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A REMARK ON RATIONAL CHEREDNIK ALGEBRAS AND DIFFERENTIAL OPERATORS ON THE CYCLIC QUIVER

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Abstract. We show that the spherical subalgebra $U_{k,c}$ of the rational Cherednik algebra associated to $S_n \wr C_\ell$, the wreath product of the symmetric group and the cyclic group of order ℓ, is isomorphic to a quotient of the ring of invariant differential operators on a space of representations of the cyclic quiver of size ℓ. This confirms a version of [**5**, Conjecture 11.22] in the case of cyclic groups. The proof is a straightforward application of work of Oblomkov [**12**] on the deformed Harish–Chandra homomorphism, and of Crawley–Boevey, [**3**] and [**4**], and Gan and Ginzburg [**7**] on preprojective algebras.

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1. Introduction.

1.1. The representation theory of symplectic reflection algebras has links with a number of subjects including algebraic combinatorics, resolutions of singularities, Lie theory and integrable systems. There is a family of symplectic reflection algebras associated to any symplectic vector space V and finite subgroup $\Gamma \leq Sp(V)$, but a simple reduction allows one to study those subgroups Γ which are generated by symplectic reflections (i.e. by elements whose set of fixed points is of codimension two in V). This essentially focuses attention on two cases:

(1) $\Gamma = W$, a finite complex reflection group, acting on $V = \mathfrak{h} \oplus \mathfrak{h}^*$ where $\mathfrak{h}$ is a reflection representation of W;
(2) $\Gamma = S_n \wr K$, where K is a finite subgroup of $SL_2(\mathbb{C})$, acting naturally on $(\mathbb{C}^2)^n$.

The representation theory in the first case is mysterious at the moment: several important results are known but there is no general theory yet. On the other hand a geometric point of view on the representation theory in the second case is beginning to emerge. A key fact is that in this case the singular space V/Γ admits a crepant resolution of singularities: the representation theory of the symplectic reflection algebra is then expected to be closely related to the resolution. In the case $\Gamma = S_n$ (i.e. K is trivial) there are two approaches to this: the first is via noncommutative algebraic geometry [**8**] the second via sheaves of differential operators [**7**]. In this paper we extend the second approach to the groups $\Gamma = \Gamma_n = S_n \wr C_\ell$.

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1.2. To state our result we need to introduce a little notation here. Let Q be the cyclic quiver with ℓ vertices and cyclic orientation. Choose an extending vertex (in this case any vertex) 0. Then let Q_∞ be the quiver obtained by adding one vertex named ∞ to Q that is joined to 0 by a single arrow.

We shall consider representation spaces of these quivers. Let $\delta = (1, 1, \ldots, 1)$ be the affine dimension vector of Q, and set $\epsilon = e_\infty + n\delta$, a dimension vector for Q_∞. Let $\mathrm{Rep}(Q, n\delta)$ and $\mathrm{Rep}(Q_\infty, \epsilon)$ be the representation spaces of these quivers with the given dimension vectors. There is an action of $G = \prod_{r=0}^{\ell-1} GL_n(\mathbb{C})$ on both these spaces. In fact, the action of the scalar matrices in G is trivial on $\mathrm{Rep}(Q, n\delta)$ (but not on $\mathrm{Rep}(Q_\infty, \epsilon)$) and so in this case the action descends to an action of $PG = G/\mathbb{C}^*$.

Let $\mathfrak{X} = \mathrm{Rep}(Q, n\delta) \times \mathbb{P}^{n-1}$. There is an action of PG on $\mathfrak{X}$.

1.3. Let $D(\mathrm{Rep}(Q_\infty, \epsilon))$ denote the ring of differential operators on the affine space $\mathrm{Rep}(Q_\infty, \epsilon)$, $D_{\mathfrak{X}}(nk)$ the sheaf of twisted differential operators on $\mathfrak{X}$ and $D(\mathfrak{X}, nk)$ its algebra of global sections. The group action of G (respectively PG) on $\mathrm{Rep}(Q_\infty, \epsilon)$ (respectively $\mathfrak{X}$) differentiates to an action of $\mathfrak{g} = \mathrm{Lie}(G)$ (respectively $\mathfrak{pg} = \mathrm{Lie}(PG)$) by differential operators. This gives mappings

$$\hat{\tau} : \mathfrak{g} \longrightarrow D(\mathrm{Rep}(Q_\infty, \epsilon)), \qquad \tau : \mathfrak{pg} \longrightarrow D_{\mathfrak{X}}(nk).$$

1.4. Let $U_{k,c}$ be the spherical subalgebra of type $S_n \wr C_\ell$. (This is defined in Section 3.4.)

THEOREM. *For all* (k, c) *there are isomorphisms of algebras*

$$\left(\frac{D(\mathrm{Rep}(Q_\infty, \epsilon))}{I_{k,c}}\right)^G \cong \left(\frac{D(\mathfrak{X}, nk)}{I_c}\right)^{PG} \cong U_{k,c},$$

where $I_{k,c}$ *is the left ideal of* $D(\mathrm{Rep}(Q_\infty, \epsilon)$ *generated by* $(\hat{\tau} - \chi_{k,c})(\mathfrak{g})$ *and* I_c *is the left ideal of* $D(\mathfrak{X}, nk)$ *generated by* $(\tau - \chi_c)(\mathfrak{pg})$ *for suitable characters* $\chi_{k,c} \in \mathfrak{g}^*$ *and* $\chi_c \in \mathfrak{pg}^*$. (*These are defined in Section* 4.)

Note that it is a standard fact that the left hand side is an algebra. The proof of the theorem has two parts. One part constructs a filtered homomorphism from the left hand side to the right hand side using as its main input the work of Oblomkov [**12**]. The other part proves that the associated graded homomorphism is an isomorphism and is a simple application of results of Crawley–Boevey [**3**] and [**4**], and of Gan and Ginzburg [**7**].

1.5. We give an application of this result in Section 4. For related pairs (k, c) and (k', c') we construct "shift functors"

$$U_{k,c}\text{-mod} \longrightarrow U_{k',c'}\text{-mod}$$

using differential operators. We expect these to be a useful tool in the representation theory of Cherednik algebras, deserving of careful study.

1.6. While writing this down, we were informed that the general version of [**5**, Conjecture 11.22] has been proved in [**6**]. This result is more general than the work presented here and requires a new approach and ideas to overcome problems that simply do not arise for the case $\Gamma = S_n \wr C_\ell$.

2. Quivers.

2.1. Once and for all fix integers ℓ and n. We assume that both are greater than 1. Set $\eta = \exp(2\pi i/\ell)$.

2.2. Let Q be the cyclic quiver with ℓ vertices and cyclic orientation. Choose an extending vertex (in this case any vertex) 0. Then let Q_∞ be the quiver obtained by adding one vertex named ∞ to Q that is joined to 0 by a single arrow. Let $\overline{Q}$ and $\overline{Q}_\infty$ denote the double quivers of Q and Q_∞ respectively, obtained by inserting an arrow a^* in the opposite direction to every arrow a in the quiver.

