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A BILINEAR APPROACH TO A PFAFFIAN SELF-DUAL YANG-MILLS EQUATION

Y. OHTA

Department of Applied Mathematics, Faculty of Engineering, Hiroshima University, 1-4-1 Kagamiyama, Higashi-Hiroshima 739-8527, Japan

J. J. C. NIMMO and C. R. GILSON

Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK
e-mail: jjcn@maths.gla.ac.uk and crg@maths.gla.ac.uk

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Abstract. By using the bilinear technique of soliton theory, a pfaffian version of the SU(2) self-dual Yang-Mills equation and its solution is constructed.

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1. Introduction. It is well known that the solutions of most of the soliton equations are given in terms of the determinants with wronskian or grammian structure. The bilinear forms of the equations are reduced to algebraic identities for determinants, the so-called Plücker relations. On the other hand, the determinant can be regarded as a special case of the pfaffian, i.e., a special choice of the elements of pfaffian recovers the determinant. Moreover the determinant and pfaffian have quite similar properties. This means that for a given soliton equation whose solution is written in determinant form, we can replace the determinant by a pfaffian keeping the appropriate structure and construct a new soliton equation whose solution is given by the pfaffian instead of determinant. We term this process *pfaffianisation*. By applying this procedure to the KP hierarchy and its determinant solution, the hierarchy of the coupled KP equation and its pfaffian solution were derived [**1**]. Instead of the Plücker relations, we use the pfaffian analogue of the Plücker relations to construct the corresponding bilinear equations in the pfaffian case.

In this paper we apply the pfaffianisation technique to the SU(2) self-dual Yang-Mills (SDYM) equation and its determinant solution, and propose a new equation with pfaffian solutions. Many articles have been written on methods of finding solutions of the SDYM equation (for example [**2**]–[**7**]), and it has been revealed that a large class of solution admits the determinant expressions [**5**,**8**] whose structure is quite similar to that of the τ function of KP hierarchy. Hence it is naturally expected that the pfaffianisation process is appropriate for the SDYM equation.

2. Persymmetric determinants and pfaffians. The SU(2) SDYM equation,

$$(J_y J^{-1})_{\bar{y}} + (J_{\bar{z}} J^{-1})_z = 0 \tag{1}$$

where J is 2×2 matrix satisfying $\det J = 1$, admits the solution [**5**],

$$J = \frac{1}{\tau_N^N}\begin{pmatrix} \tau_N^{N-1} & \tau_{N-1}^N \\ \tau_{N+1}^N & \tau_N^{N+1} \end{pmatrix} \tag{2}$$

where the entries in J are given expressed in terms of the persymmetric determinants

$$\tau_N^n = \det(\varphi_{i+j+n-N-1})_{1\le i,j\le N} = \begin{vmatrix} \varphi_{n-N+1} & \cdots & \varphi_n \\ \vdots & \iddots & \vdots \\ \varphi_n & \cdots & \varphi_{n+N-1} \end{vmatrix}$$

and φ_n satisfies the linear differential equations,

$$\partial_y \varphi_n = \partial_z \varphi_{n+1} \quad \partial_{\bar z}\varphi_n = -\partial_{\bar y}\varphi_{n+1}. \tag{3}$$

In addition, let the φ_n depend on an auxiliary variable and let ∂ denote the derivative with respect to this variable. Suppose then that φ_n satisfies

$$\partial\varphi_n = \varphi_{n+1}.$$

Then the τ_N^n may be written as bidirectional wronskian determinants with respect to this auxiliary derivative. For example,

$$\tau_N^{N-1} = \begin{vmatrix} \varphi_0 & \partial\varphi_0 & \dots & \partial^{N-1}\varphi_0 \\ \partial\varphi_0 & \partial^2\varphi_0 & \dots & \partial^N\varphi_0 \\ \vdots & \vdots & & \vdots \\ \partial^{N-1}\varphi_0 & \partial^N\varphi_0 & \dots & \partial^{2N-2}\varphi_0 \end{vmatrix}. \tag{4}$$

