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ON THE OUTER AUTOMORPHISM GROUPS OF FINITELY GENERATED, RESIDUALLY FINITE GROUPS

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ABSTRACT. Bumagin–Wise posed the question of whether every countable group can be realised as the outer automorphism group of a finitely generated, residually finite group. We give a partial answer to this problem for recursively presentable groups.

1. INTRODUCTION

Every group can be realised as the outer automorphism group of some group [Mat89]. One can ask what restrictions can be placed on the groups involved. Notably, Bumagin–Wise proved that every countable group Q can be realised as the outer automorphism group of a finitely generated group G_Q [BW05]. Several other authors have achieved results in a similar vein (see, for example, [Koj88], [GP00], [DGG01], [BG03], [FM05], [Min09]).

To prove their result, Bumagin–Wise construct G_Q as the kernel of a short exact sequence using a version of a Rips’ construction [Rip82]. Their proof also shows that if Q is finitely presented then G_Q can be taken to be residually finite. They then pose the question: can every countable group Q be realised as the outer automorphism group of a finitely generated, residually finite group G_Q ?

In this paper we give a partial answer to this question of Bumagin–Wise. Our proof is based upon the construction of Bumagin–Wise and utilises an embedding of Sapir [Sap14].

Theorem A. *If Q is a finitely generated, recursively presented group then either Q or $Q \times C_2$ can be realised as the outer automorphism group of a finitely generated, residually finite group G_Q .*

This theorem admits a possible improvement: Osin asked if every finitely generated, recursively presentable group can be embedded as a malnormal subgroup of a finitely presented group, and a positive answer to this question would allow us to dispense with the $Q \times C_2$ possibility. Sapir has recently stated that his embedding yields such a positive solution.

Theorem B. *Suppose every finitely generated, recursively presented group Q can be embedded as a malnormal subgroup of a finitely presented group H_Q . Then every finitely generated, recursively presented group Q can be realised as the outer automorphism group of a finitely generated, residually finite group G_Q .*

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Outline of the paper. In Section 2 we prove a technical theorem, Theorem 2.6, which classifies the outer automorphism group of mapping tori $H_\phi = H \rtimes_\phi \mathbb{Z}$ where H has trivial centre and has no epimorphisms onto \mathbb{Z} . In Section 3 we use this technical result to obtain a way of “grabbing” a finitely generated subgroup of $\text{Out}(H)$, which is applied to prove our main theorems, Theorems A and B.

2. THE OUTER AUTOMORPHISM GROUPS OF MAPPING TORI

For $\phi \in \text{Aut}(H)$ an automorphism of H we shall write $H_\phi = H \rtimes_\phi \mathbb{Z}$ for the mapping torus $\langle H, t; tht^{-1} = \phi(h), h \in H \rangle$ associated to ϕ . In this section we prove Theorem 2.6, which is the main technical result of this paper. For H a group with no epimorphisms onto \mathbb{Z} and with trivial center, this theorem gives a description of the outer automorphism group $\text{Out}(H_\phi)$ of a mapping torus H_ϕ . Theorem 2.6 forms the basis of the proof of Theorems A and B, which are the main theorems of this paper.

The layout of this current section is as follows. We begin by proving, in Lemma 2.1, that, because H has no epimorphisms onto \mathbb{Z} , every automorphism of the mapping torus H_ϕ fixes the subgroup H . We then use this to prove, in Lemma 2.4, that the representatives for elements of $\text{Out}(H_\phi)$ can be taken to have a specific form. In Section 2.4 we use the representatives given by Lemma 2.4 to prove Theorem 2.6.

2.1. Automorphisms fix the base group. Consider a mapping torus $H_\phi = H \rtimes_\phi \mathbb{Z}$ such that H has no epimorphisms onto \mathbb{Z} . The following lemma appears in a paper of Arzhantseva–Lafont–Minasyan [ALM14], although it is somewhat hidden in the proof of their Proposition 2.1.

Lemma 2.1 (Arzhantseva–Lafont–Minasyan). *Suppose H has no epimorphisms onto \mathbb{Z} . Then every automorphism of a mapping torus $H_\phi = H \rtimes_\phi \mathbb{Z}$ maps H to itself.*

Proof. Consider the following composition of maps, where the first embedding is the natural one of H into H_ϕ , where the map $\psi : H_\phi \rightarrow H_\phi$ is an automorphism of H_ϕ , and where the final surjection is the natural one of H_ϕ onto \mathbb{Z} by quotienting out H .

$$H \hookrightarrow H_\phi \xrightarrow{\psi} H_\phi \twoheadrightarrow \mathbb{Z}$$