We shall consider representation spaces of these quivers. Let $\delta = (1, 1, \ldots, 1)$ be the affine dimension vector of Q, and set $\epsilon = e_\infty + n\delta$, a dimension vector for Q_∞. Recall that

$$\mathrm{Rep}(Q, n\delta) = \bigoplus_{r=0}^{\ell-1} \mathrm{Mat}_n(\mathbb{C}) = \{(X_0, X_1, \ldots, X_{\ell-1}) : X_r \in \mathrm{Mat}_n(\mathbb{C})\} = \{(X)\}$$

and

$$\begin{aligned}\mathrm{Rep}(Q_\infty, \epsilon) &= \bigoplus_{r=0}^{\ell-1} \mathrm{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n = \{(X_0, X_1, \ldots, X_{\ell-1}, i) : X_r \in \mathrm{Mat}_n(\mathbb{C}), i \in \mathbb{C}^n\} \\ &= \{(X, i)\}.\end{aligned}$$

Let $G = \prod_{r=0}^{\ell-1} GL_n(\mathbb{C})$ be the base change group. If $g = (g_0, \ldots, g_{\ell-1})$, then g acts on $\mathrm{Rep}(Q, n\delta)$ by

$$g \cdot (X_0, X_1, \ldots, X_{\ell-1}) = \left(g_0X_0g_1^{-1}, g_1X_1g_2^{-1}, \ldots, g_{\ell-1}X_{\ell-1}g_0^{-1}\right)$$

and on $\mathrm{Rep}(Q_\infty, \epsilon)$ by

$$g \cdot (X_0, X_1, \ldots, X_{\ell-1}, i) = \left(g_0X_0g_1^{-1}, g_1X_1g_2^{-1}, \ldots, g_{\ell-1}X_{\ell-1}g_0^{-1}, g_0i\right).$$

The action of the scalar subgroup $\mathbb{C}^*$ is trivial in the first action (but not the second) and so we can consider the first action, as a PG-action where $PG = G/\mathbb{C}^*$. Let $\mathfrak{g}$ and $\mathfrak{pg}$ be the Lie algebras of G and PG, respectively.

2.3. Let $\mathfrak{h}^{\mathrm{reg}} \subset \mathbb{C}^n$ be the affine open subvariety consisting of points $x = (x_1, \ldots, x_n)$ such that

(i) if $i \neq j$ then $x_i \neq \eta^m x_j$ for all $m \in \mathbb{Z}$,
(ii) for each $1 \leq i \leq n$ $x_i \neq 0$.

This is the subset of $\mathbb{C}^n$ on which $\Gamma_n = S_n \wr C_\ell$ acts freely.

2.4. We can embed $\mathfrak{h}^{\text{reg}}$ into $\text{Rep}(Q, n\delta)$ by first considering a point $x = (x_1, \ldots, x_n) \in \mathfrak{h}^{\text{reg}}$ as a diagonal matrix $X = \text{diag}(x_1, \ldots, x_n)$ and then sending this to $\underline{X} = (X, X, \ldots, X)$. We denote the image of $\mathfrak{h}^{\text{reg}}$ in $\text{Rep}(Q, n\delta)$ by $\mathcal{S}$.

Let T_Δ be the subgroup of G with elements $(T, T, \ldots, T)$ where T is a diagonal matrix in $GL_n(\mathbb{C})$. Then T_Δ is the stabiliser of $\mathcal{S}$. Now consider the mapping

$$\pi : G/T_\Delta \times \mathfrak{h}^{\text{reg}} \longrightarrow \text{Rep}(Q, n\delta)$$

given by $\pi(gT_\Delta, x) = g \cdot \underline{X}$. If we let G act on $G/T_\Delta \times \mathfrak{h}^{\text{reg}}$ by left multiplication, then π is a G-equivariant mapping.

LEMMA. *π is an étale mapping with covering group Γ_n. In fact, its image* $\text{Rep}(Q, n\delta)^{reg}$ *is open in* $\text{Rep}(Q, n\delta)$ *and we have an isomorphism*

$$\omega : G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{reg} \longrightarrow \text{Rep}(Q, n\delta)^{reg}.$$

Proof. Let $\mathcal{S} = \{\underline{X} : x \in \mathfrak{h}^{\text{reg}}\}$. Set $N_G(\mathcal{S}) = \{g \in G : g \cdot \mathcal{S} = \mathcal{S}\}$ and

$$Z_G(\mathcal{S}) = \{g \in G : g \cdot \underline{X} = \underline{X} \text{ for all } \underline{X} \in \mathcal{S}\}.$$

Suppose that $g \cdot \underline{X} = \underline{Y}$ for some $\underline{X}, \underline{Y} \in \mathcal{S}$. This implies that for each $0 \leq i \leq \ell - 1$

$$g_i \,\text{diag}(x)^\ell g_i^{-1} = \text{diag}(y)^\ell.$$

The hypotheses on $\mathfrak{h}^{\text{reg}}$ imply that both $\text{diag}(x)^\ell$ and $\text{diag}(y)^\ell$ are regular semisimple in $\text{Mat}_n(\mathbb{C})$. Two such elements are conjugate if and only if $g_i \in N_{GL_n(\mathbb{C})}(T) = T \cdot S_n$, where T is the diagonal subgroup of $GL_n(\mathbb{C})$. Hence there exists $\sigma \in S_n$ such that for all i we have $g_i = t_i\sigma$ for some $t_i \in T$, and for all $1 \leq r \leq n$ we have that $x_{\sigma(r)}^\ell = y_r^\ell$. Hence $x_{\sigma(r)} = \eta^{m_r} y_r$ for some $m_r \in \mathbb{Z}$. Now we find that $\underline{Y} = g \cdot \underline{X}$ implies that $\text{diag}(y_r) = t_i t_{i+1}^{-1} \text{diag}(\eta^{m_r} y_r)$. Since $y_r \neq 0$ this shows that $t_{i+1} = \text{diag}(\eta^{m_r}) t_i$ for each i. Hence we find that $gT_\Delta = (\sigma, \text{diag}(\eta^{m_r})\sigma, \ldots, \text{diag}(\eta^{m_r})^{\ell-1}\sigma)T_\Delta$.

In particular, if $\underline{X} = \underline{Y}$ we see from above that each $m_r = 0$, so that $Z_G(\mathcal{S}) = T_\Delta$. Thus the group Γ_n is isomorphic to $N_G(\mathcal{S})/Z_G(\mathcal{S})$ via the homomorphism that sends $(\eta^{m_1}, \ldots, \eta^{m_r})\sigma$ to $(\sigma, \text{diag}(\eta^{m_r})\sigma, \ldots, \text{diag}(\eta^{m_r})^{\ell-1}\sigma)T_\Delta$.

Now suppose that $\pi(gT_\Delta, x) = \pi(hT_\Delta, y)$. Then $(h^{-1}g) \cdot \underline{X} = \underline{Y}$ and so we see that $h^{-1}g \in N_G(\mathcal{S})$. This shows that π is the composition

$$G/T_\Delta \times \mathfrak{h}^{\text{reg}} \longrightarrow G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{\text{reg}} \xrightarrow{\sim} \text{Rep}(Q, n\delta)^{reg}.$$

The first mapping factors out the action of Γ_n, and since Γ_n acts freely on $\mathfrak{h}^{\text{reg}}$ this is an étale mapping. Hence, to prove the lemma, it suffices to show that $\text{Rep}(Q, n\delta)^{reg}$ is open in $\text{Rep}(Q, n\delta)$.

