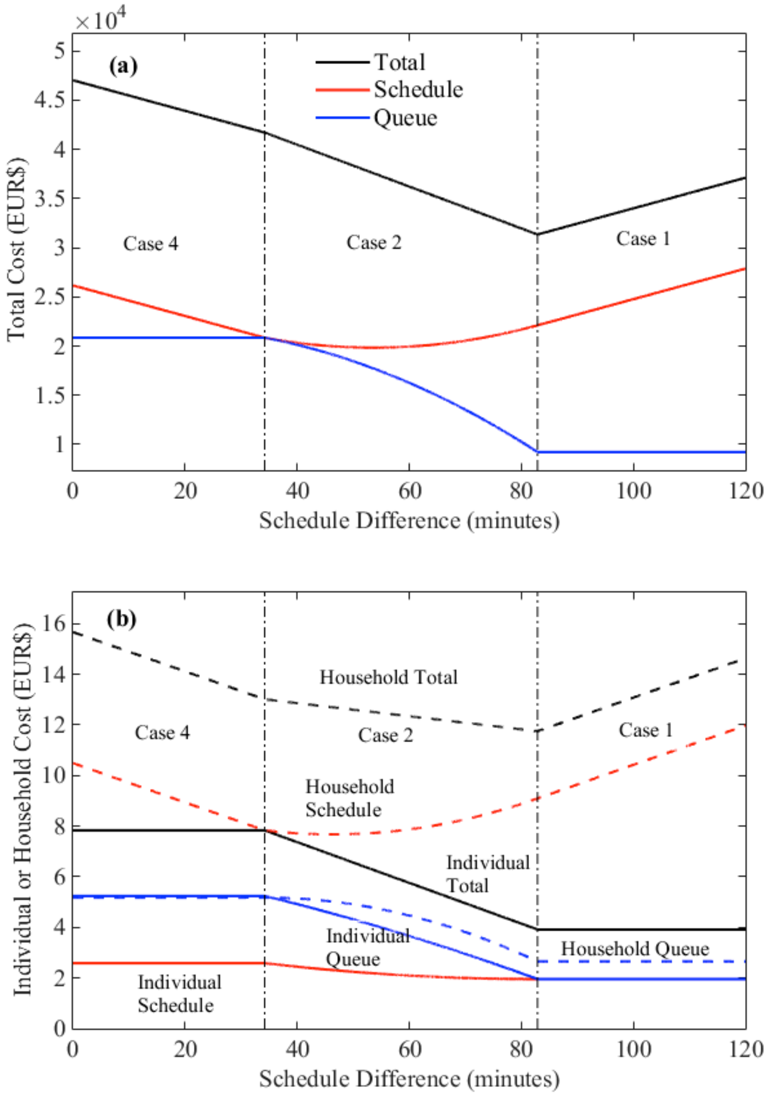


Figure 8. Costs vary with the schedule difference Δt : the case with $N_2 < \frac{\gamma}{\beta} N_1$

Similarly, Figure 8(a) repeats Figure 4(b) numerically. As already discussed about Figure 4(b), in Case 4, the total schedule delay cost decreases with Δt (this is now verified numerically in Figure 8(a)). If we further look at Figure 8(b), we notice that (in Case 2) the total schedule delay cost of individual travelers decreases with Δt . This is because, as discussed in Section 4, a larger Δt allows more individual travelers of early arrival. Moreover, we see that in Case 2, total schedule delay cost of households can decrease with Δt in the beginning. This is explained as follows. When Δt is larger, households are pushed to arrive at work further away from t_2^* (there are more individual travelers arriving early for work), and their scheduled delay cost associated with work becomes larger. But the saving in schedule delay cost associated with school might be even larger (when Δt is larger, indeed the two classes of travelers affect each other less).

5.3. Household and individual costs vary with schedule different Δt

Given total travel demand N and Δt (we use 30 minutes here), Figure 9 displays how the household and individual travel costs and the average cost of all the travelers vary with the proportion of the household travelers, i.e., $x = N_2/N$. And Figure 9(a), Figure 9(b) and Figure 9(c) represent the three situations with different demand levels, i.e., $N = 1000$, $N = 1500$, and $N = 2500$. The average cost is defined as the total travel cost divided by N, i.e., TC/N, where TC is given by Eq.(12).

