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Truncated Leximin solutions

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HIGHLIGHTS

- We consider Thomson and Lensberg's (1989) characterization of the Leximin bargaining solution.
- We remove Pareto Optimality from their axiom set.
- The remaining axioms characterize a class of Truncated Leximin solutions.
- These truncate agents' Leximin solution payoffs at a given utility level α.
- We discuss efficiency-free characterizations of the Leximin solution.

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ABSTRACT

This paper shows that three classic properties for bargaining solutions in an environment with a variable number of agents – *Anonymity* (AN), *Individual Monotonicity* (IM), and *Consistency* (CONS) – characterize a one-parameter family of *Truncated Leximin solutions*. Given a non-negative and possibly infinite α, an α-Truncated Leximin solution gives each agent the minimum of α and their Leximin solution payoff.

1. Introduction

Real-life conflict situations in which the parties involved cannot appeal to an outside arbitrator, are often resolved non-optimally, for instance due to delay in reaching an agreement, or because agents do not know each other's utilities. The consequence for the axiomatic treatment of bargaining solutions, a literature initiated by Nash (1950), is that it may not always be appropriate to impose the axiom *Pareto Optimality* on proposed solutions. For solutions whose characterizations do involve Pareto Optimality, it is thus of interest to examine what kind of violations of efficiency occur when this property is dropped from the corresponding axiom set. If much freedom is gained, it may be taken as an indication of the importance of Pareto Optimality in the composition of the studied solution, and thus, in absence of any other efficiency-free characterizations, that the solution may not be appropriate in environments where such optimality is not guaranteed. Conversely, if only few implausible solutions emerge, it means that the solution is robust against violations of efficiency.

This paper examines Thomson and Lensberg's (1989) characterization of the *Leximin solution* (Imai, 1983; Chun and Peters, 1989; Nieto, 1992; Chen, 2000), which aside from *Pareto Optimality* involves the three properties *Individual Monotonicity*, *Anonymity*, and *Consistency*. In any bargaining problem, the Leximin solution picks the unique outcome that is obtained by iteratively maximizing the payoffs of the worst-off agents, then those of the second worst-off, and so on, until no agent's utility can be further increased within the feasible set. In so doing, it embodies two different principles of distributive justice: Pareto Optimality on the one hand, and on the other Rawls's (1971) proposition that inequalities between individuals should only be allowed if they work towards the benefit of those that are less advantaged (i.e., the Difference Principle).

The main result of the paper is that dropping Pareto Optimality from Thomson and Lensberg's axiom set gives rise to a family of *Truncated Leximin solutions*: each member of this family truncates agents' Leximin solution payoffs at a given utility level. It can be argued that this is a large class of bargaining solutions. Hence, in view of Thomson and Lensberg's (1989) axiomatization, the exercise in this paper puts into question the appropriateness of the Leximin solution in non-arbitrated settings where efficiency is doubtful. On the positive side, an efficiency-free characterization of the Leximin solution follows as a simple corollary from our main result. Furthermore, the inefficient solutions that emerge from Thomson and Lensberg's axioms minus Pareto Optimality, continue to uphold Rawls's conception of egalitarianism: if the best-off agents achieve utility level α, then an α-Truncated Leximin solution yields the "most just" feasible distribution of income under this constraint, in the same way the Leximin solution does in general.

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The work most closely related to the present paper is (Thomson, 1984). That paper considers Thomson's (1983) axiomatization of the Egalitarian solution, and shows that when there are at least three bargainers, the only additional solutions that become admissible when *Weak Pareto Optimality* is removed from the axiom set, are *Truncated Egalitarian solutions*.[1] While similar to those considered in this paper, Thomson's solutions are different in two significant ways. Most obviously, Truncated Egalitarian solutions truncate the Egalitarian solution outcome rather than the Leximin solution outcome. While the Egalitarian and Leximin solutions coincide on the domain of strictly comprehensive problems, this is not generally the case. Secondly, the utility level at which Truncated Egalitarian solutions truncate the solution outcome may depend on (the size of) the coalition of bargaining agents, while for Truncated Leximin solutions the cut-off point is independent of the problem.

Other literature related to this paper deals with the Nash bargaining solution (Nash, 1950), and characterizations thereof. Roth (1977) provided a characterization that does not rely on Pareto Optimality, by removing that property from Nash's axiom set, and replacing it by *Strong Individual Rationality*. He further demonstrated that without Pareto Optimality, the only additional solution that becomes admissible is the Disagreement solution (Roth, 1979a). Lensberg (1988) characterized the Nash solution by replacing Nash's independence axiom by Consistency. Lensberg and Thomson (1988) subsequently proved a result similar to Roth's (1979a): by removing Pareto Optimality from Lensberg's axiom set, only the Disagreement solution becomes admissible. Adding Strong Individual Rationality rules out the Disagreement solution, so Lensberg and Thomson's result too gives rise to a characterization of the Nash solution that does not rely on Pareto Optimality. For the Kalai–Smorodinsky solution (Kalai and Smorodinsky, 1975), a number of efficiency-free characterizations have recently been obtained by Rachmilevitch (2014).

The paper proceeds as follows. Section 2 introduces the framework and all relevant definitions. In Section 3 we prove our main result, and Section 4 discusses its relation to other results in the literature. Section 5 concludes.

2. Preliminaries

2.1. The bargaining problem

There is an infinite population I of agents, indexed by the positive integers. The set of all finite subsets of I is denoted $\mathcal{P}$. Given $P \in \mathcal{P}$, $\mathbb{R}^P_+$ denotes the Cartesian product of $|P|$ copies of $\mathbb{R}_+$, indexed by the elements of P. For $x, y \in \mathbb{R}^P_+$, $x > y$ means $x - y \in \mathbb{R}^P_{++}$, $x \geqq y$ means $x - y \in \mathbb{R}^P_+$, and $x \geq y$ means $x \geqq y$ and $x \neq y$. The set Σ^P contains all sets $S \subseteq \mathbb{R}^P_+$ that are compact, convex and comprehensive,[2] and that further contain at least one vector z with $z > \mathbf{0} \equiv (0, \ldots, 0)$. An element S of Σ^P is a **problem for** P. The set S represents the utilities agents in P can realize by cooperation; failure to cooperate leads to the unfavorable **disagreement outcome** $\mathbf{0} \in \mathbb{R}^P_+$. The set $\Sigma \equiv \bigcup_{P \in \mathcal{P}} \Sigma^P$ contains *all* problems. A **solution** is a map φ defined on Σ, that associates for all $P \in \mathcal{P}$, and with all $S \in \Sigma^P$, a unique $\varphi(S) \in S$. The **solution outcome** $\varphi(S)$ represents the compromise the agents in P reach when faced with the problem S.

Bargaining problems and solutions can be given a *normative* or a *descriptive* interpretation. Under the normative interpretation, a bargaining problem is the representation of an arbitrated dispute: rather than engaging in negotiations with one another, the agents turn to an impartial outsider to settle the dispute for them; a solution is accordingly interpreted as an arbitration scheme, i.e. a rule by which this arbiter generally adjudicates the conflicts he is presented with. In this paper, we adopt the contrasting descriptive point of view: a bargaining problem is thus considered to be the representation of actual negotiations between rational, intelligent, and self-serving economic agents; accordingly, a solution summarizes all strategic interactions that lead those agents to an unanimous agreement.