We claim first that $\text{Rep}(Q, n\delta)^{reg}$ is the set O of representations of Q that decompose into n simple modules of dimension δ and whose endomorphism ring is n-dimensional. To prove this, observe that any element of $\text{Rep}(Q, n\delta)^{reg}$ is isomorphic to a representation of the form $\underline{X}$ and so it decomposes into the n indecomposable modules $\underline{X}_1, \ldots \underline{X}_n$ of dimension δ, where $\underline{X}_i = (x_i, x_i, \ldots, x_i)$. (The condition $x_i \neq 0$ implies simplicity.) Now the representation $\underline{X}_i$ is isomorphic to the representation $(1, 1, \ldots, 1, x_i^\ell)$. By hypothesis $x_i^\ell \neq x_j^\ell$ and so we deduce that the representations $\underline{x}_i$ are pairwise non-isomorphic which ensures that the endomorphism ring of $\underline{X}$ is n-dimensional. This proves the inclusion $\text{Rep}(Q, n\delta)^{reg} \subseteq O$. On the other hand, if V belongs to O then $V = V_1 \oplus \ldots \oplus V_n$, where each V_i is isomorphic to a representation

$(1, 1, \dots, 1, \nu_i)$, for some non-zero scalars ν_i. Moreover, since $\dim \operatorname{End}(V) = n$ the ν_i must be pairwise distinct. Now, let η_i be an ℓ-th root of ν_i. Then V_i is isomorphic to $(\eta_i, \dots, \eta_i)$. Therefore V is isomorphic to the representation $\underline{X}$, where $x = (\eta_1, \dots, \eta_n)$.

Now we must show that O is open in $\operatorname{Rep}(Q, n\delta)$. We use first the fact that the canonical decomposition of the vector $n\delta$ is $\delta + \delta + \cdots + \delta$, [**13**, Theorem 3.6]. This means that the representations of $\operatorname{Rep}(Q, n\delta)$ whose indecomposable components all have dimension δ form an open set. Now, consider the morphism f from $\operatorname{Rep}(Q, \delta)$ to $\mathbb{C}$ that sends the representation $(\lambda_1, \dots, \lambda_\ell)$ to the product $\lambda_1 \dots \lambda_\ell$. The open set $f^{-1}(\mathbb{C}^*)$ consists of the simple representations of dimension vector δ. Therefore the subset of $\operatorname{Rep}(Q, n\delta)$ consisting of representations which decompose as the sum of n simple representations of dimension vector δ is open. On the other hand, the function from $\operatorname{Rep}(Q, n\delta)$ to $\mathbb{N}$ that sends a representation V to $\dim \operatorname{End}(V)$ is upper semi-continuous. Thus $\{V : \dim \operatorname{End}(v) \le n\}$ is an open set in $\operatorname{Rep}(Q, n\delta)$. Intersecting these two sets shows that O is open, as required. $\square$

2.5. Now we are going to move from Q to Q_∞ and so we start with the following inclusion:

$$\left\{([gT_\Delta, x], i) : g_0^{-1} i \text{ is a cyclic vector for } \operatorname{diag}(x)\right\} \subset (G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{\mathrm{reg}}) \times \mathbb{C}^n.$$

By applying $\omega^{-1} \times \mathrm{id}_{\mathbb{C}^n}$, the left-hand side corresponds to an open subset of $\operatorname{Rep}(Q, n\delta) \times \mathbb{C}^n = \operatorname{Rep}(Q_\infty, \epsilon)$. Call that set U_∞. This is a G-invariant open set since the G-action on triples is given by

$$h \cdot ([gT_\Delta, x], i) = ([hgT_\Delta, x], h_0 i)$$

so that $g_0^{-1} i$ is cyclic for $\operatorname{diag}(x)$ if and only if $(h_0 g_0)^{-1} h_0 i$ is cyclic for $\operatorname{diag}(x)$. Observe too that U_∞ is an affine variety. Indeed it is defined by the non-vanishing of the morphism

$$s : (G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{\mathrm{reg}}) \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

which sends $([gT_\Delta, x], i)$ to $(g_0^{-1} i) \wedge \operatorname{diag}(x) \cdot (g_0^{-1} i) \wedge \cdots \wedge \operatorname{diag}(x)^{n-1} \cdot (g_0^{-1} i)$.

LEMMA. *The G-action on U_∞ is free and projection onto the second component*

$$\pi_2 : U_\infty \longrightarrow \mathfrak{h}^{reg}/\Gamma_n$$

is a principal G-bundle.

Proof. Suppose that $h \cdot ([gT_\Delta, x], i) = ([gT_\Delta, x], i)$.Then $[g^{-1}hgT_\Delta, x] = [T_\Delta, x]$ and so, by Lemma 2.4, $g^{-1}hg \in T_\Delta$.

We have that $h_0 i = i$. Setting $i' = g_0^{-1} i$ implies that $g_0^{-1} h_0 g_0 i' = i'$. By hypothesis i' is a cyclic vector for $\operatorname{diag}(x)$. Hence with respect to the standard basis $\{e_j\}$, i' decomposes as $\sum \lambda_j e_j$, where each λ_j is non-zero. Therefore the only diagonal matrix that fixes i' is the identity element. In other words $g_0^{-1} h_0 g_0 = I_n$. Since $g^{-1}hg \in T_\Delta$ this implies that $g^{-1}hg = \mathrm{id}$. Thus $h = \mathrm{id}$ and this proves that the action is free.

It remains to prove that each fibre of π_2 is a G-orbit. We take $([gT_\Delta, x], i) \in \pi_2^{-1}([x])$. This equals $g \cdot ([T_\Delta, x], g_0^{-1} i)$. Now $g_0^{-1} i$ is a cyclic vector for $\operatorname{diag}(x)$ and so it has the form $\sum \lambda_j e_j$ with each λ_j non-zero. Let $t = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and consider

$\underline{t} = (t, \dots, t) \in T_\Delta$. We have

$$([gT_\Delta, x], i) = g\underline{t}\underline{t}^{-1}([T_\Delta, x], g_0^{-1} i) = g\underline{t}\left([T_\Delta, x], \sum_{j=1}^{n} e_j\right).$$

This proves that each fibre of π_2 is indeed a G-orbit. $\square$

2.6. Consider the representation space for the doubled quiver $\overline{Q}_\infty$:

$$\begin{aligned}\mathrm{Rep}(\overline{Q}_\infty, \epsilon) &= \{(X_0, \dots, X_{\ell-1}, Y_0, \dots, Y_{\ell-1}, i, j) : X_r, Y_r \in \mathrm{Mat}_n(\mathbb{C}), i \in \mathbb{C}^n, j \in (\mathbb{C}^*)^n\} \\ &= \{(X, Y, i, j)\}.\end{aligned}$$

We can naturally identify it with $T^* \mathrm{Rep}(Q_\infty, \epsilon)$. The group G acts on the base and hence on the total space of the cotangent bundle. The resulting moment map

$$\mu : \mathrm{Rep}(\overline{Q}_\infty, \epsilon) \longrightarrow \mathfrak{g}^* \cong \mathfrak{g}$$

is given by

$$\mu(X, Y, i, j) = [X, Y] + ij.$$

THEOREM (Gan–Ginzburg, Crawley–Boevey). *Let $\mu^{-1}(0)$ denote the scheme-theoretic fibre of μ.*

(1) $\mu^{-1}(0)$ *is reduced, equidimensional and a complete intersection.*
(2) *The moment map μ is flat.*
(3) $\mathbb{C}[\mu^{-1}(0)]^G \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}$.

Proof. (i) This is proved in [**7**, Theorem 3.2.3].

(ii) This follows from [**3**, Theorem 1.1] and the dimension formula in [**7**, Theorem 3.2.3(iii)].

(iii) This is [**4**, Theorem 1.1]. $\square$

2.7. Let $\mathfrak{X} = \{(X, i) \in \mathrm{Rep}(Q, n\delta) \times \mathbb{P}^{n-1}\}$. This space is the quotient of the (quasi-affine) open subvariety

$$U = \{(X, i) : i \neq 0\} \subset \mathrm{Rep}(Q_\infty, \epsilon)$$

by the scalar group $\mathbb{C}^*$. Thus there is an action of PG on $\mathfrak{X}$.

