Corrigendum to: "Discounted Stochastic Games with No Stationary Nash Equilibrium: Two Examples"*

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Abstract

Levy (2013) presents examples of discounted stochastic games that do not have stationary equilibria. The second named author has pointed out that one of these examples is incorrect. In addition to describing the details of this error, this note presents a new example by the first named author that succeeds in demonstrating that discounted stochastic games with absolutely continuous transitions can fail to have stationary equilibria.

1 Introduction

The paper "Discounted Stochastic Games with No Stationary Nash Equilibrium: Two Examples" (Levy (2013)) presents two constructions of discounted stochastic games with continuous state spaces that do not possess stationary equilibria. One is in the class of stochastic games with deterministic transitions, while the other is in the class of games in which all transitions are absolutely continuous with respect to a fixed measure.

The construction of the game from the second class is accomplished in three steps. 1. An example presented in Kohlberg and Mertens (1986) has a circular set of Nash equilibria, and a map from the set of equilibria to small perturbations of the game is constructed. 2. An intricate construction concerning realising piece-wise linear functions on the square as outcomes of strategic games is presented. 3. These techniques are combined to define the desired stochastic game.

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The second named author has pointed out a serious error in Step 1 of the construction. Roughly, it is desired that each point in the set of equilibria be distant from the set of equilibria of the associated perturbed game, but for the given construction this is not the case. Section 4.6 of Levy (2013) describes conditions on a "base" strategic game (instead of the game from Kohlberg and Mertens (1986)) that would allow the construction to be carried out. However, topological obstructions preclude the existence of such a game; an accompanying note (McLennan (2014)) gives a theorem (which is stated in Section 5) that implies this. (We remark that Proposition 4.3 of Levy (2013), which is the core of Step 2 mentioned above, is correct, and both it and the techniques used to prove it will hopefully prove useful in future work.)

In addition to describing the details of the error, we present a new example, due to the first named author, of a stochastic game with absolutely continuous transitions that has no stationary equilibria for any positive discount factor, thereby confirming a negative answer to the question of existence of stationary equilibria for such games. The new example resembles the earlier erroneous construction, but is significantly shorter and simpler, and it is well behaved in the following senses: the state space is compact, payoffs are continuous, and the transitions are norm-continuous. It retains the mechanism that transmits information around a circle, but instead of the "base game" taken from Kohlberg and Mertens (1986)[1] there is a new base game, which shares some similarities to the game from Kohlberg and Mertens (1986),[2] but which achieves the relevant conditions (as summarized in Lemma 3.1) in perhaps the simplest possible manner. The underlying topological "engine" of the previous example has been replaced by a phenomenon drawn from measure theory, namely the existence of functions having a type of erratic behaviour that is exhibited, e.g., by the path of a Brownian motion.

The stochastic game model is recalled in Section 2. The new example is presented in Section 3. The proof that it possesses no stationary equilibria is in Section 4. Section 5 gives details of the error in Levy (2013).

2 Stochastic Game Model

A stochastic game $\Gamma = \langle \Omega, \mathcal{P}, I, r, q \rangle$ with a continuum of states and finitely many actions has the following components:

- A standard Borel space[3] Ω of *states*.
- A nonempty finite set $\mathcal{P}$ of *players*.
- $I = \prod_{\ell \in \mathcal{P}} I^\ell$, where each I^ℓ is a nonempty finite set of *actions* for ℓ.

[1] The game from Kohlberg and Mertens (1986) was used in an earlier version of this corrigendum.

[2] In particular, its set of equilibria form an infinite connected set, but none of them are stable.

[3] That is, a space that is homeomorphic to a Borel subset of a complete, metrizable space.

- A bounded Borel-measurable *stage payoff function* $r : \Omega \times I \to \mathbb{R}^{\mathcal{P}}$.
- A Borel-measurable[4] *transition function* $q : \Omega \times I \to \Delta(\Omega)$.

For a discount factor $\beta \in (0,1)$, $\Gamma(\beta)$ is the associated discounted stochastic game. Throughout (except in the statements of some results) we work with a fixed β.

Definition 2.1. *An Absolutely Continuous (A.C.) stochastic game is a stochastic game $\Gamma = \langle \Omega, \mathcal{P}, I, r, q \rangle$ for which there is a $\nu \in \Delta(\Omega)$ such that for all $z \in \Omega$ and $a \in I$, $q(z,a)$ is absolutely continuous with respect to ν.*

A *stationary strategy* for player ℓ is a Borel-measurable mapping $\sigma^\ell : \Omega \to \Delta(I^\ell)$. Let Σ_0^ℓ denote the set of stationary strategies for player ℓ, and let $\Sigma_0 = \prod_{\ell \in \mathcal{P}} \Sigma_0^\ell$ be the set of stationary strategy profiles. Together with the transition function and an initial state z, a stationary strategy profile σ induces a probability measure P_z^σ on the space $H^\infty := (\Omega \times I)^{\mathbb{N}}$ of infinite histories in a canonical way (see, e.g., Bertsekas and Shreve (1996)). Let

$$\gamma_\sigma(z) := E_z^\sigma\big(\sum_{n=1}^{\infty} \beta^{n-1} r(z_n, a_n)\big)$$

be the expected payoff vector under σ in the game starting from state $z = z_1$. A profile $\sigma \in \Sigma_0$ is a *stationary equilibrium* of $\Gamma(\beta)$ if

$$\gamma_\sigma^\ell(z) \geq \gamma_{(\tau,\sigma^{-\ell})}^\ell(z)$$

for all $z \in \Omega$, $\ell \in \mathcal{P}$, and $\tau \in \Sigma_0^\ell$.