As H does not map onto \mathbb{Z} , these maps compose to give the trivial map. Therefore, $H\psi \leq H$. Using the same argument with ψ^{-1} , we see that $H\psi^{-1} \leq H$ and so $H\psi = H$ as required. \square

Note that for H an arbitrary group, the automorphisms of a mapping torus $H_\phi = H \rtimes_\phi \mathbb{Z}$ which fix H form a subgroup $\text{Aut}_H(H_\phi)$ of $\text{Aut}(H_\phi)$, and this subgroup contains all the inner automorphisms so there is an analogous subgroup $\text{Out}_H(H_\phi)$ of $\text{Out}(H_\phi)$. The work in the remainder of Section 2 can be viewed as studying this subgroup $\text{Out}_H(H_\phi)$ of $\text{Out}(H_\phi)$. Lemma 2.1 proves that $\text{Out}_H(H_\phi) = \text{Out}(H_\phi)$ in our particular case.

2.2. The form of (outer) automorphisms. Consider a mapping torus $H_\phi = H \rtimes_\phi \mathbb{Z} = \langle H, t; tht^{-1} = \phi(h), h \in H \rangle$ such that H has no epimorphisms onto \mathbb{Z} and H has trivial center. Our main technical theorem, Theorem 2.6, follows from a classification of the elements of $\text{Out}(H_\phi)$, that is, to prove the main technical theorem we begin by finding representative automorphisms for elements of $\text{Out}(H_\phi)$.

The purpose of this current section, Section 2.2, is to prove Lemma 2.4, which gives this classification.

We begin by proving that certain maps, which are used as representatives for elements of $\text{Out}(H_\phi)$ in Lemma 2.4, define automorphisms of H_ϕ . There are two forms these representatives take, and Lemma 2.2 considers the first form while Lemma 2.3 considers the second form.

We shall write $[h, k] = h^{-1}k^{-1}hk$, and by $\delta\phi(h)$ we mean $\phi(\delta(h))$. We shall write multiplication in G as gh and as $g \cdot h$, with the latter notation being used to ease any ambiguity occurring when considering the images of elements under automorphisms, for example we would write $g \cdot \phi\psi(h) \cdot k$. For K some group with automorphism $\psi \in \text{Aut}(K)$, we shall write $\widehat{\psi}$ for the element of $\text{Out}(K)$ with representative ψ . We shall write $C_K(g)$ to denote the centraliser of the element $g \in K$.

Lemma 2.2. *Let H_ϕ be a mapping torus of H . If $\delta \in \text{Aut}(H)$ is such that $\widehat{\delta} \in C_{\text{Out}(H)}(\widehat{\phi})$ then δ induces an automorphism of H_ϕ in the following way, where g is such that $[\delta, \phi] = \gamma_g$.*

$$\begin{aligned} \alpha_\delta : h &\mapsto \delta(h) & \forall h \in H \\ t &\mapsto gt \end{aligned}$$

Proof. To see that α_δ is a homomorphism note that it satisfies all the relators of H , as $\alpha_\delta|_H \in \text{Aut}(H)$, so it is sufficient to prove that $\alpha_\delta(th) = \alpha_\delta(\phi(h) \cdot t)$ for all $h \in H$. So, the left hand side is as follows.

$$\begin{aligned} \alpha_\delta(th) &= gt \cdot \delta(h) \\ &= g \cdot \delta\phi(h) \cdot t & \forall h \in H \end{aligned}$$

We now evaluate the right hand side as follows. Note that (1), below, is obtained because g is such that $\delta\phi\gamma_g^{-1} = \phi\delta$.

$$\begin{aligned} \alpha_\delta(\phi(h) \cdot t) &= \phi\delta(h) \cdot gt \\ (1) \quad &= \delta\phi\gamma_g^{-1}(h) \cdot gt \\ &= g \cdot \delta\phi(h) \cdot t & \forall h \in H \end{aligned}$$

The left and right hand sides are equal, so α_δ is a homomorphism.

To see that α_δ is surjective, note that its restriction to H is surjective, and further note that $t \mapsto gt$ for some $g \in H$ so t is in the image.