The Hirota bilinear form of (1) is expressed in terms of three 2×2 matrices G, H and A and a scalar F. These relate to the solution as follows: $J = G/F$, $J^{-1} = H/F$ and A is an auxiliary matrix required to effect bilinearisation. This bilinear form is written in matrix form as

$$GH = F^2, \tag{5}$$

$$D_y G\cdot H = 2D_z A\cdot F, \tag{6}$$

$$D_{\bar z} G\cdot H = -2D_{\bar y} A\cdot F. \tag{7}$$

It may be shown [**4**,**9**] that the persymmetric determinants τ_N^n satisfy

$$\tau_N^{N+1}\tau_N^{N-1} - \tau_{N+1}^N\tau_{N-1}^N = \tau_N^{N2}, \tag{8}$$

$$D_y\tau_{N-1}^N\cdot\tau_N^{N-1} = D_z\tau_{N-1}^{N-1}\cdot\tau_N^N, \tag{9}$$

$$D_y\tau_{N+1}^N\cdot\tau_N^{N+1} = D_z\tau_{N+1}^{N+1}\cdot\tau_N^N, \tag{10}$$

$$D_y\left(\tau_N^{N-1}\cdot\tau_N^{N+1} - \tau_{N-1}^N\cdot\tau_{N+1}^N\right) = -DD_z\tau_N^N\cdot\tau_N^N, \tag{11}$$

and those obtained by the symmetry $y\to\bar z$, $z\to -\bar y$, where D is Hirota bilinear operator corresponding to ∂. Consequently, the bilinear equations (5)–(7) are satisfied by taking

$$F = \tau_N^N, \qquad G = \begin{pmatrix} \tau_N^{N-1} & \tau_{N-1}^N \\ \tau_{N+1}^N & \tau_N^{N+1} \end{pmatrix}, \qquad H = \begin{pmatrix} \tau_N^{N+1} & -\tau_{N-1}^N \\ -\tau_{N+1}^N & \tau_N^{N-1} \end{pmatrix}$$

and

$$A = \begin{pmatrix} -\partial\tau_N^N & \tau_{N-1}^{N-1} \\ \tau_{N+1}^{N+1} & \partial\tau_N^N \end{pmatrix}.$$

In what follows, we will show how the entries in these matrices may also be expressed in terms of pfaffians and how this representation leads, in a natural way, to a pfaffian version of SU(2) SDYM.

There are several ways to write a determinant as a pfaffian. Here we utilise the persymmetric structure of the above τ function. Our starting point is the following theorem.

THEOREM 1. **[10]** *Given quantities ψ_i, for each $k = 0, \ldots, 2N-2$ define*

$$\varphi_k = \sum_{p=0}^{k} \binom{k}{p} \psi_{k-2p}$$

and

$$\rho_k = \sum_{p=0}^{k} \psi_{k-2p}.$$

The following identity relating a persymmetric determinant and a persymmetric pfaffian holds

$$\det(\varphi_{i+j-2})_{1\le i,j\le N} = \mathrm{pf}(\rho_{j-i-1})_{1\le i<j\le 2N}.$$

Here and throughout this paper we will use standard results and notation for pfaffians as described in **[11]**. Before giving a proof of this theorem we illustrate it by writing the result explicitly in the case $N = 2$. We have

$$\begin{vmatrix} \psi_0 & \psi_{-1}+\psi_1 \\ \psi_{-1}+\psi_1 & \psi_{-2}+2\psi_0+\psi_2 \end{vmatrix} = \begin{vmatrix} \psi_0 & \psi_{-1}+\psi_1 & \psi_{-2}+\psi_0+\psi_2 \\ & \psi_0 & \psi_{-1}+\psi_1 \\ & & \psi_0 \end{vmatrix}.$$