Given our descriptive interpretation of the problem, the natural next question is how the utilities over which the agents negotiate are to be interpreted. Nash (1950) contended that in strategic settings comparisons of utility across agents are meaningless, and can thus not play a role in the negotiations. However, it is arguable that in many instances, the resolution of a conflict does in fact – at least in part – rely on such comparisons. Suppose for instance that two agents bargain by first staking out initial positions that are irreconcilable, and then find compromise by gradually making mutual concessions with respect to these initial claims; in such situations it is not inconceivable that the concessions the one agent makes depend on his perceived value of the concessions made by the other. The assumption of interpersonal utility comparisons also has empirical support: Nydegger and Owen (1974) tested Nash's (1950) axioms experimentally, and found that subjects did in fact compare utilities. The assumption that utilities are cardinally measurable and comparable across agents thus seems to be defensible in a descriptive theory of bargaining, and will henceforth be made without further qualifications.[3]

2.2. Further definitions and notation

Next, several definitions and notations are introduced.

An **interval** is defined as a non-empty connected subset of $\mathbb{R}_+$. Consider an interval A. If there are $a, b \in \mathbb{R}_+ \cup \{\infty\}$ such that for all $x \in \mathbb{R}_+$, $x \in A$ if and only if $a < x < b$, then A is an **open interval**, and denoted by (a, b). Similarly, if there are $a, b \in \mathbb{R}_+ \cup \{\infty\}$ such that for all $x \in \mathbb{R}_+$, $x \in A$ if and only if $a \leqq x \leqq b$, then A is a **closed interval**, and denoted by $[a, b]$. **Half-open intervals** are similarly defined, and respectively denoted by $[a, b)$ and $(a, b]$. An interval A is **non-degenerate** if there exist $x, y \in A$ such that $x > y$. Given a non-degenerate interval A, the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be **non-decreasing on** A if for all $x, y \in A$, $x > y$ implies $f(x) \geqq f(y)$; f is said to be **increasing on** A if $x > y$ implies $f(x) > f(y)$. Whenever $A = \mathbb{R}_+$, the part "on A" is omitted—i.e. f is then respectively said to be "non-decreasing" or "increasing".

For $P \in \mathcal{P}$ and $S \in \Sigma^P$, the **utopia point** is defined as $a(S) \in \mathbb{R}^P_+$ with $a_i(S) \equiv \max\{x_i \mid x \in S\}$ for all $i \in P$. The **Pareto optimal set** and the **Weak Pareto optimal set** of S are respectively defined as $PO(S) \equiv \{x \in S \mid y \geq x \text{ implies } y \notin S\}$ and $WPO(S) \equiv \{x \in S \mid y > x \text{ implies } y \notin S\}$. The **convex comprehensive hull** of S, denoted $cch\, S$, is defined as the smallest convex and comprehensive set in $\mathbb{R}^P_+$ that contains S.

For $P \in \mathcal{P}$, let Λ^P be the family of functions $\hat{\lambda} : \mathbb{R}^P_+ \to \mathbb{R}^P_+$ defined as follows: for each $i \in P$ there exists a strictly positive λ_i such that for all $x \in \mathbb{R}^P_+$, $\hat{\lambda}(x) = (\lambda_i x_i)_{i \in P}$. For $S \in \Sigma^P$, define $\hat{\lambda}(S) \equiv \{\hat{\lambda}(x) \mid x \in S\}$, and note that $\hat{\lambda}(S) \in \Sigma^P$. If there is a strictly positive constant λ such that for all $i \in P$ and $x \in \mathbb{R}^P_+$, $\hat{\lambda}_i(x) = \lambda x_i$, we write $\hat{\lambda}(x)$ and $\hat{\lambda}(S)$ respectively as λx and λS. For $P \in \mathcal{P}$, let

[1] For two-person problems there exist other solutions satisfying Thomson's (1984) axioms. This issue does not occur in our result, a matter that is addressed in Section 4.1.

[2] For all $x, y \in \mathbb{R}^P_+$, if $x \in S$ and $y \leqq x$ then $y \in S$.

[3] For further discussion of this assumption, see Luce and Raiffa (1957, p. 131–132), Shapley (1969), and Kalai (1977).

e_P be the vector in $\mathbb{R}^P$ with unit entries. Any vector $x \in \mathbb{R}^P_+$ such that $x = \lambda e_P$ for some $\lambda \in \mathbb{R}_{++}$ is said to lie **in the egalitarian direction**.

For $P, P' \in \mathcal{P}$ with $|P| = |P'|$, let $\Gamma^{P,P'}$ be the family of one-to-one functions from P to P' (note that when $P = P'$, this reduces to the set containing all permutations of P). With a slight abuse of notation, functions $\gamma \in \Gamma^{P,P'}$ can be treated as functions from $\mathbb{R}^P$ to $\mathbb{R}^{P'}$: for $x \in \mathbb{R}^P$, $\gamma(x)$ is defined as the vector y in $\mathbb{R}^{P'}$ with $y_{\gamma(i)} \equiv x_i$ for all $i \in P$. For $S \in \Sigma^P$, $\gamma(S) \equiv \{\gamma(z) \mid z \in S\}$.

For $P, Q \in \mathcal{P}$ with $P \subsetneq Q$, let $Q \setminus P$ be the set in $\mathcal{P}$ that contains all the agents in Q that are not in P. If $P = \{i\}$ with $i \in I$, then we write $Q \setminus i$ rather than $Q \setminus \{i\}$. Given $y \in \mathbb{R}^Q_+$, let y_P be the projection of y onto $\mathbb{R}^P_+$, and for $S \subseteq \mathbb{R}^Q_+$, define $S_P \equiv \{y_P \mid y \in S\}$. Finally, for $x \in S$, define $t^x_P(S)$ as the intersection of S and a hyperplane through x and parallel to $\mathbb{R}^P$, i.e. $t^x_P(S) \equiv \{y \in \mathbb{R}^P_+ \mid (y, x_{Q\setminus P}) \in S\}$.

2.3. Axioms

This section discusses a number of axioms for bargaining solutions. The fact that solutions summarize the entire strategic exchange between the bargainers has implications for whether an axiom is appropriate or not. Thus, the properties are given some further context with this interpretation in mind. To avoid repetition, the phrase "For all $P \in \mathcal{P}$, for all $S \in \Sigma^P$" is omitted.

The first axiom reflects the idea that no agent agrees to an outcome that does not make him strictly better off than disagreement.

Strong Individual Rationality (SIR): $\varphi(S) > \mathbf{0}$.

As argued in the introduction, the outcome of negotiations may well be Pareto dominated. The following two axioms are normative in nature, and rule out any such inefficiencies. In particular, the first says that there is no feasible outcome that makes *all* agents strictly better off than the solution outcome; the second says that there are no feasible outcomes that make *any* agent strictly better off.

Weak Pareto Optimality (WPO): $\varphi(S) \in WPO(S)$.

Pareto Optimality (PO): $\varphi(S) \in PO(S)$.

Suppose that two same-sized groups of agents are facing bargaining problems that are represented by the same geometrical object. If all relevant information pertaining to the conflict is contained in the geometry of the feasible set, then these two groups resolve their conflict in the same way. This intuition is contained in the following axiom.

Anonymity (AN): For all $P' \in \mathcal{P}$ with $|P| = |P'|$, for all $\sigma \in \Gamma^{P,P'}$, $\varphi(\sigma(S)) = \sigma(\varphi(S))$.

If all information on the underlying conflict is contained in the geometry of the problem, and this problem is further symmetric, then no rational agent will agree to an outcome that gives him less than his opponent(s); hence, in such case, the solution outcome is symmetric as well.

Symmetry (SY): If for all $\sigma \in \Gamma^{P,P}$, $S = \sigma(S)$, then for all $i, j \in P$, $\varphi_i(S) = \varphi_j(S)$.

Note that SY is implied by AN: if $S = \sigma(S)$ for all permutations $\sigma \in \Gamma^{P,P}$, then by AN, $\varphi(S) = \sigma(\varphi(S))$ for all $\sigma \in \Gamma^{P,P}$. Since for $x \in \mathbb{R}^P_+$, $x = \sigma(x)$ for all permutations $\sigma \in \Gamma^{P,P}$ if and only if $x_i = x_j$ for all $i, j \in P$, SY follows.