There are indeed two critical values of N, i.e., $\frac{\beta+\gamma}{\gamma} \cdot \Delta t \cdot s = 1190$ and $\frac{\beta+\gamma}{2\beta} \cdot \Delta t \cdot s = 1848$, both of which are depicted in Figure 3, as well as Figure 6. Note that for given N, varying x from zero to one corresponds to the line $N_1 + N_2 = N$ in the domain of (N_1, N_2). When $N < \frac{\beta+\gamma}{\gamma} \cdot \Delta t \cdot s$, e.g., $N = 1000$ in Figure 9(a), the traffic pattern will always be Case 1; when $\frac{\beta+\gamma}{\gamma} \cdot \Delta t \cdot s < N < \frac{\beta+\gamma}{2\beta} \cdot \Delta t \cdot s$, e.g., $N = 1500$ in Figure 9(b), the traffic pattern will vary from Case 4 to Case 2 and then to Case 1 if we increase x from zero to one; when $N > \frac{\beta+\gamma}{2\beta} \cdot \Delta t \cdot s$, e.g., $N = 2500$ in Figure 9(c), the traffic pattern varies from Case 4 to Case 2 and then to Case 3 (Case 1 is no longer possible). These results are more obvious if one also refers to Figure 3 in Section 3.

In all three situations in Figure 9, individual travel cost always decreases or at least does not increase with x, and household travel cost increases or at least does not decrease over x, which are also stated in Proposition 4-3. However, we notice that the average cost (as well as the total travel cost, since N is constant in each situation) might decrease with x. This is somehow counterintuitive as we might expect that household travel cost is larger than individual travel cost (there are two persons in a household, household travel cost is “double cost”), and a larger x means that there are more in the total demand N with “double cost”.

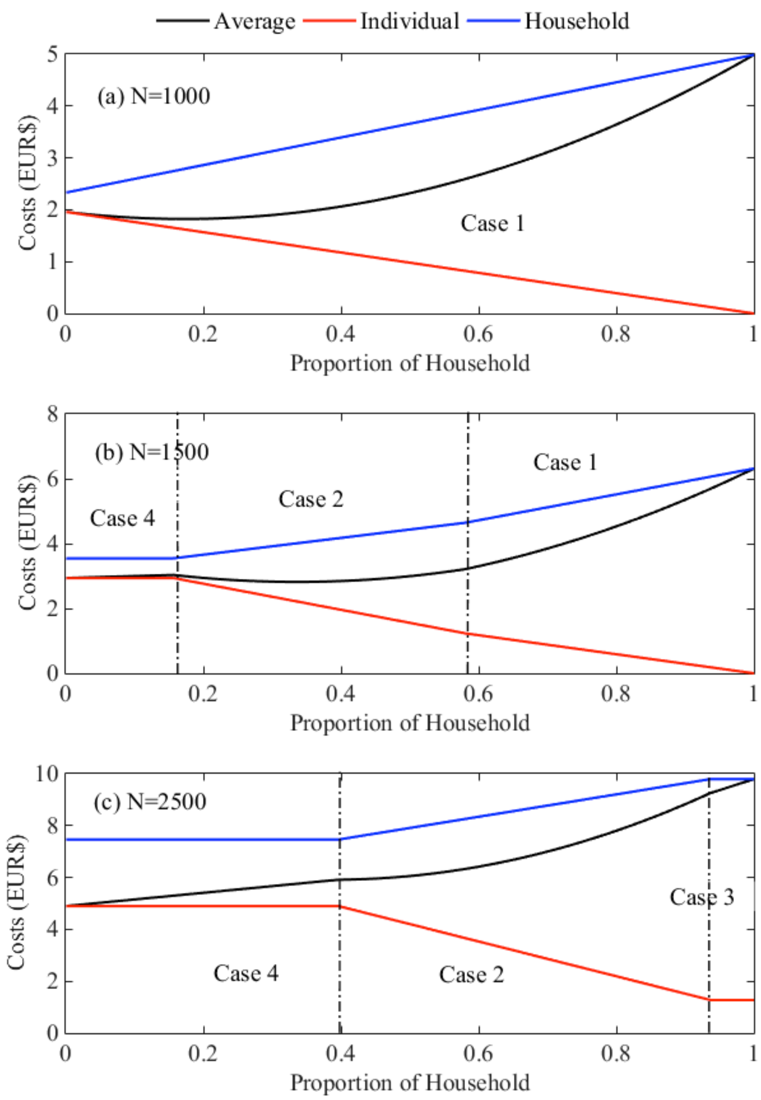


Figure 9. Costs vary with $x = N_2/N$

6. Conclusions

In this study, we examined how the rush-hour traffic pattern with household and individual travels differs from that generated by the conventional standard bottleneck model, and how it changes with the proportions of the two classes of commuters, and the time difference between school schedule and work schedule. This paper incorporates the model in Jia et al. (2016) as a special or extreme case, where the number of individual commuters is equal to zero.