Proof. We have already seen in (4) that the determinant on the left hand side may be expressed as a bidirectional wronskian $\tau_N^{\det}$, say, with respect to the auxiliary derivative ∂. This is consistent with letting the quantities ψ_k satisfy

$$\partial\psi_k = \psi_{k-1} + \psi_{k+1}.$$

The pfaffian on the right hand side is

$$\tau_N^{\mathrm{pf}} = (1, 2, \ldots, 2N)$$

where the (i, j) element

$$(i, j) = \rho_{j-i-1} = \sum_{p=0}^{j-i-1} \psi_{j-i-1-2p}$$

satisfies

$$\partial(i, j) = (i+1, j) + (i, j+1), \quad (i, j) = (i+1, j+1). \tag{12}$$

It is known that both τ_N^{det} and τ_N^{pf} satisfy the one dimensional Toda molecule equation,

$$D^2 \tau_N \cdot \tau_N = 2\tau_{N+1}\tau_{N-1} \quad N \geq 1, \qquad \tau_0 = 1,$$

where D denotes the Hirota bilinear operator corresponding to ∂. Then, since $\tau_1^{\mathrm{det}} = \tau_1^{\mathrm{pf}} = \psi_0$, the bilinear Toda equation may be used recursively to prove the identification $\tau_N^{\mathrm{det}} = \tau_N^{\mathrm{pf}}$ for all $N \geq 2$. □

It is clear from this theorem that

$$\tau_N^{N-1} = (1, 2, \ldots, 2N) \quad \text{and} \quad \tau_{N+1}^{N} = (1, 2, \ldots, 2N+2), \tag{13}$$

where $(i, j) = \rho_{j-i-1}$. It is consistent with (3) to let the quantities ψ_i satisfy the linear equations

$$\partial_y \psi_n = \partial_z(\psi_{n-1} + \psi_{n+1}), \quad \partial_{\bar{z}} \psi_n = -\partial_{\bar{y}}(\psi_{n-1} + \psi_{n+1}),$$

and then it follows from (12) and the above that

$$\partial \rho_n = \rho_{n-1} + \rho_{n+1}, \quad \partial_y \rho_n = \partial_z(\rho_{n-1} + \rho_{n+1}) \quad \text{and} \quad \partial_{\bar{z}} \rho_n = -\partial_{\bar{y}}(\rho_{n-1} + \rho_{n+1}). \tag{14}$$

The other determinants in the bilinear form of (1) may not be written as pfaffians so readily. In the next section we will study bordered pfaffians related to those described above. These pfaffians satisfy a system, which, for appropriate choices of the bordering terms, is SU(2) SDYM. Using this result we obtain not only a complete description of the solutions described above in terms of pfaffians, but also a generalisation of SU(2) SDYM itself.

3. Bordered persymmetric pfaffians. We now introduce a sequence of labels a_k to be used in defining bordered pfaffians. The pfaffian elements are defined by

$$(i, j) = \rho_{j-i-1}, \ (i, a_j) = \alpha_{j-i+1} \text{ and } (a_i, a_j) = 0, \tag{15}$$

where α_k satisfies the same linear equations (14) as ρ_k.

Let a, b denote any pair of the labels a_k. We will use the following notation

$$\sigma_N = (1, 2, \ldots, 2N), \quad \sigma_N^a = (1, 2, \ldots, 2N+1, a), \quad \sigma_N^{ab} = (1, 2, \ldots, 2N, a, b). \tag{16}$$

Further, we denote by a^+ (a^-) the label with index one greater (less) than that of label a. Because of this, and the defining properties of the pfaffian entries (15), the pfaffians are invariant under a uniform shift of all labels up or down. For example,

$$\sigma_N = (0, 1, \ldots, 2N-1) = (2, 3, \ldots, 2N+1),$$

and

$$\sigma_N^a = (0, 1, \ldots, 2N, a^-) = (2, 3, \ldots, 2N+2, a^+).$$

We now state the simplest quadratic identities satisfied by these pfaffians.

