Borrowing in Excess of Natural Ability to Repay[☆]

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Abstract

The paper aims at improving our understanding of self-enforcing debt in competitive dynamic economies with lack of commitment when default induces a permanent loss of access to international credit markets. We show, by means of examples, that a sovereign's creditworthiness is not necessarily limited by the ability to repay out of its future resources. Self-enforcing debt grows at the same rate as interest rates. If a sovereign's endowment growth rates are lower than interest rates, then debt limits eventually exceed the natural debt limits. This implies that there is asymptotic borrowing in present value terms. We show that this can be compatible with lending incentives when credible borrowers facilitate inter-temporal exchange, acting as pass-through intermediaries that alleviate the lenders' credit restrictions.

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1. Introduction

An important issue that arises in dynamic, infinite horizon economies with sequential financial markets is the specification of borrowing constraints to prevent Ponzi games. Debt constraints should limit the rate at which agents accumulate debt, but they must be sufficiently loose to permit the maximum expansion of risk-sharing without introducing unjustified financial frictions.

When there is full commitment, the only requirement imposed on debt limits is that they should be non-binding at equilibrium.[1] This implies that agents' wealth–defined as the present value of future endowments–is finite and equilibrium debt, contingent to any event, is bounded from above by the "natural debt limit" (see for instance Ljungqvist and Sargent (2004), Acemoglu (2009) and Miao (2014)). Natural debt limits correspond to what an agent can pay at a contingency by never

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[1]See Magill and Quinzii (1994) for a detailed discussion.

consuming again and using all future income for repayment. They reflect two relevant aspects of the borrowing capacity: the future resources for repayment and the market value of time (i.e., the interest rates).

When there is lack of commitment and markets are complete, debt constraints should in addition be consistent with repayment incentives. Formally, the debt limits must be self-enforcing in the sense that they are tight enough to prevent default at equilibrium. Intuition suggests that in this setting, debt should be lower than in the full commitment counterpart. In particular, one expects that borrowers should be unable to issue debt in excess of their natural ability to repay, captured by their natural debt limits.

The objective of this paper is to show that this intuition is not correct. In the context of a general equilibrium model with lack of commitment and credit market exclusion upon default, we argue (by means of several examples) that equilibrium self-enforcing debt limits can exceed a borrower's present value of future resources for repayment.[2] As in the example of debt sustainability proposed by Hellwig and Lorenzoni (2009), the present value of aggregate resources is infinite in all of our examples. However, it may be the case that the present value of endowments is finite for some agents and infinite for some others. These are the typical environments we are interested in.

The role of sufficiently low interest rates, first documented by Hellwig and Lorenzoni (2009), is still important for our result, but we identify a new channel that makes possible self-enforcing debt to exceed the natural debt limits. We argue that the mechanism comes into play only when there are at least three agents. To give some insight why this is the case, consider a deterministic economy with three agents or countries (two of them are referred as *rich* while the third one as *poor*) trying to smooth consumption over periods by trading a one-period bond subject to lack of commitment. The endowment of one of the rich countries is constant over even periods and grows at a constant rate over odd periods (and vice versa for the other rich country). The poor country starts with no endowment at the initial period and enjoys a constant endowment thereafter.[3] We show that there are primitives (preferences and endowments) for which the economy admits a competitive equilibrium where the poor country faces strictly positive self-enforcing debt limits despite the fact that its wealth is finite. Moreover, the poor country's equilibrium debt levels eventually exceed its natural debt limits.

Why potential creditors ever accept to lend in excess of a debtor's natural ability to repay? We argue that this is possible because the poor country provides an intermediation service that reduces the financial frictions, due to the lack of commitment, imposed on the rich countries. This service is not related to the country's future resources and interest rates, so there is no reason that the borrowing capacity reflects solely the country's wealth. More precisely, at equilibrium, the rich countries are not creditworthy (i.e., their debt limits are equal to zero) but they have strong incentives to trade with each other. The poor country acts as a pass-through intermediary, borrowing from one country and repaying the other one. In this way, debt is rolled over indefinitely and eventually exceeds the country's natural debt limit, with the difference reflecting the market value of the financial intermediation service the country provides.

[2] Following the seminal work of Bulow and Rogoff (1989) there is a growing literature studying models with limited enforceability of risk-sharing contracts where debt repudiation induces exclusion from borrowing but not from saving. It includes, among others, Cole and Kehoe (1995), Cole and Kehoe (1998), Kletzer and Wright (2000), Dutta and Kapur (2002), Kehoe and Perri (2002), Gul and Pesendorfer (2004), Krueger and Uhlig (2006), Hellwig and Lorenzoni (2009), Amador (2012). We refer to Wright (2011) and Aguiar and Amador (2014) for a thorough discussion of this literature.

[3] This justifies why we refer to this country as *poor*.

Our work complements Hellwig and Lorenzoni (2009)'s analysis of repayment incentives. They show that debt limits are self-enforcing if, and only if, borrowers can exactly roll over these debt limits period by period. However, in the two-agents example they analyze, there is no issue on whether or not the equilibrium debt limits exceed the natural debt limits. This is because in their example both agents have infinite wealth at equilibrium. We could infer from this that a sovereign can sustain positive levels of debt only if its wealth is infinite, or equivalently, only if interest rates are lower than its endowment growth rates.[4] Our examples show that a debtor's "good reputation" for repayment is endogenously determined at equilibrium and is not necessarily dependent on whether interest rates are lower than the debtor's endowment growth rates. Indeed, repayment incentives are ensured by the bubble property of equilibrium debt limits independently of the sovereign's wealth being finite or infinite. However, the level of equilibrium interest rates does have a preeminent role: the exact roll-over property of debt limits is consistent with the asymptotic supply of credit only if interest rates are lower than the lenders' endowment growth rates.[5]

The analysis brings additional insights on issues that may be of independent interest. For instance, Example 4.2, where one of the agents has zero endowment but is able to issue positive levels of debt, allows to connect the intermediation channel to the way money is valued in overlapping generations economies. Moreover, intermediation is compatible with strictly positive (effective) risk-less interest rates as shown in Example 4.3.

The paper is organized as follows: Section 2 presents a general stochastic dynamic economy with lack of commitment where default amounts to exclusion from credit markets forever. Section 3 discusses (in a general setting) two necessary conditions for sustaining debt in excess of the natural borrowing capacity: some agents must have infinite wealth and there must be at least three agents. Section 4 contains our examples and delivers the intuition on how the need for intermediation allows debtors to borrow more than their natural debt limits. Section 5 concludes. Some technical results–the one related to the necessity of a *market transversality condition* may be of independent interest–are presented in the appendix.

2. Fundamentals and Markets

We present an infinite horizon general equilibrium model with lack of commitment and self-enforcing debt limits along the lines of Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009). Time and uncertainty are both discrete and there is a single non-storable consumption good. The economy consists of a finite set I of infinitely lived agents (countries) sharing risks in an environment where debtors cannot commit to their promises.

2.1. Uncertainty

We use an event tree Σ to describe time, uncertainty and the revelation of information over an infinite horizon. There is a unique initial date-0 event $s^0 \in \Sigma$ and for each date $t \in \{0, 1, 2, \dots\}$ there is a finite set $S^t \subset \Sigma$ of date-t events s^t. Each s^t has a unique predecessor $\sigma(s^t)$ in S^{t-1} and a finite number of successors s^{t+1} in S^{t+1} for which $\sigma(s^{t+1}) = s^t$. We use the notation $s^{t+1} \succ s^t$ to specify that s^{t+1} is a successor of s^t. Event $s^{t+\tau}$ is said to follow event s^t, also denoted $s^{t+\tau} \succ s^t$,

[4] This is because if a non-negative and non-zero process satisfies exact roll-over, then it cannot be tighter than a process with finite present value.

[5] In our examples, the rich countries (which are the creditors) have infinite wealth at equilibrium.

if $\sigma^{(\tau)}(s^{t+\tau}) = s^t$. The set $S^{t+\tau}(s^t) := \{s^{t+\tau} \in S^{t+\tau} : s^{t+\tau} \succ s^t\}$ denotes the collection of all date-$(t+\tau)$ events following s^t. Abusing notation, we let $S^t(s^t) := \{s^t\}$. The subtree of all events starting from s^t is then

$$\Sigma(s^t) := \bigcup_{\tau \geq 0} S^{t+\tau}(s^t).$$

We use the notation $s^\tau \succeq s^t$ when $s^\tau \succ s^t$ or $s^\tau = s^t$. In particular, we have $\Sigma(s^t) = \{s^\tau \in \Sigma : s^\tau \succeq s^t\}$.

Remark 2.1. When the environment is deterministic, we abuse notation and denote processes of the form $x = (x(s^t))_{s^t \in \Sigma}$ by $x = (x_t)_{t \geq 0}$.

2.2. Endowments and Preferences

Agents' endowments are subject to random shocks. We denote by $y^i = (y^i(s^t))_{s^t \in \Sigma}$ the agent i's process of positive endowments $y^i(s^t) > 0$ of the consumption good contingent to event s^t. Preferences over (non-negative) consumption processes $c = (c(s^t))_{s^t \in \Sigma}$ are represented by the lifetime discounted utility functional

$$U(c) := \sum_{s^t \in \Sigma} \beta^t \pi(s^t) u(c(s^t)),$$

where $\beta \in (0,1)$ is the discount factor, $\pi(s^t)$ is the unconditional probability of s^t and $u : \mathbb{R}_+ \to [-\infty, \infty)$ is a Bernoulli function assumed to be strictly increasing, concave, continuous on $\mathbb{R}_+$, differentiable on $(0,\infty)$, bounded from above and satisfying Inada's condition at the origin.

Given an event s^t, we denote by $U(c|s^t)$ the lifetime continuation utility conditional on s^t, defined by

$$U(c|s^t) := \sum_{s^{t+\tau} \in \Sigma(s^t)} \beta^\tau \pi(s^{t+\tau}|s^t) u(c(s^{t+\tau})),$$

where $\pi(s^{t+\tau}|s^t) := \pi(s^{t+\tau})/\pi(s^t)$ is the conditional probability of $s^{t+\tau}$ given s^t.

A collection $(c^i)_{i \in I}$ of consumption processes is said to be resource feasible if $\sum_{i \in I} c^i = \sum_{i \in I} y^i$.