Besides modeling and analyzing the possible dynamic traffic patterns at the departure/arrival equilibrium with mixed travelers, we propose to coordinate the work and school schedules to temporally relieve traffic congestion and thus reduce the total travel cost of travelers. Efficiency of such schedule coordination has been evaluated and bounded. Moreover, we found that total travel cost can decrease with the proportion of household travelers in the population. In addition, our numerical experiment suggests that the schedule delay cost does not always increase with the difference between two desired arrival times (for work and school).

As mentioned in the paper, the value of time and schedule delay penalties for school (usually this is associated with the child in the household) is considered to be identical to those for work (usually this is associated with the adult in the household). This assumption or simplification makes the algebra in the paper much less tedious (while still quite complex). However, the main idea and modeling framework in the paper would be still valid even if we consider different VOTs and schedule penalties for school and for work. Our future study will consider that all travelers (both households and individual travelers) are heterogeneous in their VOTs and schedule penalties.

Kuwahara (1990) is one of the leading researches to examine the dynamic traffic equilibrium under two tandem bottlenecks. The network in this paper (as shown in Figure 1) would be similar to Kuwahara (1990) if a bottleneck between school and workplace is added. Our future work will try to consider such a network with a bottleneck between home and school, and with a bottleneck between school and work. Under this network setting, we will try to incorporate three different types of trips: individual work trip, household trip, and school trip (only goes to school but not work). It is worth mentioning that, our work here focuses more on how household travel (departure time choices are governed by both work and school schedules) is different from individual travel, and how interactions between household travel and individual travel re-shapes the morning commute, while we ignore the network topology's (e.g., two tandem bottlenecks) impact.[4]

Besides, this study only considers the shared ride of household (family) members. Our future study will extend to ride sharing of non-family members. In that case, schedule coordination might not only help to reduce congestion, but also help to encourage ride sharing of travelers, which can further benefit the system (e.g., fewer vehicles, less fuel consumption).

Acknowledgements. The authors thank the anonymous referees for their helpful comments, and also thank Prof. Robin Lindsey at University of British Columbia for his suggestions on an earlier draft of this paper. Any errors that remain are the authors' sole responsibility. The work described in this paper was supported by a grant from Hong Kong's Research Grants Council (HKUST16205715).

[4] We also do not consider that household members may travel together through public transit. Future work may take this into account and consider shared-ride of families in multi-modal systems such as those in e.g., Zhang et al. (2014), Zhang et al. (2016).

Appendix A. The General Case

In Section 2, we assumed that the free-flow travel times and delay at school are zero, which helps to reduce the burden of the tedious algebra and also helps us to focus on the central idea in the paper, i.e., mixed travelers, schedule coordination. Now we turn to briefly discuss the more general case where the free-flow travel times between home and bottleneck, and between bottleneck and work place, and the delay at school are not zero, which are denoted by t_1^f, t_2^f, and t_{sd}, respectively. Specifically, for both individual and household travelers, they need to spend a time of t_1^f to reach the highway bottleneck (from home). After passing the highway bottleneck, the individual travelers will go to work place directly, where the free-flow time between highway bottleneck and work place is t_2^f; and household travelers have to endure an extra delay t_{sd} to reach school and then go to work with a free-flow travel time of t_2^f.[5] Similar to the major analysis in the paper, we still consider the on-time travelers for school can be early for work. Thus, $t_2^\# - t_1^\# \geq t_{sd}$, where we let $t_2^\# = t_1^* - t_{sd}$ and $t_2^\# = t_2^* - t_2^f$.

Given the above, we can derive all the commuting equilibrium traffic patterns, which are shown in Figure 10. Furthermore, the conditions for occurrence of each flow pattern are summarized in Table 5. In presence of the non-zero free-flow times and delay at school, the arrival at bottleneck is different from the departure from home, and the arrivals at school/work are later than the departure from the bottleneck (as shown in Figure 10). Figure 10(a), Figure 10(b), Figure 10(c) and Figure 10(d) correspond to Figure 2(a), Figure 2(b), Figure 2(c) and Figure 2(d) respectively. Note that similar to Figure 2(c) and Figure 2(d), Figure 10(c) and Figure 10(d) are illustrative where indeed the departure and arrival of some households can be mixed with the individual travelers.