THEOREM 2. *For the pfaffians defined in (16), the following identities, and those obtained by the transformation $y \to \bar{z}$, $z \to -\bar{y}$, hold.*

$$\sigma_N^{ab}\sigma_N - \sigma_N\sigma_N^{a^-b^-} + \sigma_N^a\sigma_{N-1}^{b^-} - \sigma_{N-1}^{a^-}\sigma_N^b = 0, \tag{17}$$

$$\sigma_{N+1}^{ab}\sigma_N - \sigma_{N+1}\sigma_N^{a^-b^-} + \sigma_N^a\sigma_N^{b^-} - \sigma_N^{a^-}\sigma_N^b = 0, \tag{18}$$

$$\sigma_{N+1}^{ab}\sigma_N - \sigma_{N+1}\sigma_N^{ab} + \sigma_N^{a^+}\sigma_N^b - \sigma_N^a\sigma_N^{b^+} = D\,\sigma_N^a \cdot \sigma_N^b, \tag{19}$$

$$\sigma_{N+1}^{a^+b^+}\sigma_N - \sigma_{N+1}\sigma_N^{a^-b^-} + \sigma_N^a\sigma_N^{b^-} - \sigma_N^{a^-}\sigma_N^b = -D\,\sigma_N^a \cdot \sigma_N^b, \tag{20}$$

$$\begin{aligned} 2D_y\sigma_N^{ab} \cdot \sigma_N = D_z\big\{&(\sigma_N^{a^+b} \cdot \sigma_N - \sigma_N \cdot \sigma_N^{ab^-} + \sigma_{N-1}^{b^-} \cdot \sigma_N^{a^+} - \sigma_{N-1}^b \cdot \sigma_N^a) \\ &- (\sigma_N^{b^+a} \cdot \sigma_N - \sigma_N \cdot \sigma_N^{ba^-} + \sigma_{N-1}^{a^-} \cdot \sigma_N^{b^+} - \sigma_{N-1}^a \cdot \sigma_N^b)\big\}, \end{aligned} \tag{21}$$

$$\begin{aligned} 2D_y\sigma_N^a \cdot \sigma_N^b = D_z\big\{&(\sigma_{N+1}^{ab} \cdot \sigma_N + \sigma_{N+1} \cdot \sigma_N^{a^-b^-} - \sigma_N^{b^-} \cdot \sigma_N^a - \sigma_N^b \cdot \sigma_N^{a^-}) \\ &- (\sigma_N \cdot \sigma_{N+1}^{b^+a^+} + \sigma_N^{ba} \cdot \sigma_{N+1} - \sigma_N^a \cdot \sigma_N^{b^+} - \sigma_N^{a^+} \cdot \sigma_N^b)\big\}, \end{aligned} \tag{22}$$

$$\begin{aligned} &2\big(D_y(\sigma_{N+1}^{ab} \cdot \sigma_N - \sigma_{N+1} \cdot \sigma_N^{a^-b^-}) + DD_z(\sigma_N^a \cdot \sigma_N^{b^-} - \sigma_N^{a^-} \cdot \sigma_N^b)\big) \\ &\quad = D_z\big\{(\sigma_{N+1}^{a^+b} \cdot \sigma_N - \sigma_{N+1} \cdot \sigma_N^{ab^-} + \sigma_N^{a^+} \cdot \sigma_N^{b^-} - \sigma_N^a \cdot \sigma_N^b) \\ &\quad - (\sigma_{N+1}^{b^+a} \cdot \sigma_N - \sigma_{N+1} \cdot \sigma_N^{ba^-} + \sigma_N^{b^+} \cdot \sigma_N^{a^-} - \sigma_N^b \cdot \sigma_N^a) \\ &\quad + (\sigma_{N+1}^{a^-b} \cdot \sigma_N - \sigma_{N+1} \cdot \sigma_N^{a^{--}b^-} + \sigma_N^{a^-} \cdot \sigma_N^{b^-} - \sigma_N^{a^{--}} \cdot \sigma_N^b) \\ &\quad - (\sigma_{N+1}^{b^-a} \cdot \sigma_N - \sigma_{N+1} \cdot \sigma_N^{b^{--}a^-} + \sigma_N^{b^-} \cdot \sigma_N^{a^-} - \sigma_N^{b^{--}} \cdot \sigma_N^a)\big\}. \end{aligned} \tag{23}$$