The utilities over which the agents bargain may be interpreted as vNM utilities: any positive linear transformation of an agent's utility function then represents the same preferences on the part of that agent, and accordingly, any positive linear transformation of the bargaining problem represents the same conflict. A rescaling of the problem must then result in a rescaling of the solution outcome. This idea is reflected by the following axiom.[4]

Scale Invariance (SI): For all $\hat{\lambda} \in \Lambda^P$, $\hat{\lambda}(\varphi(S)) = \varphi(\hat{\lambda}(S))$.

Suppose that agents discount utilities exponentially with factor $\delta \in (0, 1)$ and that $S \in \Sigma$ is attainable at time $t + 1$. Then there is no difference between bargaining at time t over the utilities feasible at time $t + 1$ (i.e., $\varphi(\delta S)$) and bargaining at time $t + 1$ and discounting back the obtained payoffs (i.e., $\delta\varphi(S)$). This type of time-consistence is reflected by a weakening of SI that imposes the same invariance, but only when all agents' utility functions are rescaled using the same positive linear transformation.

Homogeneity (HOM): For all $\lambda \in \mathbb{R}_{++}$, $\varphi(\lambda S) = \lambda\varphi(S)$.

Nash (1950) introduced an axiom which says that if agents find agreement on some feasible outcome y, and after a contraction of the feasible set this outcome remains available, then negotiations in this smaller problem also lead to unanimous agreement on y.

Independence of Irrelevant Alternatives (IIA): For all $S' \in \Sigma^P$, if $\varphi(S) \in S' \subseteq S$ then $\varphi(S') = \varphi(S)$.

While IIA is normatively appealing, as a descriptive axiom it has been the subject of much criticism. In particular, Luce and Raiffa (1957) argued that it renders the solution outcome too unresponsive to changes in the geometry of the feasible set: if in the two-person problem $T \equiv cch\{e_1, e_2\}$ negotiations lead to $(1/2, 1/2)$, then IIA dictates that in $S \equiv \{e_1, (1/2, 1/2)\}$ the solution outcome remains $(1/2, 1/2)$, even though the ability of agents to demand higher payoffs has been asymmetrically curtailed. This criticism led Kalai and Smorodinsky (1975) to propose an alternative property. Let $P \in \mathcal{P}$, $i \in P$, and $S, S' \subseteq \Sigma^P$, and suppose that for any set of demands made by the agents in $P \setminus i$, agent i can feasibly claim a higher payoff in S' than in S. Then if i is rational, he will not agree to an outcome in S' that gives him less than the outcome that is unanimously accepted in S.

Individual Monotonicity (IM): For all $S' \in \Sigma^P$, for all $i \in P$, if $S \subseteq S'$ and $S_{P\setminus i} = S'_{P\setminus i}$, then $\varphi_i(S) \leqq \varphi_i(S')$.

Note that Kalai and Smorodinsky only formulated IM for two-agent problems. While several multilateral generalizations exist in the literature (Thomson, 1980; Imai, 1983; Chen, 2000), the present version is due to Kalai (1977).

The following is a normative axiom that says that small changes in the description of the problem should not lead to radical changes in the solution outcome.

Continuity (CONT): For all sequences $\{S^k\} \subseteq \Sigma^P$ converging to S (in the Hausdorff topology), $\varphi(S^k) \to \varphi(S)$.

Suppose that the agents in $Q \in \mathcal{P}$ face the problem $T \in \Sigma^Q$, and that the proposal $y \in T$ has been put on the table. If an agent $i \in Q$ is rational, and he has reason to believe that he could favorably renegotiate this outcome with a subgroup of agents – i.e. a set of agents $P \in \mathcal{P}$ with $i \in P$ and $P \subsetneq Q$ – then i would not deem y an acceptable compromise in T, and reject it accordingly. A necessary condition for an outcome $y \in T$ to be the solution outcome is then that no agent has an incentive to reject it on the basis of such reasoning. This is the underlying intuition of the following axiom.

Consistency (CONS): For all $Q \in \mathcal{P}$ with $P \subsetneq Q$, for all $T \in \Sigma^Q$, if $S = t^y_P(T)$ with $y = \varphi(T)$, then $\varphi(S) = y_P$.

[4] Note that in this paper, interpersonal comparisons of utility are allowed. See the above discussion.

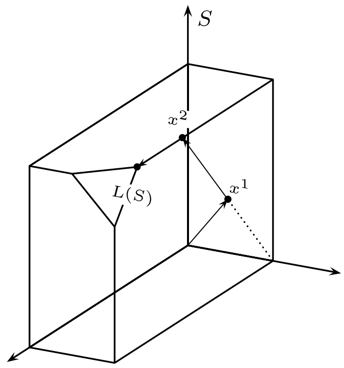

Fig. 1. A visual representation of L.

Note that Consistency was first introduced by Harsanyi (1959). The above formulation is due to Lensberg (1988).[5]

Consider again a situation in which a group of agents $Q \in \mathscr{P}$, when confronted with the problem $T \in \Sigma^Q$, can find unanimous agreement on the outcome $x = \varphi(T)$. However, prior to reaching this compromise, some of the agents – say, $P \subsetneq Q$ with $P \in \mathscr{P}$ – are, for whatever reason, prepared to relinquish all their claims to utility, and to leave the negotiations empty-handed. Then in this new situation, no rational agent $i \in Q \setminus P$ would be prepared to accept an outcome that gives him less than x_i. Thomson (1983) introduced an axiom that reflects precisely this idea.

Population monotonicity (MON): For all $Q \in \mathscr{P}$ with $P \subsetneq Q$, for all $T \in \Sigma^Q$, if $S = T_P$, then $\varphi(S) \geqq \varphi_P(T)$.

2.4. The Leximin solution

In order to define the Leximin solution, it is useful to first introduce the lexicographic maximin (in short, leximin) ordering from which it derives its name. The leximin ordering is a lexicographic extension of the maximin ordering: when comparing two outcomes x and y, it ranks highest the outcome that gives the worst-off agent the highest utility. If x and y are indistinguishable in terms of how they treat the worst-off, then it ranks highest the outcome that gives the second worst-off agent the highest utility. And so on. Note that this reflects the idea of Rawls (1971) that the worth of an allocation depends on its treatment of the less well-off (i.e. the Difference Principle). The Leximin solution embodies this principle in that it chooses from any problem $S \in \Sigma$ the unique outcome that is maximal with respect to the leximin ordering.

Given $P \in \mathscr{P}$ and $x \in \mathbb{R}^P_+$, let $\overline{P} \equiv \{1, \dots, |P|\}$, and define $\bar{x} \equiv \mu(x)$, where $\mu \in \Gamma^{P,\overline{P}}$ is a relabeling of the agents that puts the entries of x in ascending order, i.e., $\bar{x}_1 \leqq \cdots \leqq \bar{x}_{|P|}$. Then the **leximin ordering**, denoted $\succeq^l_P$, is the ordering of $\mathbb{R}^P_+$ such that for all $x, y \in \mathbb{R}^P_+$, $x \succ^l_P y$ iff either $\bar{x}_1 > \bar{y}_1$, or there is an $i \in \overline{P}$ such that $\bar{x}_i > \bar{y}_i$ and $\bar{x}_j = \bar{y}_j$ for all $j = 1, \dots, i-1$; and $x \sim^l_P y$ iff $\bar{x} = \bar{y}$. The family of all orderings $\succeq^l_P$, where $P \in \mathscr{P}$, is denoted $\succeq^l$. Given $S \in \Sigma^P$, the **Leximin solution** is defined as the unique maximizer in S w.r.t. $\succeq^l_P$, i.e., $L(S) \equiv \{x \in S \mid x \succeq^l_P y \text{ for all } y \in S\}$.