While the free-flow times t_1^f and t_2^f indeed do not change the equilibrium traffic pattern (i.e. the arrival at bottleneck and the departure from bottleneck in Figure 10), the delay due to traveling to school, i.e., t_{sd}, can slightly affect the equilibrium, which leads to Case (e) and Case (f) shown in Figure 10(e) and Figure 10(f). This is because, the delay at school can postpone the arrival of the adult in the household at work, thus he or she might be late for work even if an individual traveler departing from home at the same time can be early for

[5] While we assume that the households arrive at school after the delay t_{sd}, analysis for the situations where arrival at school occur before the delay t_{sd} or in the middle of t_{sd} would be very similar. Also, one may argue that free-flow between school and workplace might be different from free-flow time between bottleneck and workplace. However, this would make negligible impacts on the model.

work. These two cases will never occur in the original situation where the delay due to school is not considered. Even though it can still be shown that appropriate schedule coordination can reduce travel cost by separating departure/arrival of households and individual travelers. Also note that if $t_{sd} \rightarrow 0$, Case (e) and Case (f) in Figure 10 approaches Case (a) and Case (b).

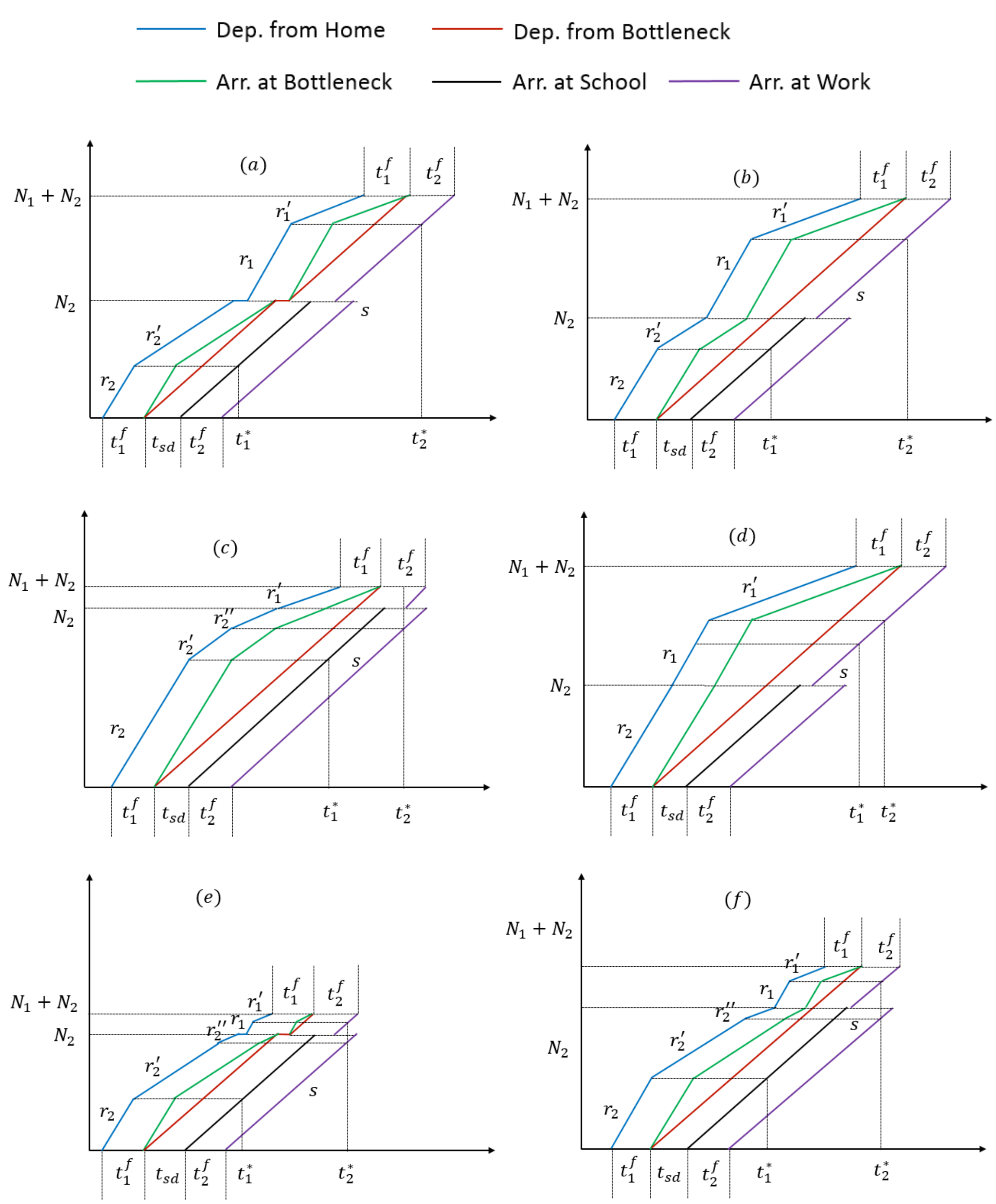


Figure 10. Possible commuting patterns when free-flow times and delay due to school are non-zero

Table 5. Conditions for the cases depicted in Figure 10

Equilibrium	Conditions
Case (a)	$t_2^\# - t_1^\# \geq \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$; $t_2^\# - t_1^\# \geq t_{sd} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$
Case (b)	$t_2^\# - t_1^\# < \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$; $t_2^\# - t_1^\# > 3t_{sd} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$; $t_2^\# - t_1^\# > \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\beta}{\beta+\gamma}\frac{N_2}{s}$