To see that α_δ is right-invertible, and so injective, we note that $\alpha_{\delta^{-1}}$ is also a homomorphism and then prove that $\alpha_\delta\alpha_{\delta^{-1}}$ is trivial. So, $\alpha_{\delta^{-1}}$ is a homomorphism as $\widehat{\delta^{-1}} \in C_{\text{Out}(H)}(\widehat{\phi})$ because $\widehat{\delta} \in C_{\text{Out}(H)}(\widehat{\phi})$. Now, because $[\delta, \phi] = \gamma_g$ we have that $[\delta^{-1}, \phi] = \gamma_{\delta^{-1}(g^{-1})}$, which means that $\alpha_{\delta^{-1}}(t) = \delta^{-1}(g^{-1}) \cdot t$. Then, $\alpha_{\delta^{-1}}$ is the right inverse of α_δ as clearly $\alpha_\delta\alpha_{\delta^{-1}}$ fixes h for all $h \in H$ while we have the following.

$$\begin{aligned} \alpha_\delta\alpha_{\delta^{-1}}(t) &= \alpha_{\delta^{-1}}(gt) \\ &= \delta^{-1}(g) \cdot \delta^{-1}(g^{-1}) \cdot t \\ &= t \end{aligned}$$

Therefore, α_δ is injective. The proof of the lemma is complete. \square

The second form which automorphisms can take is given by Lemma 2.3.

Lemma 2.3. *Let H_ϕ be a mapping torus of H . If $\delta \in \text{Aut}(H)$ is such that $\widehat{\delta}^{-1}\widehat{\phi}\widehat{\delta} = \widehat{\phi}^{-1}$ then δ induces an automorphism of H_ϕ in the following way, where g is such that $\delta^{-1}\phi\delta = \phi^{-1}\gamma_g$.*

$$\begin{aligned}\zeta_\delta : h &\mapsto \delta(h) & \forall h \in H \\ t &\mapsto g^{-1}t^{-1}\end{aligned}$$

Proof. To see that ζ_δ is a homomorphism note that it satisfies all the relators of H , as $\zeta_\delta|_H \in \text{Aut}(H)$, so it is sufficient to prove that $\zeta_\delta(th) = \zeta_\delta(\phi(h) \cdot t)$ for all $h \in H$. So, the left hand side is as follows.

$$\begin{aligned}\zeta_\delta(th) &= g^{-1}t^{-1} \cdot \delta(h) \\ &= g^{-1} \cdot \delta\phi^{-1}(h) \cdot t^{-1}\end{aligned}$$

We now evaluate the right hand side as follows. Note that (2), below, is obtained because g is such that $\delta\phi^{-1}\gamma_g = \phi\delta$.

$$\begin{aligned}\zeta_\delta(\phi(h)t) &= \phi\delta(h) \cdot g^{-1}t^{-1} \\ (2) \quad &= \delta\phi^{-1}\gamma_g(h) \cdot g^{-1}t^{-1} \\ &= g^{-1} \cdot \delta\phi^{-1}(h)t^{-1}\end{aligned}$$

The left and right hand sides are equal, so ζ_δ is a homomorphism.

To see that ζ_δ is surjective, note that its restriction to H is surjective, and further note that $t \mapsto g^{-1}t^{-1}$ for some $g \in H$ so t is in the image.

In order to prove that ζ_δ is right-invertible, and so injective, we shall prove that $\alpha_{(\delta^2\gamma_g^{-1})}$ is an automorphism of H_ϕ and that $\zeta_\delta^2\gamma_g^{-1} = \alpha_{(\delta^2\gamma_g^{-1})}$. We begin by evaluating $[\delta^2\gamma_g^{-1}, \phi]$ as follows, where (3) is obtained because $\delta^{-1}\phi^{-1}\delta = \gamma_g^{-1}\phi$, while $\phi\delta\phi\gamma_{\phi(g)}^{-1} = \delta$ yields (4).

$$\begin{aligned}[\delta^2\gamma_g^{-1}, \phi] &= \gamma_g\delta^{-2}\phi^{-1}\delta^2\gamma_g^{-1}\phi \\ &= \gamma_g\delta^{-1}(\delta^{-1}\phi^{-1}\delta)\delta\phi\gamma_{\phi(g)}^{-1} \\ (3) \quad &= \gamma_g\delta^{-1}\gamma_g^{-1}(\phi\delta\phi\gamma_{\phi(g)}^{-1}) \\ (4) \quad &= \gamma_g\delta^{-1}\gamma_g^{-1}\delta \\ &= \gamma_g\gamma_{\delta(g^{-1})} = \gamma_{g \cdot \delta(g^{-1})}\end{aligned}$$

This implies that $\widehat{\delta^2\gamma_g^{-1}} \in C_{\text{Out}(H)}(\widehat{\phi})$, so by Lemma 2.2 we have that $\alpha_{(\delta^2\gamma_g^{-1})} \in \text{Aut}(H_\phi)$. Note that it also implies the following.