Given $P \in \mathscr{P}$ and $S \in \Sigma^P$, the Leximin solution $L(S)$ can be obtained through a specific iterative procedure. First let x_1 be the point in S that is obtained by maximally increasing agents' utilities in the egalitarian direction. That is, $x_1 \equiv \alpha^1 e_P$, and $\alpha^1 \equiv \max\{\alpha \mid \alpha e_P \in S\}$. There is a number of agents whose utilities could further be increased within the feasible set. Let P^1 be the set of those agents, that is, $P^1 \equiv \{i \in P \mid \exists x \in S \text{ s.t. } x \geqq x^1 \text{ and } x_i > x^1_i\}$. Fixing the utilities of agents in $P \setminus P^1$ at their x^1-levels, we may continue increasing those of agents in P^1 in the egalitarian direction, until again a point – say x^2 – is reached in the boundary, now of $t^{x^1}_{P^1}(S)$. More precisely, $x^2_{P\setminus P^1} \equiv x^1_{P\setminus P^1}$, and $x^2_{P^1} \equiv x^1_{P^1} + (\alpha^2 - \alpha^1)e_{P^1}$, where $\alpha^2 \equiv \max\{\alpha \mid x^1_{P^1} + (\alpha - \alpha^1)e_{P^1} \in t^{x^1}_{P^1}(S)\}$. The set P^2 contains all the agents in P^1 whose utilities could feasibly be increased beyond their x^2-levels, i.e., $P^2 \equiv \{i \in P \mid \exists x \in S \text{ s.t. } x \geqq x^2 \text{ and } x_i > x^2_i\}$. In general, at the kth iteration of this procedure, we set $x^k_{P\setminus P^{k-1}} \equiv x^{k-1}_{P\setminus P^{k-1}}$, and $x^k_{P^{k-1}} \equiv x^{k-1}_{P^{k-1}} + (\alpha^k - \alpha^{k-1})e_{P^{k-1}}$, where $\alpha^k \equiv \max\{\alpha \mid x^{k-1}_{P^{k-1}} + (\alpha - \alpha^{k-1})e_{P^{k-1}} \in t^{x^{k-1}}_{P^{k-1}}(S)\}$ and $P^k \equiv \{i \in P \mid \exists x \in S \text{ s.t. } x \geqq x^k \text{ and } x_i > x^{k-1}_i\}$. Since $P^k \subsetneq P^{k-1}$ for all k, and since P is finite, there is an iteration k^* at which P^{k^*} is empty, and the procedure terminates. The Leximin solution $L(S)$ is given by x^{k^*}, the point reached at this stage (see Fig. 1).

The Leximin solution is characterized as follows.

Proposition 2.1 (*Thomson and Lensberg, 1989*)**.** *A solution satisfies PO, IM, AN, and CONS if and only if it is the Leximin solution.*

2.5. Truncated Leximin solutions

We will be concerned with truncated versions of the Leximin solution. Given $P \in \mathscr{P}$, $x \in \mathbb{R}^P_+$, and a non-negative, possibly infinite α, define $x \wedge \alpha$ as the vector $y \in \mathbb{R}^P_+$ with $y_i \equiv \min\{\alpha, x_i\}$ for all $i \in P$. For $S \in \Sigma^P$, an α-**Truncated Leximin solution** is defined by $L^\alpha(S) \equiv L(S) \wedge \alpha$. The class $\mathcal{T}$ contains all Truncated Leximin solutions—i.e., $\mathcal{T} \equiv \{L^\alpha \mid \alpha \text{ non-negative and possibly infinite}\}$.

For $\alpha = \infty$ and for $\alpha = 0$, L^α respectively coincides with L and with the **Disagreement solution**—i.e., the solution D that for all $P \in \mathscr{P}$ and all $S \in \Sigma^P$ yields $D(S) = \mathbf{0}$. While the exact value differs from one problem to another, it is clear that there is always a strictly positive and finite α such that L^α coincides with the Egalitarian solution. More generally, for finite $\alpha > 0$, L^α is equivalently defined as the Leximin solution in the α-truncated version of the considered problem (see Fig. 2). For $S \subseteq \mathbb{R}^P_+$, define $S \wedge \alpha \equiv \{x \wedge \alpha \mid x \in S\}$, and note that if $S \in \Sigma^P$ and α is strictly positive and finite, then $S \wedge \alpha$ is a well-defined problem in Σ^P.

Observation 2.2. *For all $P \in \mathscr{P}$, for all $S \in \Sigma^P$, and for all finite $\alpha > 0$, $L^\alpha(S) = L(S \wedge \alpha)$.*

Proof. Let $P \in \mathscr{P}$ and $S, T \in \Sigma^P$ with $T \equiv S \wedge \alpha$ for some finite $\alpha > 0$ be given. Let $x \equiv L^\alpha(S)$, $y \equiv L(T)$ and $z \equiv L(S)$. Since $T \subseteq S$, it follows by the definition of L that $z \succeq^l_P y$. Furthermore, since $x = L(S) \wedge \alpha$ and $L(S) \in S$, $x \in T$; then by the definition of L, $y \succeq^l_P x$. Hence, by transitivity of $\succeq^l_P$,

$$z \succeq^l_P y \succeq^l_P x. \tag{1}$$

Let $Q \subseteq P$ be the set of agents $i \in P$ for whom $z_i < \alpha$. We argue that the first $|Q|$ coordinates of $\bar{x}$, $\bar{y}$ and $\bar{z}$ coincide. Since this is trivial if $Q = \emptyset$, assume $Q \neq \emptyset$. By the definition of L^α, $x_i = z_i < \alpha$ for all $i \in Q$, so the first $|Q|$ coordinates of $\bar{x}$ and $\bar{z}$ coincide. Assume, contrary to what we want, that there is an integer $k \leqq |Q|$ such that $\bar{y}_k \neq \bar{x}_k$, while $\bar{y}_{k'} = \bar{x}_{k'}$ for all $k' = 1, \dots, k-1$. Then either $\bar{y}_k > \bar{x}_k$ implying $y \succ^l_P z$, or $\bar{y}_k < \bar{x}_k$ implying $y \prec^l_P x$. Both are in contradiction with (1).

Since this concludes the proof for $Q = P$, assume $Q \subsetneq P$. By the definition of L^α, the last $|P \setminus Q|$ coordinates of $\bar{x}$ are all equal to α. Suppose there is an integer $k > |Q|$ such that $\bar{y}_k > \bar{x}_k$; then there is $i \in P$ such that $y_i > \alpha$, contradicting $y \in T$. Hence, $\bar{y} \leqq \bar{x}$, implying $x \succeq^l_P y$, and thus by (1), $x \sim^l_P y$. Since y is the *unique* maximizer in T w.r.t. $\succeq^l_P$, $x = y$, as desired. ■

[5] Lensberg (1988) referred to this property as *Multilateral Stability*.

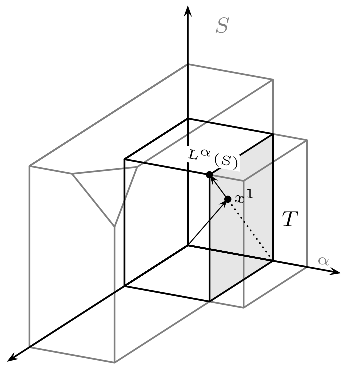

Fig. 2. $L^{\alpha}(S)$ when $\alpha > 0$ and finite.