Case (c)	$t_2^\# - t_1^\# \leq -t_{sd} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$
Case (d)	$t_2^\# - t_1^\# \leq \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\beta}{\beta+\gamma}\frac{N_2}{s}$
Case (e)	$t_2^\# - t_1^\# < t_{sd} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$; $t_2^\# - t_1^\# \geq -t_{sd} + \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$
Case (f)	$t_2^\# - t_1^\# \leq 3t_{sd} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$; $t_2^\# - t_1^\# < -t_{sd} + \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$; $t_2^\# - t_1^\# > -t_{sd} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$

Note: $t_2^\# - t_1^\# \geq t_{sd}$ always holds.

We can further translate the conditions in Table 5 into the domain of (N_1, N_2) in Figure 11, which is similar to Figure 3 (we omit the details). One can also see from Figure 11 that once $t_{sd} \to 0$, Case (e) and Case (f) will disappear. In practice, we can expect that t_{sd} is relatively small when compared to the whole journey time for the morning trips.

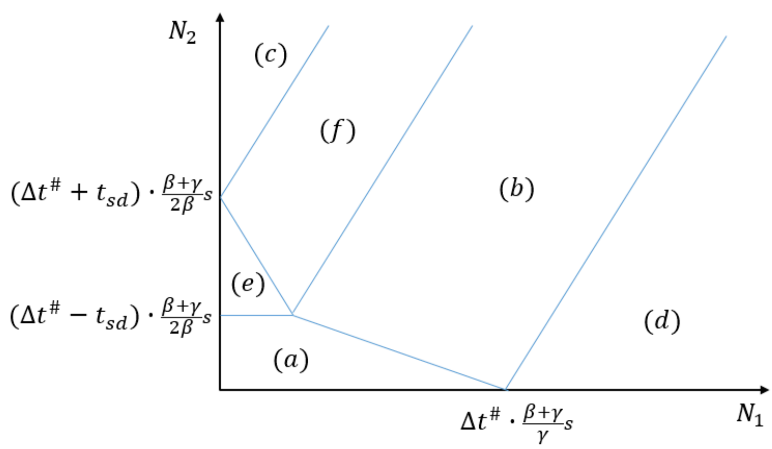


Figure 11. Domain of (N_1, N_2) for commuting equlibrium traffic patterns

Appendix B. Proof of Lemma 3-1.

Proof. When $N_2 \geq \frac{\gamma}{\beta} N_1$, based on Proposition 3-1, we know that the traffic pattern varies from Case 3 to Case 2, and then to Case 1 when we increase Δt from 0^+ to ∞. With Table 2, we then have

$$\frac{dt_1}{d\Delta t} = \begin{cases} 0 & 0 < \Delta t < \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} \\ 1/6 & \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} < \Delta t < \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} \\ -1/2 & \Delta t > \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} \end{cases} \tag{30}$$

Note that t_1 is continuous at $\Delta t = \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s}$ and $\Delta t = \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$ (however, it is not differentiable at these two points).

