THE CURVES NOT CARRIED

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ABSTRACT. Suppose τ is a train track on a surface S. Let $\mathscr{C}(\tau)$ be the set of isotopy classes of simple closed curves carried by τ. Masur and Minsky [2004] prove that $\mathscr{C}(\tau)$ is quasi-convex inside the curve complex $\mathscr{C}(S)$. We prove that the complement, $\mathscr{C}(S) - \mathscr{C}(\tau)$, is quasi-convex.

1. INTRODUCTION

The curve complex $\mathscr{C}(S)$, of a surface S, is deeply important in low-dimensional topology. One foundational result, due to Masur and Minsky, states that $\mathscr{C}(S)$ is Gromov hyperbolic [3, Theorem 1.1].

Suppose τ is a train track on S. The set $\mathscr{C}(\tau) \subset \mathscr{C}(S)$ consists of all curves α carried by τ: we write this as $\alpha \prec \tau$. Another striking result of Masur and Minsky is that $\mathscr{C}(\tau)$ is quasi-convex in $\mathscr{C}(S)$. This follows from hyperbolicity and their result that splitting sequences of train tracks give rise to quasi-convex subsets in $\mathscr{C}(S)$ [4, Theorem 1.3].

We prove a complementary result.

Theorem 3.1. *Suppose $\tau \subset S$ is a train track. Then the curves not carried by τ form a quasi-convex subset of $\mathscr{C}(S)$.*

This supports the intuition that, for a maximal birecurrent track τ, the carried set $\mathscr{C}(\tau)$ is like a half-space in a hyperbolic space.

When S is the four-holed sphere or once-holed torus the proof is an exercise in understanding how $\mathscr{C}(\tau)$ sits inside the Farey graph. In what follows we suppose that S is a connected, compact, oriented surface with $\chi(S) \leq -2$, and not a four-holed sphere. Here is a rough sketch of the proof of Theorem 3.1. Suppose γ and γ' are simple closed curves, not carried by τ. Let $[\gamma, \gamma']$ be a geodesic in $\mathscr{C}(S)$. Suppose α and α' are the first and last curves of $[\gamma, \gamma']$ carried by τ. Fix splitting sequences from τ to α and α', respectively. For each splitting sequence, the vertex sets form a K_1–quasi-convex subset inside $\mathscr{C}(S)$. Since $\mathscr{C}(S)$ is Gromov hyperbolic, the geodesic segment $[\alpha, \alpha']$ is $\mathsf{K}_1 + \delta$–close to the union of vertex sets. Proposition 6.1 completes the proof by showing each vertex cycle, along each splitting sequence, is uniformly close to a non-carried curve.

Before stating Proposition 6.1 we recall a few definitions. A train track $\tau \subset S$ is *large* if all components of $S - \tau$ are disks or peripheral annuli. A track τ is *maximal* if it is not a proper subtrack of any other track. The *support*, $\operatorname{supp}(\alpha, \tau)$, of a carried curve $\alpha \prec \tau$ is the union of the branches of τ along which α runs.

Proposition 6.1. *Suppose $\tau \subset S$ is a train track and $\alpha \prec \tau$ is a carried curve. Suppose* $\operatorname{supp}(\alpha, \tau)$ *is large, but not maximal. Then there is an essential, non-peripheral curve β so that $i(\alpha, \beta) \leq 1$ and any curve isotopic to β is not carried by τ.*

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The idea behind Proposition 6.1 is as follows. Since $\sigma = \operatorname{supp}(\alpha, \tau)$ is large all components of $S - \sigma$ are disks or peripheral annuli. Since σ is not maximal there is a component $Q \subset S - \sigma$ which is not an ideal triangle or a once-holed ideal monogon. Hence, there is a *diagonal* δ of Q that is not carried by τ. We then extend δ, in a purely local fashion, to a simple closed curve β. By construction β is in *efficient position* with respect to τ and meets α at most once. Finally, we appeal to Criteria 4.2 or 4.4 to show that β is not isotopic to a carried curve.

2. Background

We review the basic definitions needed for the rest of the paper. Throughout we suppose S is a compact, connected, smooth, oriented surface.

2.1. **Corners and index.** Suppose $R \subset S$ is a subsurface with piecewise smooth boundary. The non-smooth points of ∂R are the *corners* of R. We require that the exterior angle at each corner be either $\pi/2$ or $3\pi/2$, giving *inward* and *outward* corners. Let $c_\pm(R)$ count the inward and outward corners of R, respectively. The *index* of R is

$$\operatorname{index}(R) = \chi(R) + \frac{c_+(R)}{4} - \frac{c_-(R)}{4}.$$

For example, if R is a rectangle then its index is zero. In general, if $\alpha \subset R$ is a properly embedded, separating arc, avoiding the corners of R, and orthogonal to ∂R, and if P and Q are the closures of the components of $R - \alpha$, then we have $\operatorname{index}(P) + \operatorname{index}(Q) = \operatorname{index}(R)$.

2.2. **The curve complex.** Define $i(\alpha, \beta)$ to be the geometric intersection number between a pair of simple closed curves. The *complex of curves* $\mathcal{C}(S)$ is, for us, the following graph. Vertices are essential, non-peripheral isotopy classes of simple closed curves. Edges are pairs of distinct vertices α and β where $i(\alpha, \beta) = 0$. When $\chi(S) \le -2$ (and S is not the four-holed sphere) it is an exercise to show that $\mathcal{C}(S)$ is connected. We may equip $\mathcal{C}(S)$ with the usual edge metric, denoted d_S. Here is a foundational result due to Masur and Minsky.