It was argued above that the Leximin solution embodies the Difference Principle of Rawls (1971), in the sense that it prioritizes the utilities of those who are worse off over the utilities of those who are better off. In particular, the Leximin solution outcome is **Hammond equitable** (Hammond, 1976): no agent can be made better off at the expense of a better-off agent—i.e., for $P \in \mathcal{P}$ and $S \in \Sigma^P$, if $x = L(S)$, then there is no $y \in S$ such that $x_i < y_i < y_j < x_j$ for some $i, j \in P$, and $x_k = y_k$ for all $k \in P \setminus \{i, j\}$. Observation 2.2 reveals that this aspect of the Leximin solution is not driven by Pareto Optimality: Truncated Leximin solution outcomes too are Hammond equitable, and thus embody the Difference Principle.[6]

3. A characterization of $\mathcal{T}$

The aim of this paper is to characterize the class of Truncated Leximin solutions on the basis of IM, AN, and CONS. It is first proven in Proposition 3.1 that they indeed satisfy these axioms, then in Proposition 3.2 that they are the *only* such solutions.

Proposition 3.1. *Truncated Leximin solutions satisfy IM, AN, and CONS.*

Proof. In view of Proposition 2.1, it is easily demonstrated that Truncated Leximin solutions satisfy IM and AN. Furthermore, CONS is satisfied by D and L. To see that the latter is true for *all* Truncated Leximin solutions, let $P, Q \in \mathcal{P}$ with $P \subsetneq Q$, let $T \in \Sigma^Q$, and let $S \in \Sigma^P$ with $S \equiv t_P^y(T)$ and $y \equiv L^{\alpha}(T)$ for some finite $\alpha > 0$. By Observation 2.2, $L^{\alpha}(S) = L(S \wedge \alpha)$ and $y = L(T \wedge \alpha)$. Since L satisfies CONS, the latter implies $L(t_P^y(T \wedge \alpha)) = y_P$. Then since $S \wedge \alpha = t_P^y(T \wedge \alpha)$, $L^{\alpha}(S) = y_P$, as desired. ■

Proposition 3.2. *If a solution satisfies IM, AN, and CONS, then it is a Truncated Leximin solution.*

Throughout the rest of this paper, let φ be a solution that satisfies IM, AN, and CONS. To prove that then $\varphi \in \mathcal{T}$ we now first define a particular function f that maps the utopia values of single-agent problems into their associated solution outcomes. More precisely, given $i \in I$, let f^i be a function on $\mathbb{R}_+$ with $f^i(0) \equiv 0$ and $f^i(x) \equiv \varphi(S)$ where $S \in \Sigma^i$ and $a(S) = x$; since φ satisfies AN, there is a function f on $\mathbb{R}_+$ such that $f^i = f$ for all $i \in I$.

Lemma 3.3. *f satisfies the following properties:*

(i) *$0 \leqq f(x) \leqq x$ for all $x \in \mathbb{R}_+$;*

[6] Note that Truncated Leximin solutions do not yield the *only* Hammond equitable outcomes. However, for any $P \in \mathcal{P}$ and $S \in \Sigma^P$, if x is *not* a Truncated Leximin solution outcome of S, then there is a $y \in S$, an $i \in P$, and a $J \subseteq P \setminus i$, such that $x_j > y_j > y_i > x_i$ for all $j \in J$, while $x_k = y_k$ for all $k \in P \setminus (J \cup i)$.

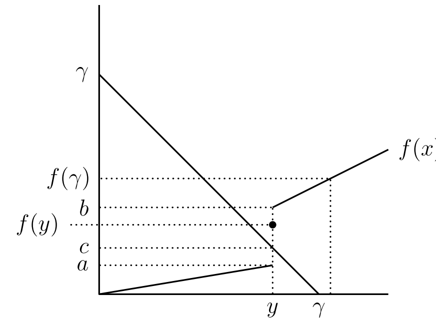

Fig. 3. There is no $x^* \in \mathbb{R}_+$ such that $f(x^*) = \gamma - x^*$.

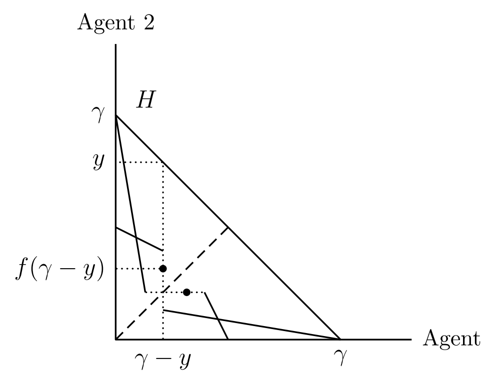

Fig. 4. If f is not continuous, AN is violated.

(ii) *f is non-decreasing;*

(iii) *f is continuous.*

Proof. Observation (i) follows directly from the definition of f. To see (ii), note that for single-agent problems S and T with $a(S) < a(T)$, $\varphi(S) \leqq \varphi(T)$ by IM. Furthermore, for $x > 0, f(x) \geqq 0 = f(0)$ by (i).

To establish (iii), note first that f is continuous at 0—i.e., for any $\epsilon > 0$, one can choose $\delta = \epsilon/2$ such that $|f(x) - f(0)| < \epsilon$ for all $x \in (0, \delta)$. Thus, assume f has a discontinuity at $y > 0$. Since f is non-decreasing by (ii), this is a jump discontinuity—i.e., there exists an open bounded interval $(a, b) \subseteq \mathbb{R}_+$ such that $f(x) \leqq a$ for all $x < y$ and $f(x) \geqq b$ for all $x > y$ (see e.g. Theorem 1 on p. 108 of Royden and Fitzpatrick, 2010). Consider $c \in (a, b)$ with $c \neq f(y)$, and define $\gamma \equiv c + y$. Then $x < y$ implies $f(x) \leqq a < c + (y - x) = \gamma - x$, and $x > y$ implies $f(x) \geqq b > c + (y - x) = \gamma - x$. Furthermore, if $x = y$, then by the choice of c, $f(x) \neq c + (y - x) = \gamma - x$. Hence, there is no $x^* \in \mathbb{R}_+$ such that $f(x^*) = \gamma - x^*$ (see Fig. 3).

Let $P \in \mathcal{P}$ with $|P| = 2$, and without loss of generality, assume $P = \{1, 2\}$. Furthermore, let $H \in \Sigma^P$ with $H = cch\{\gamma e_1, \gamma e_2\}$, and suppose that $z \equiv \varphi(H)$. Consider $S \in \Sigma^2$ with $S = t_2^z(H)$, and observe that $a(S) = \gamma - z_1$. Hence, by CONS, $z_2 = \varphi_2(H) = \varphi(S) = f(\gamma - z_1)$. By AN and symmetry of H, $z_1 = z_2$, and thus $z_1 = f(\gamma - z_1)$. Defining $x^* \equiv \gamma - z_1$, we obtain $\gamma - x^* = f(x^*)$, contradicting the above (see Fig. 4). ■

Lemma 3.4. $f(x) = \min\{\alpha, x\}$ *for some non-negative, possibly infinite α.*

Proof. Let $[a, b] \subseteq \mathbb{R}_+$ be a non-degenerate closed interval on which f is increasing. It is first demonstrated, by contradiction, that there exists no $y \in (a, b)$ such that f is differentiable in y and $f(y) < y$. Thus assume, contrary to what we want, that there exists such a $y \in (a, b)$. First note that since f is non-decreasing on $\mathbb{R}_+$ and increasing on $[a, b]$, $f(x) \geqq f(y)$ if and only if $x \geqq y$. Furthermore, since f is bounded on $[f(y), y + f(y)]$ and differentiable in y, there exists a finite $\lambda > 1$ such that

$$f(y) < f(y+h) < f(y) + \lambda h \quad \text{for all } h \in (0, f(y)], \tag{2}$$

$$f(y) > f(y+h) > f(y) + \lambda h \quad \text{for all } h \in [f(y) - y, 0), \tag{3}$$

$$f(y) \leqq \lambda(y - f(y)). \tag{4}$$

Consider $P \in \mathcal{P}$ with $|P| = 2$, and without loss of generality, let $P = \{1, 2\}$. Define $S, T \in \Sigma^P$ as

$$S \equiv cch\{(f(y), y), (y, f(y)), (y + \beta f(y), 0)\}$$
$$T \equiv cch\{(f(y), y), (y + \beta f(y), f(y))\}$$

where $\beta \equiv 1/\lambda^2$. Since $S \subseteq T$ and $S_{-i} = T_{-i}$ for $i \in P$, $\varphi(S) \leqq \varphi(T)$ by a two-fold application of IM. We will show next that $\varphi_2(S) = f(y) > \varphi_2(T)$, a contradiction.