$$\alpha_{(\delta^2\gamma_g^{-1})}(t) = g \cdot \delta(g^{-1}) \cdot t$$

Then, to prove that $\zeta_\delta^2\gamma_g^{-1} = \alpha_{(\delta^2\gamma_g^{-1})}$, note that as their restriction to H is identical and because ζ_δ^2 is a homomorphism, it is sufficient to prove that $\zeta_\delta^2\gamma_g^{-1}(t) = g \cdot \delta(g^{-1}) \cdot t$. We have the following.

$$\begin{aligned}\zeta_\delta^2\gamma_g^{-1}(t) &= \zeta_\delta\gamma_g^{-1}(g^{-1}t^{-1}) \\ &= \gamma_g^{-1}(\delta(g^{-1}) \cdot tg) \\ &= g \cdot \delta(g^{-1}) \cdot t\end{aligned}$$

We conclude that $\zeta_\delta^2\gamma_g^{-1} = \alpha_{(\delta^2\gamma_g^{-1})}$, so the lemma holds. \square

Classifying the elements of $\text{Out}(H_\phi)$. We shall now prove Lemma 2.4, which classifies the coset representatives for $\text{Out}(H_\phi)$. Proving this lemma is the purpose of this current section, Section 2.2.

Lemma 2.4. *Suppose $H_\phi = H \rtimes_\phi \mathbb{Z}$ is a mapping torus such that H has no epimorphisms onto \mathbb{Z} . Then every element $\widehat{\psi}$ of $\text{Out}(H_\phi)$ has a representative in $\text{Aut}(H_\phi)$ of the form α_δ or of the form ζ_δ . Moreover, every map α_δ and ζ_δ defines an automorphism of H_ϕ .*

Proof. By Lemma 2.2 and Lemma 2.3, each of the prospective representatives α_δ and ζ_δ define automorphisms of H_ϕ . Therefore, we prove, below, the first part of this lemma, that every element of $\text{Out}(H_\phi)$ has a representative in $\text{Aut}(H_\phi)$ of one of the stipulated forms.

We begin by proving that if $\widehat{\psi} \in \text{Out}(H_\phi)$ then there is a representative $\psi \in \text{Aut}(H_\phi)$ of the following form, where $g \in H$ and $\delta \in \text{Aut}(H)$.

$$(5) \quad \begin{aligned} \psi : h &\mapsto \delta(h) & h &\in H \\ t &\mapsto gt^\epsilon \end{aligned}$$

To see this, consider a representative $\psi \in \widehat{\psi}$. Note that $\psi(H) = H$ by Lemma 2.1, thus the restriction of ψ to H is an automorphism δ of H . Therefore, as $H_\phi = H \rtimes_\phi \mathbb{Z}$ is a semidirect product, the representative $\psi \in \text{Aut}(H_\phi)$ can be chosen to be such that $\psi(h) = \delta(h)$ for all $h \in H$, and $\psi(t) = gt^i$ where $g \in H$ and $\delta \in \text{Aut}(H)$. We shall now prove that the number i has absolute value one, $|i| = 1$. This completes our proof that a representative $\psi \in \widehat{\psi}$ can be chosen to have the form (5). To see that $|i| = 1$, note that, because ψ is an automorphism, there exists a word W over H and gt^i which represents t , $W(gt^i, H) = t$. However, as H_ϕ is a semidirect product this word can be written as $t^{ij}k$ for some $k \in H$, $j \in \mathbb{Z}$. Thus, $t = t^{ij}k$, and so $|i| = 1$ as required.

We shall use the form (5) to prove the lemma. We investigate the cases $\epsilon = 1$ and $\epsilon = -1$ separately.

Suppose $\epsilon = 1$. It is sufficient to prove that $\delta\phi = \phi\delta\gamma_g$ holds. We have the following.

$$\begin{aligned} \psi(th) &= \psi(\phi(h) \cdot t) & \forall h \in H \\ gt \cdot \delta(h) &= \phi\delta(h) \cdot gt & \forall h \in H \\ g \cdot \delta\phi(h) \cdot t &= \phi\delta(h) \cdot gt & \forall h \in H \end{aligned}$$

Then, $\delta\phi = \phi\delta\gamma_g$ holds, so $\psi = \alpha_\delta$.