Since

$$T^* \mathbb{P}^{n-1} = \{(i, j) : i \neq 0, ji = 0\}/\mathbb{C}^*$$

we have

$$T^*\mathfrak{X} = \{(X, Y, i, j) \in \mathrm{Rep}(\overline{Q}_\infty, \epsilon) : i \neq 0, ji = 0\}/\mathbb{C}^*.$$

The PG action on $\mathfrak{X}$ gives rise to a moment map

$$\mu_{\mathfrak{X}} : T^*\mathfrak{X} \longrightarrow \mathfrak{pg}^* \cong \mathfrak{pg}\,.$$

Let

$$\mu_{\mathfrak{X}}^{-1}(0) = \{(X, Y, i, j) \in \mathrm{Rep}(\overline{Q}_\infty, \epsilon) : i \neq 0, ji = 0, \quad [X, Y] + ij = 0\}/\mathbb{C}^*$$

denote the scheme theoretic fibre of 0.

PROPOSITION. *There is an isomorphism* $\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}$.

Proof. Consider the G-equivariant open subvariety of $\mu^{-1}(0)$ given by the non-vanishing of i. The variety $\mu^{-1}(0)$ is determined by the conditions $[X, Y] + ij = 0$, and so if we take the trace of this equation then we see that $0 = Tr(ij) = Tr(ji) = ji$. Thus $\{(X, Y, i, j) \in \mathrm{Rep}(\overline{Q}_\infty, \epsilon) : i \neq 0, ji = 0\} \cap \mu^{-1}(0)$ is an open subvariety of $\mu^{-1}(0)$ and so, in particular, is reduced by Theorem 2.6(1). Hence factoring out by the action of $\mathbb{C}^* \leq G$ shows that $\mu_{\mathfrak{X}}^{-1}(0)$ is reduced and that there is a PG-equivariant morphism

$$\mu_{\mathfrak{X}}^{-1}(0) \longrightarrow \mu^{-1}(0)//\mathbb{C}^*.$$

This induces an algebra map

$$\alpha : \mathbb{C}[\mu^{-1}(0)]^G \longrightarrow \mathbb{C}\left[\mu_{\mathfrak{X}}^{-1}(0)\right]^{PG}.$$

We now follow some of the proof of [**7**, Lemma 6.3.2]. Write O_1 for the conjugacy class of rank one nilpotent matrices in $\mathfrak{gl}(n)$, and let $\overline{\mathsf{O}}_1$ denote the closure of O_1 in $\mathfrak{gl}(n)$. The moment map $\upsilon : T^*\mathbb{P}^{n-1} \longrightarrow \mathfrak{gl}(n)^* \cong \mathfrak{gl}(n)$ that sends (i, j) to ij gives a birational isomorphism $T^*\mathbb{P}^{n-1} \longrightarrow \overline{\mathsf{O}}_1$. Let $J \subset \mathbb{C}[\mathfrak{gl}(n)] = C[Z]$ be the ideal generated by all 2×2 minors of the matrix Z and also by the trace function. Then J is a prime ideal whose zero scheme is $\overline{\mathsf{O}}_1$ and the pullback morphism $\upsilon^* : \mathbb{C}[\mathfrak{gl}(n)]/J \longrightarrow \mathbb{C}[T^*\mathbb{P}^{n-1}]$ is a graded isomorphism.

Now the moment map $\mu_{\mathfrak{X}} : T^*\mathfrak{X} \longrightarrow \mathfrak{g}^*$ factors as the composite

$$T^*\mathfrak{X} = T^*\mathrm{Rep}(Q, n\delta) \times T^*\mathbb{P}^{n-1} \longrightarrow T^*\mathrm{Rep}(Q, n\delta) \times \overline{\mathsf{O}}_1 \xrightarrow{\theta} \mathfrak{pg}^*,$$

where the first mapping is $\mathrm{id} \times \upsilon$ and the second mapping θ sends (X, Y, Z) to $[X, Y] + Z_0$, where Z_0 indicates that we place the matrix Z on the copy of $\mathfrak{gl}(n)$ associated to the vertex 0. We have a graded algebra isomorphism

$$\mathbb{C}[T^*\mathrm{Rep}(Q, n\delta)] \otimes \mathbb{C}[\mathfrak{gl}(n)]/J \longrightarrow \mathbb{C}[T^*\mathfrak{X}].$$

Now write $\mathbb{C}[X, Y, Z] = \mathbb{C}[T^*\mathrm{Rep}(Q, n\delta) \ \times \mathfrak{gl}(n)]$, and let $\mathbb{C}[X, Y, Z]([X, Y] + Z_0)$ denote the ideal in $\mathbb{C}[X, Y, Z]$ generated by all matrix entries of the ℓ matrices $[X, Y] + Z_0$. Let $\mathbf{I}$ denote the ideal $\mathbb{C}[X, Y, Z]([X, Y] + Z_0) + \mathbb{C}[X, Y] \otimes J \subset \mathbb{C}[X, Y, Z]$. From the above we have

$$\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)] \cong \mathbb{C}[T^*\mathrm{Rep}(Q, n\delta) \times \overline{\mathsf{O}}_1]/\mathbb{C}[T^*\mathrm{Rep}(Q, n\delta) \times \overline{\mathsf{O}}_1]\theta^*(\mathfrak{gl}(n)) = \mathbb{C}[X, Y, Z]/\mathbf{I}.$$

Define an algebra homomorphism $r : \mathbb{C}[X, Y, Z] \longrightarrow \mathbb{C}[X, Y]$ by sending $P \in \mathbb{C}[X, Y, Z]$ to the function $(X, Y) \mapsto P(X, Y, -[X, Y]_0)$. Obviously r induces an isomorphism $\mathbb{C}[X, Y, Z]/\mathbb{C}[X, Y, Z]([X, Y] + Z_0) \cong \mathbb{C}[X, Y]/I_1$, where I_1 is the ideal of $\mathbb{C}[\mathrm{Rep}(\overline{Q}, n\delta)] = \mathbb{C}[X, Y]$ generated by the elements

$$\sum_{h(a)=i} X_a X_{a^*} - \sum_{t(a)=i} X_{a^*} X_a$$

for all i not equal to zero. Observe that the linear function $P : (X, Y, Z) \mapsto TrZ = Tr([X, Y] + Z_0)$ belongs to the ideal $\mathbb{C}[X, Y, Z]([X, Y] + Z_0)$. We deduce that the mapping r sends $\mathbb{C}[X, Y] \otimes J$ to the ideal generated by

$$\operatorname{rank}\left(\sum_{h(a)=0} X_a X_{a^*} - \sum_{t(a)=0} X_{a^*} X_a\right) \leq 1.$$

Thus we obtain algebra isomorphisms

$$\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)] \cong \mathbb{C}[X, Y, Z]/\mathbf{I} \cong \mathbb{C}[T^* \operatorname{Rep}(Q, n\delta)]/I_2,$$

where I_2 is the ideal generated by the elements

$$\sum_{h(a)=i} X_a X_{a^*} - \sum_{t(a)=i} X_{a^*} X_a,$$

for all $1 \leq i \leq \ell - 1$, and

$$\operatorname{rank}\left(\sum_{h(a)=0} X_a X_{a^*} - \sum_{t(a)=0} X_{a^*} X_a\right) \leq 1.$$

By [**10**, Theorem 1] the G-invariant (respectively PG-invariant) elements of $\mathbb{C}[\operatorname{Rep}(\overline{Q}_\infty, \epsilon)]$ (respectively $\mathbb{C}[\operatorname{Rep}(\overline{Q}, n\delta)]$) are generated by traces along oriented cycles. Since all oriented cycles in $\overline{Q}$ are oriented cycles in $\overline{Q}_\infty$ we have a surjective composition of algebra homomorphisms

$$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n} \cong \mathbb{C}[\mu^{-1}(0)]^G \longrightarrow \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG} \longrightarrow \left(\frac{\mathbb{C}[\operatorname{Rep}(\overline{Q}, n\delta)]}{I_2}\right)^{PG}, \tag{2.7.1}$$

where the first isomorphism is Theorem 2.6(3). The left hand side is a domain of dimension $2\dim\mathfrak{h}$ and so, to see that the mapping is an isomorphism, it suffices to prove that the right hand side also has dimension $2\dim\mathfrak{h}$.