2.3. Markets

At every event s^t, agents can issue and trade a complete set of one-period contingent bonds which promise to pay one unit of the consumption good contingent on the realization of any successor event $s^{t+1} \succ s^t$. Let $q(s^{t+1}) > 0$ denote the price, in units of consumption, at event s^t of the s^{t+1}-contingent bond. Agent i's holding of this bond is $a^i(s^{t+1})$. The amount of state-contingent debt agent i can issue is observable and subject to state-contingent (non-negative) upper bounds (or debt limits) $D^i = (D^i(s^t))_{s^t \succ s^0}$. Given an initial financial claim $a^i(s^0)$, we denote by $B^i(D^i, a^i(s^0)|s^0)$ the budget set of all pairs (c^i, a^i) of consumption and bond holdings satisfying the following constraints: for every event $s^t \succeq s^0$,

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a^i(s^{t+1}) \leq y^i(s^t) + a^i(s^t) \tag{2.1}$$

and

$$a^i(s^{t+1}) \geq -D^i(s^{t+1}). \tag{2.2}$$

Given some initial claim $b \in \mathbb{R}$ at an event s^τ, we denote by $J^i(D^i, b|s^\tau)$ the largest continuation utility defined by

$$J^i(D^i, b|s^\tau) := \sup\{U(c^i|s^\tau) : (c^i, a^i) \in B^i(D^i, b|s^\tau)\},$$

where $B^i(D^i, b|s^\tau)$ is the set of all plans (c^i, a^i) satisfying $a^i(s^\tau) = b$, together with Equations (2.1) and (2.2) for every successor event $s^t \succeq s^\tau$.

Recall that the Euler equations and the transversality condition are sufficient conditions for the optimality of agents' choices. Formally, consider a budget feasible plan (c^i, a^i) satisfying the flow constraints (2.1) with equality. If c^i is strictly positive, satisfies the Euler equations at every event $s^t \succ s^0$, i.e.,

$$q(s^t) \geq \beta\pi(s^t|\sigma(s^t))\frac{u'(c^i(s^t))}{u'(c^i(\sigma(s^t))}, \quad \text{with an equality if } a^i(s^t) > -D^i(s^t), \tag{2.3}$$

and the transversality condition, i.e.,

$$\liminf_{t\to\infty} \sum_{s^t\in S^t} \beta^t\pi(s^t)u'(c^i(s^t))[a^i(s^t) + D^i(s^t)] = 0, \tag{2.4}$$

then (c^i, a^i) is optimal in the budget set $B^i(D^i, a^i(s^0)|s^0)$.

2.4. Default Punishment

We consider an environment where there is no commitment. Agents might not honor their debt obligations and decide to default. Such a decision depends on the consequences of default. Following Bulow and Rogoff (1989) (see also Hellwig and Lorenzoni (2009)), we assume that a defaulting agent starts with neither assets nor liabilities, is excluded from future credit but retains the ability to save (i.e., the agent can purchase bonds). Therefore, agent i's default option at an event s^t is

$$V^i(s^t) := J^i(0, 0|s^t).$$

Lenders have no incentives to provide credit contingent to some event if they anticipate that the borrower will default.[6] The maximum amount of debt $D^i(s^t)$ at any event $s^t \succ s^0$ should reflect this property. If agent i's initial financial claim at event s^t corresponds to the maximum debt $-D^i(s^t)$, then the agent prefers to repay its debt if, and only if, $J^i(D^i, -D^i(s^t)|s^t) \geq V^i(s^t)$. When a process of bounds satisfies the above inequality at every event $s^t \succ s^0$, it is called self-enforcing.[7] Competition among lenders naturally leads to consider the largest self-enforcing bound $D^i(s^t)$ defined by the equation

$$J^i(D^i, -D^i(s^t)|s^t) = V^i(s^t). \tag{2.5}$$

Since the seminal contribution of Alvarez and Jermann (2000), the literature refers to such debt limits as "not-too-tight".

2.5. Competitive Equilibrium

Bonds are in zero net supply. Fix an allocation $(a^i(s^0))_{i\in I}$ of initial financial claims that satisfies market clearing, i.e.,

$$\sum_{i\in I} a^i(s^0) = 0.$$

[6] Since the default punishment is independent of the default level, an agent either fully repays his debt or defaults totally. There is no partial default.

[7] Indeed, since the function $J^i(D^i, \cdot|s^t)$ is increasing, for any bond holding $a^i(s^t)$ satisfying the restriction $a^i(s^t) \geq -D^i(s^t)$, agent i prefers honoring his obligation than defaulting on $a^i(s^t)$.

A competitive equilibrium $(q,(c^i,a^i,D^i)_{i\in I})$ consists of state-contingent bond prices q, a resource feasible consumption allocation $(c^i)_{i\in I}$, a market clearing allocation of bond holdings $(a^i)_{i\in I}$ and an allocation of debt limits $(D^i)_{i\in I}$ such that, for each agent i, the plan (c^i,a^i) is optimal among budget feasible plans in $B^i(D^i,a^i(s^0)|s^0)$. A competitive equilibrium with self-enforcing debt is a competitive equilibrium for which debt limits are not-too-tight.

2.6. Natural Ability to Borrow

Consider for a moment the benchmark environment with full commitment. In order to prevent Ponzi schemes, we need to impose debt limits on bond holdings, however, these limits need not be self-enforcing. To ensure that the debt constraints do not introduce an additional imperfection into the model, the debt limits should be sufficiently large to permit all justified transfers of income. In other words, debt limits should never bind at equilibrium. When this is the case, the wealth of each agent–defined as the present value of future endowments–is finite at equilibrium. To state this result formally we need to introduce some notation.

Given state-contingent bond prices $q=(q(s^t))_{s^t\succ s^0}$, we denote by $p(s^t)$ the associated date-0 price of consumption at s^t defined recursively by $p(s^0)=1$ and $p(s^{t+1})=q(s^{t+1})p(s^t)$ for every $s^{t+1}\succ s^t$. We use $\mathrm{PV}(x|s^t)$ to denote the present value (at event s^t) of a process x restricted to the subtree $\Sigma(s^t)$ and defined by

$$\mathrm{PV}(x|s^t):=\frac{1}{p(s^t)}\sum_{s^\tau\in\Sigma(s^t)}p(s^\tau)x(s^\tau).$$

Agent i's wealth $W^i(s^t)$ at event s^t is then defined as the present value of future income, i.e.,

$$W^i(s^t):=\mathrm{PV}(y^i|s^t).$$

Observe that $W^i(s^t)$ could be infinite.[8] However, if $(q,(c^i,a^i,D^i)_{i\in I})$ is a competitive equilibrium in which debt limits never bind, i.e.,

$$\forall i\in I,\quad \forall s^t\succ s^0,\quad a^i(s^t)>-D^i(s^t),$$

then $W^i(s^t)$ is finite and $a^i(s^t)>-W^i(s^t)$ for every agent i and any event s^t. It turns out that the wealth process $(W^i(s^t))_{s^t\succ s^0}$ is the natural candidate for the (non-binding) debt limits in an environment with full commitment. Such debt limits are known as the "natural debt limits" or the "natural ability to repay" (see for instance Ljungqvist and Sargent (2004), Acemoglu (2009) and Miao (2014)).

Remark 2.2. Assume that agent i could trade at event s^t a complete set of bonds with all possible maturities. No arbitrage would imply that the price at event s^t of the bond with maturity at a successor event $s^\tau\succ s^t$ is $p(s^\tau)/p(s^t)$. In a such environment, country i could sell at event s^t the whole process of future endowments $(y(s^\tau))_{s^\tau\in\Sigma(s^t)}$. The proceeds would then be

$$\frac{1}{p(s^t)}\sum_{s^\tau\in\Sigma(s^t)}p(s^\tau)y^i(s^\tau)=y^i(s^t)+\sum_{s^{t+1}\succ s^t}q(s^{t+1})W^i(s^{t+1}).$$

The term $q(s^{t+1})W^i(s^{t+1})$ is the "natural borrowing limit" interpreted as the maximum amount country i can borrow at event s^t by selling his future income conditional to the successor event s^{t+1}. This is different from the wealth level $W^i(s^{t+1})$ corresponding to the maximum amount of debt country i can issue contingent to event s^{t+1}.

[8] However, if $W^i(s^0)$ is finite, then $W^i(s^t)$ is also finite at every $s^t\succ s^0$.

3. Necessary Conditions for Rolling Over Debt

The main goal of this paper is to show, by means of several examples, that there are economies having competitive equilibria in which an agent can borrow more than his wealth without violating repayment incentives. Before presenting our examples we review the existing literature and discuss two necessary conditions for sustaining debt in excess of natural debt limits.

3.1. Related Literature

Bulow and Rogoff (1989) show that debt limits cannot be simultaneously self-enforcing and tighter than natural debt limits. Formally, we have the following result.[9]

Theorem 3.1 (Bulow and Rogoff). *Assume that agent i's wealth is finite. If a debt limit process D^i is self-enforcing and tighter than natural debt limits, i.e.,*

$$\forall s^t \succ s^0, \quad J^i(D^i, -D^i(s^t)|s^t) \geq V^i(s^t) \quad \text{and} \quad D^i(s^t) \leq W^i(s^t) < \infty,$$

then $D^i(s^t) = 0$ at every event $s^t \succ s^0$.

In other words, if an agent's wealth is finite and his debt capacity is bounded from above by his natural ability to repay, then the threat of credit exclusion is not sufficient to induce repayment incentives. Hellwig and Lorenzoni (2009) (see also Bidian and Bejan (2015)) went further and characterized an agent's repayment incentives without assuming a priori that his wealth is finite. They proved the following connection between debt sustainability and rational bubbles on debt limits.

Theorem 3.2 (Hellwig and Lorenzoni). *A debt limit process D^i is not-too-tight, i.e.,*

$$\forall s^t \succ s^0, \quad J^i(D^i, -D^i(s^t)|s^t) = V^i(s^t),$$

if, and only if, it allows for exact roll-over, i.e.,

$$\forall s^t \succ s^0, \quad D^i(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1}). \tag{ER}$$

In other words, an agent can credibly promise to repay a positive amount of debt if, and only if, this debt can be rolled over indefinitely. A direct consequence of the above characterization result is that if a process D^i of not-too-tight debt limits satisfies the following transversality condition

$$\lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t) D^i(s^t) = 0, \tag{TC}$$

then debt cannot be rolled over at infinite, and we must have $D^i(s^t) = 0$ at every event $s^t \succ s^0$.[10] Observe that if D^i is tighter than natural debt limits, then it satisfies the transversality condition (TC).

[9] The model in Bulow and Rogoff (1989) is slightly different than the one presented here. They analyzed repayment incentives of a small open economy borrowing from competitive, risk neutral foreign investors. In that respect, the statement of Theorem 3.1 is slightly more general than the original result proved in Bulow and Rogoff (1989). We show in Martins-da-Rocha and Vailakis (2016) that Theorem 3.1 can be obtained as a corollary of the result in Bulow and Rogoff (1989).

[10] This is because $D^i(s^0) = \sum_{s^t \in S^t} p(s^t) D^i(s^t)$ and D^i is non-negative.

