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SOME EXTREMAL ALGEBRAS FOR HERMITIANS

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Abstract. We study three extremal Banach algebras: (a) generated by two hermitian unitaries; (b) generated by an element of norm 1 all of whose odd positive powers are hermitian; (c) generated by an element of norm 1 all of whose even positive powers are hermitian. In all three cases the numerical range is found for various elements. The second algebra is shown to be isometrically isomorphic to a subalgebra of the first. The third algebra is identified with a space of functions.

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1. Introduction. The extremal Banach algebra generated by a single normalised hermitian has been studied extensively (see [**4**], [**5**], [**6**], [**7**], [**8**]). Its dual space is a Banach space of entire functions with exponential growth and, consequently, most of the deeper properties come down to delicate questions about such entire functions, often coefficient problems. For two commuting hermitians, the extremal algebra is just the projective tensor product of the two one generator algebras (see [**5**]). We begin here a study of the case of two non-commuting hermitians. For hermitian elements h, k, the Jordan product $hk + kh$ fails to be hermitian in general, whereas the imaginary Lie product $i(hk - kh)$ is always hermitian. Hence the extremal algebra on two non-commuting hermitians will contain amongst its hermitians the free Lie algebra under this imaginary Lie product. This is a very large complicated algebra, and we shall consider here only the quotient algebra when the generators are idempotents. We get the same algebra by replacing the two hermitian idempotents with two hermitian unitaries; we take this viewpoint since it has nicer algebraic features. Our study is thus reduced here to a very tractable algebra.

Let $J = [-1, 1]$. In Section 2 we study the extremal algebra $Ea(J^2; unit)$ on two hermitian unitaries u, v. There is a natural representation for this algebra as 2×2 matrices of functions (see [**10**]). We calculate the norm and the numerical range for various elements. The hermitian element $k = \frac{1}{2}i(uv - vu)$ is obviously of special interest here. In fact, k^n is hermitian for all odd $n \in \mathbb{N}$ (but for no even $n \in \mathbb{N}$). We show in Section 3 that the subalgebra of $Ea(J^2; unit)$ generated by k (and 1) is $Ea(J; odd)$, the extremal algebra generated by a normalised hermitian all of whose odd positive powers are hermitian. It is then natural to ask for $Ea(J; even)$, the extremal algebra generated by a normalised element all of whose even positive

powers are hermitian; this is a much easier task and is dealt with in Section 4. If all powers of h are hermitian, then the extremal algebra is $C[-1, 1]$ by the Vidav-Palmer theorem.

On the face of it, we might consider hermitian generators of order other than 2. In fact, no cases arise other than of order 2. Suppose first that h is hermitian and $P(h) = 0$ for some polynomial P. Since h has real spectrum, every factor $h - \zeta$ of $P(h)$, with ζ not real, is invertible. Thus, without loss of generality, we may suppose that P has only real roots. Now let $h^n = 1$ with $n \in \mathbb{N}$, n odd. The only real factor of $h^n - 1$ is $h - 1$, and hence $h = 1$. Now let $h^n = 1$ with $n \in \mathbb{N}$, n even. The only real factors of $h^n - 1$ are $h - 1$ and $h + 1$, and hence $h^2 = 1$.

We shall write $\mathbb{D}^-$, $\mathbb{T}$ and $\mathbb{H}^-$ for the sets

$$\{z \in \mathbb{C} : |z| \leq 1\}, \quad \{z \in \mathbb{C} : |z| = 1\}, \quad \{z \in \mathbb{C} : |z - \tfrac{1}{2}| \leq \tfrac{1}{2}\},$$

respectively. For a Banach algebra element a, we denote the spectrum by $\mathrm{Sp}(a)$ and the spectral radius by $r(a)$. We use $|\cdot|_\infty$ to denote the supremum norm, taken over appropriate compact sets. For standard notation and results on numerical ranges, the reader is referred to [**1**] and [**2**].

2. The extremal algebra *Ea*(J^2; *unit*). Let $Ea(J^2; unit)$ denote the extremal Banach algebra on two generators u, v with $u^2 = v^2 = 1$ and u, v hermitian. We note that $\|u\| = \|v\| = 1$, since the spectral radius agrees with the norm for any hermitian. Then $\mathbb{Z}_2 * \mathbb{Z}_2$, the free product of the group $\mathbb{Z}_2$ with itself, is given by

$$\mathcal{I}_2 = \{1, u, v, uv, vu, uvu, vuv, \ldots\}.$$

Write $x = uv$. Then, as is well known, $\mathcal{I}_2$ may be regarded as the infinite dihedral group generated by u, x subject to $u^2 = 1$, $ux = x^{-1}u$. Give the group algebra $\mathbb{C}[\mathcal{I}_2]$ its usual involution: $(\sum \alpha_n g_n)^* = \sum \overline{\alpha_n} g_n^{-1}$. Let $\mathcal{J}$ be the subgroup of $\mathbb{C}[\mathcal{I}_2]$ consisting of all finite products of

$$\cos\tau + i(\sin\tau)u, \quad \cos\tau + i(\sin\tau)v \quad (\tau \in \mathbb{R}).$$

Define an algebra seminorm on $\mathbb{C}[\mathcal{I}_2]$ by

$$\|a\| = \inf\left\{\sum_{j=1}^{N} |\alpha_j| : N \in \mathbb{N},\ \alpha_j \in \mathbb{C},\ a_j \in \mathcal{J},\ \sum_{j=1}^{N} \alpha_j a_j = a\right\}.$$

For $a = \sum \alpha_n g_n \in \mathcal{J}$, we have $a^* a = 1$. From the coefficient of x^0, $\|a\|_2 = (\sum |\alpha_n|^2)^{1/2} = 1$, where $\|\cdot\|_2$ denotes the ℓ^2-norm. It follows that, for all $a \in \mathbb{C}[\mathcal{I}_2]$, $\|a\| \geq \|a\|_2$. Hence $\|\cdot\|$ is a norm.