Theorem 2.3. [3, Theorem 1.1] *The curve complex $\mathcal{C}(S)$ is Gromov hyperbolic.*

2.4. **Train tracks.** A *pre-track* $\tau \subset S$ is a non-empty finite embedded graph with various properties as follows. The vertices (called *switches*) are all of valence three. The edges (called *branches*) are smoothly embedded. Any point x lying in the interior of a branch $A \subset \tau$ divides A into a pair of *half-branches*. At a switch $s \in \tau$, we may orient the three incident half-branches A, B, and C away from s. After renaming the branches, if necessary, their tangents satisfy $V(s, A) = -V(s, B) = -V(s, C)$. We say A is a *large* half-branch and B and C are *small*. This finishes the definition of a pre-track. See Figures 2.5 and 2.6 for various local pictures of a pre-track.

A branch $B \subset \tau$ is either *small*, *mixed*, or *large* as it contains zero, one, or two large half-branches. We may *split* a pre-track τ along a large branch, as shown in Figure 2.5, to obtain a new track σ. Conversely, we *fold* σ to obtain τ. If a branch is mixed then we may *shift* along it to obtain σ, as shown in Figure 2.6. Note shifting is symmetric; if σ is a shift of τ then τ is a shift of σ.

Suppose $\tau \subset S$ is a pre-track. We define $N = N(\tau)$, a *tie neighborhood* of τ as follows. For every branch B we have a rectangle $R = R_B = B \times I$. For all $x \in B$ we call $\{x\} \times I$ a *tie*. The two ties of $\partial B \times I$ are the *vertical boundary* $\partial_v R$ of R. The boundaries of all of the ties form the *horizontal boundary* $\partial_h R$ of R. Any tie $J \subset R$, meeting the interior of

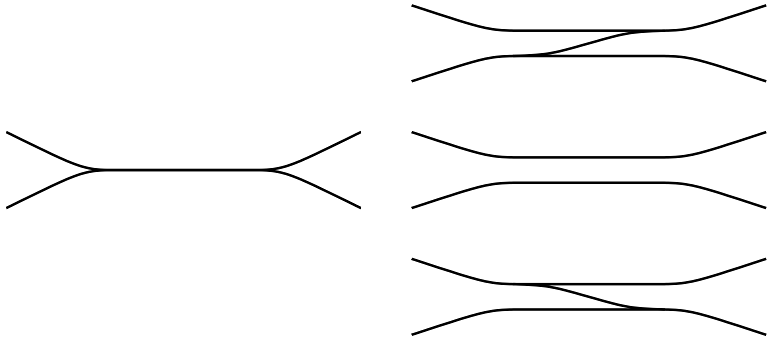

FIGURE 2.5. A large branch admits a left, central, or right splitting.

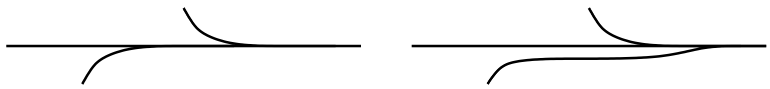

FIGURE 2.6. A mixed branch admits a shift.

R, cuts R into a pair of *half-rectangles*. The points $\partial\partial_v R = \partial\partial_h R$ are the corners of R; all four are outward corners.

We embed all of the rectangles R_B into S as follows. Suppose A (large) and B and C (small) are the half-branches incident to the switch s. The vertical boundary of R_B (respectively R_C) is glued to the upper (lower) third of the vertical boundary of R_A. See Figure 2.7. The resulting tie neighborhood $N = N(\tau)$ has horizontal boundary $\partial_h N = \bigcup \partial_h R_B$. The vertical boundary of N is the closure of $\partial N - \partial_h N$. Again $\partial\partial_v N$ is the set of corners of N; all of these are inward corners. We use $n(\tau)$ to denote the interior of $N(\tau)$. We may now give our definition of a train track.

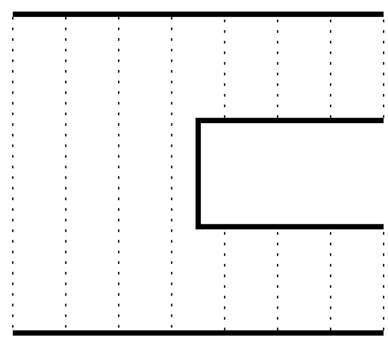

FIGURE 2.7. The local model for $N(\tau)$ near a switch. The dotted lines are ties.

Definition 2.8. Suppose $\tau \subset S$ is a pre-track and $N(\tau)$ is a tie neighborhood. We say τ is a *train track* if every component of $S - n(\tau)$ has negative index.

A track $\tau \subset S$ is *large* if every component of $S - n(\tau)$ is either a disk or a peripheral annulus. A track τ is *maximal* if every component of $S - n(\tau)$ is either a hexagon or a once-holed bigon.

2.9. **Carried curves and transverse measures.** Suppose $\alpha \subset S$ is a simple closed curve. If $\alpha \subset N(\tau)$ and α is transverse to the ties of $N(\tau)$ then we say α is *carried* by τ. We write this as $\alpha \prec \tau$. It is an exercise to show that if α is carried then α is essential and non-peripheral. We define $\mathscr{C}(\tau) = \{\alpha \in \mathscr{C}(S) \mid \alpha \prec \tau\}$. Note that $\mathscr{C}(\tau)$ is non-empty.

Let $\mathscr{B} = \mathscr{B}_\tau$ be the set of branches of τ. Fix a switch s and suppose that the half-branches A, B, and C are adjacent to s, with A being large. A function $\mu\colon \mathscr{B} \to \mathbb{R}_{\geq 0}$ satisfies the *switch equality* at s if

$$\mu(A) = \mu(B) + \mu(C).$$

We call μ a *transverse measure* if μ satisfies all switch equalities. For example, any carried curve $\alpha \prec \tau$ gives an integral transverse measure μ_α. This permits us to define $\sigma = \operatorname{supp}(\alpha, \tau)$, the *support* of α in τ: a branch $B \subset \tau$ lies in σ if $\mu_\alpha(B) > 0$.

Here is a "basic observation" from [3, page 117].