For $z \in \Omega$ and $a \in I$ let

$$X_\sigma(z,a) := r(z,a) + \beta \int_\Omega \gamma_\sigma \, dq(z,a). \tag{2.1}$$

Note that $\gamma_\sigma(z) = X_\sigma(z, \sigma(z))$. We recall the following classical dynamical programming criterion for a stationary equilibrium, which is called the *one-shot deviation principle.*

Proposition 2.2. *A profile $\sigma \in \Sigma_0$ of stationary strategies is a stationary equilibrium of $\Gamma(\beta)$ if and only if, for all $z \in \Omega$, $\sigma(z)$ is a Nash equilibrium of the game $X_\sigma(z,\cdot)$.*

3 The Example

This section presents the example (or class of examples, insofar as there is a given function that is a parameter) of an A.C. stochastic game that does not possess stationary Nash equilibria for any positive discount factor.

[4] Where $\Delta(\Omega)$, the space of regular Borel probability measures on Ω, possesses the Borel structure induced from the topology of narrow convergence.

3.1 Notations

Recall that $\langle \cdot, \cdot \rangle$ denotes the inner product of vectors. In addition the following notational conventions will be used:

- Throughout $\| \cdot \|$ denotes the L_∞ norm. That is, for a vector or bounded real-valued function f, $||f|| = \sup |f|$, where the supremum is taken over the set of indices or the domain of f.
- If p is a mixed action over an action space I and $i \in I$, then $p[i]$ denotes the probability that p chooses i.
- In connection with a tuple c indexed by the elements of some set $T \subset \mathcal{P}$ of players, if $\ell_1, \ldots, \ell_k \in T$, then $c^{\ell_1,\ldots,\ell_k}$ will denote $(c^{\ell_1}, \ldots, c^{\ell_k})$.

3.2 The Base Game

Our construction has three phases: a) selecting four perturbations of a "base" game; b) specification of a rescaled version of the stage game; c) the stochastic game itself.

The base game G has four players, A, B, C, and D. The pure strategies of player A are U and D, the pure strategies of B are L, M, and R, and players C and D are dummy players, because their sets of pure strategies are singletons. The payoffs of players A and B are shown below.

$A\backslash B$	L	M	R
U	$(1,1)$	$(1,1)$	$(0,0)$
D	$(0,0)$	$(1,1)$	$(1,1)$

Figure 3.1: The Payoffs to A and B ($G^{A,B}$)

The Nash equilibria are the pure strategy profiles (U, L), (U, M), (D, M), and (D, R), as well as all "convex combinations" of successive pairs of elements of this list. The payoffs to C and D are as follows.

$A\backslash B$	L	M	R
U	$(-1,1)$	$(1,1)$	$(0,0)$
D	$(0,0)$	$(1,-1)$	$(-1,-1)$

Figure 3.2: The Payoffs to C and D ($G^{C,D}$)

We state the properties of G that figure in the subsequent analysis. For a mixed strategy profile x, let $G(x)$ be the vector of expected payoffs.

Lemma 3.1.

(a) For each $(j, k) \in \{-1, 1\}^2$, any neighborhood of G contains a game $G_{j,k}$ whose unique Nash equilibrium x satisfies $G^{C,D}(x) = (j, k)$.

(b) For any equilibrium x of G, $||G^{C,D}(x)|| = 1$.

Proof. Obvious. $\square$

In view of (b) and the bounds on payoffs for C and D, the upper semicontinuity of the Nash equilibrium correspondence implies that there is an $\eta_0 > 0$ such that

$$\tfrac{7}{8} \leq ||G^{C,D}(x)|| \leq 1$$

whenever x is an equilibrium of a game G' such that $||G' - G|| \leq \eta_0$. For each $(j,k) \in \{-1,1\}^2$ we fix such a perturbation $G_{j,k}$ of G such that the unique Nash equilibrium x of $G_{j,k}$ satisfies $G^{C,D}(x) = (j,k)$. (The payoffs of A and B in $G_{j,k}$ play no role in our analysis after Lemma 3.1 has been established.) For each $z = (z^E, z^F) \in [-1,1]^2$ let G_z be the convex combination of the $(G_{j,k})$ given by

$$G_z = \sum_{(j,k)\in\{-1,1\}^2} \frac{(1 + j \cdot z^E)(1 + k \cdot z^F)}{4} G_{j,k}.$$

3.3 The Stage Game

Next we describe a second strategic form game; in our stochastic game the stage game in each state will be a rescaling of this game. The set of players is $\mathcal{P} = \{A, B, C, C', D, D', E, F\}$. As above, player A has the pure strategies U and D, and player B has the pure strategies L, M, and R, but in this game players C and D have pure strategies 0 and 1. Players C' and D' also have pure strategies 0 and 1, and players E and F have pure strategies -1 and 1. Pure and mixed strategy profiles will be denoted by

$$a = (a^A, a^B, a^C, a^{C'}, a^D, a^{D'}, a^E, a^F) \text{ and } x = (x^A, x^B, x^C, x^{C'}, x^D, x^{D'}, x^E, x^F).$$