An important question is whether the roll-over property (ER) is compatible with market clearing at equilibrium. Indeed, if an agent is rolling over his debt, then there must be other agents lending at infinite in present value terms. In the benchmark environment with full commitment and non-binding debt limits, this is not consistent with lenders' necessary transversality conditions. When there is lack of commitment, we present below two conditions that are necessarily satisfied if rolling over debt is compatible with lenders' incentives.

3.2. Low Interest Rates

We show that equilibrium borrowing requires interest rates to be low enough such that the wealth of some agents–but not necessarily the wealth of debtors–is infinite.

Proposition 3.1. Debt cannot be self-enforced if interest rates are such that the aggregate wealth of the economy is finite. Formally, if $(q, (c^i, a^i, D^i)_{i\in I})$ is a competitive equilibrium with self-enforcing debt such that $W^i(s^0)$ is finite for each agent i, then $D^i = 0$ and there is no trade.

Proof of Proposition 3.1. Let $(q, (c^i, a^i, D^i)_{i\in I})$ be a competitive equilibrium with self-enforcing debt. Assume that for each agent i the wealth $W^i(s^0)$ is finite. Since consumption markets clear, the present value of each individual consumption process is finite: $\mathrm{PV}(c^i|s^0) < \infty$ for each i. Applying Lemma A.1 (see Appendix A) we get the following *market transversality condition*

$$\lim_{t\to\infty} \sum_{s^t\in S^t} p(s^t)[a^i(s^t) + D^i(s^t)] = 0. \tag{3.1}$$

The bond market clearing condition then implies that

$$\sum_{i\in I} \lim_{t\to\infty} \sum_{s^t\in S^t} p(s^t) D^i(s^t) = 0.$$

Non-negativity of debt limits implies that each D^i satisfies the transversality condition (TC) which (together with Theorem 3.2) suffices to prove the desired result. □

The above result strengthens Proposition 3 in Hellwig and Lorenzoni (2009) since we do not assume a priori that debt limits are tighter than the natural debt limits. We instead show that this is a necessary condition when the wealth of each agent is finite. The crucial step in the proof is the observation that the *market transversality condition* (3.1) is always satisfied when the optimal consumption of an agent has finite present value.[11]

3.3. At Least Three Agents

We show that if an agent can borrow more than his natural ability to repay, then there must be at least three agents in the economy. We provide below the details of the proof for the deterministic case since the argument is easy to follow and delivers the economic insight that drives the result. The proof for the general stochastic environment is more convoluted, and therefore is postponed to Appendix B.

[11] In the full commitment environment where debt limits never bind, the *market transversality condition* (3.1) coincides with the individual transversality condition (2.4). This is not necessarily the case when there is lack of commitment since debt limits may bind.

Proposition 3.2. If there are two agents, debt is self-enforced only if both agents have infinite wealth. Formally, if $(q, (c^i, a^i, D^i)_{i\in\{i_1,i_2\}})$ is a competitive equilibrium with self-enforcing debt such that $W^{i_1}(s^t) < \infty$ at some event s^t, then $D^{i_1}(s^t) = 0$.

Proof of the deterministic case. Assume that one agent, say agent i_1, has finite wealth $W_0^{i_1} = \text{PV}_0(e^{i_1}) < \infty$. Assume, by way of contradiction, that $D_0^{i_1} > 0$. Using the exact roll-over property of each debt limit process D^i we get that

$$\sum_{s=0}^{t-1} p_s c_s^i + p_t[a_t^i + D_t^i] = \sum_{s=0}^{t-1} p_s e_s^i + a_0^i + D_0^i, \quad \text{for all } t \geq 1.$$

Since agent i_1 has finite wealth, we can deduce from the above equalities that

$$\text{PV}_0(c^{i_1}) := \lim_{t\to\infty} \sum_{s=0}^{t-1} p_s c_s^{i_1}$$

is also finite. Applying Lemma A.1 we get the following *market transversality condition*

$$\lim_{t\to\infty} p_t[a_t^{i_1} + D_t^{i_1}] = 0.$$

Combining the above condition with the clearing of bond markets and the exact roll-over property of D^{i_1}, we deduce that

$$\lim_{t\to\infty} p_t a_t^{i_2} = -\lim_{t\to\infty} p_t a_t^{i_1} = \lim_{t\to\infty} p_t D_t^{i_1} = D_0^{i_1} > 0.$$

Therefore, there must exist some date $\tau > 0$ such that $a_t^{i_2} > 0$ for every $t \geq \tau$. In particular, Euler equations (for all periods $t \geq \tau$) are satisfied with equality and we deduce that

$$\frac{p_t}{p_\tau} = \frac{\beta^t u'(c_t^{i_2})}{\beta^\tau u'(c_\tau^{i_2})}, \quad \text{for all } t \geq \tau.$$

This implies that

$$\lim_{t\to\infty} \beta^t u'(c_t^{i_2}) a_t^{i_2} = \frac{\beta^\tau u'(c_\tau^{i_2})}{p_\tau} \lim_{t\to\infty} p_t a_t^{i_2} > 0,$$

which contradicts agent i_2's individual transversality condition (2.4). □

The intuition for Proposition 3.2 rests on the observation that if an agent borrows more than his natural debt limit at some period τ, then he will indefinitely be a debtor in present value terms. Market clearing then implies that the other agent must indefinitely be a creditor in present value terms, a situation which is incompatible with the individual transversality condition.

It follows from Proposition 3.2 that supporting debt in excess of an agent's natural borrowing capacity requires the presence of at least two lenders. Combining this result with Proposition 3.1 we get that at least one lender must have infinite wealth. Actually, adapting the arguments in the proof of Proposition 3.2 we can show that both lenders must have infinite wealth (see Appendix B). This is precisely what happens in the examples we present in the next section. Two agents with infinite wealth need the intermediation service of a third agent to alleviate the financial frictions due to lack of commitment.

4. Borrowing in Excess of Natural Debt Limits

We present below economies with three agents (countries) having a competitive equilibrium where one country has the same repayment incentives as in Bulow and Rogoff (1989)–interest rates are sufficiently high to imply finite wealth levels at every contingency–but succeeds to sustain positive levels of debt. The celebrated critique of Bulow and Rogoff (1989) to models of reputational debt does not apply since we exhibit equilibria where one country borrows more than its natural ability to repay. That is, the country faces not-too-tight debt limits that exceed its wealth levels. This illustrates that the country's ability to borrow is not necessarily bounded from above by the present value of its future income. This is also in sharp contrast with the full commitment environment where equilibrium debt levels are necessarily tighter than the natural debt limits. We show that the excess of debt to wealth levels reflects the market value of a financial intermediation service the country provides to the other two countries.

The first example delivers the basic intuition of the intermediation mechanism and illustratess how the analysis departs from Bulow and Rogoff (1989)'s no-sustainability result. It also highlights how the level of interest rates affects not only the debtors' incentives (as stressed by Hellwig and Lorenzoni (2009)) but also the creditors' incentives. The second example helps to understand how the intermediation mechanism connects to the way money is valued in overlapping generations economies. The third example extends the analysis to a stochastic economy and shows that zero (effective) risk-less interest rates (as it is the case in the previous examples) is not an essential feature for our results.

4.1. Intermediation Mechanism

Our starting point is a deterministic economy with three agents. More precisely, we consider an environment with two rich countries, say R_1 and R_2, that grow unboundedly and a third country, say P, that is poor (it has no growth). We prove that there exists a competitive equilibrium where the asset pricing kernel is risk-neutral, and interest rates are constant and positive. Country P sustains debt in excess of its natural ability to borrow despite the fact that its repayment incentives are exactly as in Bulow and Rogoff (1989).

Why Theorem 3.1 does not apply in this setting? This is because Bulow and Rogoff (1989) restrict the sovereign's debt limits to be tighter than the natural debt limits. We show that if we do not impose this add-hoc restriction, debt can be rolled over indefinitely and eventually can exceed the natural debt limits.

At first glance this may appear obvious. However, given that interest rates are positive and time-independent, running such a *Ponzi scheme* requires unbounded supply of credit in the long run. Why do investors accept to lend asymptotically in present value terms? It is well known that this is easy to accommodate in a general equilibrium environment with overlapping generations.[12] But with risk-averse, infinitely lived foreign investors, one expects the individual transversality condition to preclude asymptotic lending (as in the full commitment literature on the non-existence of bubbles).[13] However, this is not the case because potential lenders are credit constrained and

[12]This is what makes bubbles sustainable (see Samuelson (1958), Gale (1973) and Tirole (1985)).

[13]It is known since the work of Scheinkman (1977, 1988) and Brock (1979, 1982) that asset pricing bubbles are impossible in an intertemporal equilibrium with infinitely lived traders. Essentially, the argument is that the presence of a bubble would require asymptotic growth in the asset's value, and hence asymptotic growth of the wealth of at least one of the traders at a rate that is inconsistent with the optimizing behaviour of that trader. We also refer to Santos and Woodford (1997) for a formal presentation and a generalization of these no bubble results.

they need the poor country to act as a financial intermediary to facilitate inter-temporal exchange. Indeed, the rich countries R_1 and R_2 would like to share risks by trading with each other, but they are not creditworthy (their debt limits are equal to zero). Country P emerges as a credible borrower, with its creditworthiness stemmed from its intermediation role, helping countries R_1 and R_2 to smooth consumption. For this service country P extracts a surplus.

Example 4.1. Fix arbitrary non-negative numbers δ, ω^{R_1} and ω^{R_2} with $\delta > 0$. Consider a deterministic economy with three countries (P, R_1 and R_2) where the endowment sequences are specified below.

County P's endowments are defined as follows:

$$y_0^{\text{P}} := 0 \quad \text{and} \quad y_t^{\text{P}} := \delta, \quad \text{for all } t \geq 1.$$

Country R_1's endowments are defined as follows:

$$\forall t \geq 0, \quad y_{2t+1}^{\text{R}_1} := \omega^{\text{R}_1} \quad \text{and} \quad y_{2t}^{\text{R}_1} := \omega^{\text{R}_1} + \frac{\delta}{\beta^{2t}} + \frac{\delta}{\beta^{2t+1}}.$$

In other words,

$$y_0^{\text{R}_1} = \omega^{\text{R}_1} + \delta + \frac{\delta}{\beta}, \quad y_1^{\text{R}_1} = \omega^{\text{R}_1}, \quad y_2^{\text{R}_1} = \omega^{\text{R}_1} + \frac{\delta}{\beta^2} + \frac{\delta}{\beta^3}, \quad y_3^{\text{R}_1} = \omega^{\text{R}_1}, \quad \ldots$$

The endowments of country R_2 are defined as follows:

$$\forall t \geq 0, \quad y_{2t+1}^{\text{R}_2} := \omega^{\text{R}_2} + \frac{\delta}{\beta^{2t+1}} + \frac{\delta}{\beta^{2t+2}} \quad \text{and} \quad y_{2t}^{\text{R}_2} := \omega^{\text{R}_2}.$$

In other words,

$$y_0^{\text{R}_2} = \omega^{\text{R}_2}, \quad y_1^{\text{R}_2} = \omega^{\text{R}_2} + \frac{\delta}{\beta} + \frac{\delta}{\beta^2}, \quad y_2^{\text{R}_2} = \omega^{\text{R}_2}, \quad y_3^{\text{R}_2} = \omega^{\text{R}_2} + \frac{\delta}{\beta^3} + \frac{\delta}{\beta^4}, \quad \ldots$$

The choice of ω^{R_1} and ω^{R_2} is irrelevant. To fix ideas, we can set them to be equal to zero (see Figure 1 below). The initial asset positions are set equal to zero: $a_0^{\text{P}} = a_0^{\text{R}_1} = a_0^{\text{R}_2} = 0$.