Let $v \equiv \varphi(S)$, and assume first that $v_2 > f(y)$. By construction of S, this implies $a(t_1^v(S)) = y + f(y) - v_2 < y$. Then by CONS and the fact that $y \in (a, b)$, $v_1 = f(a(t_1^v(S))) < f(y)$. By construction of S, the latter implies $a(t_2^v(S)) = y$, and thus by CONS, $v_2 = f(y)$, a contradiction. Assume next that $v_2 < f(y)$. We make two claims:

Claim 1. $f(y) < v_1 < f(y) + \frac{1}{\lambda}(f(y) - v_2)$

Since $v_2 \in [0, f(y))$, it follows by construction of S that $a(t_1^v(S)) = y + \beta(f(y) - v_2)$. Then by CONS, $v_1 = f(y + \beta(f(y) - v_2))$. Note that $\beta(f(y) - v_2) \in (0, f(y)]$. Then by (2), $f(y) < v_1 < f(y) + \lambda\beta(f(y) - v_2) = f(y) + (1/\lambda)(f(y) - v_2)$.

Claim 2. $f(y) > v_2 > f(y) + \lambda(f(y) - v_1)$

Since $v_2 \geqq 0$, $f(y) < v_1 < f(y) + f(y)/\lambda$ by the previous argument. By (4), $f(y) + f(y)/\lambda \leqq y$; hence, $v_1 \in (f(y), y]$. Then by construction of S, $a(t_2^v(S)) = y + f(y) - v_1$, and thus by CONS, $v_2 = f(y + (f(y) - v_1))$. Note that $(f(y) - v_1) \in [f(y) - y, 0)$, so by (3), $f(y) > v_2 > f(y) + \lambda(f(y) - v_1)$.

Putting the two claims together yields

$$v_2 > f(y) + \lambda(f(y) - v_1)$$
$$> f(y) + \lambda\left(f(y) - f(y) - \frac{1}{\lambda}(f(y) - v_2)\right) = v_2,$$

a contradiction. It follows that $\varphi_2(S) = f(y)$ (see Fig. 5(a)).

Next consider T, and denote $w \equiv \varphi(T)$. Assume first that $w_1 \leqq f(y)$. Then $a(t_2^w(T)) = y$, and thus by CONS, $w_2 = f(y)$. But then $a(t_1^w(T)) = y + \beta f(y) > y$, which by CONS implies $w_1 > f(y)$, a contradiction. Hence, $w_1 > f(y)$, which by construction of T implies $a(t_2^w(T)) < y$, and thus by CONS, $w_2 < f(y)$. Hence, $\varphi_2(T) < f(y) = \varphi_2(S)$, the desired contradiction (see Fig. 5(b)).

Since f is increasing on (a, b), f is almost everywhere differentiable on (a, b) by Lebesgue's theorem (e.g., p. 112 of Royden and Fitzpatrick, 2010). Then by (i) of Lemma 3.3 and the above argument, $f(x) = x$ almost everywhere on (a, b). If $f(z) < z$ for some $z \in [a, b]$, then by continuity of f there is a $\delta > 0$ such that $f(x) < x$ for all x in the open interval $(z-\delta, z+\delta) \cap (a, b)$. Since this is in contradiction with the observation that $f(x) = x$ a.e. on (a, b), $f(x) = x$ for all $x \in [a, b]$. Since $f(x) = x$ on every closed non-degenerate interval $[a, b] \subseteq \mathbb{R}_+$ on which f is increasing, the lemma follows by continuity of f. ■

Note that Lemma 3.4 establishes Proposition 3.2 for single-agent problems. Since φ satisfies CONS, the value of α in the definition of f has general implications for which solution is obtained.

Proof of Proposition 3.2. Let $P \in \mathcal{P}$, $S \in \Sigma^P$, and $y \equiv \varphi(S)$, and for $i \in P$, let $T \in \Sigma^i$ with $T \equiv t_i^y(S)$. By Lemma 3.4, $y_i = \min\{\alpha, a(T)\}$, implying $y \leqq \alpha e_P$. Hence, if $\alpha = 0$, then feasibility of y implies $y = \mathbf{0} = D(S)$.

If α is infinite, then for all $i \in P$, $y_i = \max\{z_i \mid z \in S \text{ and } z \geqq y\}$, that is, $y \in PO(S)$. Hence, φ satisfies PO, and thus by Proposition 2.1, $\varphi = L$.

If $0 < \alpha < \infty$, then $y \in S \wedge \alpha$. Furthermore, for all $i \in P$, $y_i = \max\{z_i \mid z \in S \wedge \alpha \text{ and } z \geqq y\}$, so that $z \in PO(S \wedge \alpha)$. Then by some trivial modifications of the arguments of Thomson and Lensberg (1989, p. 133–138), $\varphi = L^\alpha$. ■

Combining Propositions 3.1 and 3.2 leads to the main result of this paper.

Theorem 3.5. *A solution satisfies IM, AN, and CONS, if and only if it is a Truncated Leximin solution.*

We conclude this Section with three corollaries of Theorem 3.5. First, adding HOM to the axioms of Theorem 3.5 excludes all solutions from $\mathcal{T}$, except for the Leximin solution and the Disagreement solution.[7]

Corollary 3.6. *A solution satisfies HOM, IM, AN, and CONS, if and only if it is either the Disagreement solution or the Leximin solution.*

Proof. First note that both the Disagreement solution and the Leximin solution satisfy HOM, IM, AN, and CONS.

Let φ be a solution satisfying these axioms. Then by Theorem 3.5, there is a non-negative and possibly infinite α such that φ is the α-Truncated Leximin solution L^α. Since $L^0 = D$ and $L^\infty = L$, it is sufficient to show that L^α violates HOM in case $0 < \alpha < \infty$. To this end, consider $i \in I$ and $S \in \Sigma^i$, and let $\lambda \in \mathbb{R}_{++}$. Since L satisfies HOM, $L^\alpha(\lambda S) = \min\{\alpha, L(\lambda S)\} = \min\{\alpha, \lambda L(S)\}$. Furthermore, $\lambda L^\alpha(S) = \min\{\lambda\alpha, \lambda L(S)\}$. If $0 < \alpha < \infty$, then by choosing $\lambda > 1$ when $\alpha \leqq L(S)$, or $\lambda > \alpha/L(S)$ when $\alpha > L(S)$, we obtain $\lambda L^\alpha(S) = \min\{\lambda\alpha, \lambda L(S)\} > \min\{\alpha, \lambda L(S)\} = L^\alpha(\lambda S)$. ■

The fact that the family of Truncated Leximin solutions is rather large, and that the inefficient solutions it contains are not obviously implausible means that based on Thomson and Lensberg's characterization result, the Leximin solution may not be appropriate when efficiency of the outcome of negotiations is not guaranteed. However, since it "essentially" characterizes the Leximin solution without relying on PO, Corollary 3.6 to some extent addresses this concern: it seems questionable that rational, self-interested agents would unanimously agree on the worst possible outcome in any situation, so provided that HOM and the other axioms of Thomson and Lensberg are unproblematic, then even in environments where efficiency is in doubt, the Leximin solution outcome is the only plausible outcome of the negotiations. Implausibility of the Disagreement solution is represented by the axiom SIR, so adding it to the axioms of Corollary 3.6 leads to an alternative efficiency-free characterization of the Leximin solution.