The payoffs in the stage game depend on a parameter $\varrho \in (-\frac{1}{2}, \frac{1}{2})$. For such a ϱ let $v_E^\varrho = (1, \varrho)$ and $v_F^\varrho = (-\varrho, 1)$. Let

$$\psi(a) = (\psi^C(a^C), \psi^D(a^D)) = (2a^C - 1, 2a^D - 1)$$

and

$$\psi(x) = (\psi^C(x^C), \psi^D(x^D)) = (2x^C[1] - 1, 2x^D[1] - 1). \tag{3.1}$$

The payoffs in the game $g(\varrho, \cdot)$ are:

$$\begin{aligned}
g^A(\varrho, a) &= G^A_{a^E, a^F}(a^A, a^B), \\
g^B(\varrho, a) &= G^B_{a^E, a^F}(a^A, a^B), \\
g^C(\varrho, a) &= \begin{cases} -G^C(a^A, a^B) - \frac{1}{16}, & a^C = a^{C'}, \\ -G^C(a^A, a^B) + \frac{1}{16}, & a^C \neq a^{C'}, \end{cases} \\
g^{C'}(\varrho, a) &= -g^C(\varrho, a), \\
g^D(\varrho, a) &= \begin{cases} -G^D(a^A, a^B) - \frac{1}{16}, & a^D = a^{D'}, \\ -G^D(a^A, a^B) + \frac{1}{16}, & a^D \neq a^{D'}, \end{cases} \\
g^{D'}(\varrho, a) &= -g^D(\varrho, a), \\
g^E(\varrho, a) &= a^E \cdot \langle v^\varrho_E, \psi(a) \rangle, \\
g^F(\varrho, a) &= a^F \cdot \langle v^\varrho_F, \psi(a) \rangle.
\end{aligned}$$

In the stochastic game given below, the transitions are controlled by C, C', D, and D', so in each period the other players will only be concerned with maximizing their stage payoffs. Players A and B are in effect playing a perturbation of the game G, as described above.

The stage game payoff to C' is the negation of the stage game payoff to C, so C and C' will have opposite views concerning the desirability of the stochastic game continuing (as opposed to transitioning to an absorbing state with zero payoffs). Leaving aside the components of the stage game payoffs for C and C' that depend only on the behavior of A and B, the conflict between C and C' at time t is a zero sum game that consists of matching pennies perturbed by these concerns about future payoffs. These perturbations will be small enough that there is always a unique equilibrium which is mixed.

The conflict between D and D' is similar to the conflict between C and C'. However, the impact of A and B's behavior on the payoffs of D and D' is different from its impact on the payoffs of C and C'. Consequently the perturbation of the matching pennies game, and the resulting stage game equilibrium, will almost surely be different.

The best responses of players E and F depend on the signs of the expectations of the inner products $\langle v^\varrho_E, \psi(a) \rangle$ and $\langle v^\varrho_F, \psi(a) \rangle$ respectively. For $\varrho \in (-\frac{1}{2}, \frac{1}{2})$ and $j, k = \pm 1$ let

$$\mathcal{D}^\varrho_{j,k} := \{ \psi \in \mathbb{R}^2 \mid j \cdot \langle v^\varrho_E, \psi \rangle > 0 \text{ and } k \cdot \langle v^\varrho_F, \psi \rangle > 0 \}.$$

Observe that v^ϱ_E and v^ϱ_F are orthogonal, so the $\mathcal{D}^\varrho_{j,k}$ are just the open quadrants of the plane under a certain rotation. (See Figure 1.) Set

$$\mathcal{D}^\varrho = \bigcup_{j,k=\pm 1} \mathcal{D}^\varrho_{j,k}.$$

In the stochastic game defined below ϱ will be a function of the state $t \in [0, 1]$, and we will see that in any stationary equilibrium, for almost all t, behavior at state t is characterized by a mixed strategy profile x such that $\psi(x)$ lies in $\mathcal{D}^{\varrho(t)}$, so that E and F play pure strategies, and consequently A and B are playing one of the perturbations $G_{j,k}$ of G. In this sense the behavior of A and B is well controlled.

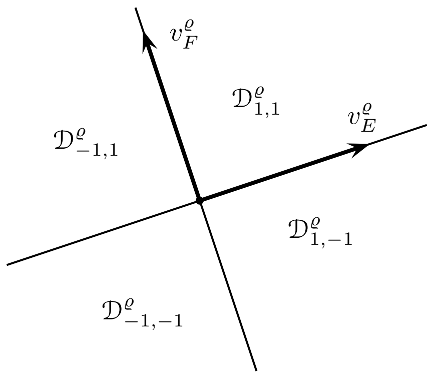

Figure 1.

3.4 The Stochastic Game

We now specify the stochastic game. Let $\varrho : [0,1] \to (-\frac{1}{2}, \frac{1}{2})$ be a measurable function. For $a \in I$ let

$$h(a) = a^C + a^{C'} + a^D + a^{D'}, \tag{3.2}$$

and for $t \in [0,1]$ let

$$\tilde{q}(t,a) = (1-t) \cdot \tfrac{1}{64} h(a).$$

Let δ_1 be the Dirac measure at 1. For $t \in [0,1]$ let U_t be the uniform distribution on $[t,1]$. (We identify U_1 with δ_1.) The stochastic game $\tilde{\Gamma}$ is as follows:

- The state space Ω is $[0,1]$, with the Borel σ-algebra.
- The set of players and their action spaces are the same as in $g(\varrho, \cdot)$.
- The stage payoff function is $r(t,a) = (1-t)g(\varrho(t), a)$.
- The transition function is $q(t,a) = \tilde{q}(t,a)U_t + (1 - \tilde{q}(t,a))\delta_1$.