Let I_3 be the ideal of $\mathbb{C}[\operatorname{Rep}(\overline{Q}, n\delta)]$ generated by the elements

$$\sum_{h(a)=i} X_a X_{a^*} - \sum_{t(a)=i} X_{a^*} X_a$$

for all i. This is the ideal of the zero fibre of the moment map for the PG-action on $\operatorname{Rep}(\overline{Q}, n\delta)$. This ideal contains I_2 since the rank condition on the matrices is implied by the commutator condition. Hence there is a surjective mapping

$$\frac{\mathbb{C}[\operatorname{Rep}(\overline{Q}, n\delta)]^{PG}}{I_2^{PG}} \longrightarrow \frac{\mathbb{C}[\operatorname{Rep}(\overline{Q}, n\delta)]^{PG}}{I_3^{PG}}.$$

We do not know yet whether the right hand side is reduced or not, but by [**4**, Theorem 1.1] the reduced quotient of the right hand side is the ring of functions of the variety $(\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n$. As this variety has dimension $2\dim\mathfrak{h}$ we deduce that the composition in (2.7.1) is an isomorphism, and hence that

$$\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}.$$

□

REMARK. In passing let us note that the commutativity of the following diagram

$$\begin{tikzcd}
\mathbb{C}[T^* \operatorname{Rep}(Q, n\delta)] \arrow[r, "\iota"] \arrow[drr, bend right, "pr"'] & \mathbb{C}[T^* \operatorname{Rep}(Q, n\delta)] \otimes \mathbb{C}[T^* \mathbb{P}^{n-1}] \arrow[r] \arrow[d, "\upsilon^*"] & \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)] \arrow[d, "\wr"] \\
& \mathbb{C}[T^* \operatorname{Rep}(Q, n\delta)] \otimes \mathbb{C}[\overline{\mathsf{O}}_1] \arrow[r, "r"] & \mathbb{C}[T^* \operatorname{Rep}(Q, n\delta)]/I_2
\end{tikzcd}$$

where $\iota(f) = f \otimes 1$, shows that im ι maps surjectively onto $\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]$.

3. Differential operators.

3.1. Symplectic reflection algebras. Let C_ℓ be the cyclic subgroup of $SL_2(\mathbb{C})$ generated by $\sigma = \operatorname{diag}(\eta, \eta^{-1})$. The vector space $V = (\mathbb{C}^2)^n$ admits an action of $S_n \wr C_\ell = S_n \ltimes (C_\ell)^n$. Here $(C_\ell)^n$ acts by extending the natural action of C_ℓ on $\mathbb{C}^2$, whilst S_n acts by permuting the n copies of $\mathbb{C}^2$. For an element $\gamma \in C_\ell$ and an integer $1 \leq i \leq n$ we write γ_i to indicate the element $(1, \ldots, \gamma, \ldots, 1) \in (C_\ell)^n$ which is non-trivial in the i-th factor.

3.2. The elements $S_n \wr C_\ell$ whose fixed points are a subspace of codimension two in V are called *symplectic reflections*. In this case their conjugacy classes are of two types.

(S) The elements $s_{ij}\gamma_i\gamma_j^{-1}$ where $1 \leq i, j \leq n$, $s_{ij} \in S_n$ is the transposition that swaps i and j, and $\gamma \in C_\ell$.

(C_ℓ) The elements γ_i for $1 \leq i \leq n$ and $\gamma \in C_\ell \setminus \{1\}$.

There is a unique conjugacy class of type (S) and $\ell - 1$ of type (C_ℓ) (depending on the non-trivial element we choose from C_ℓ). We shall consider a conjugation invariant function from the set of symplectic reflections to $\mathbb{C}$. We can identify it with a pair (k, c) where $k \in \mathbb{C}$ and c is an $(\ell - 1)$-tuple of complex numbers: the function sends elements from (S) to k and the elements $(\sigma^m)_i$ to c_m.

3.3. There is a symplectic form on V that is induced from n copies of the standard symplectic form ω on $\mathbb{C}^2$. If we pick a basis $\{x, y\}$ for $\mathbb{C}^2$ such that $\omega(x, y) = 1$, then we can extend this naturally to a basis $\{x_i, y_i : 1 \leq i \leq n\}$ of V such that the x's and the y's form Lagrangian subspaces and $\omega(x_i, y_j) = \delta_{ij}$. We let TV denote the tensor algebra on V: with our choice of basis this is just the free algebra on generators x_i, y_i for $1 \leq i \leq n$. The symplectic reflection algebra $H_{k,c}$ associated to $S_n \wr C_\ell$ is the quotient of $TV * (S_n \wr C_\ell)$ by the following relations:

$$x_i x_j = x_j x_i, \qquad y_i y_j = y_j y_i \qquad \text{(for all } 1 \leq i, j \leq n\text{)},$$

$$y_i x_i - x_i y_i = 1 + k \sum_{j \neq i} \sum_{\gamma \in C_\ell} s_{ij}\gamma_i\gamma_j^{-1} + \sum_{\gamma \in C_\ell \setminus \{1\}} c_\gamma \gamma_i \qquad \text{(for } 1 \leq i \leq n\text{)},$$

$$y_i x_j - x_j y_i = -k \sum_{m=0}^{\ell-1} \eta^m s_{ij} (\sigma^m)_i (\sigma^m)_j^{-1} \qquad \text{(for } i \neq j\text{)}.$$

(NB: my k is $-k$ for Oblomkov.)

3.4. The spherical algebra. The symmetrising idempotent of the group algebra $\mathbb{C}(S_n \wr C_\ell)$ is given by

$$e = \frac{1}{|S_n \wr C_\ell|} \sum_{w \in S_n \wr C_\ell} w.$$

The subalgebra $eH_{k,c}e$ is denoted by $U_{k,c}$ and called the *spherical algebra*. It will be our main object of study.

3.5. Rings of differential operators. Recall the definition of $\mathfrak{X}$ from 2.7. Let $D_{\mathfrak{X}}(nk)$ denote the sheaf of twisted differential operators on $\mathfrak{X}$ and let $D(\mathfrak{X}, nk)$ be its algebra of global sections. This is simply the tensor product $D(\mathrm{Rep}(Q, n\delta)) \otimes D_{\mathbb{P}^{n-1}}(nk)$. (The twisted differential operators on $\mathbb{P}^{n-1}$ can be defined as follows. Let $A_n = \mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$ be the n-th Weyl algebra. This is a graded algebra with $\deg(x_i) = 1$ and $\deg(\partial_i) = -1$. The degree zero component is the subring generated by the operators $x_i\partial_j$ which, under the commutator, generate the Lie algebra $\mathfrak{gl}(n)$. Call this subring R. Let $\mathbf{E} = \sum_{i=1}^n x_i\partial_i \in R$ be the Euler operator. Then $D(\mathbb{P}^{n-1}, nk)$ is the quotient of R by the two-sided ideal generated by $\mathbf{E} - nk$.)