Corollary 3.7. *A solution satisfies SIR, HOM, IM, AN, and CONS, if and only if it is the Leximin solution.*

Throughout the paper it is assumed that utilities are cardinally measurable and comparable across agents. There are compelling arguments in favor of this assumption, but as stated above, this view is not uncontested. For those who see SI as a minimal requirement on descriptively interpreted bargaining solutions, Theorem 3.5 is best framed as the precursor of an impossibility result: there is no scale invariant solution that satisfies the axioms IM, AN and CONS, other than the Disagreement solution.

Corollary 3.8. *A solution satisfies SI, IM, AN, and CONS, if and only if it is the Disagreement solution.*

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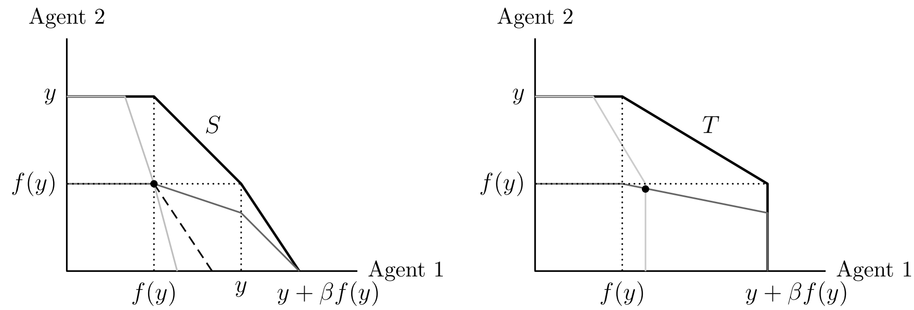
[7] Note that this also follows from Proposition 6 of Lensberg and Thomson (1988) and the characterization of L. In particular, Lensberg and Thomson (1988) show that a solution satisfying HOM, AN and CONS must either be the Disagreement solution, or satisfy PO; if IM is added to these axioms, then the solution is either the Disagreement solution, or it satisfies the characterizing axioms of the Leximin solution.



(a) The problem S.

(b) The problem T.

Fig. 5. A violation of IM.

4. Related literature

4.1. Truncated Egalitarian solutions

Given $P \in \mathscr{P}$ and $S \in \Sigma^P$, the **Egalitarian solution** is defined as $E(S) \equiv \alpha^* e_P$ where α^* is the maximal α such that $\alpha e_P \in S$. The Egalitarian solution is characterized as follows.

Proposition 4.1 (*Thomson (1983)*). *A solution satisfies WPO, SY, IIA, MON, and CONT if and only if it is the Egalitarian solution.*

Thomson (1984) defined a class of solutions akin to those considered in this paper. Given a list $\alpha \equiv \{\alpha^P \mid P \in \mathscr{P}\}$ of non-negative and possibly infinite real numbers with the property that $P \subsetneq Q$ implies $\alpha^P \geqq \alpha^Q$, the associated **Truncated Egalitarian solution** E^α is defined as follows: for all $P \in \mathscr{P}$ and $S \in \Sigma^P$, $E^\alpha(S) \equiv E(S) \wedge \alpha^P$. It is clear that these solutions satisfy all the axioms of Proposition 4.1 except for WPO. However, there are other such solutions, i.e., the axioms SY, IIA, MON, and CONT also admit solutions that allow for limited substitutability of utility in the case of two-player problems.

Proposition 4.2 (*Thomson (1984)*). *A solution φ satisfies SY, IIA, MON, and CONT if and only if*

(i) *it coincides with a Truncated Egalitarian solution E^α except perhaps when $|P| = 2$ on the subclass Σ^P_α of problems S such that $\alpha^P e_P \geqq E(S) \geqq \overline{\alpha} e_P$ where $\overline{\alpha} \equiv \sup\{\alpha^{P\cup i} \mid i \in I \setminus P\}$;*
(ii) *for each $P \in \mathscr{P}$ with $|P| = 2$, it coincides on Σ^P_α with a solution $\tilde{\varphi}$ satisfying SY, IIA, and CONT such that $\tilde{\varphi}(S) \geqq \overline{\alpha} e_P$ for all $S \in \Sigma^P_\alpha$; and*
(iii) *for all $i \in I$, $\alpha^i \geqq \sup\{\varphi_i(S) \mid P = \{i, j\}, j \in I \setminus i, S \in \Sigma^P\}$.*

In contrast, removing PO from Thomson and Lensberg's (1989) axiom set does not give rise to any other solutions than the Truncated Leximin solutions. An explanation for this difference is already partially provided by Thomson and Lensberg (1989, p. 66) in their discussion of Proposition 4.2: any utility substitution in the two-player case would be ruled out as soon as for all $P \in \mathscr{P}$ with $|P| = 2$, $\alpha^P = \overline{\alpha}$. This feature arises naturally from the axioms of Theorem 3.5: since they include CONS, the utility level α at which the solution outcome is truncated in single-agent problems, must also be the cut-off value in problems that involve multiple agents.

A further implication of Proposition 4.2 is that the role of WPO in Thomson's (1983) characterization of the Egalitarian solution is larger than that of PO in Thomson and Lensberg's (1989) characterization of the Leximin solution. However, a result similar to Corollary 3.7 can be obtained: when SIR and HOM are added to the axioms of Proposition 4.2, then the Egalitarian solution is the only one that remains admissible (See Thomson, 1984, p. 31).

4.2. Solutions of a proportional character

On the basis of Kalai's (1977) Proportional solutions, Roth (1979b) defined a class of solutions similar to Truncated Egalitarian solutions, and thus in the domain $\tilde{\Sigma}$ of strictly comprehensive problems (i.e. problems $S \in \Sigma$ with $WPO(S) = PO(S)$), also to Truncated Leximin solutions. Given $P \in \mathscr{P}, S \in \Sigma^P$ and $q \in \mathbb{R}^P_{++}$, a q-**Proportional solution** is given by the maximal point in S that is proportional to q, i.e. $F^q(S) \equiv \alpha^* q$ where $\alpha^* \equiv \max\{\alpha \mid \alpha q \in S\}$; the family of all such solutions, that is, for all $P \in \mathscr{P}$ and $q \in \mathbb{R}^P_{++}$, is referred to as the family of **Proportional solutions**. A **solution of a proportional character** is obtained by rescaling a Proportional solution outcome by a possibly problem-dependent factor $\beta(S) \in [0, 1]$. Both Truncated Egalitarian solutions and solutions of a proportional character violate WPO. However, as pointed out by Thomson (1984) there is a marked difference between the two in *how* this axiom is violated: Roth's solutions always violate it when $\beta(S) < 1$. This violation stands, irrespective of any inflation of the problem, i.e., if S is inflated by a factor $\lambda > 0$ then WPO remains violated as long as $\beta(\lambda S) < 1$. On the other hand, a Truncated Egalitarian solution outcome coincides with the symmetric Proportional (i.e., the Egalitarian) solution when the problem is small enough, but as the problem is inflated, the solution outcome is capped at a given utility level.