The game $\tilde{\Gamma}$ has the following features. First, 1 is an absorbing state, with payoff 0 for all players. The transitions from state t are mixtures of two types: U_t, which distributes uniformly in $[t,1]$, or quitting to 1. As such, the game progresses towards the right. Note that $\tilde{\Gamma}$ is A.C. because for all $t \in [0,1]$ and all $a \in I$, $q(t,a)$ is absolutely continuous w.r.t. $\frac{1}{2}(U_0 + \delta_1)$. Also, the stage payoffs of $\tilde{\Gamma}$ are continuous functions on $\Omega \times I$ when ϱ is continuous.

We now state the main step in the argument:

Proposition 3.2. *Suppose that $\varrho : [0,1] \to (-\frac{1}{2}, \frac{1}{2})$ is a Lebesgue measurable function such that $\pi(t) \neq \varrho(t)$ a.e. whenever $\pi : [0,1] \to \mathbb{R}$ is differentiable a.e. Then for any $\beta \in (0,1)$, the game $\tilde{\Gamma}(\beta)$ does not possess a stationary equilibrium.*

To see the key intuition underlying the construction, suppose that σ is a stationary equilibrium of $\tilde{\Gamma}(\beta)$. The stage game payoff is by far the largest part of $\gamma_\sigma^{C,D}(t)$, and the contribution of the perturbation of G is by far the largest part of the stage game payoff. Therefore $\gamma_\sigma^{C,D}(t) \neq (0,0)$ when $t < 1$.

For $t \in [0,1]$ let $W^{C,D}(t) = \beta \int_t^1 \gamma_\sigma^{C,D}(s)\, ds$. Since $\gamma_\sigma^{C,D}$ is a bounded measurable function, $W^{C,D}$ is absolutely continuous and consequently, for almost every t, differentiable at t with derivative equal to $-\beta$ times $\gamma_\sigma^{C,D}(t)$. Since $\gamma_\sigma^{C,D}(t) \neq (0,0)$ when $t < 1$, $W^{C,D}$ is not identically equal to the origin in $\mathbb{R}^2$.

Because ϱ is erratic, for almost all t such that $W^{C,D}(t) \neq (0,0)$, the best responses of E and F are pure, leading the perturbation of the base game to be one of the $G_{j,k}$ whose equilibrium pushes the vector of future payoffs of C and D away from the origin in $\mathbb{R}^2$ as we go forward in time (i.e., toward the right), which is to say that the derivative of $s \mapsto \|W^{C,D}(s)\|$ is positive at t, for almost all t. Since $\|W^{C,D}\|$ is absolutely continuous and $W^{C,D}(1) = (0,0)$, this is impossible, which is the desired contradiction.

An essential feature of the construction is that G does not have any equilibria that give expected utility zero to both C and D, but nonetheless the origin is in the convex hull of the set of pairs of expected payoffs for C and D induced by the equilibria of G. For this reason one cannot replace the base game with a single agent decision problem: for a decision problem the set of optimal mixed strategies, and its image in the set of pairs of expected payoffs for C and D, are both convex. In addition, one cannot replace C and D with a single agent C: if every neighborhood of G contained a game whose unique equilibrium gave agent C an expected payoff of 1, and also a game whose unique equilibrium gave an expected payoffs of -1, then (as a consequence of a theorem of Browder (1960) and Mas-Colell (1974)) every neighborhood of G would contain games with equilibria giving C an expected payoff of 0.

To pass from Proposition 3.2 to the existence of a game without a stationary equilibrium it remains to show that there is a measurable function ϱ that disagrees a.e. with any a.e. differentiable function. It turns out that it is not enough to require that ϱ be nowhere differentiable, but there is a stronger condition that works. Let λ denote Lesbesgue measure.

Definition 3.3. *If $E \subseteq \mathbb{R}$ is Lesbesgue measurable, $f : E \to \mathbb{R}$ is Lebesgue measurable, $x \in E$, and $L \in \mathbb{R}$, then f is* approximately differentiable *at x with* approximate derivative *L if, for all $\varepsilon > 0$,*

$$\frac{1}{2\delta}\lambda\Big((x-\delta, x+\delta) \cap \big\{y \in E : \Big|\frac{f(y)-f(x)}{y-x} - L\Big| < \varepsilon\big\}\Big) \to 1$$

as $\delta \to 0$.

Clearly, if f is differentiable at x with $f'(x) = L$, then f is approximately differentiable at x with approximate derivative L. The following is included in Theorem 3.3 of (Saks, 1937, Sec VII.3)[5]:

Lemma 3.4. *If $f, g : [0,1] \to \mathbb{R}$ are Lesbesgue measurable, f is approximately differentiable a.e., g is approximately differentiable almost nowhere, and $E = \{ x : f(x) = g(x) \}$, then $\lambda(E) = 0$.*

Berman (1970) shows that, with probability one, the path of a Brownian motion is nowhere approximately differentiable; by definition the path of a Brownian motion is continuous. The existence of almost nowhere approximately differentiable continuous functions is also shown more directly in Jarník (1934); see also Preiss and Zajïcek (2000) and the references within. Consequently:

Theorem 3.1. *There exists stochastic games of the form $\tilde{\Gamma}$, for continuous ϱ, such that for each $\beta \in (0,1)$, $\tilde{\Gamma}(\beta)$ does not possess a stationary equilibrium.*

4 Proof of Proposition 3.2

Let $\varrho : [0,1] \to (-\frac{1}{2}, \frac{1}{2})$ satisfy the hypothesis of Proposition 3.2. Fix a discount factor $\beta \in (0,1)$. By way of contradiction, we suppose that σ is a stationary equilibrium of $\tilde{\Gamma}(\beta)$. We first introduce a new function, along with its most basic properties, after which our analysis has two phases.