Since $\exp(i\tau u) = \cos\tau + i(\sin\tau)u$ ($\tau \in \mathbb{R}$), we have the following result.

THEOREM 2.1. *The extremal algebra $Ea(J^2; unit)$ is isometrically isomorphic to the completion of $\mathbb{C}[\mathcal{I}_2]$ with respect to $\|\cdot\|$.*

When h is hermitian, so also is $a^{-1}ha$ whenever $\|a^{-1}\| = \|a\| = 1$. It follows that all words of odd length in u, v are hermitian; thus $x^n u$ is a hermitian involution for each $n \in \mathbb{Z}$.

Now let $k = \frac{1}{2}i(uv - vu) = \frac{1}{2}i(x - x^{-1})$. Then k is hermitian, and also $\frac{1}{2}i(x^2 - x^{-2})$ since u, vuv are hermitian. More generally, the element $k_n = \frac{1}{2}i(x^n - x^{-n})$ is hermitian for each $n \in \mathbb{N}$. Thus, $\mathbb{C}[\mathcal{I}_2]$ has a basis with "three quarters" of the elements being hermitian. In contrast, we shall see that each word of even length in $\mathcal{I}_2$, other than 1, has numerical range the closed unit disc $\mathbb{D}^-$.

PROPOSITION 2.2. *We have* $\mathrm{Sp}(x) = \mathbb{T}$ *and, for all* $n \in \mathbb{N}$,

$$V(x^n) = V(x^{-n}) = V\big(\tfrac{1}{2}(x^n + x^{-n})\big) = \mathbb{D}^-.$$

Proof. Since $\|x\| = \|x^{-1}\| = 1$, we have $\mathrm{Sp}(x) \subseteq \mathbb{T}$ and, for all $n \in \mathbb{N}$, $V(x^n)$, $V(x^{-n})$ and $V\big(\frac{1}{2}(x^n + x^{-n})\big)$ are contained in $\mathbb{D}^-$. If, for $m \in \mathbb{N}$, we quotient out by $x^m = 1$ then, in the factor algebra, $\mathrm{Sp}(x)$ contains all m^{th} roots of unity. Hence in $Ea(J^2; unit)$, $\mathrm{Sp}(x)$ contains all roots of unity and, since it is closed, $\mathrm{Sp}(x) = \mathbb{T}$.

Now let $n \in \mathbb{N}$ and quotient out by $x^{2n} = 1$ so that we are working with the extremal algebra over the finite dihedral group D_{2n}. Since $x^n = x^{-n}$, to complete the proof it is enough to show that in this finite dimensional algebra $V(x^n)$ contains $\mathbb{D}^-$. The norm is determined by the fact that all products of the exponential terms $\exp(i\tau u)$, $\exp(i\tau v)$ ($\tau \in \mathbb{R}$) have norm 1. A straightforward induction argument shows that any such product w is of the form

$$w = \sum_{j=0}^{2n-1} \alpha_j x^j + i \sum_{j=0}^{2n-1} \beta_j x^j u,$$

where $\alpha_j, \beta_j \in \mathbb{R}$, and that $ww^* = 1$, where here * is the canonical involution on the group algebra $\mathbb{C}[D_{2n}]$. Equate coefficients of x^0 and x^n and we obtain

$$\sum_{j=0}^{2n-1} (\alpha_j^2 + \beta_j^2) = 1, \qquad 2\sum_{j=0}^{n-1} (\alpha_j \alpha_{j+n} + \beta_j \beta_{j+n}) = 0.$$

It follows that

$$\sum_{j=0}^{n-1} \big[(\alpha_j \pm \alpha_{j+n})^2 + (\beta_j \pm \beta_{j+n})^2\big] = 1$$

and hence $|\alpha_0| + |\alpha_n| \leq 1$. Now define a functional on $\mathbb{C}[D_{2n}]$ by $f(1) = 1, f(x^n) = \zeta$ and $f(y) = 0$ for all other elements of D_{2n}. For $|\zeta| \leq 1$ we now get $|f(a)| \leq \|a\|$, for all a. This shows that $V(x^n)$ contains $\mathbb{D}^-$. □

PROPOSITION 2.3. *For all* $n \in \mathbb{N}$,

$$V(k_n) = J \quad \textit{and} \quad V(k^{2n-1}) = J.$$

Proof. Since k is hermitian with

$$\mathrm{Sp}(k) = \big\{\tfrac{1}{2}i(z - z^{-1}) : z \in \mathbb{T}\big\} = J,$$

we have $V(k) = J$. This holds, similarly, for k_n ($n \in \mathbb{N}$). We also have

$$k^{2n-1} = (-4)^{1-n} \sum_{r=0}^{n-1} \binom{2n-1}{r} (-1)^r k_{2n-1-2r},$$

so that k^{2n-1} is hermitian for all $n \in \mathbb{N}$, with spectrum and numerical range equal to J. □

It is less obvious that, for each $n \in \mathbb{N}$, $V(k^{2n}) = \mathbb{H}^-$. This will be established later in Corollary 3.10. For the present, we have the following result.