Lemma 2.10. *Suppose τ is a maximal train track and suppose $\alpha \prec \tau$ has full support: $\tau = \operatorname{supp}(\alpha, \tau)$. Suppose β is an essential, non-peripheral curve with $i(\alpha, \beta) = 0$. Then β is also carried by τ.* □

Since the switch equalities are homogeneous the set of solutions $\mathrm{ML}(\tau)$ is a rational cone. We projectivize $\mathrm{ML}(\tau)$ to obtain $P(\tau)$, a non-empty convex polytope. All vertices of $P(\tau)$ arise from carried curves; we call such curves *vertex cycles* for τ. Thus the set $V(\tau)$ of vertex cycles is naturally a subset of $\mathscr{C}(\tau) \subset \mathscr{C}(S)$. Deduce if τ' is a shift of τ then $V(\tau') = V(\tau)$.

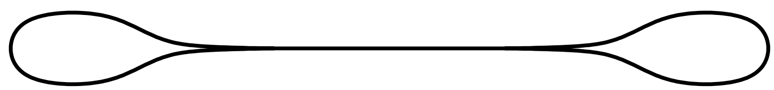

FIGURE 2.11. A *barbell*: a train track with one large branch and two small branches, where the midpoint of the large branch separates.

Lemma 2.12. *A carried curve $\alpha \prec \tau$ is a vertex cycle if and only if* $\operatorname{supp}(\alpha, \tau)$ *is either a simple close curve or a barbell (see Figure 2.11).*

Proof. The forward direction is given by Proposition 3.11.3(3) of [5]. The backward direction is an exercise in the definitions. □

The usual upper bound on distance in $\mathscr{C}(S)$, coming from geometric intersection number [6, Lemma 1.21], gives the following.

Lemma 2.13. *For any surface S there is a constant K_0 with the following property. Suppose τ is a track. Suppose σ is a split, shift, or subtrack of τ. Then the diameter of $V(\tau) \cup V(\sigma)$ inside of $\mathscr{C}(S)$ is at most K_0.* □

2.14. **Quasi-convexity.** A subset $A \subset \mathscr{C}(S)$ is *K–quasi-convex* if for every α and β in A, any geodesic $[\alpha, \beta] \subset \mathscr{C}(S)$ lies within a K–neighborhood of A. Recall if A and B are K–quasi-convex sets in $\mathscr{C}(S)$, and if $A \cap B$ is non-empty, then the union $A \cup B$ is $K + \delta$–quasi-convex. We now have a more difficult result.

Theorem 2.15. [4, Theorem 1.3] *For any surface S there is a constant K_1 with the following property. Suppose that $\{\tau_i\}$ is sequence where τ_{i+1} is a split, shift, or subtrack τ_i. Then the set $V = \bigcup_i V(\tau_i)$ is K_1–quasi-convex in $\mathscr{C}(S)$.* □

Remark 2.16. In the first statement of their Theorem 1.3 [4, page 310] Masur and Minsky assume their tracks are large and recurrent. However, as they remark after their Lemma 3.1, largeness is not necessary. Also, it is an exercise to eliminate the hypothesis of recurrence, say by using Lemma 2.13 and the subtracks $\operatorname{supp}(\alpha, \tau_i)$ (for any fixed curve $\alpha \in \bigcap P(\tau_i)$).

A more subtle point is that their Lemmas 3.2, 3.3, and 3.4 use the train-track machinery of another of their papers [3]. Transverse recurrence is used in an essential way in the second paragraph of the proof of Lemma 4.5 of that earlier paper. However the crucial "nesting lemma" [4, Lemma 3.4] can be proved without transverse recurrence. This is done in Lemma 3.2 of [1].

Thus, as stated above, Theorem 2.15 does not require any hypothesis of largeness, recurrence, or transverse recurrence.

3. Proof of the main theorem

We now have enough tools in place to see how Proposition 6.1 implies our main result.

Theorem 3.1. *Suppose $\tau \subset S$ is a train track. The curves not carried by τ form a quasi-convex subset of $\mathcal{C}(S)$.*

Proof. We may assume $\chi(S) \leq -2$, and that S is not a four-holed sphere. Suppose $\gamma, \gamma' \in \mathcal{C}(S)$ are not carried by τ. Fix a geodesic $[\gamma, \gamma']$ in $\mathcal{C}(S)$. If $[\gamma, \gamma']$ is disjoint from $\mathcal{C}(\tau)$ there is nothing to prove.

So, instead, suppose α and α' are the first and last curves, along $[\gamma, \gamma']$, carried by τ. Let β be the predecessor of α in $[\gamma, \gamma']$ and let β' be the successor of α'. Thus, β and β' are not carried by τ. The contrapositive of Lemma 2.10 now implies that the tracks $\operatorname{supp}(\alpha, \tau)$ and $\operatorname{supp}(\alpha', \tau)$ are not maximal.

For the moment, we fix our attention on α. We choose a splitting and shifting sequence $\{\tau_i\}_{i=0}^n$ with the following properties:

- $\tau_0 = \tau$,
- for all i, the curve α is carried by τ_i, and
- $\operatorname{supp}(\alpha, \tau_n)$ is a simple closed curve.

We find a similar sequence $\{\tau'_i\}$ for α'.

Let $V = \bigcup V(\tau_i)$ be the vertices of the splitting sequence $\{\tau_i\}$; define V' similarly. The hyperbolicity of $\mathcal{C}(S)$ (Theorem 2.3) and the quasi-convexity of vertex sets (Theorem 2.15) imply the geodesic $[\alpha, \alpha']$ lies within a $\mathsf{K}_1 + \delta$–neighborhood of $V \cup V'$. To finish the proof we must show that every vertex of V (and of V') is close to a non-carried curve of τ.

Using Lemma 2.12 twice we may pick vertex cycles $\alpha_i \in V(\tau_i)$ so that:

- $\alpha_n = \alpha$ and
- $\alpha_i \prec \operatorname{supp}(\alpha_{i+1}, \tau_i)$.