Suppose $\epsilon = -1$. It is sufficient to prove that $\delta^{-1}\phi\delta = \phi^{-1}\gamma_g^{-1}$ holds (note that g has been replaced with g^{-1} in the definition of ζ_δ , as $\psi(t) = gt^{-1}$). We have the following.

$$\begin{aligned} \psi(th) &= \psi(\phi(h) \cdot t) & \forall h \in H \\ gt^{-1} \cdot \delta(h) &= \phi\delta(h) \cdot gt^{-1} & \forall h \in H \\ g \cdot \delta\phi^{-1}(h) \cdot t^{-1} &= \phi\delta(h) \cdot gt^{-1} & \forall h \in H \end{aligned}$$

Then, $\delta\phi^{-1}\gamma_g^{-1} = \phi\delta$ holds, which yields the required equality, so $\psi = \zeta_\delta$. This completes the proof of the lemma. \square

2.3. The subgroup $\text{Out}^0(H_\phi)$. Having proven Lemma 2.4, we know, in a certain sense, what the elements of $\text{Out}(H_\phi)$ are, where $H_\phi = H \rtimes_\phi \mathbb{Z}$ is a mapping torus and H does not map onto \mathbb{Z} . In Section 2.4, below, we analyse the group $\text{Out}^0(H_\phi)$ consisting of the elements of the form $\widehat{\alpha}_\delta$, where α_δ was defined in Lemma 2.2, under the additional assumption that H has trivial center, and this analysis yields Theorem 2.6. Note that the purpose of Section 2 is to prove Theorem 2.6, and this result forms the basis of the proofs of the main theorems, Theorems A and B.

We shall now explain why we do not consider the automorphisms ζ_δ , but instead restrict our investigations to the subgroup $\text{Out}^0(H_\phi)$ of $\text{Out}(H_\phi)$. If there does not exist any automorphisms of the form ζ_δ then $\text{Out}^0(H_\phi) = \text{Out}(H_\phi)$. Otherwise, noting that the α_δ maps t to $g_1 t$ while $\zeta_{\delta'}$ maps t to $g_2^{-1} t^{-1}$ for some $g_1, g_2 \in H$, we see that $\text{Out}^0(H_\phi)$ is an index two subgroup of $\text{Out}(H_\phi)$. Therefore, applying Lemma 2.3, which provides conditions for the existence of a map ζ_δ , we have the following lemma.

Lemma 2.5. *The subgroup $\text{Out}^0(H_\phi)$ consisting of the outer automorphisms of the form $\widehat{\alpha}_\delta$ has index two in $\text{Out}(H_\phi)$ if $\widehat{\phi}$ is conjugate to $\widehat{\phi}^{-1}$ in $\text{Out}(H)$. Otherwise, $\text{Out}^0(H_\phi) = \text{Out}(H_\phi)$.*

This lemma is why in Theorem 2.6 we restrict our analysis to $\text{Out}^0(H_\phi)$. Note that the automorphisms of the form α_δ are such that the following hold. We use these equalities throughout the remainder of Section 2.

$$\begin{aligned}\alpha_\delta \alpha_\xi &= \alpha_{\delta\xi} \\ \alpha_\delta^{-1} &= \alpha_{\delta^{-1}}\end{aligned}$$

2.4. The outer automorphism groups of certain mapping tori. Take $H_\phi = H \rtimes_\phi \mathbb{Z}$ to be a mapping torus with base group H and associated automorphism $\phi \in \text{Aut}(H)$, and also assume that H has trivial center and has no epimorphisms onto \mathbb{Z} . In this section we prove Theorem 2.6, which gives a description of $\text{Out}(H_\phi)$ for such a group H_ϕ . Recall that for K a group and $\psi \in \text{Aut}(K)$, $\widehat{\psi}$ denotes the element of $\text{Out}(K)$ with representative ψ , and that $C_K(g)$ denotes the centraliser of the element $g \in K$.

Theorem 2.6. *Let $H_\phi = H \rtimes_\phi \mathbb{Z}$ be a mapping torus with base group H and associated automorphism ϕ . Assume H has trivial center and has no epimorphisms onto \mathbb{Z} . Then we have the following isomorphism,*

$$\text{Out}^0(H_\phi) \cong \frac{C_{\text{Out}(H)}(\widehat{\phi})}{\langle \widehat{\phi} \rangle}$$

where either $\text{Out}^0(H_\phi) = \text{Out}(H_\phi)$ or $\widehat{\phi}$ is conjugate to $\widehat{\phi}^{-1}$ in $\text{Out}(H)$, whence $\text{Out}^0(H_\phi)$ has index two in $\text{Out}(H_\phi)$.

Proof. By Lemma 2.5, $\text{Out}^0(H_\phi)$ has index one or two in $\text{Out}(G)$, and further has index two precisely when $\widehat{\phi}$ is conjugate to $\widehat{\phi}^{-1}$ in $\text{Out}(H)$, as required. We shall now prove that $\text{Out}^0(H_\phi)$ is isomorphic to $C_{\text{Out}(H)}(\widehat{\phi})/\langle \widehat{\phi} \rangle$, which completes the proof of the theorem.