LEMMA 2.4. *For all $n \in \mathbb{N}$, $V(k^{2n}) \supseteq V(k^2) = \mathbb{H}^-$.*

Proof. We have $k^2 = \frac{1}{2} - \frac{1}{4}(x^2 + x^{-2})$ and, by Proposition 2.2, $V\big(\frac{1}{2}(x^2 + x^{-2})\big) = \mathbb{D}^-$. Hence $V(k^2) = \mathbb{H}^-$. Now quotient out $x^4 = 1$. Then, for $n \in \mathbb{N}$, $k^{2n} = k^2 = \frac{1}{2}(1 - x^2)$ and it follows from the proof of Proposition 2.2 that, in the factor algebra, $V(x^2) = \mathbb{D}^-$ and $V(k^{2n}) = \mathbb{H}^-$. □

3. The extremal algebra *Ea(J; odd)*. Let $Ea(J; odd)$ denote the extremal Banach algebra generated by h subject to the conditions $\|h\| = 1$ and every odd positive power of h is hermitian. We shall show that $Ea(J; odd)$ can be identified with the closed subalgebra of $Ea(J^2; unit)$ generated by the element $k = \frac{1}{2} i(uv - vu)$.

Now let $h(t) = t$ ($t \in J$). We define

$$\mathcal{P} = \left\{ \sum_{n=0}^{\infty} \alpha_n h^{2n+1} : \alpha_n \in \mathbb{R}, \sum |\alpha_n| < \infty \right\} \subseteq C[-1, 1]$$

and then $\mathcal{G} = \exp(i\mathcal{P})$. For each $g = \exp(ip) \in \mathcal{G}$ and $t \in J$, we have $g(-t) = \overline{g(t)} = g^{-1}(t)$, $|g(t)| = 1$, $g(0) = 1$ and $g^\alpha = \exp(i\alpha p) \in \mathcal{G}$ ($\alpha \in \mathbb{R}$). We now define

$$\mathcal{S} = \left\{ \sum_{n=0}^{\infty} \alpha_n h^n : \alpha_n \in \mathbb{C}, \ \sum |\alpha_n| < \infty \right\}$$

and

$$\mathcal{W} = \left\{ \sum_{n=1}^{\infty} \alpha_n \exp(i\beta_n h) : \alpha_n \in \mathbb{C}, \ \beta_n \in \mathbb{R}, \ \sum |\alpha_n| < \infty \right\}.$$

Then $\mathcal{G} \subseteq \mathcal{S} \subseteq \mathcal{W}$. For the second inclusion, see [**5**, Theorem 2.2].

For $a \in \operatorname{lin} \mathcal{G}$, define $\|a\|$ to be the infimum of $\sum_{n=1}^{N} |\alpha_n|$ over all representations $a = \sum_{n=1}^{N} \alpha_n g_n$ with $N \in \mathbb{N}$, $\alpha_n \in \mathbb{C}$ and $g_n \in \mathcal{G}$. Since $\mathcal{G}$ is a group, this clearly defines an algebra seminorm and it is a norm since each $|g_n(t)| = 1$, which gives $|a(t)| \leq \sum |\alpha_n|$ and hence $|a|_\infty \leq \|a\|$.

LEMMA 3.1. *We have $\mathcal{S} = \operatorname{lin} \mathcal{G}$ and, for $a = \sum_{n=0}^{\infty} \alpha_n h^n \in \mathcal{S}$ with $\alpha_n \in \mathbb{R}$ and $\sum_{n=0}^{\infty} |\alpha_n| < \infty$, $\|a\| \leq \sum_{n=0}^{\infty} |\alpha_n|$.*

Proof. Certainly $\operatorname{lin}\mathcal{G} \subseteq \mathcal{S}$. Consider $p = \sum_{n=0}^{\infty} \pi_n h^{2n+1}$ with $\pi_n \in \mathbb{R}$ and $\sum |\pi_n| \leq 1$. The Maclaurin series

$$\sin^{-1} p = p + \frac{1}{2}\cdot\frac{1}{3}\, p^3 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}\, p^5 + \cdots$$

has positive coefficients with finite sum giving $\sin^{-1} p \in \mathcal{P}$. Hence $p = \sin(\sin^{-1} p) \in \operatorname{lin}\mathcal{G}$ and $\|p\| \leq 1$. In particular, $h \in \operatorname{lin}\mathcal{G}$ with $\|h\| \leq 1$. Since $\operatorname{lin}\mathcal{G}$ is an algebra, it follows that $\mathcal{S} \subseteq \operatorname{lin}\mathcal{G}$. Hence $\mathcal{S} = \operatorname{lin}\mathcal{G}$.