Define $\sigma_i = \operatorname{supp}(\alpha_i, \tau)$. By construction $\operatorname{supp}(\alpha_i, \tau_i) \subset \operatorname{supp}(\alpha_{i+1}, \tau_i)$. If we fold backwards along the sequence then, the former track yields σ_i while the latter yields σ_{i+1}. We deduce $\sigma_i \subset \sigma_{i+1}$. Recall that $\sigma_n = \operatorname{supp}(\alpha, \tau)$ is not maximal. Thus none of the σ_i are maximal.

Let $m = \max\{\ell \mid \sigma_\ell \text{ is small}\}$. Fix any curve $\omega \in \mathcal{C}(S)$ disjoint from σ_m. Using ω we deduce $d_S(\alpha_i, \alpha_m) \leq 2$, for any $i \leq m$.

If $m = n$ then Lemma 2.13 implies the set $V = \bigcup V(\tau_i)$ lies within a $\mathsf{K}_0 + 3$–neighborhood of β, and we are done.

So we may assume that $m < n$. In this case Lemma 2.13 implies the set $\bigcup_{i=0}^m V(\tau_i)$ lies within a $2\mathsf{K}_0 + 2$–neighborhood of α_{m+1}. Recall $\alpha_i \prec \tau$ and $\sigma_i = \operatorname{supp}(\alpha_i, \tau)$ is assumed to be a large, yet not maximal, subtrack of τ. Thus we may apply Proposition 6.1 to obtain a curve β_i so that:

- $\beta_i \in \mathcal{C}(S) - \mathcal{C}(\tau)$ and
- $i(\alpha_i, \beta_i) \leq 1$.

Applying Lemma 2.13 we deduce, whenever $i > m$, that $V(\tau_i)$ lies within a $\mathsf{K}_0 + 2$–neighborhood of β_i.

The same argument applies to the splitting sequence from τ to α'. This completes the proof of the theorem. $\square$

4. Efficient position

In order to prove Proposition 6.1, we here give criteria to show that a curve β *cannot* be carried by a given track τ. We state these in terms of *efficient position*, defined previously in [2, Definition 2.3]. See also [7, Definition 3.2].

Suppose τ is a train track and $N = N(\tau)$ is a tie neighborhood. A simple arc γ, properly embedded in N, is a *carried arc* if it is transverse to the ties and disjoint from $\partial_h N$.

Definition 4.1. Suppose $\beta \subset S$ is a properly embedded arc or curve which is transverse to ∂N and disjoint from $\partial\partial_v N$, the corners of N. Then β is in *efficient position* with respect to τ, written $\beta \dashv \tau$, if

- every component of $\beta \cap N(\tau)$ is carried or is a tie and
- every component of $S - n(\beta \cup \tau)$ has negative index or is a rectangle.

Here $n(\beta \cup \tau)$ is a shorthand for $n(\beta) \cup n(\tau)$, where the ties of $n(\beta)$ are either subties of, or orthogonal to, ties of $n(\tau)$. An index argument proves if $\beta \dashv \tau$ then β is essential and non-peripheral. See [2, Lemma 2.5].

Criterion 4.2. *Suppose $\beta \dashv \tau$ is a curve. Orient β. Suppose there are regions L and R of $S - n(\beta \cup \tau)$ and a component $\beta_M \subset \beta - n(\tau)$ with the following properties.*

- *L and R lie immediately to the left and right, respectively, of β_M and*
- *L and R have negative index.*

Then any curve isotopic to β is not carried by τ.

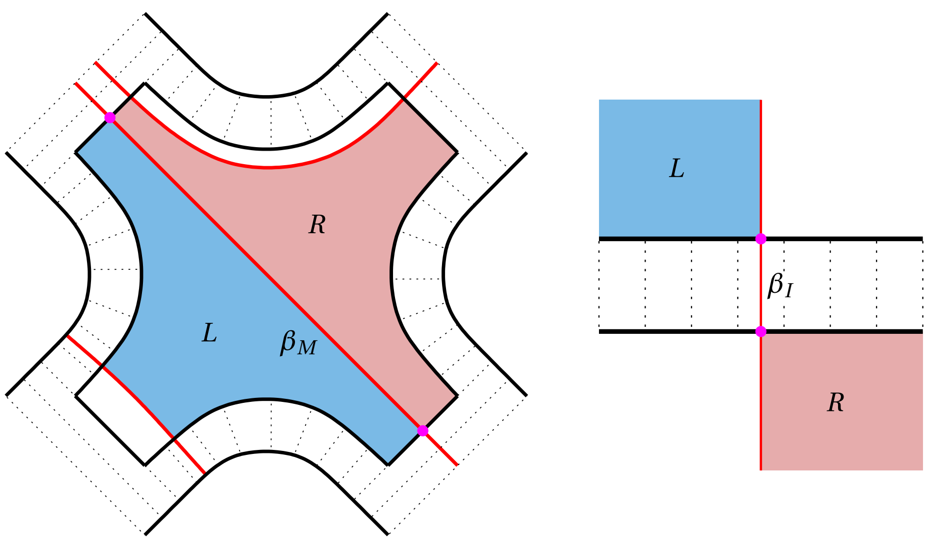

Figure 4.3. Left: The regions L and R are both adjacent to the arc $\beta_M \subset \beta - n(\tau)$. Right: A corner of L and of R meet the tie $\beta_I \subset \beta \cap N(\tau)$.

Proof. Suppose, for a contradiction, that β is isotopic to $\gamma \prec \tau$. We now induct on the intersection number $|\beta \cap \gamma|$.

In the base case β and γ are disjoint; thus β and γ cobound an annulus $A \subset S$. Since β and γ are in efficient position with respect to τ, the intersection $A \cap N(\tau)$ is a union of rectangles, so has index zero. However, one of L or R lies inside of $A - N(\tau)$. This contradicts the additivity of index.