Consider the following map. We shall prove that it is a well-defined surjective homomorphism with kernel $\langle \widehat{\phi} \rangle$, which proves the theorem.

$$\begin{aligned} \eta : C_{\text{Out}(H)}(\widehat{\phi}) &\rightarrow \text{Out}^0(H_\phi) \\ \widehat{\delta} &\mapsto \widehat{\alpha}_\delta \end{aligned}$$

Note that the map η is surjective by the definition of $\text{Out}^0(H_\phi)$, and it is a homomorphism because $\alpha_\delta \alpha_\xi = \alpha_{\delta\xi}$.

To see that η is well-defined, suppose that $\delta_2 = \delta_1 \gamma_k$. Note that $[\delta_1, \phi] = \gamma_{k g_2 \cdot \phi(k^{-1})}$ where g_2 is such that $[\delta_2, \phi] = \gamma_{g_2}$. Then $\alpha_{\delta_2}(h) = \alpha_{\delta_1} \gamma_k(h)$ for all $h \in H$, while $\alpha_{\delta_2}(t) = g_2 t$ and we have the following.

$$\begin{aligned} \alpha_{\delta_1} \gamma_k(t) &= k^{-1} k g_2 \cdot \phi(k^{-1}) \cdot t k \\ &= g_2 t \end{aligned}$$

We thus have that $\alpha_{\delta_2} = \alpha_{\delta_1} \gamma_k$, so $\widehat{\alpha}_{\delta_2} = \widehat{\alpha}_{\delta_1}$ as required.

Finally, to prove that the map η has kernel $\langle \widehat{\phi} \rangle$ begin by supposing that α_δ is inner, and so $\alpha_\delta = \gamma_{kt^i}$ for some $k \in H$ and $i \in \mathbb{Z}$. This means that $h = t^{-i} k^{-1} \cdot \delta^{-1}(h) \cdot kt^i$ for all $h \in H$, so $\phi^i(h) = \delta^{-1} \gamma_k(h)$ for all $h \in H$, and so $\widehat{\delta} = \widehat{\phi}^j$ in $\text{Out}(H)$ for some $j \in \mathbb{Z}$. Therefore, $\ker \eta \leq \langle \widehat{\phi} \rangle$. On the other hand, α_ϕ is inner because $\alpha_\phi(h) = \phi(h) = h t h^{-1}$ while $\alpha_\phi(t) = t$. Therefore, $\langle \widehat{\phi} \rangle \leq \ker \eta$. Thus, we conclude that $\alpha_\delta \in \text{Inn}(H_\phi)$ if and only if $\widehat{\delta} \in \langle \widehat{\phi} \rangle$. \square

3. PROOF OF THEOREM A

In this section we apply Theorem 2.6 to prove the main results of this paper, Theorems A and B.

Sapir's embedding. To apply Theorem 2.6 we need to have some knowledge or control over the centralisers of elements in $\text{Out}(H)$. To do this, we use an embedding of Sapir [Sap14, Theorem 5.1]. If K is a finitely generated, recursively presented group and $x \in K$, then Sapir's embedding gives a finitely presented group P containing K as a subgroup and such that $C_K(x) = C_P(x)$. It is an open problem of Osin that every recursively presented group can be embedded as a malnormal subgroup of a finitely presented group [Sap14]. Sapir remarks that in his embedding K is malnormal in P , hence the open problem of Osin has a positive solution, and that this will be proven in his next paper. The proofs of Theorems A and B both apply Sapir's embedding. Note that Theorem B can be rephrased as "if Osin's problem admits a positive solution then every finitely generated, recursively presentable group can be realised as the outer automorphism group of a finitely generated, residually finite group".

The Bumagin–Wise question. We now prove two theorems, which combine to prove Theorem A and the second of which yields Theorem B. The first theorem, Theorem 3.1, gives a partial answer to Bumagin–Wise's question for certain groups, while the second theorem, Theorem 3.2, gives a complete answer to Bumagin–Wise's question for certain groups.

The proofs of Theorems 3.1 and 3.2 both use the fact that a split extension of a finitely generated, residually finite group by a residually finite group is residually finite [Mal56]. Hence if the base group H is a finitely generated, residually finite

group then the mapping torus $H_\phi = H \rtimes_\phi \mathbb{Z}$ is also a finitely generated, residually finite group.

Recall that the group $\text{Out}^0(H_\phi)$, as defined in Section 2.3, is the subgroup of $\text{Out}(H_\phi)$ consisting of the elements of the form $\widehat{\alpha}_\delta$, where α_δ is defined in Lemma 2.2.