By linearity, the above gives $\left\|\sum_{n=0}^{\infty} \pi_n h^{2n+1}\right\| \leq \sum_{n=0}^{\infty} |\pi_n|$ when all $\pi_n \in \mathbb{R}$. Hence if $\sum_{n=0}^{\infty} |\alpha_n| < \infty$, where all $\alpha_n \in \mathbb{R}$, since $\|h\| \leq 1$, we have

$$\left\|\sum_{n=0}^{\infty} \alpha_n h^n\right\| \leq |\alpha_0| + \left\|\sum_{n=0}^{\infty} \alpha_{2n+1} h^{2n+1}\right\| + \left\|h \sum_{n=1}^{\infty} \alpha_{2n} h^{2n-1}\right\| \leq \sum_{n=0}^{\infty} |\alpha_n|,$$

as required. □

LEMMA 3.2. *Let H be a hermitian element of some Banach algebra, with $\|H\| \leq 1$.*

(a) *Let $\sum_{n=0}^{\infty} \alpha_n h^n = \sum_{n=1}^{\infty} \beta_n \exp(i\gamma_n h)$, where $\alpha_n, \beta_n \in \mathbb{C}$, $\gamma_n \in \mathbb{R}$, $\sum |\alpha_n| < \infty$ and $\sum |\beta_n| < \infty$. Then $\sum_{n=0}^{\infty} \alpha_n H^n = \sum_{n=1}^{\infty} \beta_n \exp(i\gamma_n H)$.*

(b) *Let $g = \exp(ip)$, where $p = \sum_{n=0}^{\infty} \alpha_n h^{2n+1} \in \mathcal{P}$, and $g = \sum_{n=1}^{\infty} \gamma_n \exp(i\delta_n h)$, where $\gamma_n \in \mathbb{C}$, $\delta_n \in \mathbb{R}$ and $\sum |\gamma_n| < \infty$. Then $\exp(i \sum_{n=0}^{\infty} \alpha_n H^{2n+1}) = \sum_{n=1}^{\infty} \gamma_n \exp(i\delta_n H)$.*

Proof. (a) By [**9**, the first Proposition on page 424], this holds for any polynomial $\sum_{n=0}^{N} \alpha_n h^n$. By [**5**, Theorem 2.2], we have $h = \sum_{m=1}^{\infty} \delta_m \exp(i\varepsilon_m h)$ for certain $\delta_m \in \mathbb{C}$ and $\varepsilon_m \in \mathbb{R}$ with $\sum |\delta_m| = 1$, and similar expressions for h^n $(n = 2, 3, \ldots)$. From this it follows that

$$\sum_{n=0}^{N} \alpha_n h^n = \sum_{n=1}^{\infty} \beta_n \exp(i\gamma_n h) + \sum_{n=1}^{\infty} \lambda_n \exp(i\mu_n h) \qquad (*)$$

where $\lambda_n \in \mathbb{C}$, $\mu_n \in \mathbb{R}$ and $\sum_{n=1}^{\infty} |\lambda_n| \leq \sum_{n=N+1}^{\infty} |\alpha_n|$. We may substitute H for h in $(*)$. Then, letting $N \to \infty$ gives the required equality.

(b) Write $g = \sum_{n=0}^{\infty} \eta_n h^n$ with $\eta_n \in \mathbb{C}$ and $\sum |\eta_n| < \infty$. Then $\eta_0 = 1$, $\eta_1 = i\alpha_0, \ldots$. Hence,

$$\exp\left(i \sum_{n=0}^{\infty} \alpha_n H^{2n+1}\right) = \sum_{n=0}^{\infty} \eta_n H^n = \sum_{n=1}^{\infty} \gamma_n \exp(i\delta_n H),$$

using (a) for the last equality. □

PROPOSITION 3.3. *Let H be an element of some Banach algebra, with $\|H\| \leq 1$ and all odd positive powers of H hermitian, and let $\sum_{n=0}^{\infty} |\xi_n| < \infty$, where $\xi_n \in \mathbb{C}$. Then $\|\sum_{n=0}^{\infty} \xi_n H^n\| \leq \|\sum_{n=0}^{\infty} \xi_n h^n\|$.*

Proof. Let $\sum_{n=0}^{\infty} \xi_n h^n = \sum_{n=1}^{N} \alpha_n g_n$, where $g_n = \exp(ip_n)$ and $p_n = \sum_{j=0}^{\infty} \pi_{n,j} h^{2j+1}$ with $\pi_{n,j} \in \mathbb{R}$ and $\sum_j |\pi_{n,j}| < \infty$. We can write $g_n = \sum_{j=1}^{\infty} \gamma_{n,j} \exp(i\delta_{n,j} h)$ with $\gamma_{n,j} \in \mathbb{C}$,

$\delta_{n,j} \in \mathbb{R}$ and $\sum_j |\gamma_{n,j}| < \infty$. Hence $\sum_{n=0}^{\infty} \xi_n h^n = \sum_{n=1}^{N} \sum_{j=1}^{\infty} \alpha_n \gamma_{n,j} \exp(i\delta_{n,j} h)$. By Lemma 3.2,

$$\sum_{n=0}^{\infty} \xi_n H^n = \sum_{n=1}^{N} \sum_{j=1}^{\infty} \alpha_n \gamma_{n,j} \exp(i\delta_{n,j} H) = \sum_{n=1}^{N} \alpha_n \exp(i \sum_{j=0}^{\infty} \pi_{n,j} H^{2j+1}).$$