Proposition 4.1. The economy of Example 4.1 admits a competitive equilibrium with self-enforcing debt in which country P faces positive not-too-tight debt limits $D_t^{\text{P}} = \delta/(\beta^t)$ although its natural debt limits W_t^{P} are finite at equilibrium. Moreover, for t large enough, the debt limits are strictly larger than the country's natural debt limits. More specifically, we have

$$\lim_{t\to\infty} D_t^{\text{P}} = \infty > \delta/(1-\beta) = \lim_{t\to\infty} W_t^{\text{P}}.$$

Proof. We first describe the equilibrium prices, debt limits and allocations.

Let the price sequence $q = (q_t)_{t\geq 1}$ be defined by $q_t := \beta$ for every $t \geq 1$ (i.e., the interest rate r, defined by $1 + r = \beta^{-1}$, is positive and constant).

Consider the following debt limits: $D_t^{\text{R}_1} = D_t^{\text{R}_2} = 0$ and $D_t^{\text{P}} := \delta/(\beta^t)$. These debt limits are not-too-tight under the price sequence q since they allow for exact roll-over.

Let $(c^{\text{R}_1}, a^{\text{R}_1})$ be the plan defined by $c_0^{\text{R}_1} := y_0^{\text{R}_1} - \delta$, and for every $t \geq 1$

$$c_t^{\text{R}_1} := \begin{cases} y_t^{\text{R}_1} - \delta/(\beta^t) & \text{if } t \text{ is even} \\ y_t^{\text{R}_1} + \delta/(\beta^t) & \text{if } t \text{ is odd} \end{cases} \quad \text{and} \quad a_t^{\text{R}_1} := \begin{cases} \delta/(\beta^t) & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even.} \end{cases}$$

Let also $(c^{\mathrm{R}_2}, a^{\mathrm{R}_2})$ be the plan defined by $c_0^{\mathrm{R}_2} := y_0^{\mathrm{R}_2}$, and for every $t \geq 1$

$$c_t^{\mathrm{R}_2} := \begin{cases} y_t^{\mathrm{R}_2} - \delta/(\beta^t) & \text{if } t \text{ is odd} \\ y_t^{\mathrm{R}_2} + \delta/(\beta^t) & \text{if } t \text{ is even} \end{cases} \quad \text{and} \quad a_{t+1}^{\mathrm{R}_2} := \begin{cases} \delta/(\beta^t) & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

At every even date $2t$, the rich country R_1 optimally saves the amount $\beta a_{2t+1}^{\mathrm{R}_1} = \delta/(\beta^{2t})$ in order to trade the time-varying endowments

$$(y_{2t}^{\mathrm{R}_1}, y_{2t+1}^{\mathrm{R}_1}) = (\omega^{\mathrm{R}_1} + \delta/(\beta^{2t}) + \delta/(\beta^{2t+1}), \omega^{\mathrm{R}_1})$$

in exchange of the constant consumption

$$(c_{2t}^{\mathrm{R}_1}, c_{2t+1}^{\mathrm{R}_1}) = (\omega^{\mathrm{R}_1} + \delta/(\beta^{2t+1}), \omega^{\mathrm{R}_1} + \delta/(\beta^{2t+1})).$$

Figure 1a illustrates this trade pattern when $\delta = 1$ and $\beta^{-1} = 1.25$.

The rich country R_2 follows the same strategy at odd dates 2t+1: it saves the amount $\beta a_{2t+2}^{\mathrm{R}_2} = \delta/(\beta^{2t+1})$ to trade the time-varying endowments

$$(y_{2t+1}^{\mathrm{R}_2}, y_{2t+2}^{\mathrm{R}_2}) = (\omega^{\mathrm{R}_2} + \delta/(\beta^{2t+1}) + \delta/(\beta^{2t+2}), \omega^{\mathrm{R}_2})$$

in exchange of the constant consumption

$$(c_{2t+1}^{\mathrm{R}_2}, c_{2t+2}^{\mathrm{R}_2}) = (\omega^{\mathrm{R}_2} + \delta/(\beta^{2t+2}), \omega^{\mathrm{R}_2} + \delta/(\beta^{2t+2})).$$

Figure 1b illustrates this trade pattern when $\delta = 1$ and $\beta^{-1} = 1.25$.

Finally, we define $(c^{\mathrm{P}}, a^{\mathrm{P}})$ as follows: $c_0^{\mathrm{P}} := \delta$, $c_t^{\mathrm{P}} := y_t^{\mathrm{P}} = \delta$ for every $t \geq 1$ and $a_t^{\mathrm{P}} := -\delta/(\beta^t)$ for every $t \geq 1$. The poor country borrows and consumes the amount δ at the initial period and then, instead of repaying, it rolls over this debt forever. Figure 2 illustrates this trade pattern when $\delta = 1$ and $\beta^{-1} = 1.25$ and shows how the poor country acts as a financial intermediator.

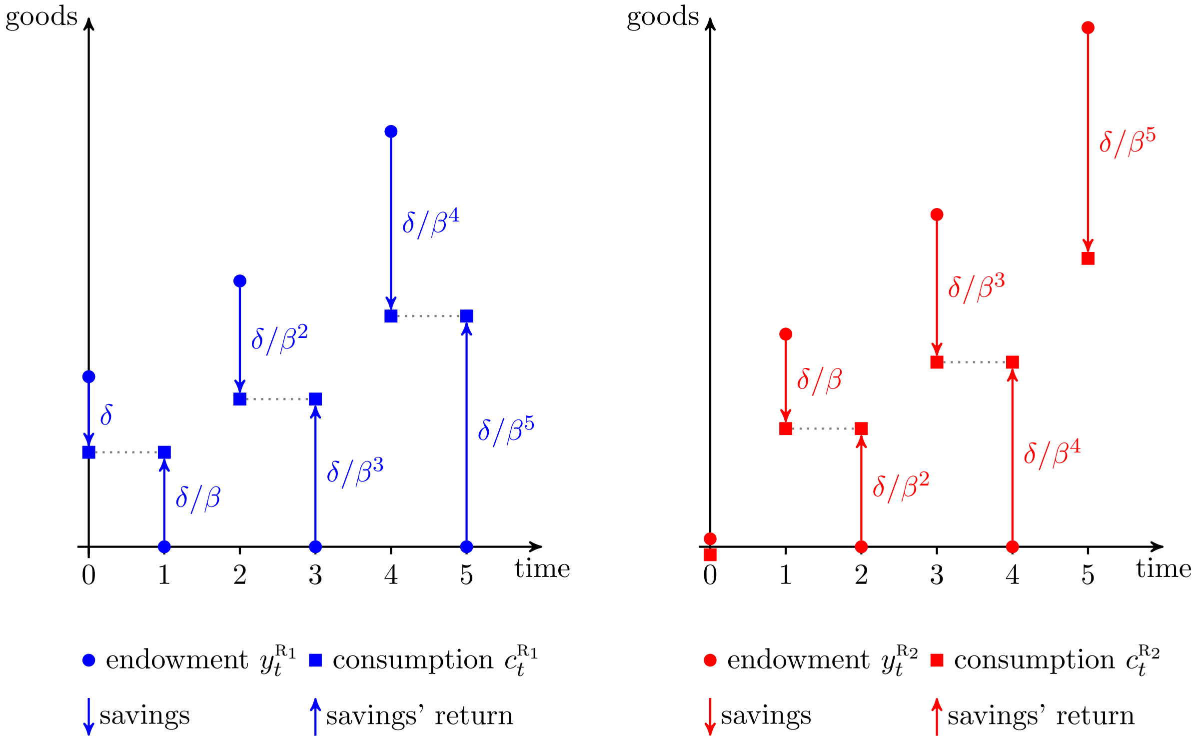
We next argue that these allocations are indeed optimal. Observe that $(c^{\mathrm{P}}, a^{\mathrm{P}})$ is optimal since it is budget feasible (with equality), it satisfies the Euler equations (2.3) (this follows from the fact that consumption is constant) and the transversality condition (2.4) (this is because debt limits bind infinitely often). Moreover, country P's wealth (and therefore any of its natural debt limits) is finite at any period since the interest rate is strictly positive and endowments are bounded from above. Formally, we have

$$W_t^{\mathrm{P}} = \delta + \beta\delta + \beta^2\delta + \ldots = \frac{\delta}{1-\beta}, \quad \text{for all } t \geq 1.$$

The plan $(c^{\mathrm{R}_k}, a^{\mathrm{R}_k})$ is also optimal since it is budget feasible (with equality) and satisfies the Euler equations (2.3). Indeed, if $a_{t+1}^{\mathrm{R}_k} > 0$ (i.e., agent R_k saves at date t), then agent R_k is financially unconstrained and we have $c_t^{\mathrm{R}_k} = c_{t+1}^{\mathrm{R}_k}$. If $a_{t+1}^{\mathrm{R}_k} = 0$, then agent R_k is financially constrained and we have $c_t^{\mathrm{R}_k} \leq c_{t+1}^{\mathrm{R}_k}$. The transversality condition (2.4) is satisfied because debt constraints bind infinitely many times.