For each $t \in [0,1]$ let

$$\omega(t) = (\omega^C(t), \omega^{C'}(t), \omega^D(t), \omega^{D'}(t)) = \beta \int_0^1 \gamma_\sigma^{C,C',D,D'}(s)\, dU_t(s).$$

Lemma 4.1. *For all $t \in [0,1]$, $\omega^{C'}(t) = -\omega^C(t)$ and $\omega^{D'}(t) = -\omega^D(t)$.*

Proof. For any t, $\gamma^C(t)$ and $\gamma^{C'}(t)$ are the expectations of random variables, each of which is the negation of the other, and similarly for $\gamma^D(t)$ and $\gamma^{D'}(t)$. □

Lemma 4.2. *For all t, $||\omega(t)|| < 2$.*

Proof. The probability of the game continuing (i.e., of the game *not* going to the absorbing state 1) is never greater than $\frac{1}{16}$, and $||r^{C,C',D,D'}|| \leq \frac{17}{16}$, so

$$||\omega(t)|| \leq \int_t^1 ||\gamma_\sigma^{C,C',D,D'}(s)||\, ds \leq \sum_{j=0}^{\infty} \frac{1}{16^j} \cdot ||r^{C,C',D,D'}|| = \frac{16}{15} \cdot \frac{17}{16} < 2.$$

□

[5] For the sake of self containment we sketch the proof. A point $x \in E$ is a *density point* of E if $\lim_{\delta \to 0} \frac{1}{2\delta} \lambda(E \cap (x - \delta, x + \delta)) = 1$. Let x be a density point at which f is approximately differentiable. Then $f|_E$ is approximately differentiable at x because f is, and g is approximately differentiable at x because $g|_E$ is. By the Lesbesque density theorem (which is a special case of the Lesbesque differentiation theorem (e.g. Federer (1969) Thm. 2.9.8)) almost every $x \in E$ is a density point.

4.1 Equilibrium in a Stage

Fix a particular $t \in [0, 1)$. For $a \in I$, $\varrho \in (-\frac{1}{2}, \frac{1}{2})$, and $\omega = (\omega^C, \omega^{C'}, \omega^D, \omega^{D'}) \in \mathbb{R}^4$ let $g_\omega(\varrho, \cdot)$ be the perturbation of $g(\varrho, \cdot)$ given by

$$g_\omega^\ell(\varrho, a) = \begin{cases} g^\ell(\varrho, a), & \ell = A, D, E, F, \\ g^\ell(\varrho, a) + \frac{1}{64} h(a) \omega^\ell, & \ell = C, C', D, D'. \end{cases}$$

where $h(\cdot)$ is defined in (3.2). Since $\gamma_\sigma(1) \equiv 0$, for $\ell = C, C', D, D'$ the definition of the transitions gives[6]

$$\begin{aligned} X_\sigma^\ell(t, a) &= r^\ell(t, a) + \tilde{q}(t, a) \cdot \beta \int_0^1 \gamma_\sigma^\ell(s)\, dU_t(s) \\ &= (1-t)\Big(g^\ell(\varrho(t), a) + \tfrac{1}{64} h(a) \beta \int_0^1 \gamma_\sigma^\ell(s)\, dU_t(s)\Big) \qquad (4.1) \\ &= (1-t)\Big(g^\ell(\varrho(t), a) + \tfrac{1}{64} h(a) \omega^\ell(t)\Big) = (1-t) g_{\omega(t)}^\ell(\varrho(t), a). \end{aligned}$$

For $\ell = A, B, E, F$ the difference between $X_\sigma^\ell(t, a)$ and $(1-t) g_{\omega(t)}^\ell(\varrho(t), a)$ is the expected future payoff, which is unaffected by a^ℓ. Therefore the one shot deviation principle implies that:

Lemma 4.3. *In the game $X_\sigma(t, \cdot) - (1-t) g_{\omega(t)}(\varrho(t), \cdot)$ no player has any effect on her own payoff. Consequently, $\sigma(t)$ is an equilibrium of $g_{\omega(t)}(\rho(t), \cdot)$.*

For the sake of more compact notation we write x in place of $\sigma(t)$ and ϱ and ω in place of $\varrho(t)$ and $\omega(t)$ in the remainder of this subsection, which extracts the relevant consequences of x being an equilibrium of $g_\omega(\varrho, \cdot)$. Recall the definition of $\psi(x)$ given in (3.1), and denote

$$z = (z^E, z^F) = (2x^E[1] - 1, 2x^F[1] - 1).$$

Equilibrium analysis for $g_\omega(\varrho, \cdot)$ has the following consequences:

Lemma 4.4.