Hence $\| \sum_{n=0}^{\infty} \xi_n H^n \| \leq \sum_{n=1}^{N} |\alpha_n|$, since $\sum_{j=0}^{\infty} \pi_{n,j} H^{2j+1}$ is hermitian. Taking the infimum of $\sum_{n=1}^{N} |\alpha_n|$ over suitable representations gives the required inequality. □

Let $\mathcal{A}$ be the completion of $\mathcal{S}$ with respect to $\| \cdot \|$. Any $a \in \mathcal{A}$ may be written in the form $\sum_{n=1}^{\infty} a_n$, where $a_n \in \mathcal{S}$ and $\sum \|a_n\| < \infty$. Then each a_n is a finite sum $\sum_j \alpha_{n,j} g_{n,j}$, with $\sum_j |\alpha_{n,j}| < \|a_n\| + 2^{-n}$. Hence $a = \sum_{n=1}^{\infty} \beta_n g_n$, with $\beta_n \in \mathbb{C}$, $g_n \in \mathcal{G}$ and $\sum |\beta_n| < \infty$, where the series for a converges in $\mathcal{A}$.

An expression such as $\exp(ih)$ now has two meanings: a given element of $\mathcal{G}$ and a power series converging in $\mathcal{A}$. These coincide. Consider, for example, $\sin h$. The remainder after the nth term is $\sum_{j=n}^{\infty} (-1)^j h^{2j+1}/(2j+1)!$ which, by Lemma 3.1, has norm at most $\sum_{j=n}^{\infty} 1/(2j+1)!$.

By construction, for $n = 0, 1, 2, \ldots$, $\| \exp(i\alpha h^{2n+1}) \| = 1$ ($\alpha \in \mathbb{R}$) so that h^{2n+1} is hermitian. Similarly, any $p \in \mathcal{P}$ is hermitian. Since $h(1) = 1$, $\|h\| = 1$. Hence, by Proposition 3.3, we have established the following result.

THEOREM 3.4. *The extremal algebra $Ea(J; odd)$ is isometrically isomorphic to* $(\mathcal{A}, \| \cdot \|)$.

Note that $\mathcal{A}$ is a commutative Banach algebra. Let ϕ be a multiplicative linear functional on $\mathcal{A}$. Since $\|h\| = 1 = \| \exp(i\alpha h) \|$ ($\alpha \in \mathbb{R}$), $\phi(h) = t \in J$. Clearly, any $t \in J$ gives a multiplicative linear functional. Consider $a = \sum_{n=1}^{\infty} \alpha_n \exp(i \sum_{j=0}^{\infty} \beta_{n,j} h^{2j+1}) \in \mathcal{A}$, where $\alpha_n \in \mathbb{C}$, $\beta_{n,j} \in \mathbb{R}$, $\sum |\alpha_n| < \infty$ and $\sum_{j=0}^{\infty} |\beta_{n,j}| < \infty$. For the above ϕ, $\phi(a) = a(t)$, where $a(t) = \sum_{n=1}^{\infty} \alpha_n \exp(i \sum_{j=0}^{\infty} \beta_{n,j} t^{2j+1})$. The function a is continuous on J. We have $\mathrm{Sp}(a) = \{a(t) : t \in J\}$, so that $r(a) = \max\{|a(t)| : t \in J\}$.

For $\alpha \in \mathbb{C}$ and $g \in \mathcal{G}$, define

$$(\alpha g)^* = \overline{\alpha} g^{-1}, \qquad (\alpha g)^\dagger = \alpha g^{-1},$$

and extend, by addition, to involutions on $\mathcal{S}$. Then, for $a \in \mathcal{S}$ and $t \in J$, $a^*(t) = \overline{a(t)}$ and $a^\dagger(t) = a(-t)$, which shows that these involutions are well defined. They are isometric on $\mathcal{S}$ and so extend to $\mathcal{A}$. Note that $h^* = h = -h^\dagger$.

Let $\mathcal{Q}$ denote the closure of $\mathcal{P}$ in $\mathcal{A}$.

LEMMA 3.5. *Let $a \in \mathcal{A}$ with $a^* = a = -a^\dagger$. Then $a \in \mathcal{Q}$.*

Proof. Let $a = \sum_{n=1}^{\infty} \alpha_n g_n$. Then

$$a = a^* = \sum_{n=1}^{\infty} \overline{\alpha_n} g_n^{-1} = -a^\dagger = -\sum_{n=1}^{\infty} \alpha_n g_n^{-1} = -(a^\dagger)^* = -\sum_{n=1}^{\infty} \overline{\alpha_n} g_n.$$

Hence $a = \sum_{n=1}^{\infty} \beta_n (g_n - g_n^{-1})$, where $\beta_n = \frac{1}{4}(\alpha_n - \overline{\alpha_n}) \in i\mathbb{R}$. For $g = \exp(ip)$ with $p \in \mathcal{P}$, we have $-\frac{1}{2} i(g - g^{-1}) = \sin p \in \mathcal{P}$. Hence $\beta_n (g_n - g_n^{-1}) \in \mathcal{P}$ and $a \in \mathcal{Q}$. □