To conclude the proof simply observe that all markets clear by construction. □

Hellwig and Lorenzoni (2009) study a stochastic economy where two agents credibly issue positive levels of debt at equilibrium. In their example, interest rates are such that the wealth of each agent is infinite. They argue that interest rates matter for debt sustainability to the extent



(a) Country R₁

(b) Country R₂

Figure 1: Rich countries

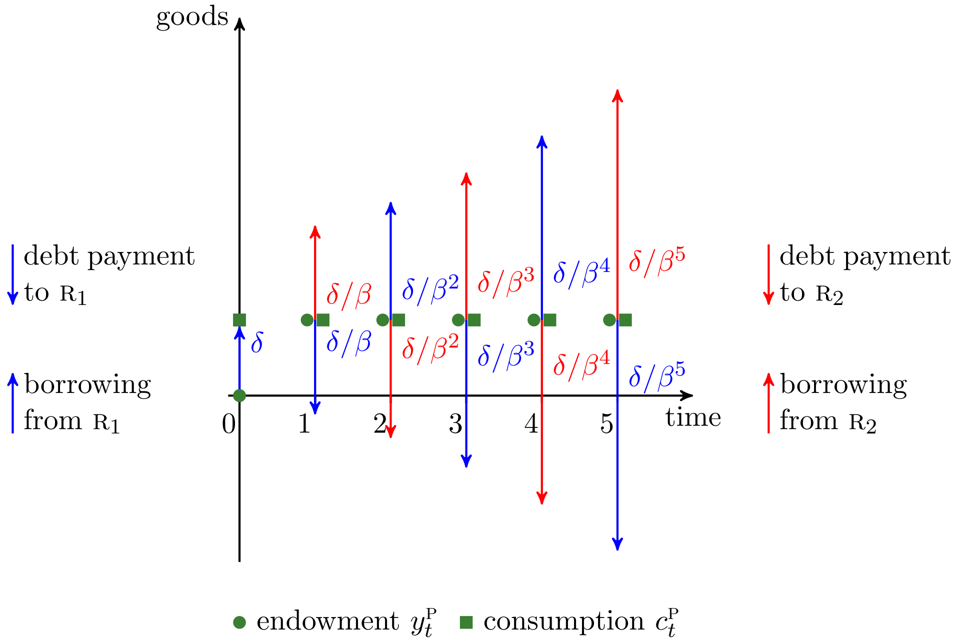


Figure 2: Country P

they are low enough to induce repayment incentives. We can easily modify their example by adding a third agent and show that the intermediation mechanism offers a different interpretation.[14] Repayment incentives depend exclusively on the roll-over property of debt limits (or, equivalently, on the presence of bubbles in the debt limits) which is independent of the level of interest rates. However, as shown by our example, the presence of bubbles in the debtor's debt limits is compatible with the supply of credit only if there is aggregate lending in the long-run. This requires sufficiently low interest rates with respect to the potential lenders' endowment growth rates. In other words, the level of interest rates also matters to induce strong lending incentives.

4.2. Borrowing Against Zero Wealth

In the competitive equilibrium described in Proposition 4.1, the *poor* country extracts surplus for its intermediation service only at the initial date and then exactly rolls over its debt consuming its endowment from date 1 onwards. In particular, debt limits bind at every period and the poor country is always on the verge of defaulting. The next example shows that our result does not hinge on these particular features. Indeed, we provide another example of a deterministic environment where the poor country has zero endowments. At first glance, one may expect that this country is irrelevant for inter-temporal exchange purposes. We show that this is not true by exhibiting a competitive equilibrium where, at every period, the country finances positive consumption by issuing debt, but never exhausts its borrowing capacity (i.e., debt limits never bind). In particular, at every period the country extracts surplus for its intermediation service and strictly prefers repaying its debt (by issuing more debt) than defaulting. In doing this, the country sustains debt because it facilitates inter-temporal trade. The reason why an agent with zero endowment matters for intertemporal trade is similar to the one that explains why money is valued in overlapping generations models like those analyzed by Samuelson (1958), Wallace (1978) or Balasko and Shell (1981).

Example 4.2. The primitives $(\beta, u(\cdot))$ are chosen such that there exists a pair $(\underline{c}, \overline{c})$ satisfying

$$0 < \underline{c} < \overline{c}, \quad \underline{c} + \overline{c} = 1 \quad \text{and} \quad \beta u'(\underline{c}) = u'(\overline{c}). \tag{4.1}$$

Let $(\underline{c}_t)_{t \geq 0}$ be a strictly decreasing sequence satisfying $\beta u'(\underline{c}_{t+1}) = u'(\underline{c}_t)$ for every $t \geq 0$. We can choose the Bernoulli function u and the initial value $\underline{c}_0$ such that[15]

$$\sum_{t \geq 0} \underline{c}_t < \underline{c}.$$

Consider the sequence $(\delta_t)_{t \geq 0}$ where $\delta_0 := 0$ and $\delta_t := \underline{c}_0 + \ldots + \underline{c}_{t-1}$ for each $t \geq 1$. Observe that

$$\delta_{t+1} = \delta_t + \underline{c}_t, \quad \text{for all } t \geq 0. \tag{4.2}$$

Moreover, the sequence $(\delta_t)_{t \geq 0}$ is strictly increasing and converges to $\delta_\infty := \sum_{t \geq 0} \underline{c}_t$.

[14] Example 4.3 presented below is in this spirit. The argument though is more involved since we look for an economy that exhibits positive risk-less interest rates. The whole exercise simplifies a lot if, as in Hellwig and Lorenzoni (2009), we look for an economy with zero risk-less interest rate.

[15] Take for instance $\underline{c}_0 < (1 - \beta)\underline{c}$ and define $u(c) := \ln(c)$ on the interval $(0, 1]$ and extend this function on $[1, \infty)$ such that the assumptions of this paper are satisfied.

There are three countries (P, R_1 and R_2) in the economy. The rich countries' endowments are defined as follows:

$$y_t^{R1} := \begin{cases} \bar{c} + \delta_{t+1} & \text{if } t \text{ is even} \\ \underline{c} - \delta_t & \text{if } t \text{ is odd} \end{cases} \quad \text{and} \quad y_t^{R2} := \begin{cases} \bar{c} + \delta_{t+1} & \text{if } t \text{ is odd} \\ \underline{c} - \delta_t & \text{if } t \text{ is even.} \end{cases}$$

Country P has no endowments, i.e., $y_t^{P} := 0$ for all $t \geq 0$.

Proposition 4.2. The economy of Example 4.2 admits a competitive equilibrium with self-enforcing debt in which country P faces positive not-too-tight debt limits $D_t^{P} = \delta_\infty$ although its natural debt limits are zero. Moreover, at every period t, country P consumes the positive amount $c_t^{P} = \underline{c}_t$ and its debt limits do not bind since $a_t^{P} = -\delta_t > -D_t^{P}$.

Proof. We first describe the equilibrium prices, debt limits and allocations.

Let the price sequence $q = (q_t)_{t\geq 1}$ be defined by $q_t := 1$ for every $t \geq 1$ (zero risk-less interest rates).

Consider the following debt limits: $D_t^{R1} = D_t^{R2} := 0$ and $D_t^{P} := \delta_\infty$. These debt limits are not-too-tight under the price sequence q since they allow for exact roll-over.

We let (c^{R_1}, a^{R_1}) be defined as follows:

$$c_t^{R1} := \begin{cases} \bar{c} & \text{if } t \text{ is even} \\ \underline{c} & \text{if } t \text{ is odd} \end{cases} \quad \text{and} \quad a_t^{R1} := \begin{cases} \delta_t & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even.} \end{cases}$$

We also let (c^{R_2}, a^{R_2}) be the plan defined by $c_0^{R2} := y_0^{R2}$, and for every $t \geq 1$,

$$c_t^{R2} := \begin{cases} \bar{c} & \text{if } t \text{ is odd} \\ \underline{c} & \text{if } t \text{ is even} \end{cases} \quad \text{and} \quad a_t^{R2} := \begin{cases} \delta_t & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

At every even date $2t$, the rich country R_1 optimally saves the amount δ_{2t+1} in order to trade the time-varying endowments $(y_{2t}^{R1}, y_{2t+1}^{R1}) = (\bar{c} + \delta_{2t+1}, \underline{c} - \delta_{2t+1})$ in exchange of the less fluctuating consumption $(c_{2t}^{R1}, c_{2t+1}^{R1}) = (\bar{c}, \underline{c})$. The rich country R_2 follows the same strategy at odd dates $2t+1$: it saves the amount δ_{2t+2} to trade the endowments $(y_{2t+1}^{R2}, y_{2t+2}^{R2}) = (\bar{c}+\delta_{2t+2}, \underline{c}-\delta_{2t+2})$ in exchange of the consumption $(c_{2t+1}^{R2}, c_{2t+2}^{R2}) = (\bar{c}, \underline{c})$.

Finally, we define (c^{P}, a^{P}) as follows: $c_t^{P} := \underline{c}_t$ for every $t \geq 0$ and $a_t^{P} = -\delta_t$ for every $t \geq 1$. The poor country borrows the amount δ_{t+1} at date t to repay the debt δ_t and finance the consumption $\underline{c}_t$.

We next show that these allocations are indeed optimal. Observe that (c^{P}, a^{P}) is budget feasible (this follows from Equation (4.2)) and satisfies the Euler equations (2.3) by construction of the sequence $(\underline{c}_t)_{t\geq 0}$. Moreover,

$$\lim_{t\to\infty} \beta^t u'(\underline{c}_t)[-\delta_t + \delta_\infty] = u'(\underline{c}_0) \lim_{t\to\infty} [-\delta_t + \delta_\infty] = 0,$$

so the transversality condition (2.4) is satisfied. This proves that (c^{P}, a^{P}) is optimal. The plan (c^{R_k}, a^{R_k}) is also optimal since it is budget feasible (with equality) and satisfies the Euler equations (2.3) (this follows from equation (4.1)) and the transversality condition (2.4) (debt limits bind infinitely many times).

To conclude the proof simply observe that all markets clear by construction. □

4.3. Positive Risk-less Interest Rates

In the previous examples, we had zero effective equilibrium risk-less interest rates (defined as the difference between the risk-less interest rates and the aggregate growth rates). We show below that our analysis is not driven by this property. More precisely, we study a stochastic economy and exhibit a competitive equilibrium where a country borrows in excess of its natural debt limits, but the risk-less interest rates are strictly positive and higher than the aggregate growth rates at every period. However, in the long run, the risk-less interest rates converge to the aggregate growth rates. This necessary condition is related to Proposition 3.1. Since the aggregate endowment is uniformly bounded from above (no aggregate risk), if the risk-less interest rates were bounded from below by some positive value, then the aggregate wealth of the economy would be finite and Proposition 3.1 would imply that debt limits are zero.

The essential lesson is that when self-enforcing debt limits allow for exact roll over, this effectively puts a lower bound on their growth rates (they should exactly coincide with interest rates). As long as the endowment process of an agent grows at a rate that is lower than the interest rates, then it is possible for the debt limits to surpass the agent's wealth. However, debt can grow as fast as interest rates only to the extend there is someone to provide credit at infinite (in present value terms). This necessarily requires that interest rates are low enough to imply an infinite wealth for some creditors.