Similar to Roth's modification of the Proportional solutions, one may also define proportional versions of the Leximin solution. Specifically, given a (constant) $\beta \in [0, 1]$, $P \in \mathscr{P}$, and $S \in \Sigma^P$, the β-**Proportional Leximin solution** PL^β is defined as $PL^\beta \equiv \beta L(S)$. Clearly, such solutions violate PO (at least for $\beta < 1$). However, while satisfying AN and IM, they also violate CONS, as illustrated by the following example.

Let $P \in \mathscr{P}$ with $|P| = 2$, and without loss of generality assume $P = \{1, 2\}$. Define $H \equiv cch\{e_1, e_2\}$, and note that $H \in \Sigma^P$. Then $z \equiv PL^{1/2}(H) = (1/4, 1/4)$. However, since $a(t^z_2(H)) = 3/4$, $TL^{1/2}(t^z_2(H)) = 3/8 \neq z_2$.

4.3. The Nash solution

For all $P \in \mathscr{P}$ and for all $S \in \Sigma^P$, the **Nash solution** (Nash, 1950) is defined as the unique outcome that maximizes the product of the agents' utilities, i.e. $N(S) \equiv \arg\max_{z \in S} \prod_{i \in P} z_i$. Following are two prominent characterizations.

Proposition 4.3. *The Nash solution is the only solution that satisfies*

(i) *PO, SY, SI, and IIA (Nash, 1950);*
(ii) *PO, AN, SI, and CONS (Lensberg, 1988).*

Both characterizations rely on PO, which, as argued above, may not be appropriate when bargaining problems are given a descriptive interpretation. Roth (1979a) therefore studied its role in Nash's characterization, and Lensberg and Thomson (1988) studied it in Lensberg's. They reached a similar answer.

Proposition 4.4. *The Nash solution and the Disagreement solution are the only solutions that satisfy*

(i) *SY, SI, and IIA (Roth, 1979a);*
(ii) *AN, SI, and CONS (Lensberg and Thomson, 1988).*

The results of Proposition 4.4 are similar to Corollary 3.7: when omitting PO from Nash's and Lensberg's respective axiom sets, the only inefficient solution that emerges is the Disagreement solution. The reason why this robustness result is in this case obtained without explicitly adding HOM is that HOM is implied by SI. Similar to Corollary 3.7, adding SIR to the axiom sets of Proposition 4.4 eliminates the Disagreement solution, and thus leads to two efficiency-free characterizations of the Nash solution.

5. Concluding remarks

Although PO is not free of criticism, the purpose of this paper is not to posit Truncated Leximin solutions as preferable alternatives to the Leximin solution, or other solutions in the literature. Indeed, the position that Truncated Leximin solutions have some advantage over other known solutions would be hard to defend. The main aim is rather to investigate to what extent PO contributes to Thomson and Lensberg's (1989) characterization of the Leximin solution. In the first place, it is revealed that removing PO from Thomson and Lensberg's axiom set does not compromise the distinguishing feature of the Leximin solution that the utilities of those who are worse off are given priority over the utilities of those who are best off. More precisely, even without PO, the remaining axioms ensure that any feasible outcome x is disqualified as a potential agreement if it features an inequality – i.e. agents i and j with $x_j > x_i$ – that could feasibly be resolved or diminished by reducing x_j and increasing x_i (Hammond, 1976). A second, more negative conclusion is that PO plays an important role in this characterization, in the sense that it rules out a large number of non-trivial inefficient solutions. This means that short of alternative efficiency-free characterizations, the Leximin solution may not be entirely appropriate when the bargaining problem represents a situation in which efficiency cannot be guaranteed.[8] Fortunately, Corollary 3.8 does provide such an alternative characterization. In particular, if instead of *removing* PO from Thomson and Lensberg's axiom set, it is replaced with SIR and HOM, then the only admissible solution is the Leximin solution.

The immediate question that then arises is whether SIR and HOM are really uncontroversial. Arguably, this is the case for SIR. After all, in which situations would the behavior of rational bargainers be accurately represented by the Disagreement solution? HOM on the other hand, could be more problematic. As stated above, the proper interpretation of utilities in our environment is that they are cardinally measurable and comparable across agents; then HOM essentially says that agents always behave in the same way, regardless of how much is at stake. Much like PO, this is a requirement that might not hold in all situations. The question whether there are unproblematic axioms that could replace PO in Thomson and Lensberg's axiom set while still characterizing L, can be answered by careful examination of our results. In particular, any such axiom would only have to rule out the existence of non-trivial intervals on which f is constant. The following would be one example.

Responsiveness to Non-trivial Inflation (RNI): For all $P \in \mathcal{P}$, for all $\lambda > 0$, and for all $S, T \in \Sigma^P$ with $T = \lambda S$, if $\varphi(S) = \varphi(T)$ then $\lambda = 1$.

RNI says that if a bargaining problem is non-trivially inflated – i.e. inflated by a factor λ different from one – then this should have an effect on the solution outcome. In other words, an inflation of the problem changes the bargaining attitudes of the agents. While to the best of my knowledge, this axiom has not appeared in the literature, it is worth pointing out that it fits into the class of axioms that Thomson (2010) refers to as **relational axioms pertaining to the feasible set**. Furthermore, RNI is implied both by PO, and by the combination of SIR and HOM.

While some may impugn the plausibility of HOM as a descriptive axiom, there may be others who are equally troubled by its absence. In particular, those who do not believe in interpersonal utility comparisons see the stronger axiom SI as a minimal requirement for bargaining solutions, and the introduction of HOM as a partial remedy against this omission of SI. Since the Leximin solution does not satisfy SI, those who hold this view may interpret Theorem 3.5 as a negative result: The only scale invariant solution that satisfies IM, AN, and CONS, is the (implausible) Disagreement solution (see Corollary 3.8).

We will conclude with a remark on *how* Truncated Leximin solutions violate HOM. As mentioned above, HOM could be interpreted as a type of time-consistency: if bargainers exponentially discount utilities with factor $\delta \in (0, 1)$, and $S \in \Sigma$ is attainable at time $t + 1$, then bargaining at time t over the utilities feasible at time $t + 1$ (i.e., $\varphi(\delta S)$) yields the same time-t outcome as bargaining at time $t + 1$ and discounting the solution outcome (i.e., $\delta\varphi(S)$). Suppose now that agents bargain with some reference point in mind, some utility level they expect or aspire to realize by engaging in the bargaining process. If their time-$(t + 1)$ reference point is α, then the corresponding time-t reference point is $\delta\alpha$. Then we may still require that bargaining at time t over the utilities feasible at time $t + 1$ (i.e., $\varphi(\delta S, \delta\alpha)$) yields the same time-$t$ outcome as bargaining at time $t + 1$ and discounting back (i.e., $\delta\varphi(S, \alpha)$).[9] While Truncated Leximin solutions violate HOM, they do satisfy this type of time-consistency. In particular, $L^{\delta\alpha}(\delta S) = \delta L^{\alpha}(S)$. Note that for L^{α} to satisfy HOM, it is required that $\alpha = \delta\alpha$, a condition that only holds if either $\alpha = 0$ or $\alpha = \infty$, i.e. respectively the conditions under which $L^{\alpha} = D$ and $L^{\alpha} = L$.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.mathsocsci.2016.07.003.

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[8] Note that the characterizations of Imai (1983), Chun and Peters (1989), Nieto (1992), and of Chen (2000) all make use of PO.

[9] Li (2007) considers a bargaining protocol in which agents have reference-dependent preferences, and in which reference points are updated in this way. In particular, his *strong assumption of history-dependent preferences* says that if an agent who discounts utility with factor $\delta \in (0, 1)$ expects to get α from the game at time t, then on any SPE path in which the state variable does not change, he must expect to get α/δ from the game at time $t + 1$.

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