In the induction step, β and γ cobound a bigon $B \subset S$. Since γ is carried, the two corners x and y of B lie inside of $N(\tau)$. Let β_x be the component of $\beta \cap N$ that contains x. We call x a *carried* or *dual* corner as β_x is a carried arc or a tie. We use the same terminology for y.

If x is a carried corner then move along $\gamma \cap \partial B$ a small amount, let I_x be the resulting tie, and use I_x to cut a triangle (containing x) off of B to obtain B'. Do the same at y to obtain B''. Now, if both x and y are dual corners then $B'' = B$ is a bigon. If exactly one of x or y is a dual corner then B'' is a triangle. In either of these cases index(B'') is positive, contradicting the assumption that β is in efficient position.

So suppose both x and y are carried corners of B; thus B'' is a rectangle. Thus B'' has index zero. Recall that β_M is a subarc of β meeting both R and L. Since neither L or R lie in B'' deduce that β_M is disjoint from B''. We now define $\beta_B = \beta \cap B$ and $\gamma_B = \gamma \cap B$, the two sides of B. We define β' to be the curve obtained from β by isotoping β_B across B, slightly past γ_B. So β' is isotopic to β, is in efficient position with respect to τ, has two fewer points of intersection with γ, and contains β_M. Thus β_M is adjacent to two regions L' and R' of $S - n(\beta' \cup \tau)$ of negative index, as desired. This completes the induction step and thus the proof of the criterion. □

Criterion 4.2 is not general enough for our purposes. We also need a criterion that covers a situation where the regions L and R are not immediately adjacent.

Criterion 4.4. *Suppose $\beta \dashv \tau$ is a curve. Orient β. Suppose there are regions L and R of $S - n(\beta \cup \tau)$ and a tie $\beta_I \subset \beta \cap N(\tau)$ with the following properties.*

- *L and R lie to the left and right, respectively, of β,*
- *the two points of $\partial\beta_I$ are corners of L and R, and*
- *L and R have negative index.*

Then any curve isotopic to β is not carried by τ. □

The proof of Criterion 4.4 is almost identical to that of Criterion 4.2 and we omit it. See Figure 4.3 for local pictures of curves $\beta \dashv \tau$ satisfying the two criteria.

5. Efficient and crossing diagonals

The next tool needed to prove Proposition 6.1 is the existence of *crossing diagonals:* efficient arcs that cannot isotoped to be carried.

Let τ be a train track. Suppose $\sigma \subset \tau$ is a subtrack. We take $N(\sigma) \subset N(\tau)$ to be a tie sub-neighborhood, as follows.

- Every tie of $N(\sigma)$ is a subarc of a tie of $N(\tau)$.
- The horizontal boundary $\partial_h N(\sigma)$ is
  - disjoint from $\partial_h N(\tau)$ and
  - transverse to the ties of $N(\tau)$.
- Every component of $\partial_v N(\sigma)$ contains a component of $\partial_v N(\tau)$.

Now suppose that Q is a component of $S - n(\sigma)$. We define $N(\tau, Q) = N(\tau) \cap Q$. We say that a properly embedded arc $\delta \subset Q$ is a *diagonal* of Q if

- $\partial\delta$ lies in $\partial_v Q$, missing the corners,
- δ is orthogonal to ∂Q, and
- all components of $Q - n(\delta)$ have negative index.

A diagonal δ is *efficient* if it satisfies Definition 4.1 with respect to $N(\tau, Q)$. An efficient diagonal δ is *short* if one component H of $Q - n(\delta)$ is a hexagon. The hexagon H meets three (or two) components of $\partial_v Q$ and properly contains one of them, say v.

In this situation we say δ *cuts* v off of Q. In the simplest example a short diagonal $\delta \subset Q$ is carried by $N(\tau, Q)$.

We say an efficient diagonal δ is a *crossing diagonal* if there is

- a subarc δ_M (or δ_I) and
- regions L and R of $Q - (\delta \cup n(\tau))$

satisfying the hypotheses of Criterion 4.2 or Criterion 4.4. Deduce, if $\beta \dashv \tau$ is a curve containing a crossing diagonal δ, that any curve isotopic to β is not carried by τ.

Lemma 5.1. *Suppose $\sigma \subset \tau$ is a large subtrack. Let Q be a component of $S - n(\sigma)$ that is not a hexagon or a once-holed bigon. Then for any component $v \subset \partial_v Q$ there is a short diagonal $\delta \subset Q$ that cuts v off of Q.*

Furthermore δ is properly isotopic, relative to the corners of Q, to a carried or a crossing diagonal.

Proof. The orientation of S induces an oriention on Q and thus of the boundary of Q. Let u and w be the components of $\partial_v Q$ immediately before and after v. (Note that we may have $u = w$. In this case Q is a once-holed rectangle.) Let h_u and h_w be the components of $\partial_h Q$ immediately before and after v.

Let N_u be the union of the ties of $N(\tau, Q)$ meeting h_u. As usual, N_u is a union of rectangles. (See Figure 5.2 for one possibility for N_u.) Let I be a tie of N_u, meeting the interior of h_u. Suppose that I contains a component of $\partial_v N_u$. Thus I locally divides N_u into a pair of half-rectangles, one large and one small. When the small half-rectangle is closer to v than it is to u (along h_u), we say I *faces* v. Among the ties of N_u facing v, let I_u be the one closest to v. (If no tie faces v we take $I_u = u$.)

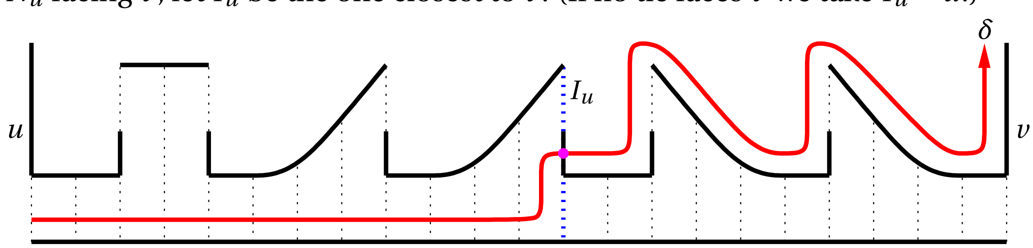

FIGURE 5.2. One possible shape for N_u, the union of all ties meeting $h_u \subset \partial_h Q$.