Example 4.3. The primitives $(\beta, u(\cdot))$ together with some probability $\pi \in (0,1)$ are chosen such that there exists a pair $(\underline{c}, \overline{c})$ satisfying

$$0 < \underline{c} < \overline{c}, \quad \underline{c} + \overline{c} = 1 \quad \text{and} \quad 1 - \beta(1-\pi) = \beta\pi \frac{u'(\underline{c})}{u'(\overline{c})}.$$

We let (q^c, q^{nc}) be defined by

$$q^c := \beta\pi \frac{u'(\underline{c})}{u'(\overline{c})} \quad \text{and} \quad q^{nc} := \beta(1-\pi).$$

Observe that $q^c + q^{nc} = 1$. We fix some arbitrary number $\delta > 0$ such that $q^c\delta < \underline{c}$.

We let $\underline{c}_0 := \underline{c}$ and $\overline{c}_0 := \overline{c}$. Since $\underline{c}_0 < \overline{c}_0$, there exist two numbers $\underline{c}_1$ and $\overline{c}_1$ such that

$$\underline{c}_0 < \underline{c}_1 < \overline{c}_1 < \overline{c}_0 \quad \text{and} \quad \underline{c}_1 + \overline{c}_1 = 1.$$

Let

$$q_1^c := \beta\pi \frac{u'(\underline{c}_1)}{u'(\overline{c}_0)} \quad \text{and} \quad q_1 := q_1^c + q^{nc}.$$

Since $q_1 < 1$, we can apply Lemma C.1 (see Appendix C) to deduce the existence of a strictly increasing sequence $(q_t)_{t\geq 1}$ such that

$$\lim_{t\to\infty} q_t = 1 \quad \text{and} \quad 0 < p_\infty := \lim_{t\to\infty} p_t \quad \text{where} \quad p_t := q_1 \ldots q_t.$$

We can construct a strictly decreasing sequence $(\underline{c}_t)_{t\geq 1}$ and a strictly increasing sequence $(\overline{c}_t)_{t\geq 1}$ such that, for every $t \geq 1$

$$q_t^c := q_t - q^{nc} = \beta\pi \frac{u'(\underline{c}_t)}{u'(\overline{c}_{t-1})} \quad \text{together with} \quad \overline{c}_t + \underline{c}_t = 1 \tag{4.3}$$

and

$$\lim_{t\to\infty}(\underline{c}_t,\overline{c}_t)=(\underline{c},\overline{c}).$$

Observe that $(q^c_t)_{t\geq 1}$ is strictly increasing and converges to q^c.

Let also $\delta_0 := \delta p_\infty$. Since $p_\infty < p_1 = q_1 < 1$, we get that $\delta_0 < \delta$. Consider the sequence $(\delta_t)_{t\geq 1}$ defined recursively by

$$q_{t+1}\delta_{t+1} := \delta_t.$$

Equivalently, we have $\delta_{t+1} = \delta(p_\infty/p_{t+1})$. Therefore, the sequence $(\delta_t)_{t\geq 1}$ is strictly increasing and converges to δ.

For each $t \geq 0$, we pose[16]

$$\overline{y}_t = \overline{c}_t + q^c_{t+1}\delta_{t+1} \quad \text{and} \quad \underline{y}_t = \underline{c}_t - q^c_{t+1}\delta_{t+1}.$$

In each period t, one of the rich countries receives the high endowment $\overline{y}_t$ and the other one receives the low endowment $\underline{y}_t$. The rich countries switch endowment with probability π from one period to the next. Formally, uncertainty is captured by the Markov process s_t with state space $\{z_1, z_2\}$ and symmetric transition probabilities

$$\pi := \text{Prob}(s_{t+1} = z_1 | s_t = z_2) = \text{Prob}(s_{t+1} = z_2 | s_t = z_1).$$

The event s^t corresponds to the sequence $(s_0, s_1, \ldots, s_t)$ and the endowments $y^{\text{R}_k}(s^t)$ only depend on the current realization of s_t, with

$$y^{\text{R}_k}(s^t) := \begin{cases} \overline{y}_t, & \text{if } s_t = z_k \\ \underline{y}_t, & \text{otherwise.} \end{cases}$$

Observe that the sequence $(\underline{y}_t)_{t\geq 0}$ is strictly decreasing and converges to $\underline{y} := \underline{c} - q^c\delta$ while the sequence $(\overline{y}_t)_{t\geq 0}$ is strictly increasing and converges to $\overline{y} := \overline{c} + q^c\delta$.

Country's P endowment is defined by $y^{\text{P}}(s^0) := \overline{y}$ while for each event $s^t \succ s^0$ we pose

$$y^{\text{P}}(s^t) := \begin{cases} y^{\text{P}}(s^{t-1}), & \text{if } s_t = s_{t-1} \\ \gamma y^{\text{P}}(s^{t-1}), & \text{otherwise} \end{cases}$$

where $\gamma \in (0,1)$ is chosen such that

$$\frac{u'(\gamma y^{\text{P}}(s^t))}{u'(y^{\text{P}}(s^t))} \leq \frac{u'(\underline{c}_1)}{u'(\overline{c}_0)}, \quad \text{for all } s^t. \tag{4.4}$$

Remark 4.1. If we let u be such that $u(c) := \ln(c)$ in the interval $(0, \overline{y}]$ while extending this function on $[\overline{y}, \infty)$ such that the assumptions of this paper are satisfied, then the inequality (4.4) is true for any γ in the interval $[\underline{c}_1/\overline{c}_1, 1)$.

To focus on an equilibrium displaying Markovian properties, we assume that the economy begins in state $s^0 = s_0 = z_1$ (the rich country R_1 has the highest endowment) and the initial asset positions are

$$a^{\text{P}}(s^0) := -\delta_0, \quad a^{\text{R}_1}(s^0) := 0 \quad \text{and} \quad a^{\text{R}_2}(s^0) := \delta_0.$$

[16] Recall that the sequence $(\underline{c}_t)_{t\geq 0}$ is strictly decreasing and converges to $\underline{c}$ while the sequence $(q_t\delta_t)_{t\geq 1}$ is strictly increasing and converges to $q^c\delta$. This implies that $\underline{y}_t \geq \lim_{\tau\to\infty}[\underline{c}_\tau - q^c_{\tau+1}\delta_{\tau+1}] = \underline{c} - q^c\delta > 0$ for all $t \geq 0$.

Proposition 4.3. The economy of Example 4.3 admits a competitive equilibrium with self-enforcing debt in which the risk-less interest rates are strictly positive, and the country P faces positive not-too-tight debt limits $D^{\mathrm{P}}(s^t) = \delta_t$ although its natural debt limits $W^{\mathrm{P}}(s^t)$ are finite. Moreover, for almost every path, the debt limits eventually exceed the country's natural debt limits. Formally, for any infinite path $(s_t)_{t\geq 0}$ displaying infinitely many switches, we have

$$\lim_{t\to\infty} D^{\mathrm{P}}(s^t) = \delta > 0 = \lim_{t\to\infty} W^{\mathrm{P}}(s^t). \tag{4.5}$$

Remark 4.2. The strict inequality (4.5) implies that for every path $(s_t)_{t\geq 0}$ displaying infinitely many switches, there exists $T \geq 0$ large enough such that, for every $t \geq T$, the not-too-tight debt limit $D^{\mathrm{P}}(s^t)$ exceeds the natural debt limit $W^p(s^t)$. Since the set of paths displaying infinitely many switches has probability one, we get that, for almost every path, the debt limits eventually exceed the country's natural debt limits.

Proof. We first describe the equilibrium prices, debt limits and allocations.

Let the price process $(q(s^t))_{s^t \succ s^0}$ be as follows:

$$q(s^t) := \begin{cases} q_t^c, & \text{if } s_t \neq s_{t-1} \\ q^{nc}, & \text{otherwise.} \end{cases}$$

Since by construction $q_t^c + q^{nc} = q_t < 1$ and $(q_t)_{t\geq 1}$ converges to 1, the risk-less interest rate is time dependent, positive and converges to zero.

Consider the following debt limits:

$$D^{\mathrm{R}_1}(s^t) = D^{\mathrm{R}_2}(s^t) := 0 \quad \text{and} \quad D^{\mathrm{P}}(s^t) := \delta_t.$$

These debt limits are not-too-tight since they allow for exact roll-over (i.e., condition (ER) holds true).

Let $(c^{\mathrm{R}_k}, a^{\mathrm{R}_k})$ be defined as follows:

$$c^{\mathrm{R}_k}(s^t) := \begin{cases} \bar{c}_t, & \text{if } s_t = z_k \\ \underline{c}_t, & \text{otherwise} \end{cases} \quad \text{and} \quad a^{\mathrm{R}_k}(s^{t+1}) := \begin{cases} \delta_{t+1}, & \text{if } s_{t+1} \neq z_k \\ 0, & \text{otherwise.} \end{cases}$$

Each rich country saves to transfer resources against the low income shock. They do not issue debt since they are credit-constrained.

The allocation $(c^{\mathrm{P}}, a^{\mathrm{P}})$ is defined as follows: $c^{\mathrm{P}}(s^t) := y^{\mathrm{P}}(s^t)$ and $a^{\mathrm{P}}(s^t) := -\delta_t$ for every event s^t. At the initial period, the poor country repays the inherited debt δ_0 by issuing the non-contingent debt δ_1. At any subsequent period t, instead of repaying its debt δ_t, it issues more debt (equal to δ_{t+1}). That is, the poor country is rolling over its debt forever.

We next show that these allocations are indeed optimal. Observe that $(c^{\mathrm{P}}, a^{\mathrm{P}})$ is budget feasible (with equality) and the transversality condition (2.4) is satisfied since debt limits always bind. Euler equations (2.3) are also satisfied. Indeed, condition (4.4) implies that

$$\beta\pi \frac{u'(\gamma y^{\mathrm{P}}(s^t))}{u'(y^{\mathrm{P}}(s^t))} \leq \beta\pi \frac{u'(\underline{c}_1)}{u'(\bar{c}_0)} \leq \beta\pi \frac{u'(\underline{c}_{t+1})}{u'(\bar{c}_t)} = q_t^c,$$

where the second inequality follows from the fact that $(\underline{c}_t)_{t\geq 0}$ is strictly decreasing and $(\bar{c}_t)_{t\geq 0}$ is strictly increasing. Most importantly, we have that

$$\sum_{s^{t+1}\succ s^t} q(s^{t+1}) y^{\mathrm{P}}(s^{t+1}) \leq (q_t^c\gamma + q^{nc}) y^{\mathrm{P}}(s^t) \leq (q^c\gamma + q^{nc}) y^{\mathrm{P}}(s^t).$$

Since $\gamma \in (0,1)$ and $q^c + q^{nc} = 1$, this implies that country P's wealth (and therefore any of its natural debt limits) is finite at any period. Indeed, if we let $\chi := (q^c\gamma + q^{nc})$, we can show that $W^{\mathrm{P}}(s^t) \leq y^{\mathrm{P}}(s^t)\sum_{t\geq 0}\chi^t = y^{\mathrm{P}}(s^t)/(1-\chi)$. Observe that $y^{\mathrm{P}}(s^t) = \gamma^\tau \bar{y}$ where τ is the number of switches in the finite path s^t. This implies that

$$\lim_{t\to\infty} W^p(s^t) \leq \frac{1}{1-\chi}\lim_{t\to\infty} y^{\mathrm{P}}(s^t) = 0$$

along any infinite path $(s_0, s_1, \ldots)$ displaying infinitely many switches.