(a) $x^{A,B}$ *is an equilibrium of* G_z;

(b) $x^C[1] = \frac{1}{2} + \frac{1}{16}\omega^C$ *and* $x^{C'}[1] = \frac{1}{2} - \frac{1}{16}\omega^{C'}$;

(c) $x^D[1] = \frac{1}{2} + \frac{1}{16}\omega^D$ *and* $x^{D'}[1] = \frac{1}{2} - \frac{1}{16}\omega^{D'}$;

(d) $z^E = 1\,(-1)$ *if* $\langle v_E^\varrho, \psi(x)\rangle > (<)\, 0$;

(e) $z^F = 1\,(-1)$ *if* $\langle v_F^\varrho, \psi(x)\rangle > (<)\, 0$;

(f) If $\omega^{C,D} \in \mathcal{D}_{j,k}^\varrho$, *then* $z = (j, k)$.

(g) If $\omega^{C,D} \in \mathcal{D}_{j,k}^\varrho$, *then* $G^{C,D}(x) = (j, k)$.

[6] Recall the definition of X_σ given in (2.1).

Proof. Here (a) follows from $g_\omega^{A,B}(\varrho,\cdot) = g^{A,B}(\varrho,\cdot)$. Observe that $g_\omega^{C,C'}(\varrho,x)$ is the sum of

$$(-G^C(x^A,x^B), G^C(x^A,x^B)) + \tfrac{1}{64}(x^D[1] + x^{D'}[0])\omega^{C,C'},$$

which is unaffected by $x^{C,C'}$, and $\frac{1}{16}$ times the payoffs resulting from applying $x^{C,C'}$ to the bimatrix game below.

$C\backslash C'$	1	0
1	$\big(-1+\frac{1}{2}\omega^C, 1+\frac{1}{2}\omega^{C'}\big)$	$\big(1+\frac{1}{4}\omega^C, -1+\frac{1}{4}\omega^{C'}\big)$
0	$\big(1+\frac{1}{4}\omega^C, -1+\frac{1}{4}\omega^{C'}\big)$	$\big(-1,1\big)$

(For example, the part of C's payoff affected by C and C''s behavior that accrues in the future is $(a^C + a^{C'})\frac{1}{64}\omega^C$.) Since $|\omega^C|, |\omega^{C'}| < 2$ (Lemma 4.2) and $\omega^{C'} = -\omega^C$ (Lemma 4.1) this bimatrix game has a unique equilibrium, which must be $x^{C,C'}$. To see that $x^C[0] = \frac{1}{2} + \frac{1}{16}\omega^{C'}$ and $x^C[1] = \frac{1}{2} - \frac{1}{16}\omega^{C'}$ one can simply compare the payoff differences for C'. Similarly, $x^{C'}[0] = \frac{1}{2} + \frac{1}{16}\omega^C$ and $x^{C'}[1] = \frac{1}{2} - \frac{1}{16}\omega^C$, and now (b) follows from Lemma 4.1. Of course (c) follows by symmetry.

Substituting into the definition of $\psi(x)$ gives $\psi(x) = \frac{1}{8}\omega$, after which (d), (e), and (f) follow from $g_\omega^{E,F}(\varrho,\cdot) = g^{E,F}(\varrho,\cdot)$, while (g) follows from (a) and (f) and the choice of the games $G_{j,k}$. □

From the specification of the payoffs we have $||g^{C,D}(\varrho,x) - (-G^{C,D}(x))|| \leq \frac{1}{16}$. Since $\|G_z - G\| < \eta_0$ we have $\frac{7}{8} \leq ||G^{C,D}(x)|| \leq 1$, which yields the following result, and if $\omega^{C,D} \in \mathcal{D}^\varrho$, then (g) of Lemma 4.4 yields (a) of Lemma 4.6 below.

Lemma 4.5. $\frac{13}{16} \leq ||g^{C,D}(\varrho,x)|| \leq \frac{17}{16}$.

Lemma 4.6. *If* $\omega^{C,D} \in \mathcal{D}^\varrho$*, then:*

(a) $\frac{15}{16} \leq |g^C(\varrho,x)| \leq \frac{17}{16}$ *and* $\frac{15}{16} \leq |g^D(\varrho,x)| \leq \frac{17}{16}$*;*

(b) if $|\omega^C| \geq \frac{1}{2}|\omega^D|$*, then* $g^C(\varrho,x)\cdot\omega^C < 0$*;*

(c) if $|\omega^D| \geq \frac{1}{2}|\omega^C|$*, then* $g^D(\varrho,x)\cdot\omega^D < 0$.

Proof. By symmetry it suffices to prove (b). As $\omega \neq 0$, $|\omega^C| \geq \frac{1}{2}|\omega^D|$ implies $\omega^C \neq 0$. Since $|\varrho| < \frac{1}{2}$, if $\omega^D \neq 0$ then $|\varrho\cdot\omega^D| < \frac{1}{2}|\omega^D| \leq |\omega^C|$, so

$$\mathrm{sign}(\langle v_E^\varrho, \omega^{C,D}\rangle) = \mathrm{sign}(\omega^C - \varrho\omega^D) = \mathrm{sign}(\omega^C).$$

Clearly this holds if $\omega^D = 0$ as well. If $\omega^{C,D} \in \mathcal{D}^\varrho_{j,k}$, then $\mathrm{sign}(\langle v_E^\varrho, \omega^{C,D}\rangle) = j = G^C(x)$ (Lemma 4.4(g)) and $|g^C(\varrho,x) - (-G^C(x))| \leq \frac{1}{16}$. □

4.2 Equilibrium Over Time

For $t \in [0,1]$ let

$$V^\ell(t) = \gamma_\sigma^\ell(t) \quad \text{and} \quad W^\ell(t) = \int_t^1 V^\ell(s)\, ds.$$

Clearly $W(1) = 0$, and since 1 is an absorbing state with payoff 0, $V(1) = 0$. Since V is measurable and bounded, W is Lipschitz. Rademacher's theorem (e.g., Federer (1969) Thm. 3.1.6) implies that W is a.e. differentiable. The main remaining step in the argument concerns the function

$$J(t) = \tfrac{1}{2}(W^C(t))^2 + \tfrac{1}{2}(W^D(t))^2.$$

Proposition 4.7. *$J' > 0$ a.e.*

Since J is absolutely continuous, the fundamental theorem of calculus holds, so $0 = J(1) > J(0) \geq 0$. This contradiction proves Proposition 3.2.