With I_u in hand, let N'_u be the closure of the component of $N_u - I_u$ that meets v. We define I_w and N'_w in the same way, with respect to h_w.

Consider the set $X = h_u \cup N'_u \cup v \cup N'_w \cup h_w$. Let $N(X)$ be a small regular neighborhood of X, taken in S, and set $\delta = Q \cap \partial N(X)$; see Figure 5.2. Note that δ cuts v off of Q. Orient δ so that v is to the right of δ.

We now prove that δ is in efficient position, after an arbitrarily small isotopy. The subarc of $\delta_u \subset \delta$ between u and I_u is carried; the same holds for the subarc δ_w between I_w and w. (If $I_u = u$ then we take $\delta_u = \emptyset$ and similarly for δ_w.) All components of $\delta \cap N(\tau)$, other than δ_u and δ_w, are ties.

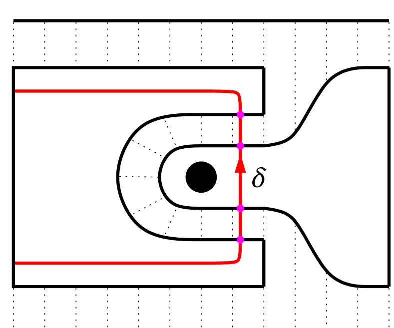

FIGURE 5.3. We have properly isotoped δ to simplify the figure.

Consider $\epsilon = \delta - n(\tau)$. If ϵ is connected, then ϵ cuts a hexagon R off of $Q - n(\tau)$. By additivity of index the region $L \subset Q - (\delta \cup$

$n(\tau)$) adjacent to R has index at most zero. If L has index zero, it is a rectangle; we deduce that δ is isotopic to a carried diagonal. If L has negative index then δ is a crossing diagonal, according to Criterion 4.2.

Suppose $\epsilon = \delta - n(\tau)$ is not connected. We deduce that the first and last components of ϵ cut pentagons off of $Q - n(\tau)$; all other components cut off rectangles. When $u \neq w$ then every region of $Q - n(\tau)$ contains at most one component of ϵ. In this case an index argument proves that δ is a crossing diagonal, according to Criterion 4.2. If $u = w$ then Q is a once-holed rectangle as shown in Figure 5.3. In this case δ is a crossing diagonal, according to Criterion 4.4. □

Lemma 5.4. *Suppose $\sigma \subset \tau$ is a large subtrack. Let Q be a component of $S - n(\sigma)$ that is not a hexagon or a once-holed bigon. Then Q has a short crossing diagonal.*

Proof. Since σ is large, Q is a disk or a peripheral annulus. Set $n = |\partial_v Q|$. According to Lemma 5.1, for every component $v \subset \partial_v Q$ there is a short diagonal δ_v cutting v off of Q. Let $H_v \subset Q - \delta_v$ be the hexagon to the right of δ_v. Also every δ_v is a carried or a crossing diagonal.

Suppose for a contradiction that δ_v is carried, for each $v \subset \partial_v Q$. Thus $K_v = H_v - n(\tau)$ is again a hexagon. If u is another component of $\partial_v Q$ then K_u and K_v are disjoint. Since index is additive, we find $\operatorname{index}(Q) \leq -\frac{n}{2}$. This inequality is strict when Q is a peripheral annulus; this is because the component of $Q - n(\tau)$ meeting ∂S must also have negative index.

On the other hand, if Q is a disk then $\operatorname{index}(Q) = 1 - \frac{n}{2}$; if Q is an annulus then $\operatorname{index}(Q) = -\frac{n}{2}$. In either case we have a contradiction. □

6. Closing up the diagonal

After introducing the necessary terminology, we give the proof of Proposition 6.1.

Suppose $\tau \subset S$ is a track, and $N = N(\tau)$ is a tie neighborhood. Suppose that $I \subset N$ is a tie, containing a component $u \subset \partial_v N$. Let R be the large half-rectangle adjacent to I. For any unit vector $V(x)$ based at $x \in \operatorname{interior}(I)$ we say $V(x)$ is *vertical* if it is tangent to I, is *large* if it points into R, and is *small* otherwise. Suppose $\alpha \prec \tau$ is a carried curve. A point $x \in \alpha \cap I$ is *innermost on I* if there is a component $\epsilon \subset I - (u \cup \alpha)$ so that the closure of ϵ meets both u and x.

Fix an oriented curve $\alpha \prec \tau$. For any $x \in \alpha$, we write $V(x, \alpha)$ for the unit tangent vector to α at x. If $x, y \in \alpha$ then we take $[x, y] \subset \alpha$ to be (the closure of) the component of $\alpha - \{x, y\}$ where $V(x, \alpha)$ points into $[x, y]$. Note that $\alpha = [x, y] \cup [y, x]$. Also, we take α^{op} to be α equipped with the opposite orientation. We make similar definitions when α is a arc.

Proposition 6.1. *Suppose $\tau \subset S$ is a train track and $\alpha \prec \tau$ is a carried curve. Suppose* $\operatorname{supp}(\alpha, \tau)$ *is large, but not maximal. Then there is a curve $\beta \dashv \tau$ so that $i(\alpha, \beta) \leq 1$ and any curve isotopic to β is not carried by τ.*

Proof. Set $\sigma = \operatorname{supp}(\alpha, \tau)$. Fix a component Q of $S - n(\sigma)$ that is not a hexagon or a once-holed bigon. By Lemma 5.4 there is a short crossing diagonal $\delta \subset Q$. Recall that $n(\delta)$ cuts a hexagon H off of Q; also, δ is oriented so that H is to the right of δ. The hexagon H meets three components $u, v, w \subset \partial_v Q$. The component v is completely contained in H; also, we may have $u = w$. Let p and q be the initial and terminal points of δ, respectively. Thus $p \in u$ and $q \in w$; also $V(p, \delta)$ is small and $V(q, \delta)$ is large. (Equivalently, $V(p, \delta)$ points into Q while $V(q, \delta)$ points out of Q.)