LEMMA 3.6. *Let $a \in \mathcal{A}$ with $a(0) = 1$, $a^*a = 1$ and $a^* = a^\dagger$. Then $a \in \exp(i\mathcal{Q})$.*

Proof. For $t \in J$, $|a(t)|^2 = |a^*(t)||a(t)| = (a^*a)(t) = 1$ and $a(-t) = a^\dagger(t) = a^*(t) = \overline{a(t)}$. Since we also have a continuous on J, there exists a real odd function b, continuous on J, such that $a = \exp(ib)$. Then there exists a real odd polynomial p in h such that $|b - p|_\infty < 1$. Let $g = \exp(ip)$ and $c = g^{-1}a$. Then $|a - g| = 2|\sin\frac{1}{2}(b - p)|$ which gives $|c - 1|_\infty = |a - g|_\infty < 1$. We have $c^*c = 1$ and $c^* = c^\dagger$. Let $d = \log c \in \mathcal{A}$, so that d is a limit of polynomials in c. Hence $d^* = \log(c^*) = \log(c^{-1}) = -\log c = -d$ and, similarly, $d^\dagger = -d$. This gives $(id)^* = id$ and $(id)^\dagger = -id$. By Lemma 3.5, $id \in \mathcal{Q}$. Hence $c = \exp d \in \exp(i\mathcal{Q})$ and $a = gc \in \exp(i\mathcal{Q})$. □

LEMMA 3.7. *Let p be a real polynomial in ih with $p(0) \neq 0$. Then there exists $g \in \exp(i\mathcal{Q})$ such that $(pg)^* = pg$.*

Proof. We have $p(-t) = \overline{p(t)}$ $(t \in J)$. Any zeros of p in J are in pairs of the form $\pm\alpha$. We may factor these out to assume, without loss, that p is never 0 on J, and hence $p^{-1} \in \mathcal{A}$. Since $(ih)^* = (ih)^\dagger$ we have $p^* = p^\dagger$. Let $q = p^*p^{-1}$. Then $q^*q = 1$ and $q^* = q^\dagger$. By Lemma 3.6, $q \in \exp(i\mathcal{Q})$. Let $g = q^{1/2}$. Then $g^2 = p^*p^{-1}$ and $pg = p^*g^{-1} = (pg)^*$. □

PROPOSITION 3.8. *Let p be a real polynomial in ih. Then $\|p\| = r(p)$.*

Proof. Assume that $r(p) < 1$. It is then enough to prove that $\|p\| \leq 1$. We have $p = \sum_{n=0}^{N} \alpha_n (ih)^n$, where $\alpha_n \in \mathbb{R}$ and $\alpha_0 \in (-1, 1)$. For $n \in \mathbb{N}$, define

$$p_n = p + (1 - \alpha_0)(1 - h^2)^n, \qquad q_n = p_n^* p_n,$$

so that q_n is a real polynomial in h^2 and $q_n(t) = |p_n(t)|^2$ $(t \in J)$. Then we have

$$q_n(0) = 1, \qquad q_n'(0) = 0 \quad \text{or} \quad q_n''(0) = 2\alpha_1^2 - 4\alpha_2 - 4n(1 - \alpha_0).$$

Choose $n_0 \in \mathbb{N}$ such that $q_{n_0}''(0) < 0$, and $\delta \in (0, 1)$ such that $q_{n_0}(t) < 1$ $(0 < |t| < \delta)$. Therefore, if $0 < |t| < \delta$ then $|p_{n_0}(t)| < 1$; if also $m > n_0$ then $p_m(t)$ is a convex combination of $p(t)$ and $p_{n_0}(t)$; and if further $r(p) + 2(1 - \delta^2)^m < 1$, then we have $|p_m(t)| < 1$ $(0 < |t| \leq 1)$. Repeat the above for p replaced with $-p$ to give $\widetilde{p}_m = -p + (1 + \alpha_0)(1 - h^2)^m$ with the same properties. We may choose, and now fix, the same m by increasing exponents. Then $2p = (1 + \alpha_0)p_m - (1 - \alpha_0)\widetilde{p}_m$ and it is enough to show that $\|p_m\| \leq 1$, since $\|\widetilde{p}_m\| \leq 1$ follows similarly. Note that $q_m(0) = 1$, $q_m''(0) < 0$ and $q_m(t) < 1$ $(0 < |t| \leq 1)$. By Lemma 3.7, there exists $g \in \exp(i\mathcal{Q})$ such that $(p_m g)^* = p_m g$. Let $s = p_m g$. Then $s^* = s = s^\dagger$, since $p_m g = p_m^* g^* = p_m^\dagger g^\dagger = (p_m g)^\dagger$. Also, $s^2 = p_m^* p_m = q_m$ and $1 - s^2 = h^2 c$, where c is a real polynomial in h^2 with $c(t) > 0$ $(t \in J)$. The functional calculus gives $\sqrt{c} \in \mathcal{A}$ as a limit of polynomials in c. Hence $(\sqrt{c})^* = \sqrt{c} = (\sqrt{c})^\dagger$. Let $d = s + ih\sqrt{c}$. Then $d^* = s - ih\sqrt{c} = d^\dagger$ and $d^*d = s^2 + h^2 c = 1$. By Lemma 3.6, $d \in \exp(i\mathcal{Q})$. Any element of $\mathcal{Q}$ is a limit of hermitians and is therefore hermitian. Hence $\|d\| = 1$. Similarly, $\|d^*\| = 1$. Hence $\|p_m\| \leq \|s\|\|g^{-1}\| = \|s\| \leq 1$. □