For the case with $N_2 < \frac{\gamma}{\beta} N_1$, the traffic pattern varies from Case 4 to Case 2, and then to Case 1 as we increase Δt from 0^+ to ∞. We then have

$$\frac{dt_1}{d\Delta t} = \begin{cases} 1/2 & 0 < \Delta t < \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\beta}{\beta+\gamma}\frac{N_2}{s} \\ 1/6 & \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\beta}{\beta+\gamma}\frac{N_2}{s} < \Delta t < \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} \\ -1/2 & \Delta t > \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} \end{cases} \tag{31}$$

Similarly, t_1 is continuous at $\Delta t = \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\beta}{\beta+\gamma}\frac{N_2}{s}$ and $\Delta t = \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$ (not differentiable).

The above analysis for the cases with either $N_2 \geq \frac{\gamma}{\beta} N_1$ or $N_2 < \frac{\gamma}{\beta} N_1$ suggests that t_1 is non-decreasing when $\Delta t < \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$, and is decreasing when $\Delta t > \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$. This proves Lemma 3-1. ■

Appendix C. Proof of Proposition 4-3.

Proof. Note that

$$\frac{dc_i}{dx} = \frac{\partial c_i}{\partial N_1} \cdot \frac{dN_1}{dx} + \frac{\partial c_i}{\partial N_2} \cdot \frac{dN_2}{dx}; \ \frac{dN_1}{dx} = -N; \ \frac{dN_2}{dx} = N.$$

With Table 4, one can verify that

$$\left(\frac{dc_1}{dx}, \frac{dc_2}{dx}\right) = \begin{cases} \left(-\frac{\beta\gamma}{\beta+\gamma}\frac{N}{s}, \frac{2\beta(\gamma-\beta)}{\beta+\gamma}\frac{N}{s}\right) & \text{Case 1} \\ \left(-\frac{\gamma}{3}\frac{N}{s}, \frac{2\beta}{3}\frac{N}{s}\right) & \text{Case 2} \\ (0,0) & \text{Case 3} \\ (0,0) & \text{Case 4} \end{cases} \tag{32}$$

As $N \geq 0$ and $\gamma > \beta$, we easily see the inequalities in Eq.(11) hold. ■

Appendix D. Proof of Proposition 4-4.

Proof. Based on Proposition 3-1, we know that when $N_2 \geq \frac{\gamma}{\beta} N_1$, the traffic pattern varies from Case 3 to Case 2 and then to Case 1 as Δt is increased from 0^+ to ∞. With Eq.(12), we then have

$$\frac{dTC}{d\Delta t} = \begin{cases} -\frac{1}{2}\gamma N_1 < 0 & 0 < \Delta t < \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} \\ -\frac{1}{3}\gamma N_1 - \frac{1}{3}\beta N_2 < 0 & \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s} < \Delta t < \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} \\ \beta N_2 > 0 & \Delta t > \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} \end{cases} \quad (33)$$

Note that TC is continuous at $\Delta t = \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} - \frac{2\gamma}{\beta+\gamma}\frac{N_1}{s}$ and $\Delta t = \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$ (however, it is not differentiable at these two points), we then conclude that in this case, TC is minimized when Eq.(13) holds.

Similarly, for the case with $N_2 < \frac{\gamma}{\beta} N_1$, the traffic pattern varies from Case 4 to Case 2 and then to Case 1 as we increase Δt from 0^+ to ∞. We then have

$$\frac{dTC}{d\Delta t} = \begin{cases} -\beta N_2 < 0 & 0 < \Delta t < \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\beta}{\beta+\gamma}\frac{N_2}{s} \\ -\frac{1}{3}\gamma N_1 - \frac{1}{3}\beta N_2 < 0 & \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\beta}{\beta+\gamma}\frac{N_2}{s} < \Delta t < \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} \\ \beta N_2 > 0 & \Delta t > \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s} \end{cases} \quad (34)$$

Again, TC is continuous at $\Delta t = \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} - \frac{\beta}{\beta+\gamma}\frac{N_2}{s}$ and $\Delta t = \frac{\gamma}{\beta+\gamma}\frac{N_1}{s} + \frac{2\beta}{\beta+\gamma}\frac{N_2}{s}$ (not differentiable). We also can conclude that in this case, TC is minimized when Eq.(13) holds. Proposition 4-4 is then proved. ■

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