Proof. Identities (17)–(20) are established by means of standard pfaffian identities together with the results

$$\partial\sigma_N^a = (1, 2, \ldots, 2N, 2N+2, a) + \sigma_N^{a^+} = (0, 2, \ldots, 2N+1, a) + \sigma_N^{a^-}. \tag{24}$$

To prove (19), for example, we consider the standard pfaffian identity

$$\begin{aligned} &(1, \ldots, 2N+2, a, b)(1, \ldots, 2N) - (1, \ldots, 2N+2)(1, \ldots, 2N, a, b) \\ &+ (1, \ldots, 2N+1, a)(1, \ldots, 2N, 2N+2, b) - (1, \ldots, 2N+1, b)(1, \ldots, 2N, 2N+2, a) \\ &= 0. \end{aligned}$$

Then using the first equality in (24) gives the required result.

The other identities are more nonstandard and as an example we show how to prove (21). We make use of the pfaffian identities

$$
\begin{aligned}
&(1,\dots,2N,a,b)_{ij}(1,\dots,2N)-(1,\dots,2N,a,b)(1,\dots,2N)_{ij}\\
&\quad=(0,\dots,2N,a)_{0i}(0,\dots,2N,b)_{0j}-(0,\dots,2N,a)_{0j}(0,\dots,2N,b)_{0i}\\
&(1,\dots,2N+1,a)_{1i}(1,\dots,2N+1,b)_{j,2N+1}-(1,\dots,2N+1,a)_{1j}(1,\dots,2N+1,b)_{i,2N+1}\\
&\quad=-(1,\dots,2N,a,b)_{ij}(2,\dots,2N+1)+(2,\dots,2N+1,a,b)(1,\dots,2N)_{ij}\\
&\quad-(1,\dots,2N+1,a)(2,\dots,2N,b)_{ij}+(2,\dots,2N,a)(1,\dots,2N+1,b)_{ij}
\end{aligned}
$$

where i and j are arbitrary integers and $(\dots)_{ij}$ means the (i,j) cofactor of the pfaffian $(\dots)$. The left hand side of (21) may be rewritten as