The group action of PG on $\mathfrak{X}$ differentiates to an action of $\mathfrak{pg}$ on $\mathfrak{X}$ by differential operators. This gives a mapping

$$\tau : \mathfrak{pg} \longrightarrow D_{\mathfrak{X}}(nk). \tag{3.5.1}$$

(One way to understand this is to start back with $U \subset \mathrm{Rep}(Q_\infty, \epsilon)$ and look at the G action on U. Differentiating the G-action gives an action of $\mathfrak{g}$ by differential operators on U, $\hat{\tau} : \mathfrak{g} \longrightarrow D_U$. Since $\mathbb{C}^*$ acts trivially on $\mathrm{Rep}(Q, n\delta)$ and by scaling on $i \in \mathrm{Rep}(Q_\infty, \epsilon)$, we find that $\hat{\tau}(\mathrm{id}) = 1 \otimes \mathbf{E}$, where $\mathrm{id} = (I_n, I_n, \dots, I_n) \in \mathbb{C} \subset \mathfrak{g}$. Thus we get an action of $\mathfrak{pg}$ on $(D_U/D_U(1 \otimes \mathbf{E} - nk))^{\mathbb{C}^*} = D_{\mathfrak{X}}(nk)$.)

3.6. Recall the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and its quotient $\mathfrak{pg} = \mathrm{Lie}(PG)$ which is simply $\mathfrak{g}/\mathbb{C} \cdot \mathrm{id}$, where $\mathrm{id} = (I_n, \dots, I_n) \in \mathfrak{g}$. Let $\chi_c : \mathfrak{g} \longrightarrow \mathbb{C}$ send an element $(X) = (X_0, \dots, X_{\ell-1}) \in \mathfrak{g}$ to

$$\chi_c(X) = \sum_{r=0}^{\ell-1} C_r \,\mathrm{Tr}(X_r),$$

where $C_r = \ell^{-1}(1 - \sum_{m=1}^{\ell-1} \eta^{mr} c_m)$ for $1 \le r \le \ell - 1$ and $C_0 = \ell^{-1}(1 - \ell - \sum_{m=1}^{\ell-1} c_m)$. Observe that

$$\chi_c(\mathrm{id}) = \mathrm{Tr}(I_n) \sum_{r=0}^{\ell-1} C_i = n \sum_{r=0}^{\ell-1} \sum_{m=0}^{\ell-1} -\eta^{rm} c_m = 0.$$

In particular χ_c is actually a character of $\mathfrak{pg}$.

Let $\chi_k : \mathfrak{g} \longrightarrow \mathbb{C}$ send an element $(X) = (X_0, \dots, X_{\ell-1})$ to $\chi_k(c) = k\,\mathrm{Tr}(X_0)$.

We shall be regularly using the character $\chi_{k,c} \in \mathfrak{g}^*$ defined by $\chi_{k,c} = \chi_c + \chi_k$.

3.7. Let us recall Oblomkov's deformed Harish–Chandra homomorphism [**12**]. By Lemma 2.4, $\mathcal{S} = \omega(\mathfrak{h}^{\mathrm{reg}}/\Gamma_n)$ is a subset of $\mathrm{Rep}(Q, n\delta)^{reg}$ which is a slice for the

PG-action on $\mathrm{Rep}(Q, n\delta)$. Let

$$W'_k = (y_1 \dots y_n)^{-k}\mathbb{C}_{(0)}\big[y_1^{\pm 1}, \dots, y_n^{\pm 1}\big],$$

a space of multivalued functions on $(\mathbb{C}^*)^n$. The Lie algebra $\mathfrak{g}$ acts on W'_k by projection onto its 0-th summand $\mathfrak{gl}(n)$, and then by the natural action of $\mathfrak{gl}(n)$ on polynomials (so that E_{ij} acts as $y_i\partial/\partial y_j$). With this action the identity matrix in $\mathfrak{gl}(n)$ becomes the Euler operator $\mathbf{E}$ which acts by multiplication by $-nk$. Thus we can make W'_k a $\mathfrak{pg}$-module by twisting W'_k by the character χ_k since then id acts trivially. If we call this module W_k, then $W_k = W'_k \otimes \chi_k$. Now define Fun' to be the space of functions on $\mathrm{Rep}(Q, n\delta)$ of the form

$$f = \tilde{f}\prod_{i=0}^{\ell-1}\det(X_i)^{r_i},$$

where $\tilde{f}$ is a rational function on $\mathrm{Rep}(Q, n\delta)^{reg}$ regular on $\mathcal{S}$, $r_i = \sum_{j=0}^{i} C_j + \sigma$ and $\sigma = \ell^{-1}\sum_{s=0}^{\ell-1} sC_s$. Then $(Fun' \otimes W_k)^{\mathfrak{pg}}$ is a space of $(\mathfrak{pg}, \chi_c)$-semi-invariant functions defined on a neighbourhood of $\mathcal{S}$ that take values in W_k. This space is a free $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]^{\Gamma_n}$-module of rank 1, the isomorphism being given by restriction to $\mathcal{S}$. (Note that the determinant of an element of the form $(X, \dots, X)$ is $\det(X)^{\sum r_i} = 1$ as $\sum r_i = 0$.) Any $\mathfrak{pg}$-invariant differential operator D acts on such a function f. Oblomkov defines his homomorphism to be the restriction of $D(f)$ to $\mathcal{S}$.

3.8. We can view the procedure above in terms of $\mathrm{Rep}(Q_\infty, \epsilon)$. By Lemma 2.5 we use $\mathcal{S}_\infty = \mathcal{S} \times (1, \dots, 1) \in U_\infty$ as a slice for the G-action. The space $\mathcal{S} \times (\mathbb{C}^*)^n$ is a closed subset of U_∞ since the condition that i be cyclic for $\mathrm{diag}(x_1, \dots, x_n)$ is equivalent to $i \in (\mathbb{C}^*)^n$. Thus functions on a neighbourhood of $\mathcal{S}_\infty$ in U_∞ can be identified with functions from a neighbourhood of $\mathcal{S}$ taking values in functions on $(\mathbb{C}^*)^n$. In particular, we can consider elements on $(Fun' \otimes W_k)^{\mathfrak{pg}}$ first as $(\mathfrak{g}, \chi_{k,c})$-semiinvariant functions from a neighbourhood of $\mathcal{S}$ taking values in W'_k and hence as $(\mathfrak{g}, \chi_{k,c})$-semiinvariant functions on an open set in a neighbourhood of $\mathcal{S}_\infty$. We can apply any element of $D \in D(U_\infty)^{\mathfrak{g}}$ to these $(\mathfrak{g}, \chi_{k,c})$-semiinvariant functions and then restrict to S_∞ to get a homomorphism

$$\mathfrak{F}_{k,c} : D(U_\infty)^{\mathfrak{g}} \longrightarrow D(\mathfrak{h}^{\mathrm{reg}}/\Gamma_n).$$

3.9. Since $\mathrm{Rep}(Q_\infty, \epsilon) = \mathrm{Rep}(Q, n\delta) \times \mathbb{C}^n$ there is a mapping

$$\mathfrak{G} : D(\mathrm{Rep}(Q, n\delta))^{\mathfrak{pg}} \longrightarrow D(U_\infty)^{\mathfrak{g}}$$

that sends $D \in D(\mathrm{Rep}(Q, n\delta))^{\mathfrak{pg}}$ to $(D \otimes 1)$. Oblomkov's homomorphism is $\mathfrak{F}_{k,c} \circ \mathfrak{G}$.