As in Section 2, we write $k = \frac{1}{2}i(x - x^{-1})$, where $x = uv$. Recall that $\|k\| = 1$ and all odd positive powers of k are hermitian. Since $Ea(J; odd)$ is the extremal algebra subject to this condition, it is immediate that $\|P(k)\| \leq \|P(h)\|$ for all polynomials P. In fact, we have the following result.

THEOREM 3.9. *The extremal algebra $Ea(J; odd)$ is isometrically isomorphic to the subalgebra of $Ea(J^2; unit)$ generated by the element $\frac{1}{2}i(uv - vu)$, where u, v are the generators of $Ea(J^2; unit)$.*

Proof. Let $\mathcal{H}$ be the subgroup of $\mathcal{I}_2$ generated by x, and let $\mathcal{B} = \mathbb{C}[\mathcal{H}]^-$ so that $\mathcal{B}$ is a commutative subalgebra of $Ea(J^2; unit)$. It is enough to prove that $\|P(k)\| = \|P(h)\|$ for all polynomials P.

Let

$$\mathcal{C} = \left\{ \sum_{n=0}^{N} \alpha_n k^n : N \in \mathbb{N},\ \alpha_n \in \mathbb{C} \right\}$$

and define an algebra norm $|\cdot|$ on $\mathcal{C}$ by $|\sum_{n=0}^{N} \alpha_n k^n| = \|\sum_{n=0}^{N} \alpha_n h^n\|$. Since $\mathrm{Sp}(k) = J$ it follows that $|\cdot|$ is well defined. It is enough to show that, for $c_0 \in \mathcal{C}$, $\|c_0\| = |c_0|$.

Let $c_0 \in \mathcal{C}$. Clearly $\|c_0\| \le |c_0|$. There exists $\phi \in \mathcal{C}'$ with $|\phi| = 1$ and $\phi(c_0) = |c_0|$. We show that ϕ has an extension $\psi \in \mathbb{C}[\mathcal{I}_2]'$ such that $\|\psi\| \le 1$. Then $\|c_0\| \ge \psi(c_0) = \phi(c_0) = |c_0|$.

Consider $a \in \mathcal{J}$. By induction, $a^*a = 1$ and $a = b + icu$ with $b, c \in \mathbb{R}[\mathcal{H}]$. For all $d \in \mathbb{R}[\mathcal{H}]$, we have $ud = d^*u$, which gives $1 = a^*a = b^*b + c^*c$. Let $\chi \in \mathcal{B}'$ be a multiplicative linear functional. Since $\chi(d^*) = \overline{\chi(d)}$ $(d \in \mathcal{B})$, we have $|\chi(b)|^2 + |\chi(c)|^2 = 1$. Hence $|\chi(b)| \le 1$ and so $r(b) \le 1$. Define an isomorphism of period 2 on $\mathbb{C}[\mathcal{H}]$ by $(x^j)^\dagger = (-x^{-1})^j$ $(j \in \mathbb{Z})$. We have $k^\dagger = k$ and so $c^\dagger = c$ $(c \in \mathcal{C})$. Since $\mathrm{Sp}(x) = \mathrm{Sp}(-x^{-1})$, we have $r(b^\dagger) = r(b)$. For $j \in \mathbb{Z}$, $x^j + (-x^{-1})^j$ is a real polynomial in $x - x^{-1} = -2ik$. Hence, if $d \in \mathbb{C}[\mathcal{H}]$ (respectively $\mathbb{R}[\mathcal{H}]$) then $d + d^\dagger$ is a polynomial (respectively real polynomial) in ik. Let $p = \frac{1}{2}(b + b^\dagger)$. We have $r(p) \le \frac{1}{2}(r(b) + r(b^\dagger)) \le 1$. By the above, $p = \sum \pi_j (ik)^j$, where $\pi_j \in \mathbb{R}$. Let $q = \sum \pi_j (ih)^j$. Since $\mathrm{Sp}(k) = \mathrm{Sp}(h)$, $r(q) = r(p) \le 1$. By Proposition 3.8, $\|q\| \le 1$. Hence $|p| = \|q\| \le 1$. Therefore $|\phi(p)| \le |\phi|\,|p| \le 1$.