$$
\begin{aligned}
&2D_y(1,\dots,2N,a,b)\cdot(1,\dots,2N)\\
&\quad=2\big((1,\dots,2N,a_y,b)+(1,\dots,2N,a,b_y)\big)(1,\dots,2N)\\
&\quad+\sum_i\sum_j(i,j)_y\big((1,\dots,2N,a,b)_{ij}(1,\dots,2N)-(1,\dots,2N,a,b)(1,\dots,2N)_{ij}\big)\\
&\quad=2\big((1,\dots,2N,a_z^+ + a_z^-,b)+(1,\dots,2N,a,b_z^+ + b_z^-)\big)(1,\dots,2N)\\
&\quad+\sum_i\sum_j\big((i+1,j)_z+(i,j+1)_z\big)\big((0,\dots,2N,a)_{0i}(0,\dots,2N,b)_{0j}\\
&\quad-(0,\dots,2N,a)_{0j}(0,\dots,2N,b)_{0i}\big)\\
&\quad=2\big((1,\dots,2N,a_z^+,b)+(2,\dots,2N+1,a_z,b^+)+(1,\dots,2N,a,b_z^+)\\
&\quad+(2,\dots,2N+1,a^+,b_z)\big)(1,\dots,2N)\\
&\quad-\sum_i\sum_j(i,j)_z\big((1,\dots,2N+1,a^+)_{1i}(1,\dots,2N+1,b)_{j,2N+1}\\
&\quad-(1,\dots,2N+1,a)_{j,2N+1}(1,\dots,2N+1,b^+)_{1i}\\
&\quad+(1,\dots,2N+1,a)_{i,2N+1}(1,\dots,2N+1,b^+)_{1j}\\
&\quad-(1,\dots,2N+1,a^+)_{1j}(1,\dots,2N+1,b)_{i,2N+1}\big)\\
&\quad=2\big\{\big((1,\dots,2N,a_z^+,b)+(1,\dots,2N,a,b_z^+)\big)(1,\dots,2N)\\
&\quad+(1,\dots,2N,a_z,b^+)(2,\dots,2N+1)\\
&\quad-(2,\dots,2N,a_z)(1,\dots,2N+1,b^+)+(1,\dots,2N+1,a_z)(2,\dots,2N,b^+)\\
&\quad+(1,\dots,2N,a^+,b_z)(2,\dots,2N+1)-(2,\dots,2N,a^+)(1,\dots,2N+1,b_z)\\
&\quad+(1,\dots,2N+1,a^+)(2,\dots,2N,b_z)\big\}\\
&\quad+\sum_i\sum_j(i,j)_z\big((1,\dots,2N,a^+,b)_{ij}(2,\dots,2N+1)\\
&\quad-(2,\dots,2N+1,a^+,b)(1,\dots,2N)_{ij}+(1,\dots,2N+1,a^+)(2,\dots,2N,b)_{ij}\\
&\quad-(2,\dots,2N,a^+)(1,\dots,2N+1,b)_{ij}+(1,\dots,2N,a,b^+)_{ij}(2,\dots,2N+1)\\
&\quad-(2,\dots,2N+1,a,b^+)(1,\dots,2N)_{ij}-(2,\dots,2N,a)_{ij}(1,\dots,2N+1,b^+)\\
&\quad+(1,\dots,2N+1,a)_{ij}(2,\dots,2N,b^+)\big)\\
&\quad=2\big((1,\dots,2N,a^+,b)_z(2,\dots,2N+1)-(2,\dots,2N+1,a^+,b)(1,\dots,2N)_z\\
&\quad+(1,\dots,2N+1,a^+)(2,\dots,2N,b)_z-(2,\dots,2N,a^+)(1,\dots,2N+1,b)_z\\
&\quad+(1,\dots,2N,a,b^+)_z(2,\dots,2N+1)-(2,\dots,2N+1,a,b^+)(1,\dots,2N)_z\\
&\quad-(2,\dots,2N,a)_z(1,\dots,2N+1,b^+)+(1,\dots,2N+1,a)_z(2,\dots,2N,b^+)\big).
\end{aligned}
$$

After symmetrising this expression using identities of the form $2X_zY = D_zX \cdot Y + (XY)_z$, it is readily shown to equal the right hand side. Hence (21) is established. The remaining identities can be proved similarly. □

Next, we will show that the identities given in Theorem 2 reduce to the ones, (8)–(11), satisfied by the persymmetric determinants which satisfy SU(2) SDYM. In this way we see that the Hirota bilinear equations (17)–(23) define a generalisation of this system.

Let the labels a_k be such that

$$a_{k+2} = -a_k,$$

so that there are just two independent labels, a_0 and a_1 say. Then one can show that

$$\sigma_N^{a_j} = \alpha_{j+N}\,\tau_N^N \quad \text{and} \quad \sigma_N^{a_0a_1} = -(\alpha_0^2 + \alpha_1^2)\tau_{N-1}^N.$$

For example,

$$\sigma_1^{a_0} = \begin{vmatrix} \rho_0 & \rho_1 & \alpha_0 \\ & \rho_0 & \alpha_{-1} \\ & & \alpha_{-2} \end{vmatrix} = \begin{vmatrix} \psi_0 & \psi_{-1}+\psi_1 & \alpha_0 \\ & \psi_0 & -\alpha_1 \\ & & -\alpha_0 \end{vmatrix} = \alpha_1|\varphi_1| = \alpha_1\tau_1^1.$$

Taking $a = a_1$ and $b = a_0$, the identities (17)–(23) reduce to (8)–(11). Hence we see that the system defined by (17)–(23) reduces to SU(2) SDYM.