3.10. Differentiating the G-action on U_∞ gives a Lie algebra homomorphism $\hat{\tau} : \mathfrak{g} \longrightarrow \mathsf{Vect}(U_\infty)$ which we extend to an algebra map

$$\hat{\tau} : U(\mathfrak{g}) \longrightarrow D(U_\infty).$$

By Lemma 2.5, U_∞ is a principal G-bundle over $\mathfrak{h}^{\mathrm{reg}}/\Gamma_n$, and so a generalisation of [**14**, Corollary 4.5] shows that the kernel of $\mathfrak{F}_{k,c}$ is $(D(U_\infty)(\hat{\tau} - \chi_{k,c})(\mathfrak{g}))^{\mathfrak{g}}$. Moreover, since the finite group Γ_n acts freely on $\mathfrak{h}^{\mathrm{reg}}$ we can identify $D(\mathfrak{h}^{\mathrm{reg}}/\Gamma_n)$ with $D(\mathfrak{h}^{\mathrm{reg}})^{\Gamma_n}$.

3.11. Recall that

$$D_{\mathfrak{X}}(nk) \cong \left(\frac{D_U}{D_U(\hat{\tau} - \chi_k)(\mathbb{C}\cdot \mathrm{id})} \right)^{\mathbb{C}^*}.$$

Hence we have

$$\left(\frac{D_U}{D_U(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right)^{G} \cong \left(\frac{D_{\mathfrak{X}}(nk)}{D_{\mathfrak{X}}(nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG}, \tag{3.11.1}$$

where $U = \{(X, i) : i \neq 0\} \subset \mathrm{Rep}(Q_\infty, n\delta)$ as in 2.7. We consider the restriction mapping $D_U \longrightarrow D(U_\infty)$. Composing the global sections of the isomorphism above with this restriction and the homomorphism $\mathfrak{F}_{k,c}$ gives

$$\mathfrak{R}'_{k,c} : \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG} \longrightarrow D(\mathfrak{h}^{\mathrm{reg}})^{\Gamma_n}.$$

3.12. Let

$$\delta_{k,c}(x) = \delta^{-k-1}\delta_\Gamma^{\sigma},$$

where $\delta = \prod_{1\le i<j\le n}(x_i^\ell - x_j^\ell)$ and $\delta_\Gamma = \prod_{i=1}^n x_i$. Define a twisted version of $\mathfrak{R}'_{k,c}$ above by

$$\mathfrak{R}_{k,c}(D) = \delta_{k,c}^{-1} \circ \mathfrak{R}'_{k,c}(D) \circ \delta_{k,c}$$

for any differential operator D.

3.13. Our main result is as follows.

THEOREM. *For all values of k and c, the homomorphism $\mathfrak{R}_{k,c}$ has image* $\mathrm{im}\,\theta_{k,c}$. *In particular we have an isomorphism*

$$\theta_{k,c}^{-1} \circ \mathfrak{R}_{k,c} : \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{\mathfrak{pg}} \xrightarrow{\ \sim\ } U_{k,c}.$$

Proof. Let us abuse notation by writing $U_{k,c}$ for the image of $U_{k,c}$ in $D(\mathfrak{h}^{\mathrm{reg}})^{\Gamma_n}$ under $\theta_{k,c}$. Since $\mathfrak{X} = \mathrm{Rep}(Q, n\delta) \times \mathbb{P}^{n-1}$, there is a mapping given by

$$D(\mathrm{Rep}(Q, n\delta))^{PG} \longrightarrow D(\mathfrak{X}, nk)^{PG} \longrightarrow D(\mathfrak{h}^{\mathrm{reg}})^{\Gamma_n},$$

that sends $D \in D(\mathrm{Rep}(Q, n\delta))^{PG}$ to $\mathfrak{R}_{k,c}(D \otimes 1)$. Recall τ from (3.5.1). Since $\mathrm{gr}\,\tau = \mu_{\mathcal{X}}^*$ we have an inclusion $\mathrm{gr}(D(\mathfrak{X}, nk))\mu_{\mathcal{X}}^*(\mathfrak{pg}) \subseteq \mathrm{gr}(D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg}))$. This gives a graded surjection

$$p : \left(\frac{\mathrm{gr}\, D(\mathfrak{X}, nk)}{\mathrm{gr}(D(\mathfrak{X}, nk))\mu_{\mathcal{X}}^*(\mathfrak{pg})} \right)^{PG} \longrightarrow \mathrm{gr} \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})} \right)^{PG}.$$

By Remark 2.7 the composition

$$\operatorname{gr} D(\operatorname{Rep}(Q, n\delta))^{PG} \longrightarrow \operatorname{gr} D(\mathfrak{X}, nk)^{PG} \longrightarrow \left(\frac{\operatorname{gr} D(\mathfrak{X}, nk)}{\operatorname{gr}(D(\mathfrak{X}, nk))\mu_{\mathcal{X}}^*(\mathfrak{pg})}\right)^{PG}$$
$$\longrightarrow \operatorname{gr}\left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})}\right)^{PG}$$

is surjective. Thus the homomorphism

$$D(\operatorname{Rep}(Q, n\delta))^{PG} \longrightarrow \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})}\right)^{PG}$$

is also surjective. In particular, by 3.9 this implies that the image of $\mathfrak{R}_{k,c}$ equals the image of Oblomkov's Harish–Chandra homomorphism, which, by [**12**, Theorem 2.5], is $U_{k,c}$.

Thus we have a filtered surjective homomorphism

$$\mathfrak{R}_{k,c} : \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})}\right)^{PG} \longrightarrow U_{k,c}.$$

Thus the dimension of the left hand side is at least $2 \dim \mathfrak{h} = \dim U_{k,c}$. By Proposition 2.7

$$\left(\frac{\operatorname{gr} D(\mathfrak{X}, nk)}{\operatorname{gr}(D(\mathfrak{X}, nk))\mu_{\mathcal{X}}^*(\mathfrak{pg})}\right)^{PG} \cong \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}.$$

Hence p is a surjection from a domain of dimension $2 \dim \mathfrak{h}$ onto an algebra of dimension at least $2 \dim \mathfrak{h}$ and so is an isomorphism. It follows that $(D(\mathfrak{X}, nk)/D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg}))^{\mathfrak{pg}}$ is a domain of dimension $2 \dim \mathfrak{h}$. This implies that $\mathfrak{R}_{k,c}$ is an isomorphism. □

4. Application: Shift functors.

4.1. The Holland-Schwarz Lemma.

We wish to understand the space

$$\frac{D(\operatorname{Rep}(Q_\infty, \epsilon))}{D(\operatorname{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})}.$$

As we observed in the proof of Theorem 3.13 there is a natural surjective homomorphism

$$\frac{\operatorname{gr} D(\operatorname{Rep}(Q_\infty, \epsilon))}{\operatorname{gr} D(\operatorname{Rep}(Q_\infty, \epsilon))\mu^*(\mathfrak{g})} \longrightarrow \operatorname{gr}\left(\frac{D(\operatorname{Rep}(Q_\infty, \epsilon))}{D(\operatorname{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})}\right). \tag{4.1.1}$$

It turns out that this is an isomorphism.