For any $a \in \mathbb{C}[\mathcal{I}_2]$, write $a = b + icu$ with $b, c \in \mathbb{C}[\mathcal{H}]$, and define $\psi(a) = \frac{1}{2}\phi(b + b^\dagger)$. If $d \in \mathcal{C}$, then $d^\dagger = d$ and $\psi(d) = \phi(d)$. If $a \in \mathcal{J}$, then $|\psi(a)| \le 1$ by the preceding paragraph. Hence $\|\psi\| \le 1$ and ψ is a suitable extension of ϕ. □

COROLLARY 3.10. *Let $k = \frac{1}{2}i(uv - vu)$, where u, v are the generators of $Ea(J^2; unit)$. Then for $n \in \mathbb{N}$, $\|k^{2n} - \frac{1}{2}\| = \frac{1}{2}$ and $V(k^{2n}) = \mathbb{H}^-$.*

Proof. Define $a \in \mathcal{A}$ by $a(t) = 1 - 2t^{2n}$. Then a satisfies the conditions of Proposition 3.8, so that $\|a\| = 1$. Hence, by Theorem 3.9, $\|1 - 2k^{2n}\| = 1$, so that $V(k^{2n}) \subseteq \mathbb{H}^-$, and applying Lemma 2.4 completes the proof. □

4. The extremal algebra *Ea(J; even)*. Let $Ea(J; even)$ denote the extremal Banach algebra generated by h (not hermitian) subject to the conditions $\|h\| = 1$ and every even positive power of h is hermitian. The Vidav-Palmer theorem shows that the subalgebra generated by 1 and h^2 is a commutative unital monogenic C^*-algebra, and so is the algebra of continuous functions on $\mathrm{Sp}(h^2)$. Certainly this spectrum is contained in J so that $\mathrm{Sp}(h)$ is contained in $[-1, 1] \cup [-i, i]$. We shall see that, in fact, equality holds in both cases. The extremal algebra may be realised as follows.

Let $K = [-1, 1] \cup [-i, i]$ and let $\mathcal{E}$ be all continuous $f : K \to \mathbb{C}$ such that

$$\widetilde{f}(0) = \lim_{t \to 0} \frac{f(t) - f(-t)}{2t}$$

exists. Then $\mathcal{E}$ is an algebra under pointwise operations. For $f \in \mathcal{E}$ and $t \in K$, define

$$f_1(t) = \frac{f(t) + f(-t)}{2}, \qquad f_2(t) = \begin{cases} \dfrac{f(t) - f(-t)}{2t} & \text{if } t \neq 0, \\ \widetilde{f}(0) & \text{if } t = 0. \end{cases}$$

For $f \in \mathcal{E}$, f_1 and f_2 are the unique even continuous functions on K such that

$$f(t) = f_1(t) + tf_2(t) \quad (t \in K).$$

Conversely, $f \in \mathcal{E}$ whenever f has such a decomposition. For $f \in \mathcal{E}$, define

$$\|f\| = |f_1|_\infty + |f_2|_\infty.$$

Then $\|\cdot\|$ is an algebra norm and $(\mathcal{E}, \|\cdot\|)$ is a Banach algebra since, as a normed space, it is the ℓ^1 direct sum of two continuous function spaces. Since the even polynomials are $|\cdot|_\infty$-dense in the even continuous functions it follows that the polynomials are $\|\cdot\|$-dense in $(\mathcal{E}, \|\cdot\|)$.

Now let h be the identity function on K. Then $h \in \mathcal{E}$ with $\|h\| = 1$. It follows directly from the definition of spectrum that $\mathrm{Sp}(h) = K$, so that all even positive powers of h have spectrum J.

Proposition 4.1. *For $n \in \mathbb{N}$,*

$$V(h^n) = J \ (n \text{ even}) \quad \text{and} \quad V(h^n) = \mathbb{D}^- \ (n \text{ odd}).$$

Generalising the odd power case, for any even function $f \in \mathcal{E}$, $V(hf)$ is the closed disc, centre 0 *and radius ρ, where $\rho = |f|_\infty$.*

Proof. For $n \in \mathbb{N}$, n even, $\|\exp(i\tau h^n)\| = 1$ ($\tau \in \mathbb{R}$), so that h^n is hermitian, and hence $V(h^n) = J$, the convex hull of the spectrum.

For $f \in \mathcal{E}$, f an even function, and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we have

$$\sup \mathrm{Re}\, V(\lambda hf) = \lim_{\alpha \to 0^+} \frac{\|1 + \alpha\lambda hf\| - 1}{\alpha} = \lim_{\alpha \to 0^+} \frac{(1 + \alpha\rho) - 1}{\alpha} = \rho,$$

giving $V(hf)$ as stated. □

Finally, we have the following identification.

Theorem 4.2. *The extremal algebra $Ea(J; \text{even})$ is isometrically isomorphic to $(\mathcal{E}, \|\cdot\|)$.*

Proof. Let $h(t) = t$ ($t \in K$). Then $\|h\| = 1$ and all even positive powers of h are hermitian. To see that $\|\cdot\|$ is extremal, let g be any Banach algebra element with

$\|g\| = 1$ and all even powers of g hermitian, let P be any polynomial and let Q and R be the even polynomials such that, for all t, $P(t) = Q(t) + tR(t)$. Then, since $Q(g)$ and $R(g)$ are in the algebra generated by 1 and g^2, and $\mathrm{Sp}(g) \subseteq K = \mathrm{Sp}(h)$, we have

$$\|P(g)\| \leq \|Q(g)\| + \|R(g)\| = r(Q(g)) + r(R(g)) \leq |Q(h)|_\infty + |R(h)|_\infty = \|P(h)\|$$

and the result follows. □

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