Theorem 3.1. *Let Q be a finitely generated, recursively presented group. Then there exists a finitely generated, residually finite group G such that $\text{Out}(G) \cong Q \times C_2$.*

Proof. Define $Q_2 = Q \times C_2$. As Q is finitely generated and recursively presented, we can use Sapir's embedding to construct a finitely presented group P which contains Q_2 and such that $C_P(k) = Q_2$ where k is the generator of the C_2 factor of Q_2 . As P is finitely presented, there exists a finitely generated, residually finite group H such that $\text{Out}(H) \cong P$ [BW05]. Note that this group H is generated by elements of finite order, and so does not map onto \mathbb{Z} , and also note that H is a non-cyclic subgroup of a finitely presented $C'(1/6)$ group and therefore has trivial center [BW05]. Thus, Theorem 2.6 is applicable to $H_\phi = H \rtimes_\phi \mathbb{Z}$ for all $\phi \in \text{Aut}(H)$.

Let $\widehat{\phi}$ be the element of $\text{Out}(H)$ associated to $k \in Q_2$. Thus, $C_{\text{Out}(H)}(\widehat{\phi}) \cong Q_2$. Form $H_\phi = H \rtimes_\phi \mathbb{Z}$ for some $\phi \in \widehat{\phi}$. Then $\text{Out}^0(H_\phi) \cong Q$ by Theorem 2.6. Note that H_ϕ is finitely generated, and residually finite [Mal56].

To complete the theorem it is sufficient to prove that $\text{Out}(G) = \text{Out}^0(H_\phi) \times C_2$. To see this, note that $k = k^{-1}$. Thus, the automorphism $\psi : h \mapsto h, t \mapsto t^{-1}$ can be taken as the coset representative for $\text{Out}(H_\phi)/\text{Out}^0(H_\phi)$. This automorphism has order two and generates a normal subgroup of $\text{Out}(H_\phi)$. Therefore, taking $G = H_\phi$, $\text{Out}(G) = \text{Out}^0(G) \times \langle \widehat{\psi} \rangle \cong Q \times C_2$, as required. \square

The following theorem, Theorem 3.2, allows us to apply a positive solution of Osin's problem to get a positive solution to Bumagin–Wise's question for finitely generated, recursively presented groups. This is because if Q is finitely generated and recursively presented then the conditions of Theorem 3.2 hold if, for example, $Q \times C_3$ embeds malnormally into a finitely presented group, and a positive solution to Osin's question gives us this embedding.

Theorem 3.2. *Let $Q' = Q \times C$ where $C = \langle k \rangle$ is cyclic of order greater than two (possibly infinite). Suppose that Q' can be embedded into a finitely presented group P where k is not conjugate to k^{-1} in P . Then there exists a finitely generated, residually finite group G such that $\text{Out}(G) \cong Q$.*

Proof. Write H for the finitely generated, residually finite group such that $\text{Out}(H) \cong P$ [BW05], and, as in the proof of Theorem 3.1, take $\widehat{\phi}$ to be the element of $\text{Out}(H)$ associated to $k \in Q'$ and form the finitely generated, residually finite group $G \cong H \rtimes_\phi \mathbb{Z}$ such that $\text{Out}^0(G) \cong Q'$. Finally, because k is not conjugate to k^{-1} in P , Lemma 2.5 allows us to conclude that $\text{Out}(G) = \text{Out}^0(G) \cong Q$, as required. \square

We shall now prove Theorems A and B.

Proof of Theorem A. Theorem A follows immediately from combining Theorems 3.1 and 3.2. \square

Note that if Osin’s open problem has a positive solution, so every finitely generated, recursively presented group is a malnormal subgroup of a finitely presented group, then we can use Theorem 3.2 and disregard Theorem 3.1 to obtain Theorem B.

Proof of Theorem B. If Q is a finitely generated, recursively presentable group then, by the assumptions of the theorem, $Q \times C_3$ embeds malnormally into a finitely presented group P . Theorem B then follows from Theorem 3.2. \square

Recursive presentability. It is natural to ask how far Theorem B goes towards solving Bumagin–Wise’s question, assuming that Osin’s problem has a positive solution and that the groups Q in the statement of the question are additionally assumed to be finitely generated. The “best possible” case would be that every finitely generated group which occurs as the outer automorphism group of a finitely generated, residually finite group is recursively presentable, and so Theorem B would be the complete solution to Bumagin–Wise’s question for finitely generated groups. However, the following proposition, Proposition 3.3, implies that this case does not happen. We then prove, in Proposition 3.5, that if the groups G_Q in the statement of Bumagin–Wise’s question are additionally assumed to be recursively presentable then Theorem B is the complete solution to Bumagin–Wise’s question for finitely generated groups.