It remains only to prove Proposition 4.7. Two lemmas prepare the main argument.

Lemma 4.8. *For all $t \in [0,1)$:*

(a) $||V^{C,D}(t) - r^{C,D}(t, \sigma(t))|| \leq \frac{1}{8}(1-t)$.

(b) $\frac{11}{16}(1-t) \leq ||V^{C,D}(t)|| \leq \frac{19}{16}(1-t)$.

Proof. Since, using (4.1),

$$V^{C,D}(t) = \gamma_\sigma^{C,D}(t) = X_\sigma^{C,D}(t, \sigma(t)) = (1-t) g_{\omega(t)}^{C,D}(\varrho(t), \sigma(t)),$$

(a) follows from $\|\omega(t)\| < 2$ (Lemma 4.2) and $h(a) \leq 4$. To obtain (b), combine (a) and the result of multiplying the inequality of Lemma 4.5 by $1-t$. □

Lemma 4.9. *For a.e. $t \in [0,1)$:*

(a) $W^{C,D}(t) \neq 0$.

(b) $\omega^{C,D}(t) \in \mathcal{D}^{\varrho(t)}$.

(c) If $|W^C(t)| \geq \frac{1}{2}|W^D(t)|$, *then* $V^C(t) \cdot W^C(t) \leq -\frac{13}{16}(1-t)|W^C(t)|$.

(d) If $|W^D(t)| \geq \frac{1}{2}|W^C(t)|$, *then* $V^D(t) \cdot W^D(t) \leq -\frac{13}{16}(1-t)|W^D(t)|$.

Proof.

(a) This follows Lemma 4.8 since $\frac{dW}{dt} = -V$ a.e.

(b) Define $\eta : [0,1] \to \mathbb{R}^2$ by $\eta(t) = \frac{\omega^{C,D}(t)}{||\omega^{C,D}(t)||} = \frac{W^{C,D}(t)}{||W^{C,D}(t)||}$. This is a.e. defined and a.e. differentiable, since W is Lipshitz and therefore both the denominator and numerator are Lipshitz, hence a.e. differentiable, and the latter is a.e. non-zero by (a). Clearly, $\eta(t) \in \mathcal{D}^{\varrho(t)}$ if and only if $\omega^{C,D}(t) \in \mathcal{D}^{\varrho(t)}$. For a.e. t, the requirement $\eta(t) \notin \mathcal{D}^{\varrho(t)}$ is equivalent (because $||\eta(\cdot)|| \equiv 1$ and $||\varrho|| < \frac{1}{2}$) to $\eta(t) \in \{\pm(-\varrho(t), 1), \pm(1, \varrho(t))\}$. Due to the assumed irregularity of $\varrho(\cdot)$, $\eta^C(t) \neq \pm\varrho(t)$ and $\eta^D(t) \neq \pm\varrho(t)$ for almost all t.

(c) Since $W^{C,D}(t)$ is a positive scalar multiple of $\omega^{C,D}(t)$, it suffices to prove the claim with $\omega^{C,D}(t)$ in place of $W^{C,D}(t)$. In view of (b) we may assume that $\omega^{C,D}(t) \in \mathcal{D}^{\varrho(t)}$, so $r^C(t, \sigma(t)) \cdot \omega^C(t) < 0$, $\frac{15}{16}(1-t) \leq |r^C(t, \sigma(t))|$, and $|r^C(t, \sigma(t)) - V^C(t)| \leq \frac{1}{8}(1-t)$. (These inequalities follow from Lemma 4.6(b), Lemma 4.6(a), and Lemma 4.8(a) respectively). Collectively these facts imply the claim.

(d) By symmetry, the proof of (c) also establishes (d).

□

Proof of Proposition 4.7. Let $t \in [0, 1)$ be such that all the properties of Lemma 4.9 hold. To simplify notation we drop the argument t. The chain rule gives $J' = -W^C \cdot V^C - W^D \cdot V^D$, so it suffices to show that $W^C \cdot V^C + W^D \cdot V^D < 0$. Either

$$|W^C| \geq \tfrac{1}{2}|W^D| \quad \text{and hence} \quad V^C \cdot W^C \leq -\tfrac{13}{16}(1-t)|W^C|$$

or

$$|W^D| \geq \tfrac{1}{2}|W^C| \quad \text{and hence} \quad V^D \cdot W^D \leq -\tfrac{13}{16}(1-t)|W^D|.$$

If both hold, then

$$V^C \cdot W^C + V^D \cdot W^D \leq -\tfrac{13}{16}(1-t) \cdot (|W^C| + |W^D|) < 0.$$