Let J_u and J_w be the ties of $N(\sigma)$ containing u and w. Rotate $V(p,\delta)$ by $\pi/2$, counterclockwise, to get an orientation of J_u. We do the same for J_w.

Since $\sigma = \operatorname{supp}(\alpha,\tau)$, there are pairs of innermost points $x_R, x_L \in \alpha \cap J_u$ and $z_R, z_L \in \alpha \cap J_w$. We choose names so that x_R, p, x_L is the order of the points along J_u and so that z_R, q, z_L is the order along J_w. Now orient α so that $V(x_R,\alpha)$ is small.

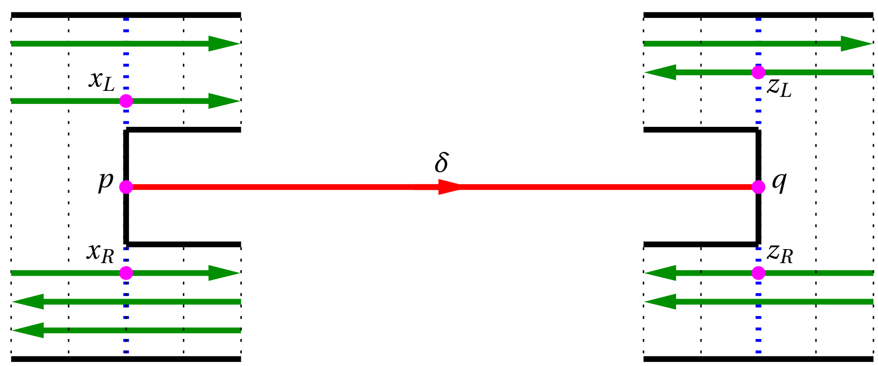

FIGURE 6.2. In this example, all of the vectors $V(x_R,\alpha)$, $V(x_L,\alpha)$, $V(z_R,\alpha)$, and $V(z_L,\alpha)$ are small.

We divide the proof into two main cases: one of $V(x_L,\alpha)$, $V(z_R,\alpha)$, $V(z_L,\alpha)$ is large, or all three vectors are small. In all cases and subcases our goal is to construct a curve β which contains δ and is, after an arbitrarily small isotopy, in efficient position with respect to $N(\tau)$. Since β contains δ one of Criterion 4.2 or Criterion 4.4 applies: any curve isotopic to β is not carried by τ.

6.3. **A tangent vector to α at x_L, z_R, or z_L is large.** This case breaks into subcases depending on whether or not $u = w$. Suppose first that $u \neq w$.

If $V(z_R,\alpha)$ is large, then consider the arcs $[q,z_R] \subset J_w^{\mathrm{op}}$, $[z_R,x_R] \subset \alpha$, and $[x_R,p] \subset J_u$. The curve

$$\beta = \delta \cup [q,z_R] \cup [z_R,x_R] \cup [x_R,p]$$

has the desired properties and satisfies $i(\alpha,\beta) = 0$.

If $V(z_L,\alpha)$ is large, then consider the arcs $[q,z_L] \subset J_w$, $[z_L,x_R] \subset \alpha$, and $[x_R,p] \subset J_u$. Then

$$\beta = \delta \cup [q,z_L] \cup [z_L,x_R] \cup [x_R,p]$$

has $i(\alpha,\beta) = 1$ because β crosses, once, from the right side to the left side of $[z_L,x_R]$.

Suppose now that $V(z_R,\alpha)$ and $V(z_L,\alpha)$ are small but $V(x_L,\alpha)$ is large. Consider the arcs $[p,x_L] \subset J_u$, $[x_L,z_L] \subset \alpha$, and $[z_L,q] \subset J_w^{\mathrm{op}}$. Then

$$\beta = [p,x_L] \cup [x_L,z_L] \cup [z_L,q] \cup \delta^{\mathrm{op}}$$

has $i(\alpha,\beta) = 0$.

We now turn to the subcase where $u = w$ and $V(x_L,\alpha)$ is large. In this case $x_R = z_L$, $x_L = z_R$, and the points x_R, p, q, x_L appear, in that order, along J_u. Consider the arcs $[q,x_L]$ and $[x_R,p] \subset J_u$ and $[x_L,x_R] \subset \alpha$. Then

$$\beta = \delta \cup [q,x_L] \cup [x_L,x_R] \cup [x_R,p].$$

has $i(\alpha,\beta) = 0$.

6.4. **The tangent vectors to α at x_L, z_R, and z_L are small.** Let $R \subset N(\tau)$ be the biggest rectangle, with embedded interior, where

- both components of $\partial_v R$ are subarcs of ties,
- $[z_R, z_L] \subset J_w$ is a component of $\partial_v R$, and
- $\partial_h R = \alpha \cap R$.

Since the interior of R is embedded, the vertical arc $(\partial_v R) - [z_R, z_L]$ contains a unique component $u' \subset \partial_v N(\sigma)$; also, the component u' is not equal to w. Pick a point p' in the interior of u'. Let $\epsilon \subset R$ be a carried arc starting at q, ending at p', and oriented away from q.

Let $J_{u'}$ be the tie in $N(\tau)$ containing u'. We orient $J_{u'}$ by rotating $V(p', \epsilon)$ by $\pi/2$, counterclockwise. Let x'_R and x'_L be the innermost points of $\alpha \cap J_{u'}$. Note that $V(x'_L, \alpha)$ and $V(x'_R, \alpha)$ are both large. Thus $u' \neq u$. We have already seen that $u' \neq w$.