The plan $(c^{\mathrm{R}_k}, a^{\mathrm{R}_k})$ is also optimal. Indeed, it is budget feasible (with equality) and satisfies the Euler equations (2.3) (this follows from the definition of asset prices, i.e., condition (4.3)). Moreover, the transversality condition (2.4) is satisfied because the equilibrium consumption is bounded from below by $\underline{c}$ and the asset position is bounded from above by δ. Formally, we have

$$\sum_{s^t\in S^t} \beta^t\pi(s^t)u'(c^{\mathrm{R}_k}(s^t))a^{\mathrm{R}_k}(s^t) \leq \beta^t u'(\underline{c})\delta_t \xrightarrow[t\to\infty]{} 0.$$

To conclude the proof simply observe that all markets clear by construction. □

5. Conclusion

In models without commitment the creditworthiness of an agent is not necessarily limited by his ability to repay out of his future resources. Indeed, we show, by means of examples, that an agent can sustain positive levels of debt by acting as a financial intermediary that alleviates the incentive compatibility constraints of some other agents. Since this financial service is not related to the agent's wealth, the borrowing capacity can exceed an agent's natural ability to repay represented by the present value of his future endowments. This is in contrast with the standard results of the full commitment literature. Moreover, our examples show that the no-borrowing result of Bulow and Rogoff (1989) hinges on the restrictive assumption that debt is bounded by the natural debt limits. They also clarify that the level of interest rates is important from the creditors' perspective: they should be low enough to provide strong lending incentives. Indeed, repayment incentives are guaranteed by the bubble property of debt limits independently of whether interest rates imply that the debtors' wealth is finite or infinite. However, the bubble property of debt limits requires asymptotic borrowing in present value terms. This is consistent with the asymptotic supply of credit only if interest rates are lower than the endowment growth rates of some potential lenders.

Appendix A. Market Transversality Condition

In this section we show that the *market transversality condition* is satisfied when the present value of an agent's optimal consumption is finite. Since we are exclusively concerned with the single-agent problem, we simplify notation by dropping the superscript i.

If c is a strictly positive consumption sequence (in the sense that $c(s^t) > 0$ for every event s^t), then the agent's marginal rate of substitution at event $s^t \succ s^0$ is denoted by

$$\mathrm{MRS}(c|s^t) := \beta\pi(s^t|\sigma(s^t))u'(c(s^t))/u'(c(\sigma(s^t))).$$

Lemma A.1. Let $b \in \mathbb{R}$ denote an initial claim at some arbitrary event s^τ and let (c, a) be optimal in $B(D, b|s^\tau)$, where D is a process of not-too-tight debt limits. If c has finite present value, i.e., $\mathrm{PV}(c|s^\tau) < \infty$, then the following *market transversality condition* is satisfied

$$\lim_{t \to \infty} \sum_{s^t \in S^t(s^\tau)} p(s^t)[a(s^t) + D(s^t)] = 0. \tag{A.1}$$

Proof. It suffices to show that for every $s^t \succ s^\tau$, we have $a(s^t) + D(s^t) \leq \mathrm{PV}(c|s^t)$. Assume, by way of contradiction, that there exists $s^t \succ s^\tau$ such that

$$a(s^t) + D(s^t) > \mathrm{PV}(c|s^t). \tag{A.2}$$

Let $\theta(s^r) := \mathrm{PV}(c|s^r)$ for every $s^r \succeq s^t$. By construction we have

$$c(s^r) + \sum_{s^{r+1} \succ s^r} q(s^{r+1})\theta(s^{r+1}) = \theta(s^r), \quad \text{for all } s^r \succeq s^t. \tag{A.3}$$

Moreover, it is easy to see that[17]

$$D(s^r) \leq y(s^r) + \sum_{s^{r+1} \succ s^r} q(s^{r+1})D(s^{r+1}), \quad \text{for all } s^r \succeq s^t. \tag{A.4}$$

Posing $\widetilde{a} := \theta - D$, it follows hat

$$c(s^r) + \sum_{s^{r+1} \succ s^r} q(s^{r+1})\widetilde{a}(s^{r+1}) = \widetilde{a}(s^r), \quad \text{for all } s^r \succeq s^t. \tag{A.5}$$

Since $\widetilde{a}(s^r) \geq -D(s^r)$, we get that $(c, \widetilde{a}) \in B(D, \widetilde{a}(s^t)|s^t))$. The bond holdings $\widetilde{a}$ finance the consumption c when the initial claim is $\widetilde{a}(s^t)$. Following Equation (A.2) we have $a(s^t) > \widetilde{a}(s^t)$. This contradicts the optimality of a.[18] Indeed, we can increase the consumption at the predecessor event $\sigma(s^t)$ by replacing $(a(s^r))_{s^r \in \Sigma(s^t)}$ with $(\widetilde{a}(s^r))_{s^r \in \Sigma(s^t)}$. □

Appendix B. Proof of Proposition 3.2

Before presenting the details of the proof, we introduce the following notations. Fix two dates $t > \tau \geq 0$. Recall that $S^t(s^\tau)$ denotes the set of all date-t events that are successors of s^τ, i.e.,

$$S^t(s^\tau) := \{s^t \in S^t \, : \, s^t \succ s^\tau\}.$$

If E^τ is a subset of S^τ, then we let $S^t(E^\tau)$ be the set of date-t events that are successors of the events in E^τ, i.e.,

$$S^t(E^\tau) := \bigcup_{s^\tau \in E^\tau} S^t(s^\tau).$$

[17] Indeed, Let $b = -y(s^r) - \sum_{s^{r+1} \succ s^r} q(s^{r+1})D(s^{r+1})$ and let (c, a) be optimal in $B(D, b|s^r)$. It is straightforward to see that we must have $c(s^r) = 0$ and $a(s^{r+1}) = -D(s^{r+1})$ implying that $U(c|s^r) = u(0) + \beta V(s^{r+1})$. We know that $V(s^r) = J(0, 0|s^r) = U(\hat{c}|s^r)$ for a consumption process $\hat{c}$ satisfying participation constraints at all successor events, i.e., $U(\hat{c}(s^{r+1})) \geq V(s^{r+1})$. In particular, we have $V(s^r) \geq u(\hat{c}(s^r)) + \beta V(s^{r+1}) \geq U(c|s^r)$ which implies that $b \leq -D(s^r)$.

[18] By the Principle of Optimality, we must have $(c, a) \in B(D, a(s^t)|s^t)$.

Proof of Proposition 3.2. Assume that one of the agents, say agent i_1, has finite wealth at some event s^τ, i.e., $W^{i_1}(s^\tau) = \mathrm{PV}(e^{i_1}|s^\tau) < \infty$. Replacing Σ by $\Sigma(s^\tau)$ if necessary, we can assume without any loss of generality, that $s^\tau = s^0$. Assume, by way of contradiction, that $D^{i_1}(s^0) > 0$.

Observe that for every $t > 0$, we have

$$\mathrm{PV}^{t-1}(c^i|s^0) + \sum_{s^t \in S^t} p(s^t)[a^i(s^t) + D^i(s^t)] = \mathrm{PV}^{t-1}(e^i|s^0) + a^i(s^0) + D^i(s^0) \tag{B.1}$$

where $\mathrm{PV}^\tau(\cdot|s^0)$ is the present value functional restricted to the subtree stopped at period τ. Since agent i_1 has finite wealth, we can deduce from (B.1) that $\mathrm{PV}(c^{i_1}|s^0)$ is also finite. Applying Lemma A.1, we get that the *market transversality condition*

$$\lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t)[a^{i_1}(s^t) + D^{i_1}(s^t)] = 0 \tag{B.2}$$

is satisfied. In particular, we have that

$$a^{i_1}(s^0) + D^{i_1}(s^0) = \mathrm{PV}(c^{i_1} - e^{i_1}|s^0).$$

Replacing the event tree Σ by $\Sigma(s^t)$, we can show that

$$a^{i_1}(s^t) = \mathrm{PV}(c^{i_1} - e^{i_1}|s^t) - D^{i_1}(s^t), \quad \text{for all } s^t \in \Sigma. \tag{B.3}$$

We let ε^{i_1} be the process defined by

$$\varepsilon^{i_1}(s^t) := \mathrm{PV}(c^{i_1} + e^{i_1}|s^t).$$

Observe that $a^{i_1} \le \varepsilon^{i_1} - D^{i_1}$. Moreover, for every event s^t, we have $c^{i_1}(s^t) + e^{i_1}(s^t) \ge 0$. This implies that

$$p(s^t)\varepsilon^{i_1}(s^t) \ge \sum_{s^{t+1} \succ s^t} p(s^{t+1})\varepsilon^{i_1}(s^{t+1}).$$

Recall that

$$\lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t)\varepsilon^{i_1}(s^t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t)D^{i_1}(s^t) = D^{i_1}(s^0) > 0$$

where the first equality follows from the property that $\mathrm{PV}(e^{i_1}|s^0)$ and $\mathrm{PV}(c^{i_1}|s^0)$ are both finite, and the second one from the roll-over property of D^{i_1}. The above equalities imply that there must exist some date $\tau \ge 0$ such that

$$\sum_{s^\tau \in S^\tau} p(s^\tau)\varepsilon^{i_1}(s^\tau) < \sum_{s^\tau \in S^\tau} p(s^\tau)D^{i_1}(s^\tau).$$

In particular, there must exist some event $s^\tau \in S^\tau$ such that $\varepsilon^{i_1}(s^\tau) < D^{i_1}(s^\tau)$. We let $E^\tau := \{s^\tau\}$ and define $E^{\tau+1} \subseteq S^{\tau+1}(s^\tau)$ as the set of successors of s^τ for which we have $\varepsilon^{i_1} < D^{i_1}$, i.e.,

$$E^{\tau+1} := \{s^{\tau+1} \in S^{\tau+1}(s^\tau) \ : \ \varepsilon^{i_1}(s^{\tau+1}) < D^{i_1}(s^{\tau+1})\}.$$

Observe that

$$\begin{aligned}
\sum_{s^{\tau+1}\in S^{\tau+1}(s^\tau)\setminus E^{\tau+1}} p(s^{\tau+1})D^{i_1}(s^{\tau+1}) &\leq \sum_{s^{\tau+1}\in S^{\tau+1}(s^\tau)\setminus E^{\tau+1}} p(s^{\tau+1})\varepsilon^{i_1}(s^{\tau+1}) \\
&\leq \sum_{s^{\tau+1}\in S^{\tau+1}(s^\tau)} p(s^{\tau+1})\varepsilon^{i_1}(s^{\tau+1}) \\
&\leq p(s^\tau)\varepsilon^{i_1}(s^\tau).
\end{aligned}$$

Combining the above inequality with the exact roll-over property of D^{i_1}, we deduce that

$$\sum_{s^{\tau+1}\in E^{\tau+1}} p(s^{\tau+1})D^{i_1}(s^{\tau+1}) \geq p(s^\tau)D^{i_1}(s^\tau) - p(s^\tau)\varepsilon^{i_1}(s^\tau) > 0 \tag{B.4}$$

which implies in particular that $E^{\tau+1} \neq \emptyset$.