LEMMA (Schwarz, Holland). *The homomorphism* (4.1.1) *is an isomorphism of* $\mathbb{C}[T^* \operatorname{Rep}(Q_\infty, \epsilon)]$*-modules.*

Proof. This is [**9**, Lemma 2.2] since, by Theorem 2.6 (2), the moment map μ is flat. □

4.2. This lets us prove the first part of the isomorphism in the statement of Theorem 1.4.

LEMMA. *There is an algebra isomorphism*

$$\left(\frac{D(\mathrm{Rep}(Q_\infty,\epsilon))}{D(\mathrm{Rep}(Q_\infty,\epsilon))(\hat{\tau}-\chi_{k,c})(\mathfrak{g})}\right)^G \longrightarrow \left(\frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau-\chi_c)(\mathfrak{pg})}\right)^{PG}.$$

Proof. We have a natural $\mathfrak{pg}$-equivariant mapping

$$D(\mathrm{Rep}(Q_\infty,\epsilon))^{\mathbb{C}^*} \longrightarrow D_U^{\mathbb{C}^*} \longrightarrow D_{\mathfrak{X}}(nk)$$

which induces a homomorphism

$$D(\mathrm{Rep}(Q_\infty,\epsilon))^G \longrightarrow \left(\frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau-\chi_c)(PG)}\right)^{\mathfrak{pg}}.$$

This is surjective since, as we observed in the proof of Theorem 3.13, the image of $D(\mathrm{Rep}(Q,n\delta))^{PG} \subset D(\mathrm{Rep}(Q_\infty,\epsilon)^G$ spans the right hand side. By (3.11.1) the kernel of this homomorphism includes the ideal $(D(\mathrm{Rep}(Q,\infty),\epsilon)(\hat{\tau}-\chi_{k,c})(\mathfrak{g}))^G$. Hence we have a surjective homomorphism

$$\left(\frac{D(\mathrm{Rep}(Q_\infty,\epsilon)}{D(\mathrm{Rep}(Q,\infty),\epsilon)(\hat{\tau}-\chi_{k,c})(\mathfrak{g})}\right)^G \longrightarrow \left(\frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau-\chi_c)(\mathfrak{pg})}\right)^{PG}. \tag{4.2.1}$$

By Lemma 4.1 and Proposition 2.7, there is an isomorphism

$$\begin{aligned}\left(\mathrm{gr}\,\frac{D(\mathrm{Rep}(Q_\infty,\epsilon)}{D(\mathrm{Rep}(Q,\infty),\epsilon)(\hat{\tau}-\chi_{k,c})(\mathfrak{g})}\right)^G &\cong \left(\frac{\mathrm{gr}\,D(\mathrm{Rep}(Q_\infty,\epsilon))}{\mathrm{gr}\,D(\mathrm{Rep}(Q_\infty,\epsilon))\mu^*(\mathfrak{g})}\right)^G = \mathbb{C}[\mu^{-1}(0)]^G \\ &= \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\Gamma_n}.\end{aligned}$$

This shows that the algebra on the left is a domain of dimension $2\dim\mathfrak{h}$ and so (4.2.1) is also injective, as required. □

4.3. Shifting. The previous two lemmas provide us with an interesting series of bimodules. Given a character Λ of G we define

$$B_{k,c}^\Lambda = \left(\frac{D(\mathrm{Rep}(Q_\infty,\epsilon))}{D(\mathrm{Rep}(Q_\infty,\epsilon))(\hat{\tau}-\chi_{k,c})(\mathfrak{g})}\right)^\Lambda$$

to be the set of (G,Λ)-semiinvariants. Thanks to Lemma 4.2 and Theorem 3.13 this is a right $U_{k,c}$-module. Now observe that if $x\in\mathfrak{g}$ and $D\in D(\mathrm{Rep}(Q_\infty,\epsilon))^\Lambda$ then

$$[\tau(x),D]=\lambda(x)D,$$

where $\lambda = d\Lambda$. Hence $B^{\Lambda}_{k,c}$ is a left $(D(\mathrm{Rep}(Q_\infty,\epsilon))/D(\mathrm{Rep}(Q_\infty,\epsilon))(\hat{\tau}-\chi_{k,c}-\lambda)(\mathfrak{g}))^G$-module and so tensoring sets up a *shift functor*

$$S^{\Lambda}_{k,c} : \left(\frac{D(\mathrm{Rep}(Q_\infty,\epsilon))}{D(\mathrm{Rep}(Q_\infty,\epsilon))(\hat{\tau}-\chi_{k,c})(\mathfrak{g})}\right)^G\text{-mod}$$
$$\longrightarrow \left(\frac{D(\mathrm{Rep}(Q_\infty,\epsilon))}{D(\mathrm{Rep}(Q_\infty,\epsilon))(\hat{\tau}-\chi_{k,c}-\lambda)(\mathfrak{g})}\right)^G\text{-mod}.$$

4.4. The character group of G is isomorphic to $\mathbb{Z}^\ell$ via

$$(i_0,\dots,i_{\ell-1}) \mapsto \left((g_0,\dots,g_{\ell-1}) \mapsto \prod_{r=0}^{\ell-1} \det(g_r)^{i_r}\right).$$

Corresponding to the standard basis element ϵ_i is the character χ_i of $\mathfrak{g}$ that sends $X \in \mathfrak{g}$ to $\mathrm{Tr}(X_i)$.

LEMMA. *The bimodule $B^{\epsilon_i}_{k,c}$ above is a $(U_{k',c'}, U_{k,c})$-bimodule, where $k' = k+1$ and $c' = c + (1-\eta^{-i}, 1-\eta^{-2i},\dots,1-\eta^{-(\ell-1)i})$.*

Proof. Recall that (k,c) corresponds to the character of $\mathfrak{g}$ we called $\chi_{k,c}$ which is defined as

$$\chi_{k,c}(X) = (C_0+k)\,\mathrm{Tr}(X_0) + \sum_{j=1}^{\ell-1} C_j\,\mathrm{Tr}(X_j),$$

where $C_r = \ell^{-1}(1-\sum_{m=1}^{\ell-1}\eta^{mr}c_m)$ for $1 \le r \le \ell-1$ and $C_0 = \ell^{-1}(1-\ell-\sum_{m=1}^{\ell-1} c_m)$. We need to calculate (k',c') so that $\chi_{k,c}+\chi_i = \chi_{k',c'}$. We have

$$\begin{aligned}(\chi_{nk,c}+\chi_i)(X) &= (C_0+k)\,\mathrm{Tr}(X_0) + \mathrm{Tr}(X_i) + \sum_{j=1}^{\ell-1} C_j\,\mathrm{Tr}(X_j)\\ &= (C_0'+k')\,\mathrm{Tr}(X_0) + \sum_{j=1}^{\ell-1} C_j'\,\mathrm{Tr}(X_j).\end{aligned}$$

Calculation shows that $k' = k+1$ and that if $i=0$ then $C_j' = C_j$ and otherwise

$$C_j' = C_j + \begin{cases} -1 & \text{if } j=0,\\ 1 & \text{if } j=i,\\ 0 & \text{otherwise.}\end{cases}$$

These unpack to give $c_m' = c_m + 1 - \eta^{-mi}$. □

4.5. Question. Thus for each $0 \le i \le \ell-1$ we have a *shift functor*

$$S_i : U_{k,c}\text{-mod} \longrightarrow U_{k+1,c'}\text{-mod}$$

where c' is as above. When is this an equivalence of categories?

REMARK. Shift functors are also constructed in [**1**] and [**15**]. Hopefully they agree with the functors here.

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