Let Q' be the component of $S - n(\delta \cup \sigma)$ that contains u'. The orientation on S restricts to Q', which in turn induces an orientation on $\partial Q'$. Let v' be the component of $\partial_v Q'$ immediately *before* u'.

If $v' = u'$ then Q' is a once-holed bigon, contradicting the fact that $V(x'_L, \alpha)$ and $V(x'_R, \alpha)$ are both large.

If $v' \subset u - \delta$ then $Q' = H \subset Q$ is the hexagon to the right of δ. Thus $u' = v$. In this case there is a curve $\alpha' \subset R \cup H$ so that

- $\alpha' \cap R$ is a properly embedded arc with endpoints z_R and x'_R and
- $\alpha' - R$ is a component of $\partial_h H$.

Thus α' is isotopic to (the right side of) α. Now, if $u \neq w$ then α' also meets the region $Q - (n(\delta) \cup H)$, near x_L. Thus α' is not contained in $R \cup H$, a contradiction. If $u = w$ then Q is a once-holed rectangle. In this case $R \cup Q$, together with a pair of rectangles, is all of S. Thus S is a once-holed torus, contradicting our standing assumption that $\chi(S) \leq -2$.

If $v' \subset w - \delta$ then $Q = Q' \cup N(\delta) \cup H$. We deduce that Q is not a once-holed rectangle; so $u \neq w$. Also, the left side of α is contained in $R \cup Q'$. However the left side of α meets the hexagon H, near the point x_R, giving a contradiction.

To recap: the arc $\delta \cup \epsilon$ enters Q' at $p' \in u' \subset \partial_v Q'$. The region Q' is not a once-holed bigon; also $Q' \cap H = \emptyset$. The component $v' \subset \partial_v Q'$ coming before u' is not contained in $w - \delta$.

Let $J_{v'}$ be the tie of $N(\tau)$ containing v'. We orient $J_{v'}$ using the orientation of Q'. Let y'_R and y'_L be the two innermost points of $\alpha \cap J_{v'}$, where y'_R comes before y'_L along $J_{v'}$. Since $V(x'_L, \alpha)$ is large, the vector $V(y'_L, \alpha)$ is small.

We now have a final pair of subcases. Either $V(y'_R, \alpha)$ is large, or it is small.

6.5. **The tangent vector to α at y'_R is large.** Consider the arcs $[p', x'_L] \subset J_{u'}$, $[x'_L, y'_L] \subset \alpha^{\mathrm{op}}$, $[y'_L, y'_R] \subset J^{\mathrm{op}}_{v'}$, $[y'_R, x_R] \subset \alpha$, and $[x_R, p] \subset J_u$. Then

$$\beta = \delta \cup \epsilon \cup [p', x'_L] \cup [x'_L, y'_L] \cup [y'_L, y'_R] \cup [y'_R, x_R] \cup [x_R, p]$$

has $i(\alpha, \beta) = 0$. (Note that after an arbitrarily small isotopy the arc $[p', x'_L] \cup [x'_L, y'_L] \cup [y'_L, y'_R]$ becomes carried.)

6.6. **The tangent vector to α at y'_R is small.** In this case we consider the component w' of $\partial_v Q'$ immediately before v'. Recall that $Q' \cap H = \emptyset$. Now, if w' is the left component of $w - \delta$ then $V(y'_R, \alpha)$ being small implies $V(z_L, \alpha)$ is large, contrary to assumption. If w' is the left component of $u - \delta$ then v' is contained in w, a contradicton.

As usual, let $J_{w'}$ be the tie in $N(\tau)$ containing w'. Since w' is not contained in u or w there are a pair of innermost points z'_R and z'_L along $J_{w'}$. Applying Lemma 5.1 there is a short diagonal δ' in Q' that

- connects $p' \in u'$ to a point $q' \in w'$ and
- cuts v' off of Q'.

Note that v' is to the *left* of δ'.

We give $J_{w'}$ the orientation coming from $\partial_v Q'$; this agrees with the orientation given by rotating $V(q', \delta')$ by angle $\pi/2$, counterclockwise. We choose names so the points z'_R, q', z'_L come in that order along $J_{w'}$.

Consider the arcs $[q', z'_L] \subset J_{w'}$, $[z'_L, x_R] \subset \alpha$, and $[x_R, p] \subset J_u$. Then

$$\beta = \delta \cup \epsilon \cup \delta' \cup [q', z'_L] \cup [z'_L, x_R] \cup [x_R, p]$$

has $i(\alpha, \beta) = 1$ because β crosses, once, from the right side to the left side of $[z'_L, x_R]$. This is the final case, and completes the proof. □

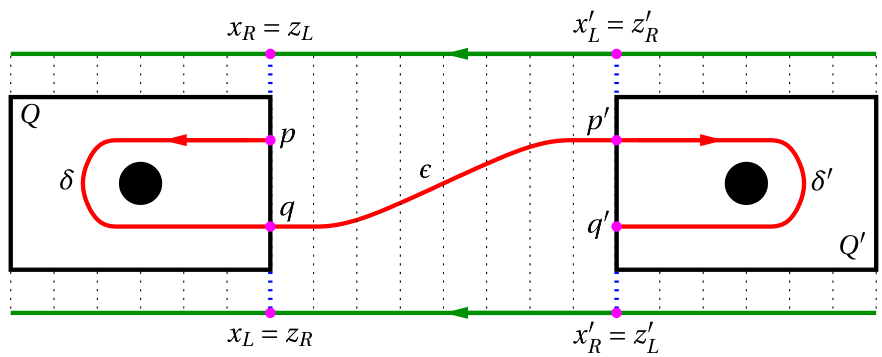

FIGURE 6.7. One of the four possibilities covered by Section 6.6. Here $u = w$ and $u' = w'$, so both of Q and Q' are once-holed rectangles.

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