We define $E^{\tau+2} \subseteq S^{\tau+2}(E^{\tau+1})$ as the set of all successors of the events in $E^{\tau+1}$ for which we have $\varepsilon^{i_1} < D^{i_1}$, i.e.,

$$E^{\tau+2} := \{s^{\tau+2} \in S^{\tau+2}(E^{\tau+1}) \ : \ \varepsilon^{i_1}(s^{\tau+2}) < D^{i_1}(s^{\tau+2})\}.$$

Using the exact roll-over property of D^{i_1} and the definition of $E^{\tau+1}$, we have

$$\begin{aligned}
p(s^\tau)D^{i_1}(s^\tau) &= \sum_{s^{\tau+2}\in S^{\tau+2}(s^\tau)} p(s^{\tau+2})D^{i_1}(s^{\tau+2}) \\
&= \sum_{s^{\tau+2}\in S^{\tau+2}(E^{\tau+1})} p(s^{\tau+2})D^{i_1}(s^{\tau+2}) + \sum_{s^{\tau+1}\in S^{\tau+1}(s^\tau)\setminus E^{\tau+1}} p(s^{\tau+1})D^{i_1}(s^{\tau+1}) \\
&\leq \sum_{s^{\tau+2}\in S^{\tau+2}(E^{\tau+1})} p(s^{\tau+2})D^{i_1}(s^{\tau+2}) + \sum_{s^{\tau+1}\in S^{\tau+1}(s^\tau)\setminus E^{\tau+1}} p(s^{\tau+1})\varepsilon^{i_1}(s^{\tau+1})
\end{aligned}$$

Observe that we can decompose the set $S^{\tau+2}(E^{\tau+1})$ as follows

$$S^{\tau+2}(E^{\tau+1}) = E^{\tau+2} \cup \left[S^{\tau+2}(E^{\tau+1}) \setminus E^{\tau+2}\right].$$

Using the definition of $E^{\tau+2}$, we get that

$$\begin{aligned}
\sum_{s^{\tau+2}\in S^{\tau+2}(E^{\tau+1})} p(s^{\tau+2})D^{i_1}(s^{\tau+2}) &\leq \sum_{s^{\tau+2}\in E^{\tau+2}} p(s^{\tau+2})D^{i_1}(s^{\tau+2}) \\
&\quad + \sum_{s^{\tau+2}\in S^{\tau+2}(E^{\tau+1})\setminus E^{\tau+2}} p(s^{\tau+2})\varepsilon^{i_1}(s^{\tau+2}) \\
&\leq \sum_{s^{\tau+2}\in E^{\tau+2}} p(s^{\tau+2})D^{i_1}(s^{\tau+2}) \\
&\quad + \sum_{s^{\tau+1}\in E^{\tau+1}} p(s^{\tau+1})\varepsilon^{i_1}(s^{\tau+1}).
\end{aligned}$$

Combining the above inequalities, we deduce that

$$p(s^\tau)D^{i_1}(s^\tau) \leq \sum_{s^{\tau+2}\in E^{\tau+2}} p(s^{\tau+2})D^{i_1}(s^{\tau+2}) + \sum_{s^{\tau+1}\in S^{\tau+1}(s^\tau)} p(s^{\tau+1})\varepsilon^{i_1}(s^{\tau+1})$$

and therefore, we get

$$\sum_{s^{\tau+2}\in E^{\tau+2}} p(s^{\tau+2})D^{i_1}(s^{\tau+2}) \geq p(s^\tau)D^{i_1}(s^\tau) - p(s^\tau)\varepsilon^{i_1}(s^\tau) > 0. \tag{B.5}$$

The above strict inequality is the counterpart of Equation (B.4) for period $\tau+2$. In particular, it implies that $E^{\tau+2} \neq \emptyset$.

By induction, we can construct a sequence $(E^{\tau+n})_{n\geq 0}$ where $E^{\tau+n} \subseteq S^{\tau+n}(E^{\tau+n-1})$ is defined by

$$\forall n > 0, \quad E^{\tau+n} := \{s^{\tau+n} \in S^{\tau+n}(E^{\tau+n-1}) \ : \ \varepsilon^{i_1}(s^{\tau+n}) < D^{i_1}(s^{\tau+n})\}.$$

Following the same arguments as above, we can use the exact roll-over property of D^{i_1} to show that

$$\sum_{s^{\tau+n}\in E^{\tau+n}} p(s^{\tau+n})D^{i_1}(s^{\tau+n}) \geq p(s^\tau)D^{i_1}(s^\tau) - p(s^\tau)\varepsilon^{i_1}(s^\tau). \tag{B.6}$$

Observe that for every $n \geq 0$, the definition of $E^{\tau+n}$ implies that

$$\forall s^{\tau+n} \in E^{\tau+n}, \quad a^{i_1}(s^{\tau+n}) \leq \varepsilon^{i_1}(s^{\tau+n}) - D^{i_1}(s^{\tau+n}) < 0.$$

Since for any $k \in \{1, \ldots, n\}$ we have $\sigma^k(E^{\tau+n}) \subseteq E^{\tau+n-k}$, we get that

$$\forall k \in \{1, \ldots, n\}, \quad a^{i_1}(\sigma^k(s^{\tau+n})) < 0.$$

Market clearing implies that agent i_2 is saving strictly positive amounts along the path

$$(s^\tau = \sigma^n(s^{\tau+n}), \sigma^{n-1}(s^{\tau+n}), \ldots, \sigma(s^{\tau+n}), s^{\tau+n}).$$

This implies that the corresponding Euler equations are satisfied with equality:

$$\forall n \geq 0, \quad \forall s^{\tau+n} \in E^{\tau+n}, \quad \frac{p(s^{\tau+n})}{p(s^\tau)} = \beta^n \pi(s^{\tau+n}|s^\tau)\frac{u'(c^{i_2}(s^{\tau+n}))}{u'(c^{i_2}(s^\tau))}.$$

If we pose $\chi := \beta^\tau \pi(s^\tau)u'(c^{i_2}(s^\tau))/p(s^\tau)$, we deduce that

$$\begin{aligned}
\chi \sum_{s^{\tau+n}\in E^{\tau+n}} p(s^{\tau+n})a^{i_2}(s^{\tau+n}) \quad &= \sum_{s^{\tau+n}\in E^{\tau+n}} \beta^{\tau+n}\pi(s^{\tau+n})u'(c^{i_2}(s^{\tau+n}))a^{i_2}(s^{\tau+n}) \\
&\leq \sum_{s^{\tau+n}\in S^{\tau+n}} \beta^{\tau+n}\pi(s^{\tau+n})u'(c^{i_2}(s^{\tau+n}))a^{i_2}(s^{\tau+n}).
\end{aligned}$$

Agent i_2's individual transversality condition implies that

$$\lim_{n\to\infty} \sum_{s^{\tau+n}\in E^{\tau+n}} p(s^{\tau+n})a^{i_2}(s^{\tau+n}) = 0.$$

Since markets clear, we deduce that

$$\lim_{n\to\infty} \sum_{s^{\tau+n}\in E^{\tau+n}} p(s^{\tau+n})a^{i_1}(s^{\tau+n}) = 0.$$

Combining with (B.2), we get that

$$\lim_{n\to\infty} \sum_{s^{\tau+n}\in E^{\tau+n}} p(s^{\tau+n})D^{i_1}(s^{\tau+n}) = 0.$$

This contradicts (B.6) since we have

$$\sum_{s^{\tau+n}\in E^{\tau+n}} p(s^{\tau+n})D^{i_1}(s^{\tau+n}) \geq p(s^\tau)D^{i_1}(s^\tau) - p(s^\tau)\varepsilon^{i_1}(s^\tau) > 0$$

where the last inequality follows from the definition of s^τ. □

We can adapt the arguments of the proof of Proposition 3.2 to prove the following result.

Proposition B.1. If an agent with finite wealth sustains debt, then there are least two other agents with infinite wealth.

Appendix C. Technical Result

We here present the proof of the simple technical result we use in Example 4.3.

Lemma C.1. For any $q_1 \in (0,1)$, there exists a strictly increasing sequence $(q_t)_{t\geq 1}$ of positive numbers converging to 1 such that the sequence $(p_t)_{t\geq 1}$ defined by

$$p_t := q_1 \dots q_t, \quad \text{for all } t \geq 1$$

converges to some positive number $p_\infty \in (0, q_1)$.

Observe that the sequence $(p_t)_{t\geq 1}$ is strictly decreasing and

$$\sum_{t=1}^{\infty} q_1 \dots q_t = \sum_{t=1}^{\infty} p_t = \infty.$$

Proof of Lemma C.1. Fix an arbitrary $\alpha > 1$. Observe that

$$\sum_{i=1}^{\infty} i^{-\alpha} < \infty.$$

Let $(\kappa_t)_{t\geq 1}$ be the sequence defined by

$$\kappa_t := \frac{1}{\exp\{t^{-\alpha}\}}.$$

For every $t \geq 1$, we have $\kappa_t \in (0,1)$, $\kappa_t \leq \kappa_{t+1}$ and $\lim_{t\to\infty} \kappa_t = 1$. Choose $\tau \geq 1$ large enough such that $\kappa_{\tau+2} > q_1$ and let $(q_t)_{t\geq 1}$ be defined by

$$\forall t \geq 2, \quad q_t := \kappa_{\tau+t}.$$

For every $t \geq 1$, we have $q_t \in (0,1)$, $q_t \leq q_{t+1}$ and $\lim_{t\to\infty} q_t = 1$. Observe moreover that

$$\lim_{t\to\infty} q_1 \dots q_t = \lim_{t\to\infty} \frac{q_1}{\exp\left\{\sum_{i=\tau+2}^{\tau+t} i^{-\alpha}\right\}} = \frac{q_1}{\exp\left\{\sum_{i=\tau+2}^{\infty} i^{-\alpha}\right\}} > 0.$$

□

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