(The final inequality is from Lemma 4.9(a).) Therefore we may suppose that one of these holds, say the first without loss of generality, and the other does not, so $|W^D| < \frac{1}{2}|W^C|$. Since $|V^D| \leq \frac{19}{16}(1-t)$ (Lemma 4.8),

$$\begin{aligned}
V^C \cdot W^C + W^D \cdot V^D &\leq -\tfrac{13}{16}(1-t)|W^C| + |W^D| \cdot |V^D| \\
&\leq (1-t) \cdot (-\tfrac{13}{16}|W^C| + \tfrac{19}{16}|W^D|) \\
&< (1-t) \cdot (-\tfrac{13}{16}|W^C| + \tfrac{19}{16} \cdot \tfrac{1}{2}|W^C|) \\
&= -\tfrac{7}{32}(1-t)|W^C| < 0.
\end{aligned}$$

□

5 Description of the Error

The error in Levy (2013) has the following description. Part (iv) of the Proposition 4.1 on page 1991 is not correct. On page 1990 there is the following game with ε positive and small:

$G_Z(\frac{1}{2}) =$

$A\backslash B$	L	M	R
L	1 - ε,1 - ε	-ε,-1	-1, 1 - ε
M	-1,-ε	ε, ε	-1 + ε, -ε
R	1 - ε, -1	-ε, -1 + ε	-2,-2

(When $\varepsilon = 0$ this is an examples from Appendix B of Kohlberg and Mertens (1986).) As the paper points out, this has the pure equilibria (L, L) and (M, M) and the mixed equilibrium

$$\left(\tfrac{2\varepsilon}{2+\varepsilon}L + \tfrac{2-\varepsilon}{2+\varepsilon}M, \tfrac{2\varepsilon}{2+\varepsilon}L + \tfrac{2-\varepsilon}{2+\varepsilon}M\right).$$

The problem, which concerns (iv) of Proposition 4.1, arises from the fact that these are not the only equilibria. Indeed, in no equilibrium can both players use the strategy R with positive probability due to dominance. However, there are equilibria in which one player uses R. Specifically, the entire set of equilibria is $NE := \{(M, M)\} \cup P_1 \cup P_2$ where

$$\begin{aligned} P_1 = \operatorname{con}\{L, \tfrac{\varepsilon}{2}L + (1 - \tfrac{\varepsilon}{2})R\} \times \{L\} \\ \cup \{\tfrac{\varepsilon}{2}L + (1 - \tfrac{\varepsilon}{2})R\} \times \operatorname{con}\{L, \tfrac{2\varepsilon}{2+\varepsilon}L + \tfrac{2-\varepsilon}{2+\varepsilon}M\} \\ \cup \operatorname{con}\{\tfrac{\varepsilon}{2}L + (1 - \tfrac{\varepsilon}{2})R, \tfrac{2\varepsilon}{2+\varepsilon}L + \tfrac{2-\varepsilon}{2+\varepsilon}M\} \times \{\tfrac{2\varepsilon}{2+\varepsilon}L + \tfrac{2-\varepsilon}{2+\varepsilon}M\} \end{aligned}$$

(here con denotes the convex hull) and P_2 is the image of this under transposition of the players.

In Proposition 4.1 $E_{x\otimes y}[\vartheta]$ is a bilinear $\mathbb{R}^2$-valued function of (x, y), and (iv) of Proposition 4.1 holds if and only if the first component of $E_{x\otimes y}[\vartheta]$ is close to 1 for all $(x, y) \in NE$ but for $\left(\frac{\varepsilon}{2}L+(1-\frac{\varepsilon}{2})R, L\right)$ and $\left(\frac{\varepsilon}{2}L+(1-\frac{\varepsilon}{2})R, \frac{2\varepsilon}{2+\varepsilon}L+\frac{2-\varepsilon}{2+\varepsilon}M\right)$ this is quite far from being the case.

As we mentioned at the outset, Section 4.6 of Levy (2013) specifies conditions on a game (which would have the role played by the Kohlberg-Mertens game in the overall construction) that would allow the construction to succeed, but Corollary 5.1 below implies that they cannot hold.

McLennan (2014) gives the following theorem.

Theorem 5.1. *Let X be a compact convex subset of a locally convex topological space, let $U \subset X$ be open with $\overline{U}$ compact, let $F : \overline{U} \to X$ be an upper semicontinuous convex valued correspondence with no fixed points in $\overline{U} \setminus U$, let P be a compact absolute neighborhood retract, and let $\rho : \overline{U} \to P$ be a continuous function. If the fixed point index of F is not zero, then there is a neighborhood V of F in the (suitably topologized) space of upper semicontinuous convex valued correspondences from $\overline{U}$ to X such that for any continuous function $g : P \to V$ there is a $p \in P$ and a fixed point x of $g(p)$ such that $\rho(x) = p$.*

To obtain the following result as a consequence of this, let X be the set of mixed strategy profiles of G, let F be its best reply correspondence, and for $e \in P$, let $g(e)$ be the best response correspondence of $h(e)$.

Corollary 5.1. *If G is a finite strategic form game,* NE *is its set of Nash equilibria, P is a compact subset of* NE *that is an absolute neighborhood retract*[7]*, U is a neighborhood of* NE *in the space of mixed strategy profiles, and $\rho : U \to P$ is a retraction, then there is a neighborhood W of G in the space of games (for the given strategic form) such that for any continuous $h : P \to W$ there is some $e \in P$ such that $\rho^{-1}(e)$ contains a Nash equilibrium of $h(e)$.*

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[7]It is always the case that NE is itself an absolute neighborhood retract—for details see McLennan (2014).