In the next section we present another constraint on the labels a_k which gives a system distinct from SU(2) SDYM.

4. A pfaffianised SU(2) self-dual Yang-Mills equation. Let us take a different reduction by imposing the simple condition,

$$a_{k+2} = a_k.$$

Choosing $a = a_0$, $b = a_1$, the identities (17)–(23) reduce to

$$2\sigma_N^{a_0a_1}\sigma_N = \sigma_N^{a_1}\sigma_{N-1}^{a_1} - \sigma_N^{a_0}\sigma_{N-1}^{a_0}, \tag{25}$$

$$2\sigma_{N+1}^{a_0a_1}\sigma_{N+1} = \sigma_{N+1}^{a_1}\sigma_N^{a_1} - \sigma_{N+1}^{a_0}\sigma_N^{a_0}, \tag{26}$$

$$\sigma_{N+1}^{a_0a_1}\sigma_N + \sigma_{N+1}\sigma_N^{a_0a_1} = \sigma_N^{a_1\,2} - \sigma_N^{a_0\,2}, \tag{27}$$

$$\sigma_{N+1}^{a_0a_1}\sigma_N - \sigma_{N+1}\sigma_N^{a_0a_1} = \sigma_N^{a_1\,2} - \sigma_N^{a_0\,2} + D\sigma_N^{a_0}\cdot\sigma_N^{a_1}, \tag{28}$$

$$D_y\sigma_N^{a_0a_1}\cdot\sigma_N = D_z(\sigma_{N-1}^{a_0}\cdot\sigma_N^{a_1} - \sigma_{N-1}^{a_1}\cdot\sigma_N^{a_0}), \tag{29}$$

$$D_y\sigma_{N+1}^{a_0a_1}\cdot\sigma_{N+1} = D_z(\sigma_N^{a_0}\cdot\sigma_{N+1}^{a_1} - \sigma_N^{a_1}\cdot\sigma_{N+1}^{a_0}), \tag{30}$$

$$D_y\sigma_N^{a_0}\cdot\sigma_N^{a_1} = D_z(\sigma_{N+1}^{a_0a_1}\cdot\sigma_N - \sigma_{N+1}\cdot\sigma_N^{a_0a_1}), \tag{31}$$

$$D_y(\sigma_{N+1}^{a_0a_1}\cdot\sigma_N + \sigma_{N+1}\cdot\sigma_N^{a_0a_1}) = DD_z(\sigma_N^{a_1}\cdot\sigma_N^{a_1} - \sigma_N^{a_0}\cdot\sigma_N^{a_0}) + 4D_z\sigma_N^{a_0}\cdot\sigma_N^{a_1}. \tag{32}$$

We now wish to obtain from these identities a system which has a similar form to SU(2) SDYM. For this reason we define the matrix

$$G = \begin{pmatrix} \sigma_N & -\sigma_N^{a_0 a_1} \\ \sigma_{N+1} & \sigma_{N+1}^{a_0 a_1} \end{pmatrix},$$

and scalars

$$F_\pm = \sigma_N^{a_1} \pm \sigma_N^{a_0}.$$

Then, by virtue of (27), the matrix

$$J = G \begin{pmatrix} F_+ & 0 \\ 0 & F_- \end{pmatrix}^{-1},$$

satisfies $\det J = 1$. The new system will be defined in terms of this matrix. Further,

$$J^{-1} = \begin{pmatrix} F_- & 0 \\ 0 & F_+ \end{pmatrix}^{-1} H,$$

where

$$H = \begin{pmatrix} \sigma_{N+1}^{a_0 a_1} & \sigma_N^{a_0 a_1} \\ -\sigma_{N+1} & \sigma_N \end{pmatrix}.$$