Proposition 3.3. *There exists a finitely generated, non-recursively presentable group Q which can be realised as the outer automorphism group of a finitely generated, residually finite group G_Q .*

We now explain the proof of Proposition 3.3. Note that there exists a finitely generated, residually finite group R which is not recursively presentable (Bridson–Wilton [BW14] point out that this follows from work of Slobodskoi[Slo81]). Using the existence of such a non-recursively presented group R , a forthcoming paper of the author (see also the author’s PhD thesis [Log14, Corollary 4.3.16]) constructs a finitely generated, residually finite group $G_{\widehat{R}}$ whose outer automorphism group is finitely generated but not recursively presentable (indeed, R is embedded with finite index into $\text{Out}(G_{\widehat{R}})$). This proves Proposition 3.3. Note, however, this group $G_{\widehat{R}}$ is itself not recursively presentable.

We now provide a positive answer to the following question: assuming Osin’s problem has a positive solution, is it true that a finitely generated group Q can be realised as the outer automorphism group of a recursively presented, finitely generated, residually finite group G_Q if and only if Q is recursively presentable? That is, is Theorem B the complete solution to this restricted version of Bumagin–Wise’s question? We provide a positive answer by combining Theorem B with following proposition, Proposition 3.4, which is due to Ashot Minasyan in a private communication with the author. Proposition 3.4 also explains why the group $G_{\widehat{R}}$ in the author’s construction, cited above, is not recursively presentable. We state the proposition, give a sketch proof of it, and then combine it with Theorem B to prove Proposition 3.5.

Proposition 3.4 (A. Minasyan). *Suppose that G is a finitely generated, recursively presentable group whose outer automorphism group $\text{Out}(G)$ is also finitely generated. Then $\text{Aut}(G)$ and $\text{Out}(G)$ are recursively presentable.*

Proof. Assume that G and $\text{Out}(G)$ are finitely generated, and that G is recursively presented. Note that this implies that $\text{Aut}(G)$ is finitely generated, as it is an extension of $\text{Inn}(G)$ by $\text{Out}(G)$, and both $\text{Inn}(G) \cong G/Z(G)$ and $\text{Out}(G)$ are assumed to be finitely generated. We shall just prove that $\text{Aut}(G)$ is recursively presentable; this implies that $\text{Out}(G)$ is recursively presentable because G and $\text{Out}(G)$ are both assumed to be finitely generated.

To prove that $\text{Aut}(G)$ is recursively presentable, we shall start with the generators of $\text{Aut}(G)$ and construct an algorithm which lists all the relators of $\text{Aut}(G)$. Let ψ_1, \dots, ψ_m be a generating set for $\text{Aut}(G)$ and let x_1, \dots, x_n be a generating set for G . We can assume that for each i, j we know words u_{ij} and v_{ij} , over the alphabet $\{x_1, \dots, x_n\}^{\pm 1}$, such that $\psi_i(x_j) = u_{ij}$ and $\psi_i^{-1}(x_j) = v_{ij}$, because this is a finite collection of words and we are only proving the existence of an algorithm. Now, since G is recursively presented, there is a (partial) algorithm A which takes on input a pair of words, (w_1, w_2) say, over $\{x_1, \dots, x_n\}^{\pm 1}$ and stops, outputting “yes” if and only if $w_1 = w_2$ in G (this is the algorithm A which re-writes w_1 in all possible ways and compares the result with w_2).

To obtain an algorithm listing all defining relators of $\text{Aut}(G)$, start enumerating all words Ψ_1, Ψ_2, \dots over $\{\psi_1, \dots, \psi_m\}^{\pm 1}$. At the same time, for every k check if $\Psi_k(x_j) = x_j$ in G for all $j = 1, \dots, n$ (by writing $\Psi_k(x_j)$ as a word W over the generators $\{x_1, \dots, x_n\}^{\pm 1}$ in terms of u_{ij} and v_{ij} , and then inputting the pair (W, x_j) into A). If $\Psi_k = 1$ in $\text{Aut}(G)$ then we will verify this in finite time, and so we can add Ψ_k to the list of defining relators of $\text{Aut}(G)$. Thus we have an algorithm listing all relators in $\text{Aut}(G)$, and conclude that $\text{Aut}(G)$ is recursively presented. \square

Proposition 3.4 and Theorem B can be combined to yield the following result. Note that combining Proposition 3.4 and Theorem A yields a similar result which is independent of Osin’s problem.

Proposition 3.5. *Suppose every finitely generated, recursively presented group Q can be embedded as a malnormal subgroup of a finitely presented group H_Q . Then a finitely generated group Q can be realised as the outer automorphism group of a recursively presented, finitely generated, residually finite group G_Q if and only if Q is recursively presentable.*

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