It is also necessary to define auxiliary matrices

$$A_\pm = \begin{pmatrix} \pm(\partial F_\pm) \pm F_\pm & \sigma_{N-1}^{a_1} \pm \sigma_{N-1}^{a_0} \\ \sigma_{N+1}^{a_1} \pm \sigma_{N+1}^{a_0} & \mp(\partial F_\pm) \mp F_\pm \end{pmatrix}.$$

Now all of the identities may be written in terms of these 4 matrices and 2 scalars as follows;

$$\begin{aligned} GH &= F_- F_+, \\ D_y G \cdot H &= D_z(A_- \cdot F_+ + F_- \cdot A_+), \\ 2GhH &= A_- F_+ + F_- A_+, \\ 2D_y G \cdot hH &= D_z F_- \cdot F_+, \end{aligned}$$

where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Next, we eliminate G and H from the final three equations in favour of J and J^{-1} to obtain

$$2J_y J^{-1} - \frac{D_y F_- \cdot F_+}{F_- F_+} JhJ^{-1} = \left(\frac{A_-}{F_-} - \frac{A_+}{F_+}\right)_z + \left(\frac{A_-}{F_-} + \frac{A_+}{F_+}\right) \frac{D_z F_- \cdot F_+}{F_- F_+},$$

$$2JhJ^{-1} = \left(\frac{A_-}{F_-} + \frac{A_+}{F_+}\right),$$

and

$$2\left(J_z h J^{-1} + J h J^{-1} J_z J^{-1} - \frac{D_z F_- \cdot F_+}{F_- F_+}\right) = \frac{D_y F_- \cdot F_+}{F_- F_+}.$$

From these equations we obtain an expression, involving J alone, as an exact z-derivative, namely

$$J_y J^{-1} - J_z J^{-1} - J h J^{-1} J_z h J^{-1} = \frac{1}{2}\left(\frac{A_-}{F_-} - \frac{A_+}{F_+}\right)_z.$$

Finally, using the transformation $y \to \bar{z},\ z \to -\bar{y}$ we get a second version of the above and eliminating the auxiliary terms by cross-differentiation gives

$$(J_y J^{-1} - J_z J^{-1} - J h J^{-1} J_z h J^{-1})_{\bar{y}} + (J_{\bar{z}} J^{-1} + J_{\bar{y}} J^{-1} + J h J^{-1} J_{\bar{y}} h J^{-1})_z = 0. \tag{33}$$

We note that this may also be written as

$$(J_y J^{-1})_{\bar{y}} + (J_{\bar{z}} J^{-1})_z + \left[[J h J^{-1}, J_z J^{-1}], [J h J^{-1}, J_{\bar{y}} J^{-1}]\right] + 2[J_z J^{-1}, J_{\bar{y}} J^{-1}] = 0, \tag{34}$$

where $[\cdot, \cdot]$ denotes the matrix commutator.

5. Concluding Remarks. By showing how the persymmetric determinant solutions of the SU(2) SDYM equation can be written in terms of pfaffians, we have been led to study a class of bordered pfaffians and the identities they satisfy. We have considered two reductions of this master system; one choice leading to SU(2) SDYM and the other leading to a new system (33). This process naturally give solutions to these systems as bordered persymmetric pfaffians.

There appears to be no straightforward connection between the SDYM equation (1) and the pfaffianised version (33). However there exists an intriguing possibility, which is still to be explored, that by making a different choice of bordering terms in the pfaffians one may be able to replace the constant matrix h with a more general one which permits the limit $h \to iI$, in which the pfaffianised version reduces to SU(2) SDYM.

In general, we expect that this pfaffianisation technique is applicable to many integrable systems which admit determinant solutions and works as a powerful tool to construct new integrable equations whose solutions are given by pfaffians.

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